Wonderful Renormalization

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Abstract

The so-called wonderful models of subspace arrangements, developed in [CP95] based on Fulton and MacPherson's seminal paper [FM94], serve as a systematic way to resolve the singularities of Feynman distributions and in this way allow to define canonical renormalization operators. In this thesis we continue the work of [BBK10] where wonderful models were introduced to solve the renormalization problem in position space. In contrast to the exposition there, instead of the subspaces in the arrangement of divergent loci we use the poset of divergent subgraphs of a given Feynman graph as the main tool to describe the wonderful construction and the renormalization operators. This is based on [Fei05] where wonderful models were studied from a purely combinatorial viewpoint. The main motivation for this approach is the fact that both, the renormalization process and the model construction, are governed by the combinatorics of this poset. Not only simplifies this the exposition considerably, but it also allows to study the renormalization operators in more detail. Moreover, we explore the renormalization group in this setting, i.e. we study how the renormalized distributions change if one varies the renormalization points. We show that a so-called finite renormalization is expressed as a sum of distributions determined by divergent subgraphs. The bottom line is that - as is well known, at the latest since the discovery of a Hopf algebra structure underlying renormalization - the whole process of perturbative renormalization is governed by the combinatorics of Feynman graphs while the calculus involved plays only a supporting role.

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Contents

Chapter 1

Introduction

Quantum field theory (QFT), the unification of quantum mechanics and special relativity, is the last century's most successful physical theory. Although plagued with infinities and ill-defined quantities all over the place, it is in astonishing agreement with data obtained from particle physics experiments. At first this seems very awkward since up until now no one has been able to prove the existence of interacting quantum fields satisfying a reasonable set of physical axioms. Instead, these "real" fields are approximated as perturbations of their free, non-interacting counterparts. Although in this perturbative world a lot of mathematical monsters hide, the theory predicts the outcome of particle physics experiments with great accuracy. The art of taming these monster, i.e. treating the infinities arising in perturbative calculations, is called renormalization. Over the years it has turned from a "black magic cooking recipe" into a well-established and rigorous formulated theory, at the latest since the 90's when Kreimer discovered a Hopf algebra structure underlying renormalization. The main implication is that (perturbative) QFT is governed by the combinatorics of Feynman diagrams. This has proven to be a very powerful tool, both in computational problems as well as in improving our understanding of QFT in general. In addition, it has revealed surprising connections to deep questions in pure mathematics, for example in number theory and algebraic geometry. For more on the Hopf algebraic formulation of renormalization and its connection to other fields we refer to the exposition in [Kre13].

The mathematical reason for divergences arising in perturbative calculations is that quantum fields are modeled by operator-valued distributions for which products are in general not well-defined. In the position space formulation of QFT renormalization translates directly into the problem of extending distributions as shown by Epstein and Glaser in [EG73]. Although they formulated and solved the renormalization problem already in the early 70's, since then no real progress has been made in this direction. This is mainly due to two reasons: Firstly, their approach was mathematical precise

but conceptually difficult. It involved a lot of functional analysis, in some sense disguising the beauty and simplicity of the idea. Secondly, it is not applicable to calculations at all. Only recently, in a first approximation to quantum gravity, physicists have started to study quantum fields on general spacetimes and in this setting one is naturally forced to work in position space [BF00].

In [BBK10] another, more geometric approach to this problem was presented. In position space the Feynman rules associate to a graph G a pair (X^G, v_G) where X^G is a product of the underlying spacetime and $v_G : X^G \to \mathbb{R}$ a rational function. One would like to evaluate this to

$$(X^G, v_G) = \int_{X^G} v_G,$$

but this fails in general as the integrand need not be an element of $L^1(X^G)$. If v_G does not vanish fast enough at infinity this is called an infrared divergence. The problem is circumvented by viewing v_G as a distribution on the space of compactly supported test functions. On the other hand, ultraviolet divergences arise from v_G having poles along certain subspaces of X^G. These subspaces are determined by $\mathcal{D}$, the set of (ultraviolet-)divergent subgraphs of G, and form the *divergent arrangement* $X_{\mathcal{D}}^G$. In this setting renormalization translates into the problem of finding an extension of v_G onto $X_{\mathcal{D}}^G$. In [BBK10] this is solved with a geometric ansatz. The idea is to resolve the divergent arrangement into a normal crossing divisor. Such a model, also called a compactification of the complement of $X_{\mathcal{D}}^G$, is provided by the *wonderful model* construction of DeConcini and Procesi [CP95]. It allows to define canonical renormalization operators $mathscr R$ that extend v_G to a distribution defined on the whole space X^G. The method of the DeConcini-Procesi construction is based on the compactification of configuration spaces introduced by Fulton and MacPherson in their seminal paper [FM94]. What makes it so well suited for renormalization is that the whole construction is governed by the combinatorics of the arrangement which translates directly into the subgraph structure of G.

The idea of employing a resolution of singularities to extend distributions is not new. It is based on a paper by Atiyah [Ati70] that highlighted the usefulness of Hironaka's famous theorem for other areas of mathematics. In addition, the same technique was applied in Chern-Simons perturbation theory independently by Kontsevich [Kon94] as well as Axelrod and Singer [AS94]. For an application of this idea to renormalization in parametric space see [BEK06].

This thesis aims to continue the work of [BBK10] emphasizing a slightly different point of view. We use another language to formulate the wonderful construction and the renormalization process. Instead of the subspaces in the divergent arrangement we express the central notions in terms of the

poset $\mathcal{D}$, formed by all divergent subgraphs of G, partially ordered by inclusion. This is inspired by [Fei05] where the wonderful model construction is studied from a combinatorial point of view. Not only does this simplify the definitions and proofs immensely, it also highlights the combinatorial flavour in the construction of both, the wonderful models and the renormalization operators. In addition, instead of the vertex set of G we use *adapted* spanning trees t to define coordinates on X^G, naturally suited to the problem. This is also mentioned in [BBK10], but not used to its full extent. The main point is that such spanning trees are stable under graph theoretic operations like contraction of subgraphs and therefore provide a convenient tool to formulate the wonderful construction. It allows to treat the definition of the renormalization operators in more detail ([BBK10] focuses mainly on the model construction for arrangements coming from graphs) and to study the renormalization group, a powerful tool (not only) in QFT, that allows even for statements beyond perturbation theory. The main result is a formula for the change of renormalization points, the parameters involved in defining the renormalization operators. It relates a so-called finite renormalization of the renormalized distribution $\mathscr{R}[v_G]$ to a sum of distributions determined by the divergent subgraphs of G.

The presentation is organized as follows. In the next chapter we start with a short introduction to QFT and renormalization, mainly aimed at mathematicians, just to introduce the playing field and motivate the problem we are considering. It finishes with an account of renormalization à la Epstein-Glaser.

The next three chapters are devoted to the central objects in this thesis, distributions, smooth models and posets. Chapter 3 gives a short overview of the theory of distributions on manifolds and shows how extension of distributions works in a toy model case. It finishes with a definition of Feynman rules, i.e. how QFT associates distributions to Feynman diagrams, and an analysis of the divergent loci of these distributions. Chapter 4 starts with a general introduction to smooth models. We continue then with the special case of models for linear arrangements and review the wonderful construction of DeConcini and Procesi. Chapter 5 is of purely combinatorial flavour. We introduce the necessary language and discuss wonderful models from a combinatorial viewpoint, emphasizing the special case of arrangements coming from graphs. The whole wonderful construction is reformulated in terms of the poset $\mathcal{D}$.

After these mostly preliminary steps we come to the main part, the definition of the wonderful renormalization process. We first study the pole structure of the pull-back of a Feynman distribution onto an associated wonderful model and then define two renormalization operators. This definition requires some choices to be made and a natural question, considered in Chapter 7, is to ask what happens if one varies these parameters. We

derive and proof a formula for these so-called finite renormalizations.

The last chapter finishes the wonderful renormalization process by showing that it is physical reasonable, i.e. it satisfies the Epstein-Glaser recursion principle, in other contexts known as locality of counterterms. After that we discuss the connection between the renormalization operation for single graphs presented in this thesis and the Epstein-Glaser method. We finish with an outlook to further studies: The treatment of amplitudes and the role of the Fulton-MacPherson compactifications in this setting, and the Hopf algebraic formulation of wonderful renormalization.

Chapter 2

Quantum Field Theory

This chapter tries to give an introduction to QFT in a somewhat mathematical precise manner. This is actually impossible, especially in such a short volume, but we try to sketch the main notions and mention the conceptual difficulties along the way. It is not needed for understanding the theory of wonderful renormalization, but should serve as a motivation why we are considering an extension problem for distributions of this special kind and what graphs have to do with it. The reader familiar with QFT can easily skip to the last section that summarizes the ideas of the Epstein-Glaser approach.

2.1   A *very* short introduction to QFT

What is a theory of quantum fields? It is a relativistic field theory compatible with the principles of quantum mechanics. We start with the definition given in most physics textbooks, then introduce the axiomatic approach due to Wightman. For an extensive exposition we refer the interested reader to [BS80] or [Reb12].

   To combine the features of both quantum mechanics and special relativity means that we are looking for a theory built on a Hilbert space H with states represented by unit rays and observables by operators, respecting the symmetry group of special relativity, the *Poincaré group*

$$\mathcal{P} = \mathbb{R}^{1,3} \rtimes SL(2, \mathbb{C}).$$

Here $SL(2, \mathbb{C})$ enters as the universal cover of $O^+(1,3)$, the component of the Minkowski spacetime's orthogonal group $O(1,3)$, that is connected to the identity - also known as the group of *proper orthochronous Lorentz transformations*. By Wigner's theorem [Wig59] we think of a particle as an irreducible representation of $\mathcal{P}$ on H. This leads to the classification of particles in terms of their mass m and spin s, or helicity h in the massless case. In this thesis we will consider only scalar particles ($s = h = 0$), so we continue focusing on this special case. It suffices to understand the general concept.

6

From the fundamental equation $E = mc^2$ it follows that in a relativistic system the number of particles is not conserved. To model this, H is replaced with the *Fock space*

$$\mathcal{H} := \bigoplus_{k=0}^{\infty} S(H^{\otimes k}),$$

the direct sum of k-particle Hilbert spaces (symmetrized because we are considering bosonic particles here - $s = 0$ - for fermions all Hilbert spaces have to be antisymmetrized according to the Spin-Statistics theorem [SW00]), completed with respect to the norm

$$||\Psi||^2 := \sum_{k=0}^{\infty} ||\Psi_k||^2_{S(H^{\otimes k})}.$$

There are two important operators on $\mathcal{H}$, A^+ and A^-, responsible for creating and annihilating particles. They enter the theory when one quantizes the classical fields. For scalar particles the classical equation of motion is given by the *Klein-Gordon equation*

$$(\Box + m^2)\phi = 0.$$

It is the Euler-Lagrange equation for the Lagrangian density

$$\mathcal{L}(\phi, \partial\phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2.$$

As is known from the theory of partial differential equations, at this point we have to leave the world of smooth functions to search for solutions. Therefore this equation is to be understood in the sense of distributions. By Fourier analysis methods we obtain the general solution (ignoring constant prefactors)

$$\phi(x) = \phi^+(x) + \phi^-(x),$$

$$\phi^{\pm}(x) = \int d\mu(k)\, e^{\pm ipx} A^{\pm}(p).$$

Here the measure $d\mu(p) := dp\, \theta(p^0)\delta(p^2 - m^2)$ fixes the p^0-component to $p^0 = \omega(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$ according to Einstein's famous equation.

Quantization then introduces the bosonic commutation relations on the Fourier coefficients $A^-(p)$ and $A^+(p)$, now viewed as operators on $\mathcal{H}$:

$$[A^{\pm}(p), A^{\pm}(p')] = 0$$
$$[A^-(p), A^+(p')] = \delta(\vec{p} - \vec{p}').$$

The Hamiltonian of the system is given by

$$H = \frac{1}{2}\int d\mu(p) : A^-(p)A^+(p) + A^+(p)A^-(p) := \int d\mu(p)\, A^+(p)A^-(p),$$

an infinite sum (integral) of Harmonic oscillators with infinitely many degrees of freedom. This leads to the interpretation of A^+ and A^- as creation and annihilation operators and the existence of a ground state, the *vacuum* $|0\rangle$. It is defined by $A^-(p)|0\rangle = 0$ for all momenta p.

In the definition of H a *normal ordering* operation $: \cdot :$ has been introduced to get rid of the physical problem of a ground state having infinite energy. Normal ordering is defined by moving all creation operators to the left and annihilation operators to the right. There is another important ordering operation, given by the *time ordering* operator T,

$$T\big(A(x)B(y)\big) := \begin{cases} A(x)B(y) & \text{if } y^0 < x^0, \\ B(y)A(x) & \text{if } x^0 < y^0. \end{cases}$$

With T we can define the *Feynman propagator* $\triangle(x-y) := \langle 0|T\big(\phi(x)\phi(y)\big)|0\rangle$. It is a Green's function for the Klein-Gordon operator,

$$(\Box + m^2)\triangle(x - y) = \delta(x - y),$$

and interpreted as the probability amplitude for a particle to travel between two points of spacetime. In principle, this finishes the discussion of the free field, but we need to mention one more important property of $\triangle$, the equation

$$T\big(\phi(x)\phi(y)\big) =: \phi(x)\phi(y) : + \triangle(x - y). \tag{2.1}$$

A generalization of this for any collection of fields is known as *Wick's theorem*. It relates time and normal ordering of a product of fields to a sum over all possible contractions of the form (2.1). This theorem is essential for the introduction of Feynman diagrams in perturbation theory.

Before we turn to this, let us take a look at the axiomatic formulation of (scalar) QFT. It is based on the *Wightman axioms* [SW00]:

1. A *quantum field theory* is a tuple $(\mathcal{H}, U, \Omega, \phi)$. $\mathcal{H}$ is a separable Hilbert space whose unit rays represent the states of the theory. Relativity is encoded by U, a continuous unitary representation of the Poincarè group on $\mathcal{H}$,

$$(a, A) \mapsto U(a, A).$$

From unitarity it follows that $U(a, \text{id}) = \exp(iP^\mu a_\mu)$ where P is an unbounded hermitian operator, the *energy momentum operator* of the theory. The eigenvalues of P lie in or on the forward light cone and the mass of the theory is given by $m^2 = P^\mu P_\mu$. In $\mathcal{H}$ there is an U-invariant, cyclic state Ω, the *vacuum*. It is unique up to a phase factor.

2. Let $\mathcal{E}(M)$ denote the space of Schwartz functions on Minkowski space. A *quantum field* is characterized by

(a) domain and continuity of the field: For each test function $f \in \mathcal{E}(M)$ there is an operator $\phi(f)$ (in the scalar case self-adjoint). It is an endomorphism on a dense subset D of $\mathcal{H}$. The set D contains the vacuum and is invariant under U. Moreover, given $\Psi, \Phi \in D$, then

$$f \mapsto (\Phi \mid \phi(f)\Psi) \qquad (2.2)$$

defines a tempered distribution on $\mathcal{E}(M)$.

(b) transformation law: The field respects the transformation law

$$U(a, A)\phi(f)U(a, A)^{-1} = \phi\big((a, A).f\big)$$

where $(a, A).f(x) := f\big(A^{-1}(x - a)\big)$.

(c) causality: Let f and g be two Schwartz functions with space-like separated support, i.e. $f(x)g(y) \neq 0$ for $(x - y)^2 < 0$. Then the following commutation relations hold,

$$[\phi(f), \phi(g)] = 0.$$

There are more technical requirements, some equivalent to the stated axioms, some are additional and needed to formulate scattering theory, but the previous definitions basically characterize a (free) quantum field theory. Note that we have two examples at hand, trivial fields and the free field constructed above. Up to now there is no example of an interacting field that satisfies these axioms (in dimension four). One of the Millenium Price Problems is to show that there exists a gauge theory satisfying the Wightman axioms, or an appropriate version of similar axioms.

There are "equivalent" axioms for quantum field theory on Euclidean space, formulated by Osterwalder and Schrader. "Equivalent" here means that a theory satisfying one set of axioms can be transformed into a theory satisfying the other set. The difficult part in this change of background metric lies in translating axiom one in the definition of the fields, i.e. transforming the *Wightman functions* defined by (2.2) into their Euclidean counterparts, the *Schwinger functions*. The point here is that distributions are related to boundary values of holomorphic functions via the Laplace transform. For more on this we refer to [SW00] and [Reb12].

2.2 Perturbation Theory and Renormalization

So far we have considered a quantum theory of free fields. The trouble starts when one tries to model interactions between particles, i.e. adds nonlinear terms to the equation of motion. For example, the interaction of $n \geq 3$ scalar particles is modeled by the equation

$$(\Box + m^2)\phi = n\lambda\phi^{n-1},$$

where λ is a constant, the coupling strength. Since there is no general theory of solving equations of this type, the ansatz is to construct solutions as perturbative expansion in the coupling λ. Although no one has been able to prove the existence of interacting quantum fields, perturbation theory around the free fields has been immensely successful - it is the most precise physical theory up to date - despite many conceptual and technical difficulties.

Let us sketch the formulation of perturbation theory by defining a scattering operator, the *S-matrix*. Let α, β be two states of the free field, α representing a configuration before interaction takes place (the "in-field"), whereas β represents the final state ("out-field"). A scattering process is then modeled by assuming that in the far away past and future the state of the system is free, i.e. there is no interaction present. Both states live in $\mathcal{H}$, the Fock space of the free field. Therefore, scattering theory predicts the existence of an unitary operator S mapping the in-state to the out-state (at this point lurks a conceptual problem of scattering theory, known as Haag's theorem; for more on this we refer to [Haa55]). The probability of this process is given by the square of the absolute value of the *amplitude*

$$\langle \beta | S\alpha \rangle.$$

In perturbation theory we assume that the Hamiltonian H is a sum $H_0 + H_I$ where H_0 is the Hamiltonian of the free theory and H_I given by the interaction terms. In the interaction picture H_0 governs the time dependence of operators, while H_I controls the evolution of states. Then S is obtained from the limit $s \to -\infty$, $t \to \infty$ of the *time evolution operator* $U(s,t)$ that is the solution of

$$i\frac{\partial}{\partial t}U(t_0, t) = H_I(t)U(t_0, t),$$
$$U(t_0, t_0) = \mathrm{id}.$$

The solution is given by an iterated integral, known as *Dyson's series*,

$$U(s,t) = T(e^{-i\int_s^t d\tau\, H_I(\tau)}).$$

Here time ordering is needed because the Hamiltonians evaluated at different times need not commute. Since time ordering is defined with the generalized function θ, this is the point where (ultraviolet) divergences are inserted into the theory; in general one cannot multiply distributions by discontinuous functions, or equivalently, in the language of distributions, the product of two distributions is not well-defined. There are two ways of dealing with this problem: Try to construct a well-defined version of T, or proceed with the calculation and try to get rid of the problems at the end. The first is the basic

idea of the Epstein-Glaser approach, the second leads to renormalization, the art of removing these divergences in a physical meaningful manner.

Expanding Dyson's series and using Wick's theorem for the products of fields in the interaction Hamiltonian we obtain a formal power series (convergence is not clear at all) in the coupling λ. Every summand is an integral over a product of Feynman propagators. Feynman's ingenious observation was that every integral can be encoded by an associated diagram. The map that translates the diagram back into an integral is called *Feynman rules* and usually denoted by Φ.

We make the following definitions and conventions. By a *Feynman diagram* we mean the "picture" that one draws to represent a term in the expansion of the S-matrix. It is a graphical object with time- and momentum arrows and external momenta or particles. In contrast, a *Feynman graph* is the resulting graph if we remove all labels and external edges (we consider only scalar theories - no tensor structure).

By a graph we mean the following combinatorial object.

Definition 2.1. A graph G is an ordered pair $G = (V, E)$ of a set V of vertices and a multiset E of unordered distinct (we do not allow loop edges, i.e. edges connecting a vertex with itself) pairs of elements of V.

Example. The *dunce's cap* graph (Figure 2.1) will serve as main example later throughout the text. Here $V = \{v_1, v_2, v_3\}$ and $E = \{e_1 = (v_1, v_2), e_2 = (v_1, v_3), e_3 = (v_2, v_3), e_4 = (v_2, v_3)\}$
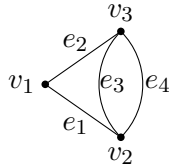
Figure 2.1: Dunce's cap

Definition 2.2. A *subgraph* g of G, denoted by $g \subseteq G$, is determined by a subset $E(g) \subseteq E(G)$.

Usually one defines the vertex set of g to be the set of vertices of $V(G)$ that are connected to edges of g, so that g is a graph itself, $g = (V(g), E(g))$. For our purposes it is more convenient to allow also for isolated vertices. Therefore we define a subgraph $g \subseteq G$ to be an equivalence class under the relation

$$g \sim g' \iff g' = g \cup (\cup_{v \in V'} v) \text{ for } V' \subseteq V(G) \setminus V(g).$$

For subgraphs $g, h \subseteq G$ we introduce the following operations:

1. Union and intersection: $g \cup h$ and $g \cap h$ are the subgraphs of G defined by the corresponding operations on the edge sets of g and h.

2. Deletion: For $g \subseteq h$ the deletion $h \setminus g$ is the graph h with all edges of g removed.

3. Contraction: For $g \subseteq h$ the contraction h/g is the graph h with all edges e in $E(g)$ removed and for every $e \in E(g)$ the two vertices connected to e identified.

Remark. In QFT one usually works with graphs defined by *half edges* to incorporate external edges, see for example [BEK06]. By the argumentation of Section 3.3 we do not need this here. Also, in applications a particular theory is defined by fixing a labelling of the edges (representing different propagators) and the valence of vertices (representing different interactions). Since we work in a scalar theory we need no labelling. Furthermore, the method presented here works for general graphs, so we do not restrict to a specific interaction.

Let us sketch the idea behind renormalization with a baby example. For a thorough treatment of renormalization from two different viewpoints we refer to [Kre13] and [Col84].

Example. For $p \in \mathbb{R}^+$ consider the logarithmically divergent integral

$$I(p) = \int_p^\infty \frac{dx}{x}.$$

To renormalize it we introduce a regularization parameter s by cutting off the domain of integration so that

$$I_s(p) := \int_p^s \frac{dx}{x}$$

is well-defined. If we subtract the *counterterm* $I_s(p)|_{p=\nu}$ for some fixed $\nu \in \mathbb{R}^+$ we obtain a finite expression for which we can take the *physical limit*, i.e. send s back to its original value. The renormalized value $R_\nu(I)(p)$ is then given by

$$\lim_{s \to \infty} I_s(p) - I_s(\nu) = \log \frac{\nu}{p}.$$

Obviously, there are many ways to renormalize such an expression. They depend on the choice of regularization and a *renormalization scheme*, which is determined by the form of the counterterms and the *renormalization point*. The common factor is the overall structure of the process.

In QFT we want to renormalize an expression $\Phi(G)$ associated to a Feynman diagram G. The first step is to identify the set of divergences. This is done by the Weinberg-Dyson convergence theorem [Wei96] that essentially

relates divergent sectors in $\Phi(G)$ to divergent subgraphs of G (cf. Definition 3.9). In general, more than one subtractions are needed and one cannot proceed in arbitrary order, but must arrange the subtractions according to Zimmermann's forest formula [Zim69]. This formula gives a recursive procedure for renormalizing $\Phi(G)$ by subtracting counterterms for all possible nested sets of divergent subgraphs in G. It is the starting point of the Hopf algebraic formulation of renormalization, which allows to put the ideas sketched here on firm ground in a compact form.

Finally, a theory is called *renormalizable* if in every order of perturbation theory the counterterms added can be put into the Lagrangian without changing its structure. In other words, if renormalizing the Lagrangian amounts to a rescaling of its parameters.

2.3   Causal perturbation theory

The other approach to renormalization goes back to Stückelberg, Bogolyubov and Shirkov [BS80]. The basic idea is to find a well-defined version of the time ordering operator to take care of divergences before they can emerge. In *causal perturbation theory* one tries to construct the S-matrix not as a perturbation series but from a set of physical axioms such as causality and Lorentz covariance. The first solution of this problem was given by Epstein and Glaser [EG73]. We will present a modern formulation which goes back to Stora and can be found in [BF00].

As a technical tool, but also to avoid the complications posed by Haag's theorem as well as infrared divergences, we use a test function $\lambda \in \mathcal{D}(M)$ that switches the interaction on in a bounded region of spacetime. Since then both the initial and final states are free (no interaction present), we propose the existence of a scattering operator $S = S(\lambda)$, now a functional of λ, that maps the in-state to the out-state. The starting point in causal perturbation theory is to assume that S is given by a formal series of operator-valued distributions,

$$\mathcal{D}(M) \ni \lambda \longmapsto \ S(\lambda) = 1 + \sum_{n \in \mathbb{N}} T^n(\lambda^{\otimes n}). \qquad (2.3)$$

Here the operator-valued distributions T^n are to be derived from a set of physical axioms the S-matrix should satisfy. These axioms allow then to construct T^n from knowledge of the T^k with $k < n$. An early formulation of this can be found in [BS80]. It differs from the Epstein-Glaser method in the crucial last step where the causal S-matrix is connected to its counterpart in conventional perturbation theory with the help of the time ordering operator T, thus leading to the well-known problem of ultraviolet divergences. Based on this observation, Epstein and Glaser showed that the axiomatic properties of S determine the T^n up to the small diagonal $D = \{x_1 = \ldots = x_n\}$

in M^n. Therefore the renormalization problem translates into finding an extension of T^n onto the full space M^n.

Let $M := \mathbb{R}^{1,3}$ denote Minkowski space (any manifold with a causal structure - to be defined below - works as well) and assume that the S-matrix is given by (2.3). The axiomatic properties S should fulfill are stated here in their respective version for the T^n:

1. All T^n are well defined operator valued distributions on M^n (using Schwartz' nuclear theorem [Sch66]).

2. T^n is symmetric under permutation of indices, in the sense of distributions.

3. Let $I \cup I^c = \{1, \ldots, n\}$ be a *causal* partition into two non-empty subsets. Then
$$T^n = T^I T^{I^c}.$$
A partition is causal if no x_i ($i \in I$) lies in the past lightcone $V^-(x_j)$ of the x_j with $j \in I^c$. The notation T^I is justified by Axiom 2.

4. T^n is translation invariant and Lorentz covariant.

Axiom 3 allows to construct T^n recursively from the T^k with $k < n$ up to the small diagonal in M^n. Translation invariance, together with a causal version of Wick's theorem, reduces the problem then to the extension of distributions onto a single point. This is done using a *causal covering* of M^n: For $\emptyset \neq I \subseteq \{1, \ldots, n\}$ define

$$U_I := \{x \in M^n \mid x_i \notin V^-(x_j) \text{ for } i \in I, \; j \in I^c\}.$$

These sets cover M^n up to the small diagonal,

$$M^n \setminus D = \bigcup_{\emptyset \neq I \subseteq \{1, \ldots, n\}} U_I.$$

Remark. This works also for more general spacetimes. The crucial point is here a causal structure on M that allows to define the U_I as above. Such structures are always present on *globally hyperbolic spacetimes*, as considered in [BF00].

By Axiom 3 we have on any U_I

$$T^n_I(x) := T^I(x_I) T^{I^c}(x_{I^c})$$

where x_J denotes $\{x_j\}_{j \in J}$. To define T^n on $M^n \setminus D$ pick a partition of unity χ_I, subordinate to the U_I, and set

$$T^n_0 := \sum_{\emptyset \neq I \subseteq \{1, \ldots, n\}} \chi_I T^n_I.$$

This defines an operator-valued distribution on $M^n \setminus D$ that satisfies all axioms of causal perturbation theory. The last remaining step is then to extend T_0^n onto the diagonal D in M^n. On Minkowski spacetime this reduces to an extension problem for numerical distributions and can be solved with the methods presented in the next chapter. On general spacetimes more sophisticated methods are required, for example the theory of microlocal analysis [Hö90].

Chapter 3

Distributions

In the last chapter we have seen how distributions arise in the formulation of quantum fields. Nevertheless, in most textbooks on QFT the distributional character of the theory is largely neglected because in the momentum space formulation it is less important. The situation is different in position space where distributions play a central role in every aspect.

We start this chapter with a short review of distribution theory. Then we study the extension problem for distributions in a toy model case and finish with a discussion of *Feynman distributions*, distributions given by Feynman rules.

3.1   General theory of distributions

Distributions were introduced by Laurent Schwartz [Sch66] who developed a theory of distributions based on earlier works by Hadamard, Dirac and Sobolev. The idea is to generalize the notion of ordinary functions by viewing them as continuous linear functionals on the space of (compactly supported) test functions. To every $f \in C^0(\mathbb{R})$ we can associate the functional

$$u_f : \varphi \longmapsto \int dx\, f(x)\varphi(x).$$

On the other hand, by a classical result of Hadamard, every continuous linear functional u can be expressed by a sequence of integrals of this type

$$u : \varphi \longmapsto \lim_{n \to \infty} \int dx\, u_n(x)\varphi(x).$$

But taking limits and integration do not commute in general, i.e. $\lim_{n \to \infty} u_n$ need not exist as a function. This is where distributions enter the game as *generalized functions*.

We write $\langle u | \varphi \rangle$ for the value of u at φ and by abuse of notation we use the same symbol f for a function and the functional u_f it represents. In the

latter case we refer to f as the kernel of u_f. The locus where u cannot be given by a function is called the *singular support* of u.

The consideration above also shows that the definition of distributions depends on the space of test functions they act on. Usually this space is chosen to be either $\mathcal{D}(\mathbb{R}^n)$, the space of smooth functions with compact support, or $\mathcal{S}(\mathbb{R}^n)$, the space of *Schwartz functions*. $\mathcal{S}(\mathbb{R}^n)$ is defined as the space of *rapidly decreasing* smooth functions:

Definition 3.1. On the space $C^\infty(\mathbb{R}^n)$ define for every pair of multi-indices α, β a seminorm

$$||f||_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|.$$

The *Schwartz space* is defined as

$$\mathcal{S}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) \mid ||f||_{\alpha,\beta} < \infty, \ \forall \alpha, \beta\}.$$

Although compactly supported functions might seem more natural to work with, the space of Schwartz functions has some advantageous features: Schwartz functions can be analytic and, most importantly, the Fourier transform restricts to a linear isomorphism on $\mathcal{S}(\mathbb{R}^n)$. Because of these nice features this space is usually used in the formulation of QFT. Distributions on $S(\mathbb{R}^n)$ are called *tempered distributions*. On the other hand, for practical calculations it is often more convenient to work with compactly supported functions. In addition, on manifolds Schwartz functions are hard to define (see for example [AG08]). Therefore, we work in this thesis with distributions on the space of compactly supported test functions.

Let X be a (smooth) d-dimensional manifold and denote by $\mathcal{D}(X) := C_0^\infty(X)$ the space of compactly supported smooth functions on X. For $X \subseteq \mathbb{R}^d$ open it would be natural to define the space of distributions $\mathcal{D}'(X) := (\mathcal{D}(X))^*$ as the space of continuous linear forms on $\mathcal{D}(X)$. To generalize this to the manifold case there are two possibilities, depending on whether distributions should generalize functions or measures (cf. [Hö90]). In the following let $\{\psi_i : U_i \to \tilde{U}_i \subseteq \mathbb{R}^d\}_{i \in I}$ be an atlas for X.

Definition 3.2. A *distribution* u on X is given by a collection of distributions $\{u_i \in \mathcal{D}'(\tilde{U}_i)\}_{i \in I}$ such that for all $i, j \in I$

$$u_j = (\psi_i \circ \psi_j^{-1})^* u_i \ \text{ in } \ \psi_j(U_i \cap U_j).$$

The space of distributions on X is denoted by $\mathcal{D}'(X)$.

This is the way to define distributions as generalized functions on X (every $u \in C^0(X)$ defines a distribution by setting $u_i := u \circ \psi_i^{-1}$). If we start from the point of view that they are continuous linear forms on $\mathcal{D}(X)$, we arrive at generalized measures on X:

Definition 3.3. A *distribution density* $\tilde{u}$ on X is a collection of distributions $\{\tilde{u}_i \in \mathcal{D}'(\tilde{U}_i)\}_{i \in I}$ such that for all $i, j \in I$

$$\tilde{u}_j = |\det D(\psi_i \circ \psi_j^{-1})|(\psi_i \circ \psi_j^{-1})^* \tilde{u}_i \ \text{ in } \ \psi_j(U_i \cap U_j).$$

The space of distribution densities on X is denoted by $\tilde{\mathcal{D}}'(X)$.

Because of their transformation properties, distribution densities are also called *pseudoforms*. They generalize differential forms in the sense that they can be integrated even on non-orientable manifolds. For more on pseudo-forms and integration on non-orientable manifolds we refer to [Nic07]. If X is orientable, there is an isomorphism $\mathcal{D}'(X) \cong \tilde{\mathcal{D}}'(X)$ via $u \mapsto u\nu$ for ν a strictly positive density (i.e. a volume form) on X. In particular, on $\mathbb{R}^d$ such a density is given by the Lebesgue measure $\nu = |dx|$ and we write $\tilde{u}$ for $u|dx|$ with $u \in \mathcal{D}'(\mathbb{R}^d)$.

For later purposes we introduce two operations on distributions and densities, the pullback and pushforward along a smooth map $f : X \to X'$.

Definition 3.4 (Pushforward)**.** Let $X \subseteq \mathbb{R}^m$ and $X' \subseteq \mathbb{R}^n$ be open and $f : X \to X'$ be surjective and proper (if u is compactly supported this requirement can be dropped). For a distribution u on X the pushforward $f_* u \in \mathcal{D}'(X')$ is defined by

$$\langle f_* u | \varphi \rangle = \langle u | f^* \varphi \rangle \ \text{ for all } \ \varphi \in \mathcal{D}(X').$$

For X and X' manifolds with atlantes $(\psi_i, U_i)_{i \in I}$ and $(\psi'_j, U'_j)_{j \in J}$ we define the pushforward $f_* \tilde{u} \in \tilde{\mathcal{D}}'(X')$ of $\tilde{u} \in \tilde{\mathcal{D}}'(X)$ by

$$(f_* u)_j := (\psi'_j \circ f \circ \psi_i^{-1})_* u_i \ \text{ in } \ U'_j \cap (\psi'_j \circ f \circ \psi_i^{-1})(U_i).$$

The question under what conditions the pullback of distributions is defined is more delicate, see [Hö90] for a detailed exposition. We state only one special case where it is possible to define a pullback: Let X and X be open subsets of $\mathbb{R}^n$ and $f : X \to X'$ a smooth submersion. Then there exists a unique linear operator $f^* : \mathcal{D}'(X') \to \mathcal{D}'(X)$ such that $f^* u = u \circ f$ if $u \in C^0(X')$. If X and X' are manifolds and $\tilde{u}$ is a density on X' then $f^* \tilde{u} \in \tilde{\mathcal{D}}'(X)$ is defined by

$$(f^* u)_i := (\psi'_j \circ f \circ \psi_i^{-1})^* u_j \ \text{ in } \ U'_j \cap (\psi'_j f \psi_i^{-1})(U_i).$$

3.2 Extension of distributions

In this section we present the theory of extending distributions. We study a toy model, distributions on $\mathbb{R} \setminus \{0\}$ given by kernels that have an algebraic singularity at 0. We follow the exposition in [GS64]. Applying this toy model to the extension problem for Feynman distributions is precisely the idea behind the geometric approach to renormalization.

3.2.1 The extension problem

The extension problem for distribution densities is formulated as follows:

Definition 3.5 (Extension problem)**.** Let X be a smooth manifold and $Y \subseteq X$ an immersed submanifold. Given a density $\tilde{u} \in \tilde{\mathcal{D}}'(X \setminus Y)$ find an extension of $\tilde{u}$ onto X, i.e. find a density $\tilde{u}_{\text{ext}} \in \tilde{\mathcal{D}}'(X)$ with

$$\langle \tilde{u}_{\text{ext}} | \varphi \rangle = \langle \tilde{u} | \varphi \rangle \text{ for all } \varphi \in \mathcal{D}(X \setminus Y).$$

In this very general formulation the problem is not always solvable. Moreover, if there is a solution, it need not be unique since by definition two extension may differ by a distribution supported on Y. Therefore additional conditions are sometimes formulated to confine the space of solutions. Usually one demands that the extension should have the same properties as u, for example scaling behaviour, Poincare covariance or solving certain differential equations.

In QFT (on flat spacetimes) the problem reduces to a solvable case, the extension of distributions $u \in \mathcal{D}'(\mathbb{R}^d \setminus \{0\})$ onto the origin. To construct a solution we need a device to measure the degree of divergence of u at a point. This is done by Steinmann's *scaling degree* [BF00] which generalizes the notion of homogeneity from functions to distributions. To define a scaling operation on distributions we introduce the scaling operator T_λ on test functions,

$$T_\lambda[\varphi](x) := \lambda^d \varphi(\lambda x) \text{ for } \varphi \in \mathcal{D}(\mathbb{R}^d) \text{ and } \lambda > 0.$$

The adjoint operation on distributions is then defined by

$$\langle T_\lambda^* u | \varphi \rangle := \langle u | T_{\lambda^{-1}}[\varphi] \rangle.$$

Definition 3.6. Let $u \in \mathcal{D}'(\mathbb{R}^d)$. The **scaling degree** $\sigma = \sigma(u)$ of u with respect to $0 \in \mathbb{R}^d$ is defined as follows,

$$\sigma := \sup_{\sigma'} \{ \sigma' \mid \lim_{\lambda \to \infty} \lambda^{\sigma'} T_\lambda^* u = 0 \}.$$

Example. 1. The Dirac distribution $\delta \in \mathcal{D}'(\mathbb{R}^d)$ has scaling degree equal to d, the derivatives $\partial^\alpha \delta$ of δ given by $\langle \partial^\alpha \delta | \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(0)$ have $\sigma = d + |\alpha|$.

2. Let $u \in \mathcal{D}'(\mathbb{R}^d)$ be homogeneous of order α, i.e. $\langle T_\lambda^* u | \varphi \rangle = \lambda^{-\alpha} \langle u | \varphi \rangle$. Then u has scaling degree $\sigma = -\alpha$.

3. Let $u \in \mathcal{D}'(\mathbb{R}_+ \setminus \{0\})$ be given by the kernel $x \mapsto \exp(\frac{1}{x})$. Then $\sigma(u) = \infty$.

With the scaling degree we are able to state the following extension theorem:

Theorem 3.7. *Let u be a distribution on $\mathbb{R}^d \setminus \{0\}$ with finite scaling degree σ. If $\sigma < d$ there exists a unique extension $u_{ext} \in \mathcal{D}'(\mathbb{R}^d)$ with the same scaling degree $\sigma(u_{ext}) = \sigma$. If $\sigma \geq d$ there exist extensions $u_{ext} \in \mathcal{D}'(\mathbb{R}^d)$ with scaling degrees equal σ. Each extension depends on a choice of test functions $\nu_{\underline{n}}$, where $\underline{n} \in \mathbb{N}^d$ is a multi-index, $0 \leq |\underline{n}| \leq \sigma - d$.*

Proof. We sketch the proof:

In the first case uniqueness follows from the fact that the difference of two extensions would be supported at $\{0\}$, hence a sum of δ and its derivatives. But they all have scaling degree $\sigma \geq d$ and the scaling degree is additive. Therefore the extension must be unique. Existence of u_{ext} is shown by constructing it as the limit of a sequence of distributions $(u_n)_{n \in \mathbb{N}}$. For this define $u_n := \mu_n u$ with $\mu_n \in \mathcal{D}(\mathbb{R}^d)$ such that $\mu_n \equiv 0$ in a ball of radius 2^{-n} around 0. This defines a distribution on $\mathbb{R}^d$ and in a last step it is shown that the scaling degree did not increase in this process.

In the second case the idea is to make the test functions vanish "fast enough" at 0. Set $\omega = \sigma - d$ and introduce the (Taylor-)subtraction operator W that maps test functions on $\mathbb{R}^d$ into $\mathcal{D}_\omega(\mathbb{R}^d)$, the space of test functions vanishing faster than $||x||^\omega$ at the origin,

$$W : \varphi \mapsto \varphi - \sum_{|\alpha| \leq \omega} \nu_\alpha \partial^\alpha \varphi(0).$$

Here $\nu_\alpha \in \mathcal{D}(\mathbb{R}^d)$ with $\partial^\alpha \nu_\beta(0) = \delta_{\alpha\beta}$. The extension u_{ext} is then defined by

$$\langle u_{\text{ext}} | \varphi \rangle := \langle u | W[\varphi] \rangle \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^d)$$

and one shows again that u_{ext} has the same scaling degree as u. $\qquad \square$

In the next section we will introduce the (Euclidean scalar, massless) propagator distribution $\triangle \in \mathcal{D}'(\mathbb{R}^d)$ given by the kernel

$$x \longmapsto \frac{1}{x^{d-2}}.$$

This function is homogeneous of order $2 - d$ (the same holds for the propagator of a massive scalar field) and therefore $\sigma(\triangle) = d - 2$. Feynman rules associate to every graph G a distribution given by a product of $\triangle$'s and for distributions of this kind $\omega = \sigma - d$ equals the superficial degree of divergence of G. Thus the extension theorem shows that the problem does not have a unique solution if G is divergent. The study of the ambiguity of solutions and how physics depends on it leads to the theory of the renormalization group which we will turn to in Chapter 7.

Recall that in the Epstein-Glaser approach the main ingredients are causality and translation invariance which allow to reduce renormalization

to the extension problem considered above. If translation invariance were not given we would need to extend distributions onto submanifolds. In this case more sophisticated methods are needed (see [BF00]). This leads to the theory of microlocal analysis and the notion of wavefront sets ([Hö90]). Motivated by various versions of the Paley-Wiener theorem the basic idea is to probe the analyticity of functions or distributions by looking at their Fourier or Laplace transforms. For example, the singular support of a tempered distribution is characterized by the following property: For $x \in \mathbb{R}^d$ choose $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\varphi(x) \neq 0$. Then $u \in \mathcal{S}'(\mathbb{R}^d)$ is smooth in a neighborhood of x if and only if the Fourier transform of the *localized* distribution φu satisfies

$$|\mathcal{F}[\varphi u](p)| \leq c_N(1 + ||p||)^{-N} \text{ for all } p \in \mathbb{R}^d \text{ and } N \in \mathbb{N}. \qquad (3.1)$$

The idea of the wavefront set is to refine this property: If u fails to fulfill (3.1), it might still be true for a certain set of directions. In this case denote by $\Sigma_x(u)$ the set of all $q \in \mathbb{R}^d$ having no conic neighborhood V such that the equation is satisfied for $p \in V$.

Definition 3.8. Let $X \subseteq \mathbb{R}^d$ be open. The *wavefront set* of $u \in \mathcal{D}'(X)$ is defined as the closed subset of $X \times (\mathbb{R}^d \setminus \{0\})$ given by

$$\mathrm{WF}(u) = \{(x,p) \in X \times (\mathbb{R}^d \setminus \{0\}) \mid p \in \Sigma_x(u)\}.$$

The wavefront set is a very convenient tool to study distributions, for example it has the property that $\mathrm{WF}(Pu) = \mathrm{WF}(u)$ for every linear differential operator P. Moreover, the conditions for the existence of a pullback operation on distributions can be neatly formulated in terms of $\mathrm{WF}(u)$.

We will take a different route though. In contrast to the recursive procedure used in [EG73] or [BF00] we will renormalize all divergences of a given distribution at once. The geometric approach via wonderful models allows to do this using an even easier extension method that we present in the next section.

3.2.2 A toy model

Let $u \in \mathcal{D}'(\mathbb{R} \setminus \{0\})$ be defined by the kernel $x \mapsto \frac{1}{|x|}$. A priori u is only defined as a distribution on the space of test functions vanishing at 0. Since $\omega(u) = 0$ we know from Theorem 3.7 that an extension exists, unique up to a δ-distribution.

The first step in the process of extending u is to regularize it by introducing a complex parameter $s \in \mathbb{C}$. Raising u to a complex power u^s is

known as *analytic regularization*. It justifies the following calculations:

$$
\begin{aligned}
\langle u^s | \varphi \rangle &= \int_{\mathbb{R}} dx \frac{1}{|x|^s} \varphi(x) \\
&= \int_{[-1,1]} dx \frac{\varphi(x)}{|x|^s} + \int_{\mathbb{R}\setminus[-1,1]} dx \frac{\varphi(x)}{|x|^s} \\
&= \int_{[-1,1]} dx \frac{\varphi(x) - \varphi(0)}{|x|^s} + \varphi(0) \int_{[-1,1]} dx \frac{1}{|x|^s} + \int_{\mathbb{R}\setminus[-1,1]} dx \frac{\varphi(x)}{|x|^s} \\
&= \int_{[-1,1]} dx \frac{\varphi(x) - \varphi(0)}{|x|^s} + \frac{2\varphi(0)}{1-s} + \int_{\mathbb{R}\setminus[-1,1]} dx \frac{\varphi(x)}{|x|^s}.
\end{aligned}
$$

The last term is defined for all $s \in \mathbb{C}$, the second term for $s \neq 1$ and the first one for $\mathrm{Re}(s) < 3$ because

$$
\int_{[0,1]} dx \frac{\varphi(x) + \varphi(-x) - 2\varphi(0)}{x^s} = 2 \int_{[0,1]} dx \sum_{k=1}^{\infty} \frac{\varphi^{2k}(0)}{(2k)!} x^{2k-s}
$$

converges for $\mathrm{Re}(2k - s) > -1$. Thus, we have found a way to split the regularized distribution $u^s = u_\infty(s, \cdot) + u_\heartsuit(s, \cdot)$ into a divergent and a convergent part. The divergent part is the principal part of the Laurent expansion of the meromorphic distribution-valued function $s \mapsto u^s$ in a punctured disc around 1 in $\mathbb{C}$.

$$
\begin{aligned}
\langle u_\infty(s, \cdot) | \varphi \rangle &= \frac{2\varphi(0)}{1-s}, \\
\langle u_\heartsuit(s, \cdot) | \varphi \rangle &= \int_{\mathbb{R}} dx \frac{\varphi(x) - \theta(1 - |x|)\varphi(0)}{|x|^s}.
\end{aligned}
$$

To continue the process of extending u we have to get rid of the divergent part in some sensible way (in physics this is the choice of a *renormalization scheme*) and take the limit $s \to 1$. The most straightforward way to do so is by subtracting the pole (*minimal subtraction*) and set

$$
\hat{u} = r_1[u^s]|_{s=1} := (u^s - \frac{2\delta}{1-s})|_{s=1} = u_\heartsuit(1, \cdot).
$$

The map r_1 is called a *renormalization operator*. Obviously this technique can be generalized to extend distributions u with higher negative powers of $|x|$ - one simply subtracts a higher order Taylor polynomial from φ.

Another renormalization scheme, *subtraction at fixed conditions*, is given by

$$
\langle r_\nu[u^s] | \varphi \rangle := \langle u^s | \varphi \rangle - \langle u^s | \varphi(0)\nu \rangle
$$

with $\nu \in \mathcal{D}(\mathbb{R})$ a smooth cutoff function with $\nu(0) = 1$. Another way to formulate the subtracted distribution is

$$
\langle u^s | \varphi(0)\nu \rangle = \langle (p_0)_*(\nu u^s) | \delta_0[\varphi] \rangle.
$$

Here $p_0 : \mathbb{R} \to \{0\}$ is the projection onto the divergent locus and δ_0 is interpreted as an operator $\mathcal{D}(\mathbb{R}) \to \mathcal{D}(0)$ mapping test functions on $\mathbb{R}$ onto test functions supported on the divergent locus. From this it is also clear that the difference between two such renormalization operators R_ν and $R_{\nu'}$ is given by a distribution supported on $\{0\}$, i.e. a linear combination of δ and its derivatives.

This formulation will be very useful later. The tricky part in applying these renormalization techniques to densities coming from Feynman graphs is the fact that they are products of divergent and smooth functions. Therefore the renormalization operators act on the divergent part only, while the smooth part is treated like a test function - this makes it hard to find a globally consistent notation for the renormalized densities.

A nice feature of these renormalization operators r is that they commute with multiplication by smooth functions, $r[fu] = fr[u]$ for $f \in \mathcal{C}^\infty(\mathbb{R})$. In addition, r belongs to the class of *Rota-Baxter operators*, a fact extensively used in the Hopf algebraic formulation of renormalization (see for example [EFG07]).

Later we will work with distributions given by kernels

$$u^s(x) = \frac{1}{|x|^{1+d(s-1)}}.$$

In this case u^s splits into

$$u^s = -\frac{2}{d}\frac{\delta_0}{s-1} + u_\heartsuit(s) \tag{3.2}$$

with $u_\heartsuit(s)$ holomorphic for $\mathrm{Re}(s) < \frac{2+d}{d}$.

In the next chapter we will introduce the wonderful models of DeConcini and Procesi. These models allow to apply the analysis of this toy model to the general case of a distribution density associated to a graph G by Feynman rules.

3.3 Feynman distributions

Feynman diagrams are convenient book-keeping devices for the terms in the perturbative expansion of physical quantities. The map that assigns to every Feynman diagram its corresponding analytical expression is called *Feynman rules* and denoted by Φ. In position space the map Φ assigns to every diagram G a pair $(X^G, \tilde{v}_G)$ where $\tilde{v}_G$ is a differential form on the space X^G, a cartesian product of the underlying spacetime M. We would like to evaluate

$$(X^G, \tilde{v}_G) \longmapsto \int_{X^G} \tilde{v}_G,$$

but this is in general not possible due to the problem of ultraviolet and infrared divergences. While we avoid the infrared problem by viewing $\tilde{v}_G$ as a distribution density, the ultraviolet problem translates into an extension problem for $\tilde{v}_G$. The ultraviolet divergences of $\tilde{v}_G$ are assembled in a certain subspace arrangement that we will describe at the end of this section, after the definition of Φ.

In this thesis we consider a massless scalar quantum field in d-dimensional Euclidean spacetime $M := \mathbb{R}^d$. The case of fields with higher spin differs only by notational complexity. On the other hand, the massive case is much harder because already the simplest examples have special functions arising as propagators of the free theory. Working in the Euclidean metric is justified by the technique of Wick rotation (see [Wei96]) that allows one to do calculations in M and transform the results back to Minkoswki spacetime. The position space propagator of a massless scalar field is given by the Fourier transform of the momentum space propagator,

$$\triangle(x) = \mathcal{F}(k \mapsto \frac{1}{k^2})(x) = \frac{1}{x^{d-2}} \ , \quad x \in M.$$

Let us look at an introductory example to motivate the abstract point of view presented further below. Let G be a connected Feynman diagram with $n+1$ internal and m external vertices labelled by $i \in \{1, \ldots, n+1\}$ and $j \in \{1, \ldots, m\}$ with associated coordinates $x_i, y_j \in M$ (cf. Figure 3.1). According to the Wightman axioms we are interested in computing $\langle w_G | \varphi \rangle$, the pairing of the Wightman distribution w_G with a test function $\varphi \in \mathcal{D}(M^m)$. The physical meaning of φ is a smeared configuration of external particles (the configuration before and after the process). As in causal perturbation theory we introduce another test function $\lambda \in \mathcal{D}(M)$, the *switching function*, that restricts the interaction to a bounded area of spacetime (at the end of the day we would like to compute the *adiabatic limit* $\lambda \to$ const, a problem related to infrared divergences, which we do not treat here).
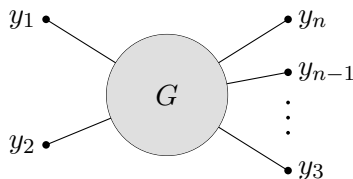

Figure 3.1: A Feynman diagram

The kernel of w_G is given by a rational function and we denote by w_G both objects, the Wightman distribution and its kernel. w_G is one part of the m-point function W_m obtained by summing over all Wightman distributions w_G for Feynman diagrams with m external vertices.

Feynman rules associate to G a distribution

$$w_G(y) = \int dx \prod_{k=1}^{m} \triangle(y_k - x_{c(k)}) \prod_{i<j} \triangle(x_j - x_i)^{e_{i,j}} \prod_{l=1}^{n+1} \lambda(x_l).$$

Here $c(k)$ is the label of the internal vertex connected to the external vertex k and $e_{i,j}$ is the number of edges between the vertices i and j. Assume y_1 is connected to x_{n+1}. Then by the transformation formula and Fubini's theorem we have (up to a constant factor)

$$
\begin{aligned}
\langle w_G | \varphi \rangle &= \int dy \ w_G(y) \varphi(y) \\
&= \int dy \int dx \prod_{k=1}^{m} \triangle(y_k - x_{c(k)}) \prod_{i<j} \triangle(x_j - x_i)^{e_{i,j}} \prod_{l=1}^{n+1} \lambda(x_l) \varphi(y) \\
&= \int dy \int dx \prod_{k=1}^{m} \triangle(y_k) \prod_{i<j} \triangle(x_j - x_i)^{e_{i,j}} \prod_{l=1}^{n+1} \lambda(x_l) \\
&\quad \times \varphi(y_1 + x_{n+1}, \ldots, y_m + x_{c(m)}) \\
&= \int dy \int dx \prod_{k=1}^{m} \triangle(y_k) \prod_{i<j} \triangle(x_j - x_i)^{e_{i,j}} \Big|_{x_{n+1}=0} \lambda(x_{n+1}) \\
&\quad \times \prod_{l=1}^{n} \lambda(x_l - x_{n+1}) \varphi(y_1 + x_{n+1}, \ldots, y_m + x_{c(m)} - x_{n+1}) \\
&=: \int dx_1 \cdots dx_n \prod_{i<j} \triangle(x_j - x_i)^{e_{i,j}} \Big|_{x_{n+1}=0} \phi(x_1, \ldots x_n) \\
&= \langle v_G | \phi \rangle.
\end{aligned}
$$

The last equation is justified because ϕ is itself a test function as $\triangle$ is in $L^1_{\mathrm{loc}}(M)$ and λ and φ are smooth and compactly supported. In physical terms it is the *n-point tree-level amplitude*, evaluated on the test function

$$y \longmapsto \prod_{l=1}^{n} \lambda(x_l - x_{n+1}) \varphi(y_1 + x_{n+1}, \ldots, y_m + x_{c(m)} - x_{n+1}).$$

The ultraviolet problem translates into the fact that v_G is not a distribution on M^n; it is only defined outside of an arrangement of subspaces of M^n. Based on this calculation we focus from now on on connected graphs obtained from Feynman diagrams by forgetting external vertices and edges.

Now we employ a more abstract point of view. Let G be a connected graph. As shown in [BBK10], Feynman rules are determined by the topology of G. Pick a labelling $V = \{v_0, \ldots, v_n\}$ of the vertices of G and an orientation

on the edges of G. For a finite set F let $\mathbb{R}^F$ denote the vector space with (fixed) basis the elements of F. The cohomology of the simplicial complex $G = (V, E)$ gives rise to an exact sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\sigma} \mathbb{R}^V \xrightarrow{\delta} \mathbb{R}^E \longrightarrow H^1(G, \mathbb{R}) \longrightarrow 0.$$

Here the map σ sends 1 to $v_0 + \cdots + v_n$ and δ is given by $\delta(v) = \sum_{e \in E}(v : e)e$ with $(v : e) = \pm 1$ if e starts/ends at v and 0 otherwise. Fix a basis of $\operatorname{coker}(\sigma)$ by an isomorphism $\varphi : V' := V \setminus \{v_0\} \to \operatorname{coker}(\sigma)$. This defines an inclusion $\iota : \mathbb{R}^{V'} \cong \operatorname{coker}\sigma \hookrightarrow \mathbb{R}^E$. Doing this component-wise on the space $X^G := M^{V'} = (\mathbb{R}^d)^{V'}$ we obtain an inclusion $I := \iota^{\oplus d} : X^G \hookrightarrow M^E$ and define $v_G : X^G \to \mathbb{R}$ by

$$v_G : \sum_{v \in V'} x_v v \longmapsto \prod_{e \in E(G)} \triangle(\sum_{v \in V'}(v : e)x_v).$$

Moreover, every edge $e \in E$ defines a linear form $\omega_e := e^* \circ \iota$ on $\mathbb{R}^{V'}$ and a linear subspace of $(X^G)^*$ by

$$A_e := < \omega_e >^{\oplus d} = \left\{ (x_1, \ldots, x_n) \mapsto \sum_{i=1}^d \alpha_i \omega_e(x_1^i, \ldots, x_n^i),\ \alpha_i \in \mathbb{R} \right\}.$$

For a subgraph $g \subseteq G$ we define $A_g := \sum_{e \in E(g)} A_e$. Families $\mathcal{P}$ of subgraphs of G give then rise to *subspace arrangements* in $(X^G)^*$,

$$\mathcal{A}_{\mathcal{P}} := \{A_g \mid g \in \mathcal{P}\}.$$

Note that two subgraphs $g, h \subseteq G$ may define the same subspace, $A_g = A_h$. Therefore we will consider only subfamilies of $\mathcal{G}$, the set of *saturated* subgraphs of G. Saturated subgraphs are maximal with respect to the property of defining their corresponding subspaces A_g. A precise definition is given in Section 5.4.

Two arrangements are especially important for our purposes. The *singular arrangement*

$$\mathcal{A}_{\mathcal{G}} := \{A_g \mid g \subseteq G \text{ is saturated}\},$$

and the arrangement coming from the family $\mathcal{D}$ of *divergent* subgraphs of G,

$$\mathcal{A}_{\mathcal{D}} = \{A_g \mid g \subseteq G \text{ is divergent}\}.$$

Definition 3.9. Let $h_1(\cdot)$ denote the first Betti number. Define the *superficial degree of divergence* ω of G by

$$\omega(G) := dh_1(G) - 2|E(G)|.$$

Then G is called *divergent* if $\omega(G) \geq 0$. G is *at most logarithmic* if $\omega(G) \leq 0$ holds for all $g \subseteq G$. If $\mathcal{D} = \{\emptyset, G\}$, then G is called *primitive*.

Lemma 3.10. *Let G be at most logarithmic and $d > 2$. Then*

$$g \in \mathcal{D} \implies g \text{ is saturated.}$$

Proof. Suppose $A_g = A_{g \cup e}$ for some $e \in E(G \setminus g)$. From this follows that $V(g) = V(g \cup e)$ because $A_g \subseteq (M^{V'})^*$. But then adding e to g must produce a cycle, so $h_1(g \cup e) = h_1(g) + 1$. Thus, $\omega(g \cup e) = \omega(g) + d - 2 > 0$, a contradiction to G at most logarithmic. $\square$

As shown in the next proposition, the divergent arrangement $\mathcal{A}_{\mathcal{D}}$ describes exactly the locus where extension is necessary.

Proposition 3.11. *Let G be connected and at most logarithmic. Set*

$$X_s := \bigcup_{e \in E} A_e^\perp \text{ and } X_{\mathcal{D}} := \bigcup_{g \in \mathcal{D}} A_g^\perp.$$

Then v_G is a well defined distribution on $X \setminus X_{\mathcal{D}}$ and the singular support of v_G is given by the complement $X_s \setminus X_{\mathcal{D}}$.

Proof. Let $V = \{v_0, \ldots, v_n\}$. Wherever defined, v_G can be written as

$$v_G(x_1, \ldots, x_n) = I^* \left(\triangle^{\otimes |E|} \right) (x_1, \ldots, x_n) = \prod_{e \in E} \triangle \left(\sum_{i=1}^n (v_i : e) x_i \right).$$

Since sing supp$(\triangle) = \{0\}$, the singular support of $\triangle^{\otimes E}$ is the set where at least one $x_e \in M$ vanishes. But this is precisely the image of $A_e^\perp$ under I. Thus, sing supp$(v_G) \subseteq X_s$.

For $K \subseteq X$ compact and χ_K the (smooth approximation of the) characteristic function of K we need to show that $\langle v_G | \chi_K \rangle = \int_K dx \ v(x) < \infty$ as long as K is disjoint from $X_{\mathcal{D}}$. Assume the contrary, $K \cap X_{\mathcal{D}} \neq \emptyset$; more precisely, K intersects $A_g^\perp$ for some $g \in \mathcal{D}$, but no other divergent loci. Moreover, assume that g is connected - otherwise $A_g^\perp = A_{g_1}^\perp \cup A_{g_2}^\perp$ and a smaller K will intersect only one of these subspaces. Then v_G splits into two factors

$$v_G(x) = \prod_{e \in E(g)} \triangle \left(\sum_{i=1}^n (v_i : e) x_i \right) \prod_{e \in E \setminus E(g)} \triangle \left(\sum_{i=1}^n (v_i : e) x_i \right),$$

with the second factor being smooth on $A_g^\perp \setminus \bigcup_{g \subsetneq g'} A_{g'}^\perp$.

Now we need some power counting: The integral $\int_K dx \, v_G(x)$ is over a dn-dimensional space. Since A_g is the sum over all A_e with $e \in E(g)$, it is already spanned by the edges in a *spanning tree* t of g (a spanning tree is a subgraph without loops meeting every vertex exactly once - see Definition 5.20). A spanning tree of a connected graph with n vertices has necessarily $n - 1$ edges, therefore dim $A_g = d(|V(g)| - 1)$. Adding an edge

to t produces a cycle, so that $h_1(g) = |E(g)| - |V(g)| + 1$. We conclude that $\dim A_g = d(|E(g)| - h_1(g))$. Each $\triangle(x)$ is of order $\mathcal{O}(x^{2-d})$ as $x \to 0$ and there are $|E(g)|$ products in the first factor expressing v_G. Thus, the whole product scales as $(2 - d)|E(g)|$ as x approaches $A_g^\perp$ in X^G:

$$\int dx \, v_G(x) \propto \int dr \, r^{\dim A_g - 1 + (2-d)|E(g)|}$$

and the integral converges if and only if

$$\dim A_g + (2-d)|E(g)| > 0 \iff d(|E(g)| - h_1(g)) + (2-d)|E(g)| > 0$$
$$\iff \omega(g) < 0$$
$$\iff g \notin \mathcal{D}.$$

$\square$

In Chapter 5 we will employ a more practical point of view. We use coordinates on X^G not given by the vertex set V', but on the edges of an *adapted* spanning spanning tree t. Since every spanning tree of G must have $|V| - 1$ vertices, reformulating everything in coordinates given by edges of t is just a change of basis for $M^{V'}$. The point here is that although it might seem to be more intuitive and "positional" to work with the vertex set of G, the formulation with t is more convenient because the combinatorics of renormalization show up in the subgraph structure of G and subgraphs are determined by subsets of E, not of V.

Putting everything together, we conclude that Feynman rules in position space are given by a map

$$\Phi : G \longmapsto (X^G, \tilde{v}_G)$$

where $\tilde{v}_G \in \tilde{\mathcal{D}}'(X^G \backslash X_\mathcal{D}^G)$. The ultimate goal would be to evaluate this at the test function given by the value of the corresponding tree-level distribution on another test function representing the external configuration (the in- and out-particles). Avoiding infrared divergences, we restrict ourselves to the problem of ultraviolet renormalization, i.e. we want to find an extension of $\tilde{v}_G$ to a distribution defined for all test functions in $\mathcal{D}(X^G)$.

Chapter 4

Resolution of singularities: Geometry

The problem of resolving singularities has been a major topic in algebraic geometry since the time of Newton who solved the problem of resolving curves in the complex plane. In its most basic form the problem can be formulated as follows.

Definition 4.1. Let X be an algebraic variety over a field k. Then a nonsingular variety Y is a *resolution* for X if there exists a proper and surjective rational map $\beta : Y \longrightarrow X$.

There are various types of resolutions, depending on additional conditions on Y and β. In this thesis we demand that β is the composition of blow-ups along smooth subvarieties of X. This allows for an explicit description of the manifold Y.

Hironaka showed in his celebrated work [Hir64] that for fields of characteristic zero a resolution always exists; for fields of non-trivial characteristic this is still an open problem. He gave a constructive proof using a sequence of blow-ups. The difficulty lies in the fact that one cannot proceed by just blowing up all singularities in X, but must choose a specific order in doing so. For an extensive treatment of this topic, including a comparison of different resolutions, we refer to [Kol07].

Since we will use the same method to resolve the singularities of a Feynman distribution, we start this chapter with a general introduction to blowups.

4.1 Blow-ups

What is meant by blowing up a subvariety of a variety X? First, we blow up the origin in $X = \mathbb{R}^n$, following [GH94]. The idea is to replace the origin

by the space of all possible directions entering it, in such a way that all directions are disjoint. To do so set $\mathcal{E} := \mathbb{P}(X)$ with homogeneous coordinates $[y_1 : \ldots : y_n]$ and define $Y \subseteq X \times \mathcal{E}$ by

$$Y := \big\{ (x_1, \ldots, x_n, [y_1 : \ldots : y_n]) \mid x_i y_j = x_j y_i \text{ for all } i \neq j \big\}.$$

The map $\beta : Y \to X$ is then simply the projection onto the first factor. Since the defining equations are smooth, Y is a smooth submanifold of $X \times \mathcal{E}$. To define an atlas for Y let for $i = 1, \ldots, n$ the maps $\rho_i : \mathbb{R}^n \to X \times \mathcal{E}$ be given by

$$(x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_n, [y_1 : \ldots : y_n])$$

where

$$y_k = \begin{cases} x_i & \text{if } k = i, \\ x_i x_k & \text{if } k \neq i. \end{cases}$$

Set $U_i = \rho_i(\mathbb{R}^n)$ and $\kappa_i := \rho_i^{-1}$. Then the collection of charts $(U_i, \kappa_i)_{i \in \{1, \ldots, n\}}$ forms an atlas for Y.

The submanifold $\mathcal{E}$, called the *exceptional divisor*, is locally given by $\{x_i = 0\}$ and covered by induced charts $(V_i, \phi_i)_{i \in \{1, \ldots, n\}}$ where $V_i := \hat{\rho}_i(\mathbb{R}^{n-1})$ and $\phi_i := \hat{\rho}_i^{-1}$ with

$$\hat{\rho}_i := \rho_i|_{x_i = 0} : \mathbb{R}^{n-1} \longrightarrow \{0\} \times \mathcal{E} \subseteq Y.$$

Blowing up along a submanifold S of $\mathbb{R}^n$ is done similarly by replacing S by the projectivization of its normal bundle. More precisely, if S is locally given by $\{x_1 = \ldots = x_k = 0\}$, then one proceeds as above but restricts the defining equation to these coordinates,

$$Y := \big\{ (x_1, \ldots, x_n, [y_1 : \ldots : y_k]) \mid x_i y_j = x_j y_i \text{ for all } i \neq j \in \{1, \ldots, k\} \big\}.$$

Since all constructions are local, this can easily be generalized to the case where S is a subvariety of a smooth variety X: Blow up locally, then globalize by patching together the local blow-ups.

If $S' \subseteq X$ is another submanifold that is distinct from S, then S' is essentially unaffected by the blow-up process. However, if it has a nonempty intersection with S, then S' has two "preimages" in Y: The *strict transform* of S' is defined as the closure of $\beta^{-1}(S' \setminus S)$ in Y, while the preimage $\beta^{-1}(S')$ is called the *total transform* of S'. Loosely speaking, the blow-up makes degenerate intersections transversal and transversal ones disjoint. Therefore, if building a resolution consists of multiple blow-ups, the order of blowing up is important - we will get back to this point later.

We introduced here the algebro-geometric version of blowing up. There is also a differential-geometric equivalent, where one replaces the locus to

be blown up by its normal spherebundle (as used in [AS94]). Both cases have drawbacks: Using the projective normal bundles leads to Y being non-orientable in general, while the differential-geometric blow-up produces a manifold with boundary.

4.2 Smooth models

To renormalize distributions $v = v_G$ (from now on we drop the index G) coming from Feynman diagrams systematically we want to arrange the loci of divergences in a "nice" way. This means, we are looking for a *compact-ification* of $X \setminus X_{\mathcal{D}}$, or, in other words, a *smooth model* for the divergent arrangement in X^G.

The general setup is the following: Let X be a finite dimensional smooth variety over a field k of characteristic zero. An *arrangement* $\mathcal{A}$ in X is a finite family of smooth subvarieties of X. Let $M(\mathcal{A})$ denote the complement of the arrangement, $M(\mathcal{A}) = X \setminus \cup_{A \in \mathcal{A}} A$.

Definition 4.2. A smooth model for the arrangement $\mathcal{A}$ is a pair $(Y_{\mathcal{A}}, \beta)$, where $Y_{\mathcal{A}}$ is a smooth variety and $\beta : Y_{\mathcal{A}} \longrightarrow X$ is a proper surjective map with the following properties:

1. β is an isomorphism outside of $\mathcal{E} := \beta^{-1}(X \setminus M(\mathcal{A}))$.

2. $\mathcal{E}$ is a normal crossing divisor, i.e. there exist local coordinates such that it is given by $\mathcal{E} = \{(x_1, \ldots, x_n) \mid x_1 \cdot \ldots \cdot x_k = 0\}$.

3. β is a composition of blowups along smooth centers.

Recall, that β is proper if and only if $\beta^{-1}(K)$ is compact for all compact sets $K \subseteq X$; this is why smooth models are sometimes also called compact-ifications. From [Hir64] we know that such a model always exists, actually in way more general situations. In their seminal paper [FM94] Fulton and MacPherson constructed a compactification of the configuration space

$$F_n(X) := \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for all } i \neq j\}$$

for a non-singular variety X. This is just an example of a smooth model for the arrangement given by all diagonals D_I in X^n,

$$\mathcal{A} = \{D_I \mid I \subseteq \{1, \ldots, n\}\} \text{ where } D_I = \{x_i = x_j \mid \forall i, j \in I\}.$$

Inspired by the techniques used in [FM94], DeConcini and Procesi developed a systematic way to construct smooth models for general linear arrangements. Since their technique is local, it can be generalized to arrangements in smooth varieties (see [Li09]), but we do not need this here and stick to the notation of [CP95].

4.3 Wonderful models

Let V be a finite dimensional k-vector space (here $k = \mathbb{R}$) and $\mathcal{A}$ be a linear arrangement in the dual V^*, i.e. a finite family $\{A_1, \ldots, A_k\}$ of linear subspaces of V^* (for the construction of DeConcini and Procesi it is more convenient to work in the dual). We first give an abstract definition of a smooth model $Y_{\mathcal{A}}$ for $\mathcal{A}$, then we construct it explicitly.

Definition 4.3 (Wonderful definition I). Let $\mathcal{A}$ be a linear arrangement in V^*. For every $A \in \mathcal{A}$ the projection $\pi_A : V \longrightarrow V/A^\perp \longrightarrow \mathbb{P}(V/A^\perp)$ is a well defined map outside of $A^\perp$. Doing this for every element in the arrangement, we obtain a rational map

$$\pi_{\mathcal{A}} : M(\mathcal{A}) \longrightarrow \prod_{A \in \mathcal{A}} \mathbb{P}(V/A^\perp).$$

The graph $\Gamma(\pi_{\mathcal{A}})$ of this map defines an open embedding of $M(\mathcal{A})$ into $V \times \prod_{A \in \mathcal{A}} \mathbb{P}(V/A^\perp)$. The *wonderful model* $Y_{\mathcal{A}}$ is defined as the closure of the image of this embedding.

The second way of defining $Y_{\mathcal{A}}$ is to explicitly construct it by a sequence of blow-ups. This sequence is actually completely determined by the combinatorics of the *intersection poset* $P(\mathcal{A})$, a point we will use extensively in the following chapters.

For the wonderful construction we need to introduce some terminology. The first notion is based on the fact that $Y_{\mathcal{A}}$ is also a wonderful model for arrangements $\mathcal{A}'$, as long as $\mathcal{A} \subseteq \mathcal{A}'$ is a *building set* for $\mathcal{A}'$. The idea is that an arrangement may carry too much information and in this case one needs only a subfamily $\mathcal{B} \subseteq \mathcal{A}$ to encode this information. While the choice of a building set controls the geometry of the wonderful model, more precisely of the exceptional divisor $\mathcal{E}$, certain subsets of $\mathcal{B}$, the $\mathcal{B}$-*nested sets*, and the choice of a $\mathcal{B}$-*adapted basis* of V are the crucial elements in the explicit construction of an atlas for $Y_{\mathcal{A}}$.

We cite the main definitions and results from DeConcini and Procesi; for the proofs we refer the reader to [CP95].

Definition 4.4 (Building sets). Let $\mathcal{A}$ be an arrangement in V^*. A subfamily $\mathcal{B} \subseteq \mathcal{A}$ is a *building set* for $\mathcal{A}$ if

1. Every $A \in \mathcal{A}$ is the direct sum $A = B_1 \oplus \cdots \oplus B_k$ of the maximal elements of $\mathcal{B}$ contained in A.

2. This decomposition property holds also for all $A' \in \mathcal{A}$ with $A' \subseteq A$, i.e. $A' = (B_1 \cap A') \oplus \cdots \oplus (B_k \cap A')$.

There are two important building sets for any arrangement $\mathcal{A}$, the *maximal building set*, given by all elements of $\mathcal{A}$, and the *minimal building set* $I(\mathcal{A})$. The latter consists of all $A \in \mathcal{A}$ that do not allow for a non-trivial decomposition. Note that every other building set $\mathcal{B}$ satisfies $I(\mathcal{A}) \subseteq \mathcal{B} \subseteq \mathcal{A}$. In [CP95] it is shown that for every building set $\mathcal{B} \subseteq \mathcal{A}$ the variety $Y_{\mathcal{B}}$ as defined above is a smooth model for $\mathcal{A}$. Moreover, the exceptional divisor $\mathcal{E}$ is the union of smooth irreducible components $\mathcal{E}_B$, one for each $B \in \mathcal{B}$.

Now we turn to the explicit description of $Y_{\mathcal{B}}$.

Definition 4.5 (Nested sets)**.** Let $\mathcal{B}$ be a building set. $\mathcal{N} \subseteq \mathcal{B}$ is $\mathcal{B}$-*nested* if the following holds: For all subsets $\{A_1, \ldots, A_k\} \subseteq \mathcal{N}$ of pairwise incomparable elements their direct sum does not belong to $\mathcal{B}$.

Nested sets are one main ingredient in the description of $Y_{\mathcal{B}}$, the second one being markings of an adapted basis of V^*. While nested sets reflect the combinatorics of the stratification of $\mathcal{E}$, the markings are related to the dimension of each submanifold in this stratification. Together they describe all components of the exceptional divisor.

Definition 4.6 (Adapted bases)**.** A basis B of V^* is $\mathcal{N}$-*adapted* if for all $A \in \mathcal{N}$ the set $B \cap A$ generates A. A *marking* of an $\mathcal{N}$-adapted basis is for every $A \in \mathcal{N}$ the choice of an element $b_A \in B$ with $p(b_A) = A$. Here $p = p_{\mathcal{N}}$ is the map assigning to $x \in V^* \setminus \{0\}$ the minimal element of $\mathcal{N} \cup \{V^*\}$ containing x (it exists because $\mathcal{N}$ is nested).

The map p and a marking define a partial order on B,

$$b \preceq b' \iff p(b) \subseteq p(b') \text{ and } b' \text{ is marked.}$$

This partial order defines a map $\rho = \rho_{\mathcal{N},B} : \mathbb{R}^B \to V$ as follows. For every $x = \sum_{b \in B} x_b b \in \mathbb{R}^B$ the image $\rho(x)$ is an element of $V = \hom(V^*, \mathbb{R})$ given by

$$B \ni b \mapsto \begin{cases} \prod_{p(b) \subseteq A} x_{b_A} & \text{if } b \text{ is marked,} \\ x_b \prod_{p(v) \subseteq A} x_{b_A} & \text{else.} \end{cases}$$

Viewing the elements of B as nonlinear coordinates on V and with $x_A := x_{b_A}$, we can write ρ as

$$\rho(x)_b = \rho(x)(b) = \begin{cases} \prod_{p(b) \subseteq A} x_A & \text{if } b \text{ is marked,} \\ x_b \prod_{p(b) \subseteq A} x_A & \text{else.} \end{cases}$$

The next proposition shows that the map ρ has all the properties of a local description of a composition of blow-ups.

Proposition 4.7. *For every nested set $\mathcal{N}$ and an adapted and marked basis B the map $\rho = \rho_{\mathcal{N},B}$ is a birational morphism with the following properties:*

It maps the subspace defined by $x_A = 0$ onto $A^\perp$ and it restricts to an isomorphism

$$V \setminus \bigcup_{A \in \mathcal{N}} \{x_A = 0\} \xrightarrow{\cong} V \setminus \bigcup_{A \in \mathcal{N}} A^\perp.$$

Furthermore, every v in $V^* \setminus \{0\}$ with $p(v) = A \in \mathcal{N}$ is mapped by $\rho(x)$ to

$$\rho(x)(v) = P_v(x) \prod_{b_A \preceq b} v_b x_b$$

where P_v is a polynomial, depending only on $\{x_b\}_{b \prec b_A}$ and linear in each variable.

Definition 4.8 (Wonderful definition II)**.** Let $\mathcal{N}$ be a $\mathcal{B}$-nested set for a building set $\mathcal{B} \subseteq \mathcal{A}$ and B an adapted, marked basis. Define $Z_A \subseteq \mathbb{R}^B$ by $Z_A = \{P_v = 0, v \in A\}$, the vanishing locus of all P_v for $v \in A$. Then for every $A \in \mathcal{B}$ the composition of ρ with the projection $\pi_A : V \setminus A^\perp \to \mathbb{P}(V/A^\perp)$ is well-defined outside of Z_A.

Composing the map ρ with $\Gamma(\pi_{\mathcal{B}}) : M(\mathcal{B}) \to V \times \prod_{A \in \mathcal{B}} \mathbb{P}(V/A^\perp)$ defines an open embedding

$$(\Gamma(\pi_{\mathcal{B}}) \circ \rho)_{\mathcal{N},B} : \mathbb{R}^B \setminus \bigcup_{A \in \mathcal{B}} Z_A \longrightarrow Y_{\mathcal{B}}.$$

Set $U_{\mathcal{N},B} := \mathrm{im}\big((\Gamma(\pi_{\mathcal{B}}) \circ \rho)_{\mathcal{N},B}\big)$ and $\kappa_{\mathcal{N},B} := (\Gamma(\pi_{\mathcal{B}}) \circ \rho)_{\mathcal{N},B}^{-1}$. Varying over all $\mathcal{B}$-nested sets $\mathcal{N}$ and adapted, marked bases B, we obtain an atlas $(U_{\mathcal{N},B}, \kappa_{\mathcal{N},B})$ for the wonderful model $Y_{\mathcal{B}}$. The map β is just the projection onto the first factor, in local coordinates given by ρ.

That this really defines a smooth model for the arrangement $\mathcal{A}$ follows from the next proposition.

Theorem 4.9 (Geometry of the wonderful model)**.** *Let $\mathcal{B}$ be a building set for $\mathcal{A}$. The wonderful model $Y_{\mathcal{B}}$ has the following properties:*

1. *The exceptional divisor $\mathcal{E}$ is normal crossing, i.e.*

$$\mathcal{E} := \beta^{-1}(\bigcup_{A \in \mathcal{B}} A^\perp) \stackrel{loc.}{=} \{\prod_{A \in \mathcal{N}} x_A = 0\}.$$

2. *$\mathcal{E}$ is the union of smooth irreducible components $\mathcal{E}_A$ where $A \in \mathcal{B}$ and $\beta(\mathcal{E}_A) = A^\perp$. A family of these components $\mathcal{E}_{A_1}, \ldots, \mathcal{E}_{A_k}$ has non-empty intersection if and only if $\{A_1, \ldots, A_k\}$ is a $\mathcal{B}$-nested set. In this case the intersection is transversal and irreducible.*

3. *For A minimal in $\mathcal{B} \setminus I(\mathcal{A})$ let $A = A_1 \oplus \ldots \oplus A_k$ be its irreducible decomposition. Set $\mathcal{B}' = \mathcal{B} \setminus \{A\}$. Then $Y_{\mathcal{B}}$ is obtained from $Y_{\mathcal{B}'}$ by blowing up $\mathcal{E}_A = \mathcal{E}_{A_1} \cap \ldots \cap \mathcal{E}_{A_k}$.*

4. *For A minimal in $\mathcal{B} = I(\mathcal{A})$ set $\mathcal{B}' = \mathcal{B} \setminus \{A\}$. Then $Y_{\mathcal{B}}$ is obtained from $Y_{\mathcal{B}'}$ by blowing up the proper transform of $A^\perp$.*

As stated before, the most famous example of a wonderful model is the Fulton-MacPherson compactification of the configuration space $F_n(X)$ in the case where X is a linear space. It is the minimal wonderful model for the arrangement of all (poly-)diagonals in X^n,

$$\mathcal{A} = \{D_\pi \mid \pi \text{ is a partition of } \{1, \ldots, n\}\},$$
$$D_\pi = \{x_i = x_j \mid i, j \text{ lie in the same partition block of } \pi \}.$$

Here the minimal building set consists of all simple diagonals in the n-fold product of X. The wonderful model for the maximal building set was studied by Ulyanov in [Uly02] and called a *polydiagonal compactification* of configuration space. The main difference, apart from the geometry of the exceptional divisor, is the blowup sequence in the construction. In [Uly02] the model is obtained by successively blowing up (the strict transforms of) all elements of the building set by increasing dimension, but in the minimal case one has to proceed with care; some strict transforms of diagonals to be blown up in the next step might still have nonempty intersection and in this case the result depends on the order of blowups. To separate them before proceeding requires additional blow-ups, exactly those given by the additional elements in the maximal building set. These are the polydiagonals, obtained by intersecting simple diagonals.

The interested reader is encouraged to study the example $F_n(X)$ for $X = \mathbb{R}$ and $n > 3$ (for smaller n minimal and maximal models coincide). It is a well studied object, the *real rank $n - 1$ braid arrangement*, see for example [Fei05].

The next step is to adapt this construction to the case of the divergent arrangement associated to a Feynman graph G. In [BBK10] this is done by examining the special structure of the elements of this arrangement, $\mathcal{A} = \{A_g \mid g \subseteq G \text{ divergent}\}$. These properties of $\mathcal{A}$ can be directly formulated in graph theoretical terms. Here we will focus even more on this combinatorial flavour of QFT and formulate everything with the help of the poset of divergent subgraphs of G. In the next chapter we will express all central notions of the wonderful construction, building sets, nested sets and adapted, marked bases in terms of this poset.

Chapter 5

Resolution of singularities: Combinatorics

So far we have constructed wonderful models for (linear) arrangements $\mathcal{A}$ from a purely geometric point of view. The notions used were formulated directly in terms of the elements of $\mathcal{A}$. This is sufficient as far as one is interested only in the geometry of $Y_{\mathcal{A}}$. In the application to graphs and the renormalization program to be presented here, combinatorics play a major role and therefore we reformulate the central objects of the last chapter in terms of a poset associated to $\mathcal{A}$. We focus on arrangements coming from graphs via Feynman rules, but note that from every given arrangement we can form the *intersection poset* to study its combinatorics.

5.1 Graphs, arrangements and posets

We start this chapter by showing how subspace arrangements and graphs give rise to partially ordered sets.

Definition 5.1. A *poset* $(\mathcal{P}, \leq)$ is a finite set $\mathcal{P}$ (we consider here only finite graphs and posets) endowed with a partial order $\leq$.

We say that p *covers* q if $p > q$ and there is no $r \in \mathcal{P}$ with $p > r > q$. The *closed interval* $[p, q] = \mathcal{P}_{[p,q]}$ is defined as the set of elements $r \in \mathcal{P}$ satisfying $p \leq r \leq q$. The *open interval* $(p, q) = \mathcal{P}_{(p,q)}$ and the subsets $\mathcal{P}_{<p}, \mathcal{P}_{\leq p}, \mathcal{P}_{>p}, \mathcal{P}_{\geq p}$ are defined similarly. We denote by $\hat{0}$ and $\hat{1}$ the unique minimal and maximal elements of $\mathcal{P}$ if they exist.

A poset is best visualized by drawing its *Hasse diagram*, a directed graph with its vertices given by the elements of $\mathcal{P}$ and edges between every pair of elements $p, q \in \mathcal{P}$ such that p covers q. Another way to encode the data of $\mathcal{P}$ is the *order complex* $\Delta(\mathcal{P})$. It is the abstract simplicial complex defined by its k-faces being the linearly ordered $k + 1$-element subsets of $\mathcal{P}$. The order

complex stores all the combinatorial information of $\mathcal{P}$ as is demonstrated by the following theorem taken from [GM87].

Definition 5.2 (Intersection lattice)**.** Let V be an n-dimensional real vector space and let $\mathcal{A} := \{A_1, \ldots, A_m\}$ be an arrangement in V. Every arrangement gives rise to a poset (actually a *lattice*, defined below) with its underlying set consisting of all possible intersections of elements in $\mathcal{A}$,

$$\mathcal{P} = \mathcal{P}(\mathcal{A}) = \{\bigcap_{i \in I} A_i \mid I \subseteq \{1, \ldots, m\}\},$$

partially ordered by reverse inclusion. It is called the *intersection poset/lattice* of $\mathcal{A}$.

In addition, $\mathcal{P}(\mathcal{A})$ is equipped with a ranking, i.e. a map $r : \mathcal{P}(\mathcal{A}) \to \mathbb{N}$ mapping each element of $\mathcal{P}(\mathcal{A})$ to the codimension of the corresponding intersection in V. Let $M(\mathcal{A}) := V \setminus \bigcup_{i=1}^m A_i$ denote the complement of the arrangement. Then the homology of $M(\mathcal{A})$ is encoded in $\Delta(\mathcal{P}(\mathcal{A}))$.

Theorem 5.3. *(Goresky, MacPherson) Let H denote the (singular) homology functor. Let $\mathcal{A}$ be an arrangement in V and let $M(\mathcal{A})$ denote the complement. Then*

$$H_k(M(\mathcal{A}), \mathbb{Z}) \cong \bigoplus_{A \in \mathcal{P}(\mathcal{A})} G_k(A),$$

where

$$G_k(A) = \begin{cases} H^{-k}(point, \mathbb{Z}) & \textit{if } A = \hat{0}, \\ H^{r(A)-k-1}(point, \mathbb{Z}) & \textit{if } A \textit{ covers } \hat{0}, \\ \tilde{H}^{r(A)-k-2}(\Delta(\mathcal{P}_{(\hat{0},A)}), \mathbb{Z}) & \textit{otherwise.} \end{cases} \qquad (5.1)$$

Recall from Chapter 3 the definition of the singular and divergent arrangements of a graph G. They give rise to corresponding intersection posets, but we can also define them directly in terms of G.

Definition 5.4. To a graph G we associate the *(saturated) graph poset* $(\mathcal{G}, \subseteq)$ consisting of the set of all saturated subgraphs of G, partially ordered by inclusion. A connected subgraph $g \subseteq G$ is *saturated* if the following holds

$$\forall t \text{ span. tree of } g : \forall e \in E(G \setminus g) : t \text{ is not a spanning for } g \cup e.$$

If g has more than one connected components, it is saturated if every component is.

In terms of the singular arrangement a saturated subgraph g is the maximal subgraph of G defining $A_g \in \mathcal{A}_{\mathcal{G}}$. This means, that adding an edge to a saturated graph necessarily changes the space A_g, while removing an edge might still define the same subspace of $(X^G)^*$.
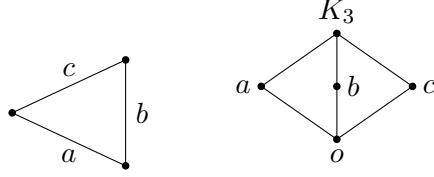
Figure 5.1: K_3 and the Hasse diagram of $\mathcal{G}(K_3)$

Example. Let K_3 be the complete graph on 3 vertices. The saturated subgraphs are the three single-edged subgraphs and K_3 itself.

Definition 5.5. The *divergent graph poset* $\mathcal{D}$ is given by the subset of $\mathcal{G}$ formed by all divergent subgraphs, partially ordered by inclusion.

Of course we can do the same for other subsets of $\mathcal{G}$, but for our purposes only $\mathcal{G}$ and $\mathcal{D}$ (and special subsets thereof) will be important. As already seen in Proposition 3.11 they carry all the information necessary for renormalization. Note that all subsets of $\mathcal{G}$ have an unique minimal element, the empty graph, which we denote by o. In our convention o is defined by $E(o) = \emptyset$.

For the divergent arrangement of a connected and at most logarithmic graph G, Theorem 5.3 allows us to compute the homology of $M(X_{\mathcal{D}}^G)$, the complement of the divergent loci in X^G. It is determined by the set of *atoms* of $\mathcal{D}$, the minimal elements in $\mathcal{D}_{>o}$. These elements are precisely the primitive subgraphs of G.

Proposition 5.6. *Let G be connected and at most logarithmic. Define n_i to be the number of atoms $g \in \mathcal{D}$ with $r(g) = \dim A_g = di$ (i.e. the primitive subgraphs on $i + 1$ vertices). Let $\alpha \in \mathbb{N}^l$ be a multi index with $\alpha_i \geq \alpha_j$ for $1 \leq i < j \leq l$ and $\|\alpha\|_1 = l$. The homology of $M(X_{\mathcal{D}}^G)$ is then given by*

$$
H_k(M(X_{\mathcal{D}}^G), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z}^{n_i} & \text{if } k = di - 1, \\ \mathbb{Z}^{\binom{n_i}{2}} & \text{if } k = 2di - 2, \\ \mathbb{Z}^{n_{i_1} n_{i_2}} & \text{if } k = d(i_1 + i_2) - 2, \\ \ldots \\ \mathbb{Z}^{\prod_{j=1}^{l} \binom{n_{i_j}}{\alpha_j}} & \text{if } k = d\sum_{j=1}^{l} \alpha_j i_j - l, \\ \ldots \end{cases} \tag{5.2}
$$

Proof. The atoms of $\mathcal{D}$ determine the topology of the complement because the corresponding subspaces $A_g^\perp$ contain all other divergent subspaces.

38

Abbreviate $H_k(M(X_{\mathcal{D}}^G), \mathbb{Z})$ by H_k. Using Theorem 5.3 we have $H_0 = \mathbb{Z}$. Moreover, there is a generator in H_k with $k = r(g) - 1 = di - 1$ for every atom g such that $r(g) = di$. For an element γ that is given by the union of atoms we have to use the third row in Equation (5.1): If γ is the union of two atoms g and h, the subcomplex $\Delta(\mathcal{D}_{(o,\gamma)})$ consists of 2 disconnected points (representing the two atoms). Therefore we have $\binom{n_i}{2}$ generators in dimension $k = 2di - 2$ if $r(g) = r(h) = di$ and $n_{i_1} n_{i_2}$ generators in dimension $k = d(i_1 + i_2) - 2$ if $r(g) = di_1$ and $r(h) = di_2$. If γ is the union of $l > 2$ atoms, the interval (o, γ) consists of these atoms and all unions thereof. It is the face poset of the standard $(l-1)$-simplex $\triangle^{l-1}$ with interior removed. Thus, $\Delta(\mathcal{D}_{(o,\gamma)}) = \Delta(\mathcal{F}(\partial \triangle^{l-1})) \cong \partial \triangle^{l-1} \cong S^{l-2}$. Since $\tilde{H}^k(S^{l-2})$ equals $\mathbb{Z}$ if $k = l - 2$ and is trivial else, we conclude that there are generators in H_k coming from such elements γ if $k = r(\gamma) - l$. Let $\alpha \in \mathbb{N}^l$ with $\alpha_i \geq \alpha_j$ for $1 \leq i < j \leq l$ and $\|\alpha\|_1 = l$. If $r(\gamma) = d \sum_{j=1}^l \alpha_j i_j$ then such γ can be formed out of l atoms in $\prod_{j=1}^l \binom{n_{i_j}}{\alpha_j}$ possible ways and (5.2) follows. $\qquad\square$

5.2 The divergent graph lattice

From now on let G always be connected. We continue studying the graph poset $\mathcal{G}$ in more detail. As it turns out it has extra structure, it is a *lattice*.

Definition 5.7 (Lattices)**.** Let $(\mathcal{P}, \leq)$ be a poset and $p, q \in \mathcal{P}$. A *least upper bound* or *join* of p and q is an upper bound r for both elements such that every other upper bound s satifies $r \leq s$. If the join of p and q exists, it is unique and denoted by $p \vee q$.

Dually one defines a *greatest lower bound* or *meet* of two elements p and q in $\mathcal{P}$, denoted by $p \wedge q$.

$\mathcal{P}$ is called a *join-semilattice* (*meet-semilattice*) if for all $p, q \in \mathcal{P}$ the join $p \vee q$ (the meet $p \wedge q$) exists. $\mathcal{P}$ is called a *lattice* if it is both a join- and a meet-semilattice.

For any arrangement $\mathcal{A}$ the intersection poset $\mathcal{P}(\mathcal{A})$ is a lattice: If one orders the elements of $\mathcal{P}(\mathcal{A})$ by reverse inclusion then the join operation is just given by set theoretic intersection. The statement then follows from the fact that every finite join-semilattice with $\hat{0}$ (represented by the empty intersection, the ambient space V) is a lattice (Proposition 3.3.1 in [Sta97]).

Regarding the definition of the partial order by inclusion or reverse inclusion there are different conventions used in the literature. Both have their advantages and can be converted into the other since the dual of any lattice, i.e. the lattice with reversed order, is a lattice as well. We use reverse inclusion because it matches the convention in [CP95] using arrangements in the dual and it fits with the natural partial order on subgraphs.

Since $\mathcal{G}$ is the intersection poset of the (dual) singular arrangement $\mathcal{A}_\mathcal{G}$ in X^G, it is a lattice. Clearly, if $\mathcal{P} \subseteq \mathcal{G}$ is closed under union and intersection, it is the intersection lattice of some corresponding arrangement. This is the case for the set of divergent subgraphs:

Proposition 5.8. *Let G be at most logarithmic. Then $(\mathcal{D}, \subseteq)$ is a lattice.*

Proof. For $g, h \subseteq G$ divergent subgraphs we define the join and meet operations in $\mathcal{D}$ by

$$g \vee h := g \cup h$$
$$g \wedge h := g \cap h$$

Suppose g and h have k shared edges and l shared loops. Moreover, assume that m new loops are created by uniting g and h. In formulae:

$$h_1(g \cup h) = h_1(g) + h_1(h) + m - l$$
$$E(g \cup h) = E(g) + E(h) - k.$$

From this we conclude that the superficial degree of divergence of $g \cup h$ is given by

$$\omega(g \cup h) = d(h_1(g) + h_1(h) + m - l) - 2(e(g) + e(h) - k)$$
$$= d(m - l) + 2k \overset{!}{\leq} 0.$$

Split $k = k_l + k_0$ into edges in the shared loops and those that are not. Then $dl \leq 2k_l$ and we conclude

$$\omega(g \cup h) \geq dm + 2k_0 \geq 0$$

Thus $m = k_0 = 0$ and

$$0 \geq \omega(g \cup h) = dl - 2k_l = \omega(g \cap h) \leq 0.$$

Therefore $g \cup h$ and $g \cap h$ are both divergent subgraphs of G. Clearly, they are the minimal (maximal) elements of $\mathcal{D}$ bounding g and h from above (below). $\qquad\square$

With the methods used in the above proof we are able to show another property of $\mathcal{G}$ and $\mathcal{D}$. They are *graded* lattices.

Definition 5.9. A poset $(\mathcal{P}, \leq)$ is *graded* if it is equipped with a map $\tau : \mathcal{P} \to \mathbb{N}$ that has the following two properties: τ is order preserving with respect to the natural order on $\mathbb{N}$ and if there are $p, q \in \mathcal{P}$ with p covering q, then $\tau(p) = \tau(q) + 1$.

Proposition 5.10. *For any connected graph G the graph lattice $\mathcal{G}$ is graded.*

Proof. The map τ associates to every saturated subgraph $g \subseteq G$ the value $d^{-1} \dim A_g = V(g) - c_g$ (c_g denoting the number of connected components of g, cf. the proof of Proposition 3.11). Clearly, τ is order preserving and $\tau(p) = \tau(q)+1$ for p covering q holds because of the saturated condition. $\square$

Proposition 5.11. *Let G be at most logarithmic. Then $\mathcal{D}$ is a graded lattice.*

To prove this we use Proposition 3.3.2 from [Sta97].

Proposition 5.12. *Let $\mathcal{L}$ be a finite lattice. The following two conditions are equivalent:*
1. $\mathcal{L}$ is graded and the map τ satisfies $\tau(x) + \tau(y) \geq \tau(x \wedge y) + \tau(x \vee y)$ for all $x, y \in \mathcal{L}$.
2. If x and y both cover $x \wedge y$, then $x \vee y$ covers both x and y.

Proof of Proposition 5.11 . We argue by contradiction: Let $g, h \subseteq G$ be divergent and suppose there is a $\gamma \in \mathcal{D}$ with $g < \gamma < g \vee h$, i.e. $g \vee h$ does not cover both g and h. First, note that $\gamma \cap h \neq \emptyset$ because otherwise γ would not be a subgraph of $g \vee h$. From Proposition 5.8 we know that $\gamma \cap h$ is divergent. But then $g \wedge h < \gamma \wedge h < h$, which means h is not covering $g \wedge h$. $\square$

We will not use this here but for the sake of completeness we mention one additional property of $\mathcal{D}$. From a combinatorial viewpoint *distributive* lattices are important because this extra structure allows one to prove many powerful theorems, for example Birkhoff's famous representation theorem [Bir67].

Proposition 5.13. *Let G be at most logarithmic. Then $\mathcal{D}$ is a distributive lattice, i.e.*

$$\begin{aligned} f \vee (g \wedge h) &= (f \vee g) \wedge (f \vee h), \\ f \wedge (g \vee h) &= (f \wedge g) \vee (f \wedge h), \end{aligned}$$

for all f, g, h in $\mathcal{D}$.

Proof. Since one of the properties implies the other ,we will only proof the first one. Moreover, the proof works exactly the same in the second case.

Let $f, g, h \subseteq G$ be divergent. Compare the edge set of the graphs on the left and the right:

$$\begin{aligned} E\big(f \vee (g \wedge h)\big) &= E\big(f \cup (g \cap h)\big) = E(f) \cup E(g \cap h) \\ &= \big(E(f) \cup E(g)\big) \cap \big(E(f) \cup E(h)\big) \\ &= E(f \cup g) \cap E(f \cup h) = E\big((f \cup g) \cap (f \cup h)\big) \\ &= E\big((f \vee g) \wedge (f \vee h)\big). \end{aligned}$$

$\square$

5.3 Wonderful models revisited

In this section we reformulate wonderful models in terms of the graph lattice $\mathcal{G}$. This is based on [Fei05] where a combinatorial version of the wonderful model construction is developed for any (finite) lattice $\mathcal{L}$.

In general we can associate to every arrangement the corresponding intersection lattice defined in the previous section. It is the combinatorics of this lattice that reflect the topological properties of the wonderful models as seen for example in Theorem 5.3. Another example is the following theorem by Feichtner that relates combinatorial and geometric wonderful models via a *combinatorial blow-up* (Definition 3.5 and Theorem 3.6 in [Fei05]).

Theorem 5.14. *Let $\mathcal{L}$ be an intersection lattice, $\mathcal{B}$ a combinatorial building set in $\mathcal{L}$, and $B_1, \ldots, B_t$ a linear order on $\mathcal{B}$ that is non-increasing with respect to the partial order on $\mathcal{L}$. Then consecutive combinatorial blowups in $B_1, \ldots, B_t$ result in the face poset of the nested set complex $\Delta_{\mathcal{N}}(\mathcal{L}, \mathcal{B})$,*

$$Bl_{G_t}(\ldots (Bl_{G_2}(Bl_{G_1}))) = \mathcal{F}(\Delta_{\mathcal{N}}(\mathcal{L}, \mathcal{B})).$$

Although the following definitions apply to any lattice $\mathcal{L}$, to connect with Section 3.3 think of $\mathcal{L}$ as being given by the singular or divergent arrangement of a connected and at most logarithmic graph G. We define the central notions of the wonderful construction in combinatorial language following [Fei05] where the interested reader finds a thorough exposition of the wonderful models from a combinatorial geometer's viewpoint. In this case building sets and nested sets are certain subposets of $\mathcal{L}$ (a subposet of a poset $(\mathcal{P}, \leq)$ is a subset of $\mathcal{P}$ with the induced partial order). In some cases these subsets are even lattices, although not necessarily sublattices since the meet and join operations need not be induced by the corresponding operations on $\mathcal{L}$.

Definition 5.15 (Combinatorial building sets)**.** Let $\mathcal{L}$ be a lattice. A non-empty subset $\mathcal{B}$ of $\mathcal{L}$ is a *combinatorial building set* for $\mathcal{L}$ if the following holds: For all $p \in \mathcal{L}_{>\hat{0}}$ and $\{q_1, \ldots, q_k\} = \max \mathcal{B}_{\leq p}$ there is an isomorphism of posets

$$\varphi_p : \prod_{i=1}^{k} [\hat{0}, q_i] \longrightarrow [\hat{0}, p] \tag{5.3}$$

with $\varphi_p(\hat{0}, \ldots, q_j, \ldots, \hat{0}) = q_j$ for $j = 1, \ldots, k$.

This defines combinatorial building sets which are more general than the building sets introduced in Chapter 4. To get the notion of a *geometric building set* according to the construction of DeConcini and Procesi we have to demand an additional geometric compatibility condition.

Definition 5.16 (Geometric building sets)**.** We call $\mathcal{B}$ a *geometric building set* for $\mathcal{L}$ if it is a combinatorial building set and

$$\dim A_p = \sum_{i=1}^k \dim A_{q_i}.$$

Note that if $\mathcal{L} \subseteq \mathcal{G}$, since $\dim A_g = d(|V(g)| - 1)$ (or $d(|E(g)| - h_1(g))$ if g is divergent), we can express this geometric condition also purely in graph theoretic terms.

Example. For every lattice $\mathcal{L}$ itself is a building set, the *maximal building set*. The *minimal building set* is given by the irreducible elements of $\mathcal{L}$. It is formed by all $p \in \mathcal{L}$ for which there is no product decomposition as in (5.3) of the interval $[\hat{0}, p]$. We denote this building set by $I(\mathcal{L})$.

The geometric condition gives a handy criterion to check whether a given element is irreducible or not.

Lemma 5.17. *Let $\mathcal{L} \subseteq \mathcal{G}$ be a lattice. Let $g \in \mathcal{L}$ be the union of irreducible subgraphs $g = g_1 \cup \cdots \cup g_k$ with non-empty overlap $h = g_1 \cap \cdots \cap g_k$. W.l.o.g. assume that the g_i are maximal with this property. Then g is irreducible.*
Vice versa, for every reducible element $g \in \mathcal{L} \setminus I(\mathcal{L})$ we have that g is the union of some $g_1, \ldots, g_k \in I(\mathcal{L})$ with

$$\bigcap_{i=1}^k g_i = \bigcup_{v \in V'} v \sim o$$

for some vertex set $V' \subseteq V(G)$.

Proof. Write $d(g)$ for $\dim A_g$. If g would be reducible, then $d(g) = \sum_{i=1}^k d(g_i)$ because the g_i form the set max $\mathcal{I}(\mathcal{L})_{\leq g}$. On the other hand, $d(g) = \sum_{i=1}^k d(g_i) - d(h)$ - the sum can not be direct because of the overlap h. Thus, $d(h) = 0$, i.e. $A_h = \{0\}$ which means $h = o$.

The second statement follows from the same argument. The geometric condition for reducibility $d(g) = \sum_{i=1}^k d(g_i)$ cannot hold if the g_i have common edges. $\qquad\square$

Recall that the choice of a building set $\mathcal{B}$ determines the structure of the exceptional divisor $\mathcal{E}$ in the wonderful model; the elements of $\mathcal{B}$ control the number of components of $\mathcal{E}$ and how they intersect. To construct $Y_\mathcal{B}$ explicitly we need another family of sub(po)sets of $\mathcal{B}$, the $\mathcal{B}$-*nested sets*.

Definition 5.18 (Nested sets)**.** Let $\mathcal{B}$ be a building set in a lattice $\mathcal{L}$. A subset $\mathcal{N} \subseteq \mathcal{B}$ is $\mathcal{B}$-*nested* if for all subsets $\{p_1, \ldots, p_k\} \subseteq \mathcal{N}$ of pairwise incomparable elements the join (in $\mathcal{L}$!) $p_1 \vee \ldots \vee p_k$ exists and does not belong to $\mathcal{B}$.

With nested sets we can build another abstract simplicial complex, the *nested set complex* $\Delta_\mathcal{N}(\mathcal{L}, \mathcal{B})$. Its k-faces consist of the $\mathcal{B}$-nested sets with $k + 1$ elements. It is the generalization of the order complex for non-maximal building sets. For the maximal building set $\mathcal{B} = \mathcal{L}$ a subset is nested if and only if it is linearly ordered in $\mathcal{B}$, so that in this case we have $\Delta(\mathcal{L}) = \Delta_\mathcal{N}(\mathcal{L}, \mathcal{B})$. By Theorem 4.9 it contains all the information about the stratification of the exceptional divisor $\mathcal{E}$ in $Y_\mathcal{B}$.

Since $\mathcal{D}$ is a graded lattice, we have proven here a little conjecture (in the case G at most logarithmic) that appears in many texts on Hopf algebraic renormalization (for example [BK08]):

Corollary 5.19. *Every maximal forest of a graph G has the same cardinality.*

Proof. In the language of posets this translates into the fact that every maximal nested set has equal cardinality. But this is equivalent to $\mathcal{D}$ being graded because the grading map τ forbids maximal linearly ordered subsets of different length. $\square$
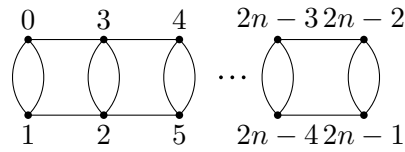
Example. Here are some examples, all in $d = 4$:

Figure 5.2: The n-bubble graph

1. Let G^n be the graph in Figure 5.2. Here the index n stands for the number of atoms, the one-loop *fish* subgraphs on two edges, and the numbering of vertices is chosen to match the most "natural" choice of an adapted spanning tree t (see Definition 5.21).

   Let g_l^k denote the full subgraph of G^n given by the vertex set $V(g_l^k) = \{2l - 2, \ldots, 2l - 2 + 2k - 1\}$. From the fact that $\mathcal{D}(G^{n+1})$ contains two copies of $\mathcal{D}(G^n)$, given by the intervals $[o, g_1^n]$ and $[o, g_2^n]$, and Lemma 5.17, it follows by induction that

   $$I(\mathcal{D}(G^n)) = \{g_l^k \subseteq G^n \mid k = 1, \ldots, n \text{ and } l = 1, \ldots, n - k + 1\}.$$

2. Next we look at the graph G_n, depicted in Figure 5.3, constructed by a sequence of n insertions of the fish into itself. Here minimal and maximal building set coincide because all divergent subgraphs are nested into each other:

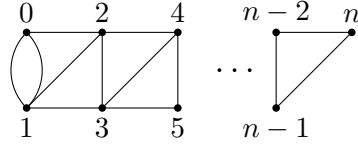 $$\mathcal{D}(G_n) = I(\mathcal{D}(G_n)) = \{g_1, g_2, \ldots, g_n = G_n\}$$

44

Figure 5.3: The n-insertions graph

where g_i is the full subgraph of G^n corresponding to the vertex set $\{0,\ldots,i\}$. The partial order is a total order. Thus, the $\mathcal{D}(G_n)$-nested sets are all non-empty subsets of the power set $\mathcal{P}\left(\mathcal{D}(G^n)\right)$.

3. Let $G^{n,m}$ be the graph obtained by inserting n bubbles on the left and m bubbles on the right into the fish graph (Figure 5.4). Here

$$I(\mathcal{D}(G^{n,m})) = \{g_1,\ldots,g_n,h_1\ldots,h_m,G^{n,m}\}$$

where g_i is the fish subgraph on the vertex set $\{i-1,i\}$ for $i \in \{1,\ldots,n\}$ and h_j is the fish subgraph on the vertex set $\{n+j,n+j+1\}$ for $j \in \{1,\ldots,m\}$. All subgraphs in $I(\mathcal{D}(G^{n,m}))\backslash\{G^{n,m}\}$ have disjoint edge sets. Therefore, as in the previous example, the $I(\mathcal{D}(G^{n,m}))$-nested sets are all non-empty subsets of $\mathcal{P}\left(I(\mathcal{D}(G^{n,m}))\right)$.
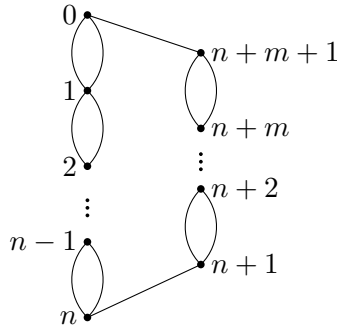

Figure 5.4: The n,m-bubble graph

4. For $n > 0$ let $G = K_{n+1}$ be the complete graph on $n+1$ vertices. By induction it follows that saturated subgraphs are either disjoint unions or complete subgraphs on their respective vertex set. Thus, if $n = 2$ then $\mathcal{G} = I(\mathcal{G})$. For $n = 3$ (Figure 5.5) the three subgraphs given by the disjoint union of edges $a \,\dot\cup\, c, b \,\dot\cup\, d$ and $e \,\dot\cup\, f$ are reducible while the four embeddings of K_3 given by $a \cup b \cup f$ etc. are irreducible ($|[o, a \cup b \cup f]| = 5$ is not divisible by two).

In general, $I(\mathcal{G}(K_{n+1}))$ consists of all subgraphs that are embeddings of K_i into K_{n+1} for $i = 1,\ldots,n$ while the reducible subgraphs are the

disjoint unions of embeddings of K_i and K_j for $i + j \leq n + 1$. These disjoint unions represent the polydiagonals that make the difference in the blow-up sequence of the Fulton-MacPherson compactification $M[n]$ and Ulyanov's polydiagonal compactification $M\langle n \rangle$.
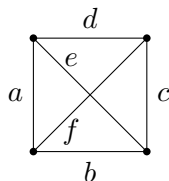

Figure 5.5: K_4

It remains to define the combinatorial version of adapted bases. For this we need *adapted spanning trees*.

Definition 5.20. Let G be a connected graph. A *spanning tree* for G is a simply-connected subgraph $t \subseteq G$ with $V(t) = V(G)$. If G is not connected, $G = G_1 \dot\cup \ldots \dot\cup G_n$, a *spanning n-forest* for G is the disjoint union $t = t_1 \dot\cup \ldots \dot\cup t_n$ of n spanning trees t_i for G_i.

Definition 5.21 (Adapted spanning trees)**.** Let G be a graph and $\mathcal{P} \subseteq \mathcal{G}$ a family of subgraphs of G. A spanning tree t of G is $\mathcal{P}$-*adapted* if for each $g \in \mathcal{P}$ the graph t_g, defined by $E(t_g) := E(t) \cap E(g)$ is a spanning tree for g. More precisely, if g is not connected, then we demand t_g to be a spanning forest for g.

Example. For dunce's cap an $\mathcal{D}$-adapted spanning tree ($d = 4$) is given by $E(t) = \{e_1, e_3\}$ or $E(t) = \{e_2, e_4\}$, while t with $E(t) = \{e_1, e_2\}$ is spanning but not adapted.

Proposition 5.22. *A $\mathcal{D}$-adapted spanning tree always exists for G at most logarithmic.*

Proof. We construct t using the fact that divergent graphs can be built from primitive ones using the insertion operation. Moreover, this process is reversible, i.e. in the dual process of contracting subgraphs no information is lost.

Start with the primitive subgraphs of G and let G_1 be the graph obtained from G by contracting all these primitive subgraphs. G_1 might have primitive subgraphs itself (the $g \in \mathcal{D}$ with *coradical degree* equal to two, cf. [Kre13]). Repeat the process. After a finite number of steps G_k will be free of subdivergences. Now choose a spanning tree t_1 for G_k and spanning trees for all subgraphs contracted in the step from G_{k-1} to G_k. Then t_2,

46

the union of all these spanning trees, is a tree in G_{k-1} visiting every vertex exactly once. Thus, it is a spanning tree for G_{k-1}. Repeat this process until after k steps we have an $\mathcal{D}$-adapted spanning tree $t = t_k$ of G. $\qquad\square$

Remark. An interesting question arising here is the following: For which families $\mathcal{P} \subseteq \mathcal{G}$ does such a $\mathcal{P}$-adapted spanning tree exist? For a counterexample just take $\mathcal{P} = \mathcal{G}$ or $I(\mathcal{G})$: In the first case every edge of G lies in $\mathcal{G}$, so there cannot exist a $\mathcal{G}$-adapted spanning tree. For the second case consider the example K_4; there is no spanning tree that generates all four irreducible "triangle" subgraphs. Another question is for which class of graphs this holds, i.e. if the assumption of G being at most logarithmic can be dropped.

With adapted spanning trees we are able to define $\mathcal{N}$-adapted bases of $(X^G)^*$ in combinatorial terms. If the divergent lattice is considered, a $\mathcal{D}$-adapted spanning tree will automatically be $\mathcal{N}$-adapted for any nested set of any building set in $\mathcal{D}$. This allows us to fix a convenient basis from the beginning on. Every spanning tree t of G has $|V| - 1$ edges (otherwise it would contain a loop, contradicting simply-connectedness). Therefore, for every spanning tree t of G (with the same orientation) we have a linear map $\psi_t : M^{E(t)} \to M^{V'}$ defined by

$$
e \mapsto \begin{cases} v_j - v_i & \text{if } e \text{ starts at } v_i \text{ and ends } v_j, \\ \pm v_k & \text{if } e \text{ connects } v_0 \text{ to } v_k. \end{cases} \tag{5.4}
$$

Pulling back v_G along ψ_t amounts to a linear change of coordinates on X^G (as well as altering the numbering of the vertices of G, its orientation or the choice of a different (adapted) spanning tree). Any automorphism of X^G will not change the topology of the arrangement and, as is shown in [CP95], induces an isomorphism on the corresponding wonderful models. Therefore the wonderful construction and renormalization do not depend on these choices and we can work in a convenient basis given by an adapted spanning tree.

In this basis v_G is given by

$$
v_G(\{x_e\}_{e \in E(t)}) = \prod_{e \in E(t)} \triangle(x_e) \prod_{e \in E(G \setminus t)} \triangle(\sum_{e' \in E(t_e)} \sigma_t(e') x_{e'})
$$

where t_e is the path in t connecting the source and target vertices of e and $\sigma_t : E(t) \to \{-1, +1\}$ is determined by the chosen orientation of G. The point is that for the divergent poset $\mathcal{D}$ in these coordinates $x = \sum_{e \in E(t)} x_e e$ we have $A_g^\perp = \{x_e = 0 \mid e \in E(t_g)\}$ for all $g \in \mathcal{D}$. Dually this means that the elements in $B|_{e \in E(t_g)}$, defined below, form a basis of A_g. In other words, we have an adapted basis in the sense of DeConcini-Procesi!

By duality B also defines a basis of X^G. By abuse of notation we will denote both bases by B - the meaning should always be clear from the context. This choice of basis will be important when we study the pullback of

v_G onto the wonderful model in the next chapter.

For $\mathcal{G}$ and other lattices there need not be an adapted spanning tree, but we can always find an $\mathcal{N}$-adapted spanning tree for any nested set.

Proposition 5.23. *Let $\mathcal{N}$ be nested for some building set $\mathcal{B}$ in some lattice $\mathcal{L} \subseteq \mathcal{G}$. Then there exists an $\mathcal{N}$-adapted spanning tree.*

Proof. The idea is the same as in Proposition 5.22. Start with the set $\mathcal{M}$ of maximal elements in $\mathcal{N}$ and contract all other elements. Pick a spanning tree for the resulting graphs. Proceed in the same manner with $\mathcal{N} \setminus \mathcal{M}$ and repeat the process until all of $\mathcal{N}$ has been exhausted. This produces a spanning forest t for $\cup_{\gamma \in \mathcal{N}} \gamma$, except if there are g, h in $\mathcal{N}$ that are non-comparable and have non-empty intersection. In this case we argue like in the proof of Lemma 5.17 to see that the join $g \vee h$ must also be in $\mathcal{B}$. But this is impossible since $\mathcal{N}$ is $\mathcal{B}$-nested. In a last step contract all elements of $\mathcal{N}$ in G and pick a spanning tree t' for the resulting graph. The union $t \cup t'$ is then an $\mathcal{N}$-adapted spanning tree for G. □

Definition 5.24. Let G be at most logarithmic and $\mathcal{N}$ a $\mathcal{B}$-nested set for some building set $\mathcal{B}$ in a lattice $\mathcal{L} \subseteq \mathcal{G}$. Given an $\mathcal{N}$-adapted spanning tree t define the map ψ_t as in Equation 5.4. Together with the linear forms ω_e introduced in Section 3.3 we define an $\mathcal{N}$-*adapted* basis of $(X^G)^*$ by

$$B := \{b_e^i := (\omega_e \circ \psi_t)^i \mid e \in E(t), \ i = 1, \ldots, d\}.$$

In such a basis the map $p : (X^G)^* \to \mathcal{N} \cup \{G\}$ from Definition 4.6 is then given by

$$p : x = \sum_{\substack{e \in E(t) \\ i = 1, \ldots, d}} x_e^i b_e^i \longmapsto \min\{g \in \mathcal{N} \cup \{G\} \mid x_e^i = 0 \text{ for all } e \in E(t \setminus t_g)\}.$$

A *marking* of an adapted basis is for every $g \in \mathcal{N}$ the choice of a $b_g^{i_g} \in B$ with $p(b_g^{i_g}) = g$. Equivalently, we can view it as a labelling on the elements of $\mathcal{N}$:

$$g \longmapsto b_e^j \text{ for } e \in E(t \cap (g \setminus \mathcal{N}_{<g})) \text{ and some } j \in \{1, \ldots, d\}$$

where $g \setminus \mathcal{N}_{<g} := g \setminus (h_1 \cup \cdots \cup h_k)$ for $\{h_1, \ldots, h_k\} = \{h \in \mathcal{N} \mid h < g\}$ denotes the graph g with all its lower bounds in $\mathcal{N}$ removed.

Finally, the partial order $\preceq$ on B that determines the local blow-up $\rho_{\mathcal{N}, \mathcal{B}}$ is given by

$$b_e^i \preceq b_{e'}^j \iff e \in E(t_g), e' \in E(t_{g'}) \text{ with } g \subseteq g' \text{ and } b_{e'}^j \text{ is marked.}$$

This finishes all necessary definitions and from here on we could repeat the construction of a wonderful model in purely combinatorial terms. In the divergent case, $\mathcal{A} = \mathcal{D}$, we thus conclude that all ingredients are already determined by the topology and subgraph structure of G. Therefore there is really no need for purely geometric data to build an atlas for $Y_{\mathcal{B}}$. However obtained, now after the planting has been done, it is time to reap the fruits and see what a wonderful model can do for us.

Chapter 6

Wonderful renormalization

Having constructed the wonderful models $(Y_{\mathcal{A}}, \beta)$ for general arrangements $\mathcal{A}$, we now focus on the divergent and singular arrangements $\mathcal{A} = \mathcal{A}_{\mathcal{D}}, \mathcal{A}_{\mathcal{G}}$ of a connected and at most logarithmic graph G. We study the pullback of v_G onto the model and the pole structure of its Laurent expansion, then define (local) renormalization operators. Since $Y_{\mathcal{A}}$ is non-orientable, we work from now on with distribution densities.

Throughout this chapter let G be connected and at most logarithmic. To keep the notation from exploding we drop the indices where possible. We write $x = (x_1, \ldots, x_n)$ for a point in $X = M^n$ where $x_i = (x_i^1, \ldots, x_i^d)$. The marking of an adapted basis assigns individual coordinates to the elements of a nested set of graphs. We denote the marked elements by $\mathcal{N} \ni g \mapsto x_g^{i_g}$. If a vector $x_g \in M$ is marked in this way, let $\hat{x}_g$ denote the vector

$$\hat{x}_g = (x_g^1, \ldots, x_g^{i-1}, 1, x_g^{i+1}, \ldots, x_g^d) \in M.$$

The first two sections follow the exposition in [BBK10], especially the proofs of Proposition 6.1 and Theorem 6.4. The difference lies in the emphasis on the combinatorics of $\mathcal{D}$ and the role of adapted spanning trees in our formulation. In addition, we correct some minor flaws and fill out missing details in the proofs.

6.1 The pullback of v_G onto the wonderful model

Let (Y, β) be a wonderful model and $v = v_G$ the Feynman distribution associated to a graph G. We start the renormalization program by disassembling the pullback of v onto Y into a regular and a singular part.

Proposition 6.1. *Let $\mathcal{N}$ be $\mathcal{B}$-nested for a building set $\mathcal{B}$ of $\mathcal{D}$ (or $\mathcal{G}$) and B an adapted, marked basis. In local coordinates on $U_{\mathcal{N},B}$ the pullback of $\tilde{v}^s := v^s |dx|$ along β is given by*

$$(\tilde{w}^s)_{\mathcal{N},B} := (\beta^* \tilde{v}^s)_{\mathcal{N},B} = f_{\mathcal{N},B}^s \prod_{g \in \mathcal{N}} u_g^{-1+d_g+s(2-d)|E(g)|} |dx| \qquad (6.1)$$

where $u_g(x_g^{i_g}) = |x_g^{i_g}|^{-1}$ and $d_g := \dim A_g = d(|E(g)| - h_1(g))$.

The map $f_{\mathcal{N},B} : \kappa_{\mathcal{N},B}(U_{\mathcal{N},B}) \longrightarrow \mathbb{R}$ is in $L^1_{loc}(\kappa_{\mathcal{N},B}(U_{\mathcal{N},B}))$ (or in $\mathcal{C}^\infty(\kappa_{\mathcal{N},B}(U_{\mathcal{N},B}))$ if the singular arrangement is considered) but smooth in the variables $x_g^{i_g}$, $g \in \mathcal{N}$.

As mentioned above, we drop indices to keep the notation minimal. Therefore, and because we always work in local coordinates, from now on we do not indicate local expressions by the subscript $\mathcal{N}, B$. In addition, local coordinates are always given by an adapted spanning tree t.

Proof. The crucial point here is that locally β is given by the map

$$\rho : X \longrightarrow X,$$

$$\sum_{i=1}^d \sum_{e \in E(t)} x_e^i b_e^i \mapsto \sum_{i=1}^d \sum_{e \in E(t)} \prod_{x_e^i \preceq x_{e'}^k} x_{e'}^k b_e^i.$$

Recall the choice of coordinates given by t (Equation 5.4). In these coordinates

$$\beta^* v^s(x) = v^s(\rho(x)) = \left(\rho^* \left(I^* \triangle^{s \otimes |E(G)|} \right) \right)(x)$$

$$= \rho^* \Big(\prod_{e \in E(t)} \triangle^s(x_e) \prod_{e \in E(G \setminus t)} \triangle^s \Big(\sum_{e' \in E(t_e)} \sigma_t(e') x_{e'} \Big) \Big)$$

$$= \prod_{e \in E(t)} \triangle^s \Big(\prod_{p(x_e) \subseteq g} x_g^{i_g} \hat{x}_e \Big) \prod_{e \in E(G \setminus t)} \triangle^s \Big(\sum_{e' \in E(t_e)} \prod_{p(x_{e'}) \subseteq g} x_g^{i_g} \sigma_t(e') \hat{x}_{e'} \Big)$$

where $\hat{x}_e := (x_e^1, \ldots, \overset{i_g}{\overbrace{1}}, \ldots, x_e^d)$ if x_e has a marked component. Since $\triangle$ is homogeneous of degree $(2 - d)$, we can pull out all the factors $x_g^{i_g}$ in the first product of $\triangle$'s, so that the kernel of $\tilde{w}^s(x)$ is given by

$$\prod_{e \in E(t)} \Big(\prod_{p(x_e) \subseteq g} x_g^{i_g} \Big)^{s(2-d)} \triangle^s(\hat{x}_e) \prod_{e \in E(G \setminus t)} \triangle^s \Big(\sum_{e' \in E(t_e)} \prod_{p(x_{e'}) \subseteq g} x_g^{i_g} \sigma_t(e') \hat{x}_{e'} \Big).$$

In the second factor we can pull out a factor $x_g^{i_g}$ if it appears in every term in the sum, i.e. if t_e is a subgraph of some $g \in \mathcal{N}$. But this is equivalent to $e \in E(g)$ because t is an adapted spanning tree. Thus, $x_g^{i_g}$ appears exactly $|E(g)|$-times and we conclude

$$w^s(x) = f^s(x) \prod_{g \in \mathcal{N}} (x_g^{i_g})^{s(2-d)|E(g)|}.$$

Under the coordinate transformation ρ every $dx_e = \bigwedge_{i=1,\ldots,d} dx_e^i$ transforms into

$$\rho^* dx_e = \begin{cases} (x_g^{i_g})^{d-1} dx_e & \text{if } x_e \text{ contains a marked component,} \\ (x_g^{i_g})^d dx_e & \text{if } x_e \text{ has no marked component.} \end{cases}$$

How many x_e are scaled by the same $x_g^{i_g}$? As many as there are edges in $E(g)$. Therefore, there are in total $(\dim A_g - 1)$ factors and the measure

$$|dx| = |\bigwedge_{e \in E(t), i=1,\ldots,d} dx_e^i|$$

tranforms into

$$\rho^* |dx| = \prod_{g \in \mathcal{N}} |x_g^{i_g}|^{\dim A_g - 1} |dx|.$$

Putting everything together we conclude

$$\tilde{w}^s(x) = f^s(x) \prod_{g \in N} |x_g^{i_g}|^{-1 + d_g + s(2-d)|E(g)|} |dx|.$$

For the divergent lattice the exponents of $|x_g^{i_g}|$ are given by $-1 + d_g(s-1)$ because $d|h_1(g)| = 2|E(g)|$ and $d_g = \dim A_g = d(|E(g)| - h_1(g))$ (cf. the proof of Proposition 3.11).

It remains to show that $f \in L^1_{\text{loc}}(\kappa(U))$ or $\mathcal{C}^\infty(\kappa(U))$, respectively. Recall the definition of $U = U_{\mathcal{N},B} = X \setminus \cup_{\gamma \in \mathcal{B}} Z_\gamma$ where Z_γ is the vanishing locus of the polynomials P_v for $v \in X^*$ such that $p(v) = \gamma$. For the singular arrangement every building set $\mathcal{B}$ must contain all subgraphs consisting of a single edge. But for these elements of $\mathcal{B}$ the $Z_\gamma = Z_e$ are precisely the sets where an entire sum $\sum_{e' \in E(t_e)} \sigma_t(e') x_{e'}$ expressing an edge e of G vanishes. Since all functions $\triangle$ are smooth off the origin it follows that f is a smooth function. The same reasoning works for the divergent arrangement. Every building set of $\mathcal{D}$ must contain all irreducible subgraphs. In addition, every element of $\mathcal{D}$ is built out of elements of $I(\mathcal{D})$ by the join operation (i.e. using $\cup$). Therefore, as in the singular case, it follows that linear combinations expressing edges in any divergent subgraph can not vanish on U. The map f fails to be smooth only at propagators of edges that do not lie in some element of $\mathcal{B}$. But by the proof of Proposition 3.11 we know that there f is still locally integrable, hence $f \in L^1_{\text{loc}}(\kappa(U))$.

Smoothness in the marked elements $x_g^{i_g}$, $g \in \mathcal{N}$, follows from the simple fact that in the definition of f already all marked elements have been pulled out of the linear combinations expressing edges in G. If one such expression would vanish at $x_g^{i_g} = 0$, then all $x_{e'}$ were scaled by $x_g^{i_g}$ and this factor would have been absorbed into the exponent of u_g. Therefore no argument in the product of $\triangle$'s can vanish at $x_g^{i_g} = 0$. $\qquad \square$

6.2 Laurent expansion

From now on we will consider the divergent lattice $\mathcal{D}$ only. In this case we define $u_g^s(x_g^{i_g}) := |x_g^{i_g}|^{-1 + d_g(1-s)}$ and for a finite product of maps F_i, $i \in I$,

we write $F_I := \prod_{i \in I} F_i$. Then we have under the assumptions of Proposition 6.1

$$\tilde{w}^s = f^s \prod_{g \in \mathcal{N}} u_g^s |dx| = f^s u_\mathcal{N}^s |dx|.$$

To define local renormalization operators we need a better understanding of the pole structure of $\tilde{w}^s$. As it turns out, this structure is already encoded in the geometry of the exceptional divisor $\mathcal{E}$ and reflects the structure of divergent lattice $\mathcal{D}$.

It will be useful to consider first the case of primitive graphs. In this case Y is the blow-up of the origin in X, covered by charts U_i where i runs from 1 to dn (corresponding to all possible markings of an adapted basis). We already know from the extension theory for distributions that the Laurent expansion around $s = 1$ of $\tilde{w}^s$ has a simple pole with its residue given by

$$\tilde{a}_{-1} = -\frac{2}{d_G} f \delta_\mathcal{E}.$$

Here $\delta_\mathcal{E}$ is a density on Y, the delta distribution *centered on* $\mathcal{E}$ (cf. [GS64]), locally in U_i given by the delta distribution in the marked coordinate x^i, i.e.

$$\langle \delta_\mathcal{E} | \varphi \rangle \overset{loc.}{=} \int dx \, \delta(x_i) \varphi(x).$$

Pairing $\tilde{a}_{-1}$ with the characteristic function χ of Y produces a projective integral

$$\langle \tilde{a}_{-1} | \chi \rangle = -\frac{2}{d_G} \int_\mathcal{E} f.$$

Recall from Section 4.1 the definition of induced charts (V_i, ϕ_i) for $\mathcal{E}$. Since any such chart covers $\mathcal{E}$ up to a set of measure zero, it suffices to do this integral in one of them. Thus,

$$\int_\mathcal{E} f = \int d\hat{x} f(\hat{x})$$

where $d\hat{x} = dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{nd}$ for some $i \in \{1, \ldots, dn\}$.

Definition 6.2 (Period of a primitive graph)**.** Let G be primitive. The *period* $\mathscr{P}(G)$ of G is defined as the projective integral

$$\mathscr{P}(G) := \langle \tilde{a}_{-1} | \chi \rangle = -\frac{2}{d_G} \int_\mathcal{E} f.$$

For more on periods see the overview in [Sch10]. Until recently it was believed that all periods in massless ϕ^4-theory (i.e. $d = 4$ and all vertices of the Feynman diagram corresponding to G are 4-valent) are rational combinations of multiple zeta values. But counterexamples [BD13] have proven this false, relating a better understanding of these periods to deep questions

in algebraic geometry [Bro10].

In the Laurent expansion of $\tilde{w}^s$ for non-primitive G terms corresponding to contracted graphs will appear. Since we work in local coordinates indexed by $\mathcal{B}$-nested sets, we need a more sophisticated (local) contraction operation on graphs:

Definition 6.3. Let $g \subseteq G$ and $\mathcal{P} \subseteq \mathcal{D}$. The *contraction relative to* $\mathcal{P}$ is defined as

$$g//\mathcal{P} := \begin{cases} g/(\bigcup_{\gamma \in \mathcal{P}_{<g}} \gamma) & \text{if } g \in \mathcal{P}, \\ g/(g \cap \bigcup_{\gamma \in \mathcal{P}_{<g}} \gamma) & \text{else.} \end{cases}$$

Especially important will be the contraction relative to nested sets. The reader should think of it as a local version of the contraction in the definition of the coproduct in the Hopf algebra of Feynman graphs. It will show up in all formulas that include the coproduct in their usual formulation, say in momentum space. Note that for $g \subseteq G$ the "normal" contraction G/g is included in this definition as contraction of G with respect to the nested set $\mathcal{N} = \{g, G\}$. Moreover, if $\mathcal{N}$ is nested and $g \in \mathcal{N}$ or all elements of $\mathcal{N}$ are contained in g, then $g//\mathcal{N}$ is at most logarithmic as well. For a general discussion of which classes of graphs are closed under contraction we refer the reader to [BK08].

We continue by studying the Laurent expansion of $\tilde{w}^s$ on Y.

Theorem 6.4. *Let Y be a wonderful model for some building set $\mathcal{B}$ of $\mathcal{D}$. Let $\tilde{w}^s = \beta^* \tilde{v}^s$ be the pullback of the density $\tilde{v}^s \in \tilde{\mathcal{D}}'(X)$ onto Y. Then:*

1. *The Laurent expansion of $\tilde{w}^s$ at $s = 1$ has a pole of order N where N is the cardinality of the largest $\mathcal{B}$-nested set.*

2. *The coefficients $\tilde{a}_k$ in the principal part of the Laurent expansion*

$$\tilde{w}^s = \sum_{-N \leq k} \tilde{a}_k (s-1)^k$$

   are densities with $(k \leq -1)$

$$\operatorname{supp} \tilde{a}_k = \bigcup_{|\mathcal{N}| = -k} \mathcal{E}_{\mathcal{N}}.$$

3. *Consider the irreducible elements $I(\mathcal{D})$ as building set. Assume $G \in I(\mathcal{D})$. Let $\mathcal{N}$ be a maximal nested set and denote by χ the constant function on the wonderful model $Y_{I(\mathcal{D})}$. Then for $N = |\mathcal{N}|$*

$$\langle \tilde{a}_{-N} | \chi \rangle = \sum_{|\mathcal{M}| = N} \prod_{\gamma \in \mathcal{M}} \mathcal{P}(\gamma//\mathcal{M}).$$

Proof. 1. This follows from the local expression for $\tilde{w}^s$. Using Formula (3.2) we have

$$\tilde{w}^s \overset{loc.}{=} f^s u_{\mathcal{N}}^s |dx| = f^s \prod_{g \in \mathcal{N}} \left(-\frac{2}{d_g} \delta_g (s-1)^{-1} + u_{g\heartsuit}^s \right) |dx|.$$

Since $u_{g\heartsuit}^s$ is regular in s, the highest pole order is given by $|\mathcal{N}|$.

2. Expand $u_\heartsuit^s \in \mathcal{D}'(\mathbb{R})$ into a Taylor series at $s = 1$,

$$u_\heartsuit^s = \sum_{k=0}^\infty u_k (s-1)^k$$

where the distributions u_k are given by

$$u_k : \varphi \mapsto \int dx\, |x|^{-1} \log^k(|x|) \big(\varphi(x) - \theta(1 - |x|)\varphi(0) \big). \qquad (6.2)$$

In the following we write θ_g for the map $x_g^{i_g} \mapsto \theta\big(1 - |x_g^{i_g}|\big)$ in all coefficients of the expansion of the regular part of u_g^s. Expanding f^s gives

$$f^s = \exp(\log(f^s)) = f \exp\big((s-1)\log(f)\big) = f \sum_{k=0}^\infty \frac{\log^k(f)}{k!}(s-1)^k.$$

Fix a $\mathcal{B}$-nested set $\mathcal{N}$. To determine the lower pole parts in the local expression for $\tilde{w}^s$ we multiply all series $u_{g\heartsuit}^s$ for $g \in \mathcal{N}$ and reorder the sum. Denote by $(u_g)_l$ the l-th order coefficient of the expansion of u_g^s. Then for i in $\{1, \ldots, N\}$ the kernel of $\tilde{a}_{-i}$ is given by

$$\sum_{k=0}^{N-i-1} \frac{f \log^k(f)}{k!} \Bigg(\sum_{j=k}^{N-i-1} \sum_{\substack{\mathcal{L} \subseteq \mathcal{N} \\ |\mathcal{L}| = i+j}} \prod_{\gamma \in \mathcal{L}} \left(-\frac{2}{d_\gamma} \right) \delta_\gamma \prod_{\substack{\eta \in \mathcal{N} \setminus \mathcal{L} \\ \{l_\eta \in \mathbb{N} | \sum_{\eta \in \mathcal{N} \setminus \mathcal{L}} l_\eta = j-k\}}} (u_\eta)_{l_\eta} \Bigg)$$

$$(6.3)$$

$$+ \frac{f \log^{N-i}(f)}{(N-i)!} \prod_{\gamma \in \mathcal{N}} \left(-\frac{2}{d_\gamma} \right) \delta_\gamma.$$

Recall that locally $\mathcal{E}_g$ is given by $x_g^{i_g} = 0$ and $\mathcal{E}_\mathcal{I} \subseteq \mathcal{E}_\mathcal{J}$ for $\mathcal{J} \subseteq \mathcal{I} \subseteq \mathcal{N}$. Therefore the support of $\tilde{a}_{-i}$ is given by the $(k = j = 0)$-summand in (6.3), carrying the product of i δ-distributions in the marked coordinates of an i-element subset of $\mathcal{N}$. Varying over all $\mathcal{B}$-nested sets $\mathcal{N}$ (and the marking of B) the same holds for all i-element subsets of any nested set. Thus, from the expansion formula (6.3) we conclude that the densities $\tilde{a}_{-i}$ are supported on

$$\bigcup_{|\mathcal{N}|=i} \Big(\bigcap_{\gamma \in \mathcal{N}} \mathcal{E}_\gamma \Big) = \bigcup_{|\mathcal{N}|=i} \mathcal{E}_\mathcal{N}.$$

3. This follows essentially from two assertions:

If we view the pairing of a product of delta distributions ($\delta_g := \delta(x_g^{i_g})$) with a function φ as an operator $\delta_{\mathcal{N}}$, locally given by

$$\delta_{\mathcal{N}} : \varphi \in L^1_{loc}(\kappa(U)) \longmapsto \prod_{\gamma \in \mathcal{N}} \delta_\gamma[\varphi] \in L^1_{loc}(\kappa(U \cap \mathcal{E}_{\mathcal{N}})),$$

then for f the regular part of the pullback of v^s we have

$$\delta_{\mathcal{N}}[f] = \prod_{\gamma \in \mathcal{N}} f_{\gamma // \mathcal{N}}. \tag{6.4}$$

Here $f_{g // \mathcal{N}}$ is obtained from f by setting all marked elements corresponding to graphs in $\mathcal{N}_{<g}$ to zero. It equals the regular part of the local pullback of $v^s_{g // \mathcal{N}}$ onto the wonderful model for the graph $g // \mathcal{N}$ in the nested set $\mathcal{N}' = \{g // \mathcal{N}\}$ ($g // \mathcal{N}$ is primitive!). For a precise definition and the proof of this assertion we refer to Chapter 7, Theorem 7.3, where this is elaborated in a much more general case. The important point here is that $\delta_{\mathcal{N}}[f]$ is a product of maps $f_{g // \mathcal{N}}$, each one depending only on the set of variables $\{x_e\}$ with e in $E(t_g) \setminus E(\mathcal{N}_{<g})$, minus all marked elements.

The second assertion is that in every maximal $I(\mathcal{D})$-nested set all contracted graphs $g // \mathcal{N}$ are primitive. Note that if G is divergent and irreducible, it must be contained in every maximal nested set. To prove the assertion let $g \in \mathcal{N}$ and assume $g // \mathcal{N}$ is not primitive. This means there is an $h \in \mathcal{D}$ with either $h \subseteq g // \mathcal{N}$ or $h // \mathcal{N} \subseteq g // \mathcal{N}$. In both cases we can assume that h is irreducible (if not, then h is the union of irreducible elements and we do the following for every irreducible component of h). Then the set $\mathcal{N}' = \mathcal{N} \cup \{h\}$ is also nested if h satisfies the following property: For all g' in $\mathcal{N}$ that are incomparable to g the join $h \vee g' = h \cup g'$ must not lie in $I(\mathcal{D})$. But if there is $g' \in \mathcal{N}$, incomparable to g, with $h \leq g'$ then g and g' have both h as common subgraph. By Lemma 5.17 this implies that $g \vee g'$ is irreducible, showing that g and g' cannot both lie in $\mathcal{N}$ because $\mathcal{N}$ is $I(\mathcal{D})$-nested. If $g // \mathcal{N}'$ is still not primitive, repeat the process until all contracted graphs are primitive. The resulting nested set $\mathcal{N}'$ is then really maximal: If adding another graph would not violate the property of being nested, then it must necessarily be disjoint from all $g \in \mathcal{N}'$ (otherwise some $g // \mathcal{N}$ is not primitive) and this is impossible due to $G \in \mathcal{N}'$.

For $G \in I(\mathcal{D})$ all edges of G lie in some divergent subgraph (if not, say for one edge e, then contract all divergent subgraphs. The resulting graph is primitively divergent and contains e). Thus, in every maximal nested set $\mathcal{N}$ all edges of an adapted spanning tree t correspond to some element of $\mathcal{N}$ and by Definition 4.8 we have $U_{\mathcal{N},B} = X$ for all maximal nested sets $\mathcal{N}$. Let $\mathcal{N}$ be such a maximal nested set. Using 2. and the first assertion we have

locally in U

$$\langle \tilde{a}_{-N} | \chi \rangle \overset{loc.}{=} \int_{\kappa(U)} dx \prod_{\gamma \in \mathcal{N}} (-\frac{2}{d_\gamma}) \delta_\gamma[f] = \prod_{\gamma \in \mathcal{N}} (-\frac{2}{d_\gamma}) \int_{\kappa(V)} d\hat{x} \, \delta_{\mathcal{N}}[f]$$

$$= \int_{\kappa(V)} d\hat{x} \prod_{\gamma \in \mathcal{N}} (-\frac{2}{d_\gamma}) f_{\gamma//\mathcal{N}}.$$

Here $\hat{x}$ denotes $\{x_e\}_{e \in E(t)}$ minus all marked elements and $V = V_{\mathcal{N},B}$ is the chart domain for local coordinates on $\mathcal{E}_{\mathcal{N}}$, obtained by restriction of the chart κ (cf. Section 4.1). Since it covers $\mathcal{E}_{\mathcal{N}}$ up to a set of measure zero (cf. Definition 6.2), integration in a single chart suffices. Moreover, two components of the exceptional divisor $\mathcal{E}_{\mathcal{N}}$ and $\mathcal{E}_{\mathcal{M}}$ have non-empty intersection if and only if $\mathcal{N} \cup \mathcal{M}$ is nested. But this is impossible due to maximality of $\mathcal{N}$. Therefore we can sum the contributions from charts given by different maximal nested sets to obtain the global result

$$\langle \tilde{a}_{-N} | \chi \rangle = \sum_{\mathcal{N}} \int_{\kappa(V)} d\hat{x} \prod_{\gamma \in \mathcal{N}} (-\frac{2}{d_\gamma}) f_{\gamma//\mathcal{N}}$$

where the sum is over all maximal nested sets.

Finally, since all $f_{\gamma//\mathcal{N}}$ depend on mutually disjoint sets of variables, the integral factorizes and since restricting $\kappa|_V$ further to $\{\hat{x}_e\}_{e \in E(t_\gamma)}$ is a local chart for $\mathcal{E}_{\gamma//\mathcal{N}}$, we conclude that

$$\sum_{\mathcal{N}} \int_{\kappa(V)} d\hat{x} \prod_{\gamma \in \mathcal{N}} (-\frac{2}{d_\gamma}) f_{\gamma//\mathcal{N}} = \sum_{\mathcal{N}} \prod_{\gamma \in \mathcal{N}} \int_{\mathcal{E}_{\gamma//\mathcal{N}}} (-\frac{2}{d_\gamma}) f_{\gamma//\mathcal{N}}$$

$$= \sum_{\mathcal{N}} \prod_{\gamma \in \mathcal{N}} \mathscr{P}(\gamma//\mathcal{N}).$$

$\square$

This theorem is a first hint at the Hopf algebraic formulation of the renormalization group (see [Kre13], [CK01]). It shows that the poles of $\tilde{w}^s$ are not arbitrary densities but reflect the combinatorics of $\mathcal{D}$. The highest order pole is completely determined by the structure of $\mathcal{D}$. For the poles of lower order the same holds in a weaker version; they are supported on components of $\mathcal{E}$ whose stratification is given by the combinatorial structure of $\mathcal{D}$ as well.

6.3   Renormalization

With the main result of the previous section we are now able to tackle the renormalization problem. Since all poles of $\tilde{w}^s$ live on the components of the exceptional divisor, we can get rid of them using local subtractions

depending on the direction such a pole is approached. These directions are encoded by nested sets, so that we will employ local versions of the previously defined renormalization maps r_1 and r_ν, depending on the chosen coordinate system given by $\mathcal{B}$-nested sets and markings of an adapted basis B. From now on we consider only the minimal or maximal building sets in the divergent lattice $\mathcal{D}$.

Definition 6.5 ((Local) minimal subtraction)**.** Let R_1 denote the collection of renormalization maps $\{R_1^{\mathcal{N},B}\}$ for $\mathcal{N}$ a $\mathcal{B}$-nested set and B an adapted basis (more precisely, the marking since the basis is fixed). $R_1^{\mathcal{N},B}$ removes the poles in the coordinates associated to the marked elements, i.e.

$$R_1[\tilde{w}^s] \stackrel{loc.}{=} R_1^{\mathcal{N},B}[f^s \tilde{u}_{\mathcal{N}}^s] := f^s \prod_{g \in \mathcal{N}} r_1[u_g^s]|dx|.$$

Recall from Chapter 3 that $r_1[u_g^s] = (u_g^s)_\heartsuit$, so there are no poles anymore and we can take the limit $s \to 1$ to obtain a well defined density on Y.

The next definition introduces a renormalization operator that produces a density for s in a complex neighborhood of 1. It should be thought of as a smooth version of minimal subtraction.

Definition 6.6 ((Local) subtraction at fixed conditions)**.** Let R_ν denote the collection of renormalization maps $\{R_\nu^{\mathcal{N},B}\}$ where $\mathcal{N}$ is a $\mathcal{B}$-nested sets and B marked. The symbol $\nu = \{\nu_g^{\mathcal{N},B}\}_{g \in \mathcal{N}}$ stands for a collection of smooth functions on $\kappa(U)$. Each $\nu_g^{\mathcal{N},B}$ depends only on the coordinates x_e with $e \in E(t) \cap E(g \setminus \mathcal{N}_{<g})$ and satisfies

$$\nu_g^{\mathcal{N},B}\big|_{x_g^{i_g}=0} = 1.$$

Furthermore, it is compactly supported in all other directions.

Similarly to R_1 the operator R_ν is defined by

$$R_\nu[\tilde{w}^s] \stackrel{loc.}{=} R_\nu^{\mathcal{N},B}[f^s \tilde{u}_{\mathcal{N}}^s] := f^s \prod_{g \in \mathcal{N}} r_{\nu_g^{\mathcal{N},B}}[u_g^s]|dx|.$$

Remark. In contrast to the definition of r_ν given in Chapter 3 the maps ν_g (we drop the index $(\mathcal{N}, B)$ from here on, as well as the dependence of the operators R on B) are not only test functions in the marked coordinate $x_g^{i_g}$, but in all $\{x_e\}_{e \in E(t) \cap E(g \setminus \mathcal{N}_{<g})}$. This is to ensure that the counterterms are really well-defined densities in a neighborhood of $s = 1$. There is some ambivalence in defining them, so it pays of to be careful at this point.

We introduce another useful expression for R_ν: For $\mathcal{K} \subseteq \mathcal{N}$ write $\nu_{\mathcal{K}}$ for the product $\prod_{\gamma \in \mathcal{K}} \nu_\gamma$. In a chart U the operators $R_\nu^{\mathcal{N}}$ act on $\tilde{w}^s$ by

$$R_\nu^{\mathcal{N}}[\tilde{w}^s] = \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} \nu_{\mathcal{K}} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{K}}}$$

where $\mathcal{E}_\mathcal{K} = \bigcap_{\gamma \in \mathcal{K}} \mathcal{E}_\gamma \subseteq \mathcal{E}$ is the component of the exceptional divisor associated to the nested set $\mathcal{K} \subseteq \mathcal{N}$. This is to be understood in the following way. First restrict the regular part f^s of $\tilde{w}^s$ and the test function φ to $\kappa(U \cap \mathcal{E}_\mathcal{K})$, then pull this product back onto U, then multiply by $u^s_\mathcal{N}$ and $\nu_\mathcal{K}$ and finally integrate, in formulae

$$\langle \nu_\mathcal{K} \cdot (\tilde{w}^s)_{\mathcal{E}_\mathcal{K}} | \varphi \rangle = \langle (p_\mathcal{K})_* (\nu_\mathcal{K} u^s_\mathcal{N} |dx|) | \delta_\mathcal{K}[f^s \varphi] \rangle.$$

Here $p_\mathcal{K}$ is the local expression of the canonical projection $\pi : Y \to \mathcal{E}_\mathcal{K}$ and $\delta_\mathcal{K}$ is the corresponding map $\mathcal{D}(Y) \to \mathcal{D}(\mathcal{E}_K)$. For $\mathcal{K} = \{g_1, \ldots, g_k\}$ both are given by

$$p_{g_1, \ldots, g_k} : x \mapsto (x^1_1, \ldots, x^{i_{g_1}}_{g_1}, \ldots, x^{i_{g_k}}_{g_k}, \ldots, x^d_n)\big|_{x^{i_{g_1}}_{g_1}, \ldots, x^{i_{g_k}}_{g_k} = 0},$$

$$\delta_{g_1, \ldots, g_k} : \varphi \mapsto \varphi\big|_{x^{i_{g_1}}_{g_1}, \ldots, x^{i_{g_k}}_{g_k} = 0}.$$

Note that $\delta_\mathcal{K}[f^s \varphi]$ remains compactly supported in the coordinates associated to $G \setminus \cup_{\gamma \in \mathcal{K}} \mathcal{N}_{<\gamma}$. On the other hand, $\nu_\mathcal{K}$ is compactly supported in the coordinates associated to $G \cap (\cup_{\gamma \in \mathcal{K}}(\gamma \setminus \mathcal{N}_{<\gamma}))$. But these sets cover G and therefore the counterterms $\nu_\mathcal{K} \cdot (\tilde{w}^s)_{\mathcal{E}_\mathcal{K}}$ are well defined densities in all coordinates except the marked elements (cf. the proof of Theorem 7.2).

The notation is chosen to suggest that $(\tilde{w}^s)_{\mathcal{E}_\mathcal{K}}$ can be thought of as the "restriction" of $\tilde{w}^s$ onto $\mathcal{E}_\mathcal{K}$ and the symbol "$\cdot$" in $\nu_\mathcal{K} \cdot (\tilde{w}^s)_{\mathcal{E}_\mathcal{K}}$ is used to highlight the fact that this expression differs from the usual product of distributions and smooth functions. We call it "product" because it is linear and multiplicative in ν. Although this notation might seem awkward, it will turn out be very useful in the next chapter!

The following lemma will be needed to compare the renormalized densities obtained by choosing different maps ν.

Lemma 6.7. *Let $g, h \in \mathcal{D}$. The maps $p_{g,h}$ and $\delta_{g,h}$ both fulfill the following "commutation rules":*

$$p_{g,h} = p^g_{g,h} \circ p_g = p^h_{g,h} \circ p_h,$$
$$\delta_{g,h} = \delta^g_{g,h} \circ \delta_g = \delta^h_{g,h} \circ \delta_h,$$

where locally $p^g_{g,h} : \kappa(U \cap \mathcal{E}_g) \longrightarrow \kappa(U \cap \mathcal{E}_{g,h})$ is defined by

$$(x^1_1, \ldots, \hat{x}^{i_g}_g, \ldots, x^d_n) \mapsto (x^1_1, \ldots, \hat{x}^{i_g}_g, \ldots, \hat{x}^{i_h}_h, \ldots, x^d_n)$$

and $\delta^g_{g,h} : \mathcal{D}(\kappa(U \cap \mathcal{E}_g)) \mapsto \mathcal{D}(\kappa(U \cap \mathcal{E}_{g,h}))$ by

$$\varphi\big|_{x^{i_g}_g = 0} \mapsto \varphi\big|_{x^{i_g}_g = x^{i_h}_h = 0}.$$

Proof. Clear from the definition of both maps. $\qquad\square$

Obviously this property generalizes to the case where instead of $\{g, h\}$ a finite subset of a nested set is considered, e.g.

$$p_{g_1, \ldots, g_k} = p_{g_1, \ldots, g_k}^{g_1, \ldots, g_{k-1}} \circ \cdots \circ p_{g_1, g_2}^{g_1} \circ p_{g_1}$$

and similarly for $\delta_{g_1, \ldots, g_k}$.

Both renormalization operations produce well-defined densities at $s = 1$ as is shown in the next proposition.

Proposition 6.8. *Let (Y, β) be the minimal or maximal wonderful model for the divergent lattice $\mathcal{D}$. Then $R_1[\tilde{w}^s]|_{s=1}$ defines a density on Y, while $R_\nu[\tilde{w}^s]$ is a density-valued holomorphic function for all s in a neighborhood of 1 in $\mathbb{C}$.*

Proof. Note that from the proof of Theorem 6.4 it follows in particular that $\tilde{w}^s$ is really a density on Y. By the same argumentation we are able to conclude from expression (6.3) for the counterterms in R_ν that they are all densities for s in a neighborhood of 1: Every subtraction term has the same combination of $u_{\mathcal{N}}^s$ and f^s, transforming under a change of coordinates according to the definition of densities.

In the case of minimal subtraction, by Theorem 6.4 and the definition of r_1, all poles of $\tilde{w}^s$ have been discarded. Therefore, $R_1[\tilde{w}^s]|_{s=1}$ is a finite density. From the Taylor expansion of $u_\heartsuit^s$ (Equation (6.2)) it follows that $R_1[\tilde{w}^s]$ fails to be a density for $s \neq 1$ because the u_g^s do not transform correctly under a change of coordinates.

It remains to show the finiteness of $R_\nu^{\mathcal{N}}[\tilde{w}^s]$ for all $\mathcal{B}$-nested sets $\mathcal{N}$. We argue by induction on the cardinality of $|\mathcal{N}|$. First consider the case where the nested set consists of a single graph, $\mathcal{N} = \{g\}$ for some $g \subseteq G$. Let $x_g^{i_g}$ denote the marked element, otherwise we drop all indices. Then by Definition 6.6

$$\langle R_\nu^{\mathcal{N}}[\tilde{w}^s] | \varphi \rangle = \langle \tilde{w}^s | \varphi \rangle - \langle \nu \cdot (\tilde{w}^s)_{\mathcal{E}_g} | \varphi \rangle,$$

$\nu = \nu_g$ depending on all x_e with $e \in E(t_g)$. We expand both summands into their Laurent series (focusing on the principle part only) to get

$$\langle \tilde{w}^s | \varphi \rangle = \int dx \; |x_g^{i_g}|^{-1 + d_g(1-s)} f^s(x) \varphi(x)$$

$$= \int dx_g^{i_g} |x_g^{i_g}|^{-1 + d_g(1-s)} \int d\hat{x} \; f^s(x_g^{i_g}, \hat{x}) \varphi(x_g^{i_g}, \hat{x})$$

$$=: \int dx_g^{i_g} |x_g^{i_g}|^{-1 + d_g(1-s)} F(s, x_g^{i_g}).$$

Using Formula (3.2) from Chapter 3,

$$\langle \tilde{w}^s | \varphi \rangle = -\frac{2}{d_g} F(s,0)(s-1)^{-1}$$
$$+ \int dx_g^{i_g} \; |x_g^{i_g}|^{-1+d_g(1-s)} \left(F(s, x_g^{i_g}) - \theta_g(x_g^{i_g}) F(s,0) \right).$$

For the counterterm we have

$$\langle \nu \cdot (\tilde{w}^s)_{\mathcal{E}_g} | \varphi \rangle = \int dx \; |x_g^{i_g}|^{-1+d_g(1-s)} \nu(x_{t_g}) \left(f^s(x) \varphi(x) \right) \big|_{x_g^{i_g}=0}$$
$$= \int dx_g^{i_g} |x_g^{i_g}|^{-1+d_g(1-s)} \int d\hat{x} \; \nu(x_{t_g}) f^s(0, \hat{x}) \varphi(0, \hat{x})$$
$$=: \int dx_g^{i_g} |x_g^{i_g}|^{-1+d_g(1-s)} G_\nu(s, x_g^{i_g}).$$

In the same way as above we get

$$\langle \nu \cdot (\tilde{w}^s)_{\mathcal{E}_g} | \varphi \rangle = -\frac{2}{d_g} G_\nu(s,0)(s-1)^{-1}$$
$$+ \int dx_g^{i_g} \; |x_g^{i_g}|^{-1+d_g(1-s)} (G_\nu(s, x_g^{i_g}) - \theta_g(x_g^{i_g}) G_\nu(s,0)).$$

Since $\nu(x_{t_g})|_{x_g^{i_g}=0} = 1$, $F(s,0) = G(s,0)$ and the pole cancels in the difference. Therefore, $\langle R_\nu^{\mathcal{N}}[\tilde{w}^s] | \varphi \rangle$ is finite for all $\varphi \in \mathcal{D}(\kappa(U))$.

Now let $\mathcal{N}$ be nested and $h \subseteq G$ such that $\mathcal{N}' := \mathcal{N} \cup \{h\}$ is also nested. For $\mathcal{K} \subseteq \mathcal{N}$ set $\mathcal{K}' := \mathcal{K} \cup \{h\}$. Assume h to be minimal in $\mathcal{N}'$ (if not choose another minimal element). We want to show finiteness of $\tilde{w}^s$ in $\kappa'(U')$ for $U' = U_{\mathcal{N}',B}$ with B marked for $\mathcal{N}$ plus an additional marking for the element h (since h is minimal, all markings of $\mathcal{N}'$ are of this form). By induction hypothesis $R_\nu^{\mathcal{N}}[\tilde{w}^s]$ is a well-defined density on $\kappa(U)$ for all s in a neighborhood of 1 in $\mathbb{C}$. In [CP95] it is shown that U' is the blow-up of the proper transform of $A_h^\perp$ in $U_{\mathcal{N},B}$. By minimality of h this blow-up β_h is locally given by ρ_h scaling all $\{x_e\}_{e \in E(t_h)}$ with $x_h^{i_h}$. Moreover, the chart κ' is just the inverse of the composition of ρ_h with $\Gamma(\pi_\mathcal{B}) \circ \rho_\mathcal{N}$. The pullback of the density $R_\nu^{\mathcal{N}}[\tilde{w}^s]$ along this blow-up has now an additional divergence in the coordinate $x_h^{i_h}$. Therefore one more subtraction is needed to obtain a finite density on U' (the index $\mathcal{N}$ stands for the local expression in U),

$$R_{\nu_h} \left[\beta_h^* R_\nu^{\mathcal{N}}[\tilde{w}_\mathcal{N}^s] \right] = R_{\nu_h} \left[\beta_h^* \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} \nu_\mathcal{K} \cdot (\tilde{w}_\mathcal{N}^s)_{\mathcal{E}_\mathcal{K}} \right] = *$$

We compute the pullbacks locally in U, the power counting that produces

u_h^s works exactly like in the proof of Proposition 6.1.

$$\langle \rho_h^*(\nu_\mathcal{K} \cdot (\tilde{w}_\mathcal{N}^s)\varepsilon_\mathcal{K})|\varphi\rangle = \int (\rho_h^* dx)(\nu_\mathcal{K} \circ \rho_h)(u_\mathcal{N}^s \circ \rho_h)p_\mathcal{K}^* \delta_\mathcal{K}[(f_\mathcal{N}^s \circ \rho_h)\varphi]$$

$$= \int dx\, \nu_\mathcal{K} u_\mathcal{N}^s u_h^s p_\mathcal{K}^* \delta_\mathcal{K}[(f_{\mathcal{N}'}^s)\varphi]$$

$$= \langle \nu_\mathcal{K} \cdot (\tilde{w}_{\mathcal{N}'}^s)\varepsilon_\mathcal{K}|\varphi\rangle.$$

Thus,

$$* = \sum_{\mathcal{K}\subseteq\mathcal{N}} (-1)^{|\mathcal{K}|}\nu_\mathcal{K} \cdot (\tilde{w}_{\mathcal{N}'}^s)\varepsilon_\mathcal{K} - \nu_h \cdot \sum_{\mathcal{K}\subseteq\mathcal{N}} (-1)^{|\mathcal{K}|}\nu_\mathcal{K} \cdot (\tilde{w}_{\mathcal{N}'}^s)\varepsilon_{\mathcal{K}\cup\{h\}}$$

$$= \sum_{\mathcal{K}\subseteq\mathcal{N}} (-1)^{|\mathcal{K}|}\nu_\mathcal{K} \cdot (\tilde{w}_{\mathcal{N}'}^s)\varepsilon_\mathcal{K} - \sum_{\mathcal{K}\subseteq\mathcal{N}} (-1)^{|\mathcal{K}|}\nu_{\mathcal{K}'} \cdot (\tilde{w}_{\mathcal{N}'}^s)\varepsilon_{\mathcal{K}'}$$

$$= \sum_{\mathcal{K}\subseteq\mathcal{N}'} (-1)^{|\mathcal{K}|}\nu_\mathcal{K} \cdot (\tilde{w}_{\mathcal{N}'}^s)\varepsilon_\mathcal{K}$$

$$= R_\nu^{\mathcal{N}'}[\tilde{w}_{\mathcal{N}'}^s]$$

where we have used minimality of h again,

$$\langle \nu_h \cdot (\nu_\mathcal{K} \cdot (\tilde{w}^s)\varepsilon_\mathcal{K})\varepsilon_h|\varphi\rangle = \int dx\, u_\mathcal{N}^s \nu_h \nu_\mathcal{K}|_{x_h^{i_h}=0}\, (f^s\varphi)\,|_{x_\mathcal{K}^{i_\mathcal{K}}=x_h^{i_h}=0}$$

$$= \int dx\, u_\mathcal{N}^s \nu_{\mathcal{K}'}\, (f^s\varphi)\,|_{x_\mathcal{K}^{i_\mathcal{K}}=x_h^{i_h}=0}$$

$$= \langle (\nu_{\mathcal{K}'}) \cdot (\tilde{w}^s)\varepsilon_{\mathcal{K}'}|\varphi\rangle.$$

We see that both densities coincide and the proposition is proven. $\square$

Both renormalization operators have another property that every sensible renormalization should have; they commute with multiplication by smooth functions.

Lemma 6.9. *Let $f \in \mathcal{C}^\infty(\kappa(U))$, then*

$$R_0^\mathcal{N}[\tilde{w}_\mathcal{N}^s f] = fR_0^\mathcal{N}[\tilde{w}_\mathcal{N}^s]$$
$$R_\nu^\mathcal{N}[\tilde{w}_\mathcal{N}^s f] = fR_\nu^\mathcal{N}[\tilde{w}_\mathcal{N}^s].$$

Proof. Clear from the definition of both operators. Under the renormalization operation the regular part of the density is treated as a test function which is also the definition of the product of smooth functions and densities. $\square$

Finally, we are able to state a solution of the renormalization problem.

Definition 6.10 (Renormalized Feynman rules). Let R denote one of the renormalization operators R_1 or R_ν on the wonderful model $(Y_\mathcal{B}, \beta)$ for $\mathcal{B} = I(\mathcal{D})$ or $\mathcal{D}$. Define the renormalized Feynman distribution by

$$\mathscr{R}[v_G] := \beta_* R[\tilde{w}_G]|_{s=1}.$$

Then the *renormalized Feynman rules* are given by the map

$$\Phi_R : G \longmapsto (X^G, \mathscr{R}(v_G)).$$

The pair $(X^G, \mathscr{R}(v_G))$ can now be evaluated at $\varphi \in \mathcal{D}(X)$,

$$\begin{aligned}
\left(\mathrm{eval}_\varphi \circ \Phi_R\right)(G) &= \langle \mathscr{R}[v_G] | \varphi \rangle \\
&= \langle \beta_* R[\tilde{w}_G]|_{s=1} \mid \varphi \rangle \\
&= \langle R[\tilde{w}_G]|_{s=1} \mid \beta^* \varphi \rangle.
\end{aligned}$$

To carry out the evaluation at φ we choose a partition of unity $\{\chi_i\}_{i \in \{(\mathcal{N},B)\}}$ on $Y_\mathcal{B}$, subordinate to the covering $\{U_i\}_{i \in \{(\mathcal{N}.B)\}}$. Write π_i for $\chi_i \circ \kappa_i^{-1}$. Then

$$\langle R[\tilde{w}_G^s]|_{s=1} \mid \beta^* \varphi \rangle = \sum_i \langle \pi_i (R[\tilde{w}_G^s])_i|_{s=1} \mid \varphi \circ \rho_i \rangle.$$

To see that this definition does not depend on the chosen partition of unity let $\{\chi_j'\}$ denote another partition, also subordinate to the $\{U_j\}_{j \in \{(\mathcal{N},B)\}}$. Then

$$\begin{aligned}
&\sum_i \langle \pi_i (R[\tilde{w}_G])_i|_{s=1} \mid \varphi \circ \rho_i \rangle \\
={} &\sum_i \int_{\mathrm{supp}(\pi_i)} dx \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} u_\mathcal{N} \nu_\mathcal{K} \delta_\mathcal{K}[f_i(\varphi \circ \rho_i)\pi_i] \\
={} &\sum_i \int_{\mathrm{supp}(\pi_i)} dx \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} u_\mathcal{N} \nu_\mathcal{K} \delta_\mathcal{K}[f_i(\varphi \circ \rho_i)\pi_i \sum_j \pi_j'] \\
={} &\sum_i \sum_j \int_{\mathrm{supp}(\pi_i) \cap \mathrm{supp}(\pi_j')} dx \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} u_\mathcal{N} \nu_\mathcal{K} \delta_\mathcal{K}[f_i(\varphi \circ \rho_i)\pi_i \pi_j'] \\
={} &\sum_i \sum_j \int_{\mathrm{supp}(\pi_i) \cap \mathrm{supp}(\pi_j')} dx \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} u_\mathcal{N} \nu_\mathcal{K} \delta_\mathcal{K}[f_j(\varphi \circ \rho_j)\pi_i \pi_j'] \\
={} &\sum_j \int_{\mathrm{supp}(\pi_j')} dx \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} u_\mathcal{N} \nu_\mathcal{K} \delta_\mathcal{K}[f_j(\varphi \circ \rho_j)\pi_j' \sum_i \pi_i] \\
={} &\sum_j \langle \pi_j' (R[\tilde{w}_G])_j|_{s=1} \mid \varphi \circ \rho_j \rangle.
\end{aligned}$$

This finishes the process of *wonderful renormalization*. From a mathematical point of view we are done, but for a physicist it is not clear yet that

we have constructed a reasonable renormalization. In addition to producing finite distributions both schemes have to fulfill another condition that is dictated by physics. It is called the *locality principle* (see [EG73]) and, roughly speaking, assures that the renormalized distributions still obey the laws of physics. There are various equivalent formulations of this; we will use a version for single graphs from [BBK10]. It is fulfilled only in the minimal case $\mathcal{B} = I(\mathcal{D})$. We will get back to this point in Chapter 8.

Before that we turn our attention to the dependence of the operators R on the *renormalization points*, i.e. we study what happens if we change the collection of maps $\{\nu\}$ or the cutoff in the definition of $u_\heartsuit^s$, respectively.

Chapter 7

Renormalization group

In this chapter we take a closer look at the renormalized distribution densities. First we consider (local) subtraction at fixed conditions. The case of minimal subtraction then follows by similar arguments since it can be thought of as a "non-smooth" version of the former.

What happens if we change the cutoff functions ν_g in the definition of the operator R_ν? Clearly, for primitive graphs the difference is a density supported on the exceptional divisor $\mathcal{E}$ and after blowing down the renormalized density it becomes a density supported on $\{0\} \in X$. To get an idea what happens in the general case it is useful to start with an example.

Example. Let G be the dunce cap graph (Figure 2.1). Locally (in $d = 4$ dimensions and $\mathcal{N} = \{g, G\}$, g denoting the divergent one loop subgraph) we have for $\varphi \in \mathcal{D}(\kappa(U))$

$$
\begin{aligned}
\langle R_\nu^{\mathcal{N}}[\tilde{w}^s] | \varphi \rangle &= \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} \langle \nu_{\mathcal{K}} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{K}}} | \varphi \rangle \\
&= \langle \tilde{w}^s | \varphi \rangle - \langle \nu_g \cdot (\tilde{w}^s)_{\mathcal{E}_g} | \varphi \rangle - \langle \nu_G \cdot [\tilde{w}^s]_{\mathcal{E}_G} | \varphi \rangle \\
&\quad + \langle \nu_g \nu_G \cdot [\tilde{w}^s]_{\mathcal{E}_{g,G}} | \varphi \rangle \\
&= \langle \tilde{w}^s | \varphi \rangle - \langle (p_G)_* (\nu_G u_{\mathcal{N}}^s | dx|) | \delta_G [f^s \varphi] \rangle \\
&\quad - \langle (p_g)_* (\nu_g u_{\mathcal{N}}^s | dx|) | \delta_g [f^s \varphi] \rangle \\
&\quad + \langle (p_{g,G})_* (\nu_g \nu_G u_{\mathcal{N}}^s | dx|) | \delta_{g,G} [f^s \varphi] \rangle.
\end{aligned}
$$

Changing the renormalization point, i.e. the collection of test functions $\{\nu\}$, by linearity the difference of the two renormalized expressions is again a sum of this form. However, it will contain a mixture of ν and ν' as renormalization points. But we can express the terms with ν' again by ν-terms only and obtain a finite sum of ν-renormalized expressions. Another

way to see this is by Taylor expansion using the calculus of variations,

$$\frac{d}{dt}\big|_{t=0}\,\langle R_{\nu+t\mu}[\tilde{w}^s]|\varphi\rangle = -\langle (p_g)_*(\mu_g u_{\mathcal{N}}^s|dx|)|\delta_g[f^s\varphi]\rangle$$
$$-\langle (p_G)_*(\mu_G u_{\mathcal{N}}^s|dx|)|\delta_G[f^s\varphi]\rangle$$
$$+\langle (p_{g,G})_*\big((\nu_g\mu_G+\nu_G\mu_g)u_{\mathcal{N}}^s|dx|\big)|\delta_{g,G}[f^s\varphi]\rangle,$$

$$\frac{d^2}{dt^2}\big|_{t=0}\,\langle R_{\nu+t\mu}[\tilde{w}^s]|\varphi\rangle = 2\langle (p_{g,G})_*(\nu_g\nu_G u_{\mathcal{N}}^s|dx|)|\delta_{g,G}[f^s\varphi]\rangle,$$

$$\frac{d^k}{dt^k}\big|_{t=0}\,\langle R_{\nu+t\mu}[\tilde{w}^s]|\varphi\rangle = 0 \text{ for all } k>2.$$

Thus, for $\mu_\gamma := \nu'_\gamma - \nu_\gamma$, $\gamma\in\{g,G\}$,

$$\langle R_{\nu'}[\tilde{w}^s]-R_\nu[\tilde{w}^s]|\varphi\rangle = -\Big(\langle (p_g)_*(\mu_g u_{\mathcal{N}}^s|dx|)|\delta_g[f^s\varphi]\rangle$$
$$-\langle (p_{g,G})_*(\nu_G\mu_g u_{\mathcal{N}}^s|dx|)|\delta_{g,G}[f^s\varphi]\rangle\Big)$$
$$-\Big(\langle (p_G)_*(\mu_G u_{\mathcal{N}}^s|dx|)|\delta_G[f^s\varphi]\rangle-$$
$$\langle (p_{g,G})_*(\nu_g\mu_G u_{\mathcal{N}}^s|dx|)|\delta_{g,G}[f^s\varphi]\rangle\Big)$$
$$+\langle (p_{g,G})_*(\nu_g\nu_G u_{\mathcal{N}}^s|dx|)|\delta_{g,G}[f^s\varphi]\rangle$$
$$= -\langle R_{\nu_G}[\mu_g\cdot(\tilde{w}^s)_{\mathcal{E}_g}]|\varphi\rangle - \langle R_{\nu_g}[\mu_G\cdot(\tilde{w}^s)_{\mathcal{E}_G}]|\varphi\rangle$$
$$+\langle \nu_g\nu_G\cdot(\tilde{w}^s)_{\mathcal{E}_{g,G}}|\varphi\rangle.$$

The last equality holds because of the properties of δ_γ and p_γ (Lemma 6.7).

$$\langle (p_{g,G})_*(\nu_G\mu_g u_{\mathcal{N}}^s|dx|)|\delta_{g,G}[f^s\varphi]\rangle$$
$$=\langle (p_{g,G}^g\circ p_g)_*(\nu_G\mu_g u_{\mathcal{N}}^s|dx|)|(\delta_{g,G}^g\circ\delta_g)[f^s\varphi]\rangle$$
$$=\langle (p_{g,G}^g)_*\big((p_g)_*(\nu_G\mu_g u_{\mathcal{N}}^s|dx|)\big)|\delta_{g,G}^g[\delta_g[f^s\varphi]]\rangle.$$

We see, as expected, that the difference is a sum of densities supported on the components of the exceptional divisor, given by subsets of the nested set $\mathcal{N}$. Since $\mu_\gamma = 0$ for $x_\gamma^{i_\gamma} = 0$, they are finite, except if a point approaches the intersection of two components $\mathcal{E}_{\mathcal{J}}\cap\mathcal{E}_{\mathcal{K}}$ for $\mathcal{J},\mathcal{K}\subseteq\mathcal{N}$. But in this case the necessary subtractions are already provided by the counterterms associated to the set $\mathcal{J}\cup\mathcal{K}$.

We first state this property in the general case, then dive deeper into the structure of these densities by studying their form more closely.

Proposition 7.1. *Let $\mathcal{N}$ be a nested set for the building set $I(\mathcal{D})$ or $\mathcal{D}$ and let B be marked accordingly. For two collections of renormalization points $\{\nu\}$ and $\{\nu'\}$ set $\mu_\gamma := \nu'_\gamma - \nu_\gamma$ for $\gamma\in\mathcal{N}$. Locally in U the difference between the operators $R_{\nu'}$ and R_ν acting on $\tilde{w}^s$ is given by*

$$(R_{\nu'}^{\mathcal{N}}-R_\nu^{\mathcal{N}})[\tilde{w}^s] = \sum_{\emptyset\neq\mathcal{K}\subseteq\mathcal{N}}(-1)^{|\mathcal{K}|}R_\nu^{\mathcal{N}\setminus\mathcal{K}}[\mu_{\mathcal{K}}\cdot(\tilde{w}^s)_{\mathcal{E}_{\mathcal{K}}}]$$

with $R_\nu^\emptyset := id_{\tilde{\mathcal{D}}'(\kappa(U))}$.

Proof. Induction on $n = |\mathcal{N}|$. The statement holds in the cases $n = 1$ and 2 (see example above). Let $\mathcal{N}$ be a nested set of cardinality n and $h \notin \mathcal{N}$ an additional divergent subgraph such that $\mathcal{N}' = \mathcal{N} \cup \{h\}$ is also nested. For $\mathcal{K} \subseteq \mathcal{N}$ set $\mathcal{K}' := \mathcal{K} \cup \{h\}$.

$$
\begin{aligned}
(R_{\nu'}^{\mathcal{N}'} - R_\nu^{\mathcal{N}'})[\tilde{w}^s] &= \sum_{\emptyset \neq \mathcal{K} \subseteq \mathcal{N}'} (-1)^{|\mathcal{K}|}(\nu'_\mathcal{K} - \nu_\mathcal{K}) \cdot (\tilde{w}^s)_{\mathcal{E}_\mathcal{K}} \\
&= \sum_{\emptyset \neq \mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|}(\nu'_\mathcal{K} - \nu_\mathcal{K}) \cdot (\tilde{w}^s)_{\mathcal{E}_\mathcal{K}} \\
&\quad + \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}'|}(\nu'_{\mathcal{K}'} - \nu_{\mathcal{K}'}) \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{K}'}} \\
&=: A + B.
\end{aligned}
$$

By induction hypothesis

$$
A = \sum_{\emptyset \neq \mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} R_\nu^{\mathcal{N} \setminus \mathcal{K}}[\mu_\mathcal{K} \cdot (\tilde{w}^s)_{\mathcal{E}_\mathcal{K}}].
$$

Using

$$
\prod_{i=1}^n (a_i + b_i) - \prod_{i=1}^n a_i = \sum_{\emptyset \neq J \subseteq \{1,\dots,n\}} b_J a_{\{1,\dots,n\} \setminus J}
$$

we expand B into two parts, depending on whether ν or μ carries an index h,

$$
\begin{aligned}
B = B_1 + B_2 &= \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}'|} \sum_{\mathcal{J} \subseteq \mathcal{K}} \mu_{\mathcal{J}'} \nu_{\mathcal{K} \setminus \mathcal{J}} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{K}'}} \\
&\quad + \sum_{\emptyset \neq \mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}'|} \sum_{\emptyset \neq \mathcal{J} \subseteq \mathcal{K}} \mu_\mathcal{J} \nu_{\mathcal{K}' \setminus \mathcal{J}} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{K}'}}.
\end{aligned}
$$

Note that in A all densities $\mu_\mathcal{L} \cdot (\tilde{w}^s)_{\mathcal{E}_\mathcal{L}}$ have an additional, not yet renormalized, divergence corresponding to the subgraph $h \in \mathcal{N}'$. In order to renormalize them we have to add the counterterms associated to h, i.e. all terms in B containing ν_h. For non-empty $\mathcal{L} \subseteq \mathcal{N}$ fixed

$$
\begin{aligned}
&(-1)^{|\mathcal{L}|} R_\nu^{\mathcal{N} \setminus \mathcal{L}}[\mu_\mathcal{L} \cdot (\tilde{w}^s)_{\mathcal{E}_\mathcal{L}}] + \sum_{\mathcal{L} \subseteq \mathcal{J} \subseteq \mathcal{N}} (-1)^{|\mathcal{J}'|} \mu_\mathcal{L} \nu_{\mathcal{J}' \setminus \mathcal{L}} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{J}'}} \\
&= (-1)^{|\mathcal{L}|} R_\nu^{\mathcal{N} \setminus \mathcal{L}}[\mu_\mathcal{L} \cdot (\tilde{w}^s)_{\mathcal{E}_\mathcal{L}}] + \sum_{\mathcal{I} \subseteq \mathcal{N} \setminus \mathcal{L}} (-1)^{|\mathcal{L}| + |\mathcal{I}| + 1} \mu_\mathcal{L} \nu_{\mathcal{I}'} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{L} \cup \mathcal{I}'}} \\
&= (-1)^{|\mathcal{L}|} \sum_{\mathcal{I} \subseteq \mathcal{N} \setminus \mathcal{L}} (-1)^{|\mathcal{I}|} \left(\mu_\mathcal{L} \nu_\mathcal{I} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{L} \cup \mathcal{I}}} - \mu_\mathcal{L} \nu_{\mathcal{I}'} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{L} \cup \mathcal{I}'}} \right) \\
&= (-1)^{|\mathcal{L}|} \sum_{\mathcal{I} \subseteq \mathcal{N}' \setminus \mathcal{L}} (-1)^{|\mathcal{I}|} \mu_\mathcal{L} \nu_\mathcal{I} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{L} \cup \mathcal{I}}} = (-1)^{|\mathcal{L}|} R_\nu^{\mathcal{N}' \setminus \mathcal{L}}[\mu_\mathcal{L} \cdot (\tilde{w}^s)_{\mathcal{E}_\mathcal{L}}]
\end{aligned}
$$

is then a finite expression. Doing this for every non-empty $\mathcal{L} \subseteq \mathcal{N}$ covers the whole sum B_2 because every term $\mu_\mathcal{L} \nu_{\mathcal{I}'} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{L} \cup \mathcal{I}'}}$ appears exactly once and the signs match since

$$\sum_{\mathcal{I} \subseteq \mathcal{N} \setminus \mathcal{L}} (-1)^{|\mathcal{L}| + |\mathcal{I}| + 1} \mu_\mathcal{L} \nu_{\mathcal{I}'} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{L} \cup \mathcal{I}'}}$$
$$= \sum_{\emptyset \neq \mathcal{K} \subseteq \mathcal{N}, \mathcal{L} \subseteq \mathcal{K}} (-1)^{|\mathcal{K}'|} \mu_\mathcal{L} \nu_{\mathcal{K}' \setminus \mathcal{L}} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{K}'}}.$$

The same argumentation works for B_1. Fix $\mathcal{L} \subseteq \mathcal{N}$ and consider all terms in B_1 containing $\mu_{\mathcal{L}'}$:

$$\sum_{\mathcal{K} \subseteq \mathcal{N}, \mathcal{L} \subseteq \mathcal{K}} (-1)^{|\mathcal{K}'|} \mu_{\mathcal{L}'} \nu_{\mathcal{K} \setminus \mathcal{L}} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{K}'}}$$
$$= \sum_{\mathcal{I} \subseteq \mathcal{N} \setminus \mathcal{L}} (-1)^{|\mathcal{L}'| + |\mathcal{I}|} \mu_{\mathcal{L}'} \nu_{\mathcal{I}} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{L}' \cup \mathcal{I}}}$$
$$= (-1)^{|\mathcal{L}'|} R_\nu^{\mathcal{N} \setminus \mathcal{L}} [\mu_{\mathcal{L}'} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{L}'}}] = (-1)^{|\mathcal{L}'|} R_\nu^{\mathcal{N}' \setminus \mathcal{L}'} [\mu_{\mathcal{L}'} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{L}'}}].$$

Putting everything together we have shown that locally the difference between two renormalization operators $R_{\nu'}^{\mathcal{N}}$ and $R_\nu^{\mathcal{N}}$ is expressible as a sum of densities, supported on the components $\mathcal{E}_\mathcal{K}$ for $\mathcal{K} \subseteq \mathcal{N}$ and renormalized in the remaining directions according to subsets of $\mathcal{N} \setminus \mathcal{K}$:

$$(R_{\nu'}^{\mathcal{N}} - R_\nu^{\mathcal{N}})[\tilde{w}^s] = \sum_{\emptyset \neq \mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} R_\nu^{\mathcal{N}' \setminus \mathcal{K}} [\mu_\mathcal{K} \cdot (\tilde{w}^s)_{\mathcal{E}_\mathcal{K}}]$$
$$+ \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}'|} R_\nu^{\mathcal{N}' \setminus \mathcal{K}'} [\mu_{\mathcal{K}'} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{K}'}}]$$
$$= \sum_{\emptyset \neq \mathcal{K} \subseteq \mathcal{N}'} (-1)^{|\mathcal{K}|} R_\nu^{\mathcal{N}' \setminus \mathcal{K}} [\mu_\mathcal{K} \cdot (\tilde{w}^s)_{\mathcal{E}_\mathcal{K}}].$$

$\square$

This is a nice formula showing that a *finite renormalization* (i.e. changing the renormalization point $\{\nu\}$) amounts to adding a density supported on the exceptional divisor, like expected from the toy model case on $\mathbb{R}$ (or $\mathbb{R}^d$ for homogeneous distributions). But we can do even better and physics tells us what to expect: The Hopf algebraic formulation of the renormalization group predicts that the densities appearing in $(R_{\nu'}^{\mathcal{N}} - R_\nu^{\mathcal{N}})[\tilde{w}^s]$ should correspond to graphs showing up in the coproduct of G (for more on this we refer to [CK01] and [Kre13]). In the local formulation presented here the coproduct translates into local contractions, i.e. contractions with respect to nested sets $\mathcal{N}$.

Turning back to the example at the beginning of this chapter where we calculated $\langle R_{\nu'}^{\mathcal{N}}[\tilde{w}^s] - R_{\nu}^{\mathcal{N}}[\tilde{w}^s]|\varphi\rangle$ to be

$$-\langle R_{\nu_G}[\mu_g \cdot (\tilde{w}^s)_{\mathcal{E}_g}]|\varphi\rangle - \langle R_{\nu_g}[\mu_G \cdot (\tilde{w}^s)_{\mathcal{E}_G}]|\varphi\rangle + \langle \nu_g \nu_G \cdot (\tilde{w}^s)_{\mathcal{E}_{g,G}}|\varphi\rangle,$$

we now examine the individual terms in more detail. Eventually we are interested in the pairing with test functions φ that are pullbacks of test functions on X. Recall that such φ are then locally given by $\varphi = \beta^*\psi = \psi \circ \rho$ for $\psi \in \mathcal{D}(\beta(U))$. In $\kappa(U) = X^G = M^2$, corresponding to $\mathcal{N} = \{g, G\}$, an adapted spanning tree chosen as in the example in Section 5.3 and marked elements x_G, y_g $(x, y \in \mathbb{R}^4)$,

$$\begin{aligned}
\langle R_{\nu_G}[\mu_g \cdot (\tilde{w}^s)_{\mathcal{E}_g}]|\varphi\rangle &= \int_{M^2} d^4x\, d^4y\, \mu_g |x_G|^{7-8s} |y_g|^{3-4s} \left(\delta_g[f^s\varphi] - \nu_G \delta_{g,G}[f^s\varphi] \right) \\
&= \int_{M^2} d^4x\, d^4y\, \mu_g |x_G|^{7-8s} |y_g|^{3-4s} \\
&\quad \times \left((f^s\varphi)|_{y_g=0} - \nu_G (f^s\varphi)|_{y_g, x_G=0} \right) \\
&= \int_{M^2} d^4x\, d^4y\, \mu_g |x_G|^{7-8s} |y_g|^{3-4s} \\
&\quad \times \left(\frac{\psi(x_G\hat{x}, 0)}{\hat{x}^{4s}\hat{y}^{4s}} - \frac{\nu_G(x)\psi(0,0)}{\hat{x}^{4s}\hat{y}^{4s}} \right) \\
&= c_g \int_M d^4x\, |x_G|^{7-8s} \left(\frac{\psi(x_G\hat{x}, 0)}{\hat{x}^{4s}} - \frac{\nu_G(x)\psi(0,0)}{\hat{x}^{4s}} \right) \\
&= c_g \langle R_{\nu_G}[\tilde{w}^s_{G/g}]|\delta_g[\varphi]\rangle.
\end{aligned}$$

Here $\tilde{w}^s_{G/g}$ is the density associated to the contracted graph G/g (more precise, its local expression in $U_{B',\mathcal{N}'}$ with B' spanned by $x = \{x_e^i\}_{e \in E(t)\backslash E(g)}$ and $\mathcal{N}' = \{G/g\}$ - the exponent $7-8s$ in $|x_G|$ does not match but we neglect this little technical problem here; see below for the general argument). The coefficient c_g is given by

$$\begin{aligned}
c_g &= \int_M d^4y\, \frac{|y_g|^{3-4s}}{\hat{y}^{4s}} \left(\nu'_g(y) - \nu_g(y) \right) \\
&= \int_M d^4y\, \frac{|y_g|^{3-4s}}{\hat{y}^{4s}} \left(\nu'_g(y) - \nu_g(y)\delta_g[\nu_g(y)'] \right) \\
&= \langle R_{\nu_g}[\tilde{w}^s_g]|\nu'_g\rangle
\end{aligned}$$

because $\nu'_g|_{y_g=0} = 1$. In the same manner we calculate

$$\langle R_{\nu_g}[\mu_G \cdot (\tilde{w}^s)_{\mathcal{E}_G}]|\varphi\rangle = c_G\psi(0,0) = c_G\langle \delta_G|\varphi\rangle$$

with

$$c_G = \int_{M^2} d^4x d^4y \, \mu_G |x_G|^{7-8s} |y_g|^{3-4s} \left(\frac{1}{\hat{x}^{2s}\hat{y}^{4s}(\hat{x}+y_g\hat{y})^{2s}} - \frac{\nu_g(y)}{\hat{x}^{4s}\hat{y}^{4s}} \right)$$

$$= \int_{M^2} d^4x d^4y \, |x_G|^{7-8s} |y_g|^{3-4s}$$

$$\times \left(\frac{\nu'_G}{\hat{x}^{2s}\hat{y}^{4s}(\hat{x}+y_g\hat{y})^{2s}} - \frac{\nu_G}{\hat{x}^{2s}\hat{y}^{4s}(\hat{x}+y_g\hat{y})^{2s}} - \frac{\nu_g\nu'_G}{\hat{x}^{4s}\hat{y}^{4s}} + \frac{\nu_g\nu_G}{\hat{x}^{4s}\hat{y}^{4s}} \right)$$

$$= \int_{M^2} d^4x d^4y \, |x_G|^{7-8s} |y_g|^{3-4s}$$

$$\times \left(f^s\nu'_G - \nu_G\delta_G[f^s\nu'_G] - \nu_g\delta_g[f^s\nu'_G] + \nu_g\nu_G\delta_{g,G}[f^s\nu'_G] \right)$$

$$= \langle R^{\mathcal{N}}_\nu [\tilde{w}^s_G] | \nu'_G \rangle.$$

The last term $\langle [\nu_g\nu_G \cdot \tilde{w}^s]_{\mathcal{E}_{g,G}} | \varphi \rangle$ evaluates to $c_{g,G}\psi(0,0)$ with

$$c_{g,G} = \int_{M^2} d^4x d^4y \, \frac{\mu_G(x)\mu_g(y)|x_G|^{7-8s}|y_g|^{3-4s}}{\hat{x}^{4s}\hat{y}^{4s}}$$

$$= \langle R_{\nu_g}[\tilde{w}^s_g] | \nu'_g \rangle \langle R_{\nu_G}[\tilde{w}^s_{G/g}] | \nu'_G \rangle.$$

To formulate this in the general case it will be useful to define the contraction operation $/\!/$ not only on single graphs but also on nested sets.

Definition 7.2. Let $\mathcal{N}$ be a nested set for some building set $\mathcal{B} \subseteq \mathcal{D}$ and let $\mathcal{J} \subseteq \mathcal{N}$. The contraction $\mathcal{N}/\!/\mathcal{J}$ is defined as the poset with underlying set

$$\mathcal{N}/\!/\mathcal{J} := \{ g/\!/\mathcal{J} \mid g \in \mathcal{N} \},$$

partially ordered by inclusion. Since the inclusion operation differs from the one in $\mathcal{N}$ (contracted graphs may not be subgraphs of G anymore, although we can identify them with subgraphs via their edge sets), we denote this partial order by $\sqsubseteq$.

The partial order $\sqsubseteq$ is most easily understood by looking at the Hasse diagram of $\mathcal{N}$. Replace every $g \in \mathcal{N}$ by $g/\!/\mathcal{J}$, remove all lines that connect elements of $\mathcal{J}$ to "above" and draw a new line from o to every element that became disconnected in the process. Note that in particular all elements of $\mathcal{J}$ have become maximal in $\mathcal{N}/\!/\mathcal{J}$.

In addition, we denote by abuse of notation the corresponding contractions on adapted spanning trees by the same symbol, i.e. we define

$$t/\!/\mathcal{J} := t/t_{\mathcal{J}} \text{ where } t_{\mathcal{J}} := \bigcup_{\gamma \in \mathcal{J}} t_\gamma,$$

$$t_g/\!/\mathcal{J} := t_g/\!/\mathcal{J}_{<g}.$$

Example. Let G be the graph shown in Figure 7.1. Denote by γ_1, γ_2 and γ_3 the three fish subgraphs from left to right, and let g and h be the full subgraphs on the vertex set $V(g) = \{0, 1, 2, 3\}$ and $V(h) = \{2, 3, 4, 5\}$. In Figure 7.2 we depict an $I(\mathcal{D}(G))$-nested set $\mathcal{N}$ and the poset $(\mathcal{N}//\mathcal{J}, \sqsubseteq)$ for $\mathcal{J} = \{\gamma_1, \gamma_3, g\}$.
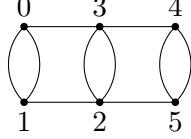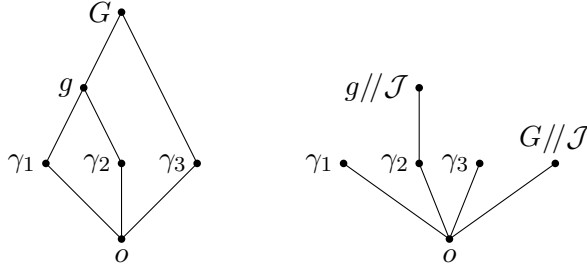

Figure 7.1: The graph G

Figure 7.2: The posets $\mathcal{N}$ and $(\mathcal{N}//\mathcal{J}, \sqsubseteq)$

In the general case we have the following structure for finite renormalizations:

Theorem 7.3. *Consider the collection of renormalization operators R_ν for two sets of subtraction points $\{\nu'\}$ and $\{\nu\}$. Let $\mathcal{N}$ be a nested set for the building set $\mathcal{D}$ or $I(\mathcal{D})$ and B be an adapted, marked basis. Then the local expression for the difference $(R_{\nu'}^{\mathcal{N}} - R_\nu^{\mathcal{N}})[\tilde{w}^s]$ applied on a test function $\varphi = \beta^* \psi$ for $\psi \in \mathcal{D}(\beta(U))$ is given by*

$$\langle (R_{\nu'}^{\mathcal{N}} - R_\nu^{\mathcal{N}})[\tilde{w}^s] | \varphi \rangle = \sum_{\emptyset \neq \mathcal{K} \subseteq \mathcal{N}} c_{\mathcal{K}} \langle R_\nu[\tilde{w}_{G//\mathcal{K}}^s] | \delta_{\mathcal{K}}[\varphi] \rangle \tag{7.1}$$

where

$$c_{\mathcal{K}} = \prod_{\gamma \in \mathcal{K}} \langle R_\nu[\tilde{w}_{\gamma//\mathcal{K}}^s] \mid \nu'_\gamma \rangle \tag{7.2}$$

and $\langle R_\nu[\tilde{w}_{\emptyset}^s] | \delta_G[\varphi] \rangle$ is to be understood as $\langle \delta | \psi \rangle = \psi(0)$.

We have dropped the indices in R_ν for simplicity. Define $\mathcal{H} := \cup_{\gamma \in \mathcal{K}} \mathcal{H}_\gamma$ with $\mathcal{H}_\gamma := \{h \in \mathcal{N} \mid h//\mathcal{K} \in (\mathcal{N}//\mathcal{K})_{\sqsubset \gamma//\mathcal{K}}\}$. Then it is shown below that in (7.1) the index is given by $\mathcal{N} \setminus (\mathcal{K} \cup \mathcal{H})$. Likewise, in (7.2) $\mathcal{H}_\gamma \cup \{\gamma\}$ is the index in the factor associated to $\gamma \in \mathcal{K}$.

Proof. Using Proposition 7.1 we examine all terms in $\langle (R_{\nu'}^{\mathcal{N}} - R_{\nu}^{\mathcal{N}})[\tilde{w}^s]|\varphi \rangle$ separately. The proof consists of two steps. First we study how $\delta_{\mathcal{K}}$ acts on the maps f and $\varphi = \beta^* \psi$. This allows then in the second step to show that the integral arising in the evaluation of $\langle (R_{\nu'}^{\mathcal{N}} - R_{\nu}^{\mathcal{N}})[\tilde{w}^s]|\varphi \rangle$ factorizes into a product of integrals according to (7.1) and (7.2).

Recall that B is given by an $\mathcal{D}$-adapted spanning tree t and $\tilde{w}^s = u_{\mathcal{N}}^s f^s |dx|$. Claim: For $\mathcal{J} \subseteq \mathcal{N}$ the map $\delta_{\mathcal{J}}$ operates on f and $\varphi = \beta^* \psi \overset{loc.}{=} \psi \circ \rho$ by

$$\varphi \mapsto \delta_{\mathcal{J}}[\varphi] = \varphi|_{x_{\mathcal{J}}=0},$$
$$f \mapsto \delta_{\mathcal{J}}[f] = \prod_{\gamma \in \mathcal{J} \cup \{G\}} f_{\gamma//\mathcal{J}}.$$

Here $f_{g//\mathcal{J}}$ is defined as follows: Contracting t_g with respect to $\mathcal{J}$ defines an adapted spanning tree for $g//\mathcal{J}$ (contracting graphs in $\mathcal{N}$ and t accordingly does not change the properties of t being spanning and adapted - cf. the construction in Proposition 5.22). Define

$$X^{g//\mathcal{J}} := \{(x_{e_1}, \ldots, x_{e_k}) \mid \{e_1, \ldots, e_k\} = E(t_g//\mathcal{J})\} \tag{7.3}$$

with adapted basis $B' = B|_{e \in E(t_g//\mathcal{J})}$. The set $\mathcal{N}' = \mathcal{N}//\mathcal{J}_{\sqsubseteq g//\mathcal{J}}$ is nested for the minimal building set $I(\mathcal{D}(g//\mathcal{J}))$ in the divergent arrangement of $g//\mathcal{J}$. Mimicing the construction of the wonderful models for this case, we obtain an open set $U_{\mathcal{N}',B'}$ that is a local piece of the (minimal) wonderful model for the graph $g//\mathcal{J}$. The function $f_{g//\mathcal{J}}^s$ is then the regular part of the local pullback of $\tilde{v}_{g//\mathcal{J}}^s$ onto this model. The factor $f_{G//\mathcal{J}}$ collects all the remaining parts and is defined in the same way, except for one special case: If G does not lie in $\mathcal{N}$, or even not in $\mathcal{D}$ (locally in $U_{\mathcal{N},B}$ this is the same!), and $G//\mathcal{J}$ is primitive, then $\mathcal{N}' = \emptyset$ and we do not have a local model to pullback $\tilde{v}_{G//\mathcal{J}}^s$ onto. But in this case $v_{G//\mathcal{J}} = f_{G//\mathcal{J}}$ is already regular and no model is required. Also note that if $G \in \mathcal{N}$, the operation δ_G does not alter f since it does not depend on the variable $x_G^{i_G}$.

Recall that in coordinates given by an adapted spanning tree the distribution kernel v is a product of factors $(y_e)^{2-d}$ with $e \in E(G)$ and

$$y_e = \begin{cases} x_e & \text{if } e \text{ is an edge of } t, \\ \sum_{e' \in E(t_e)} \sigma_t(e') x_{e'} & \text{if } e \text{ is an edge in } G \setminus t. \end{cases}$$

Moreover, the pullback under β onto the wonderful model Y is locally given by the map $\rho = \rho_{\mathcal{N},B}$ that scales all x_e with $e \in E(t_g)$ and $g \in \mathcal{N}$ by $x_g^{i_g}$.

To prove the claimed properties of $\delta_{\mathcal{J}}$ we argue like in the proof of Theorem 6.4:

1. Since $\varphi = \psi \circ \rho$, we have that $\delta_{\mathcal{J}}[\varphi]$ is equivalent to $\varphi|_{\{x_h^{i_h}=0\}}$ for h in $\max \mathcal{J}$, the set of maximal elements of $\mathcal{J}$. This means that the resulting map only depends on the variables x_e with $e \in E(t \cap \mathcal{N}_{>\max \mathcal{J}})$. All other vectors are scaled by the $x_h^{i_h}$ and therefore vanish after $\delta_{\mathcal{J}}$ is applied. Another way to put this is that $\delta_{\mathcal{J}}[\varphi]$ depends only on the x_e with $e \in E(t//\mathcal{J})$. In particular, if $G \in \mathcal{J}$ then $\delta_{\mathcal{J}}[\varphi]$ is just a constant, $\delta_{\mathcal{J}}[\varphi] = \langle \delta | \psi \rangle = \psi(0)$.

2. For the second claim start with $\mathcal{J} = \{g\}$ consisting only of a single subgraph $g \subsetneq G$. The part of f that depends only on the vectors associated to edges of g is unaffected by setting $x_g^{i_g} = 0$ because all x_e with $e \in E(t_g)$ get scaled and so the factor $x_g^{i_g}$ pulls out (it is already absorbed into the definition of u_g^s). On the other hand, the remaining part of f depends on x_e with $e \in E(t_g)$ only through special linear combinations. These linear combinations express vectors representing edges e' that do not lie in g but are connected to a vertex of g such that $E(t_{e'}) \cap E(t_g) \neq \emptyset$. They become independent of x_e after setting $x_g^{i_g}$ to zero. Therefore, $\delta_{\mathcal{J}}[f]$ splits into a product of two factors depending on the mutual disjoint sets of vectors $\{x_e\}_{e \in E(t_g)}$ or $\{x_e\}_{e \in E(t/t_g)}$, i.e.

$$\delta_g[f] = f_g f_{G/g} = f_{g//\mathcal{J}} f_{G//\mathcal{J}}.$$

Adding another graph $h \neq G$ from $\mathcal{N}$ to $\mathcal{J}$ and using Lemma 6.7 we can express $\delta_{\mathcal{J}}$ as

$$\delta_{\mathcal{J}}[f] = \delta_{g,h}^g[\delta_g[f]] = \delta_h[f_g f_{G/g}].$$

There are three possible cases (due to Lemma 5.17 there cannot be two incomparable g, h with non-empty overlap in an $I(\mathcal{D})$-nested set; if the maximal building set $\mathcal{D}$ is considered only case 2 and 3 are possible):

1. g and h are incomparable. Then f_g does not depend on any x_e with $e \in E(t_h)$ and

$$\delta_h[f_g f_{G/g}] = f_g \delta_h[f_{G/g}] = f_{g//\mathcal{J}} \delta_h[f_{G/g}].$$

2. h is contained in g, $h \subsetneq g$. Then all $\{x_e\}_{e \in E(t_h)}$ are scaled by $x_g^{i_g}$ and $f_{G/g}$ is independent of these. Thus, only f_g is affected by contracting h,

$$\delta_h[f_g f_{G/g}] = \delta_h[f_g] f_{G/g} = \delta_h[f_g] f_{G//\mathcal{J}}.$$

3. h contains g, $g \subsetneq h$. Then all $\{x_e\}_{e \in E(t_g)}$ are scaled by $x_h^{i_h}$ and f_g is not affected by setting $x_h^{i_h} = 0$. Therefore

$$\delta_h[f_g f_{G/g}] = f_g \delta_h[f_{G/g}] = f_{g//\mathcal{J}} \delta_h[f_{G/g}].$$

In all three cases we argue like in the first step to carry out the operation of δ_h and conclude

$$\delta_{\mathcal{J}}[f] = f_{g//\mathcal{J}} f_{h//\mathcal{J}} f_{G//\mathcal{J}}.$$

For general $\mathcal{J} \subseteq \mathcal{N}$ we repeat this procedure for a finite number of steps to show

$$\delta_{\mathcal{J}}[f] = \prod_{\gamma \in \mathcal{J} \cup \{G\}} f_{\gamma//J}.$$

With the help of these two assertions we are now able to examine the integrals

$$\langle R_\nu^{\mathcal{N}\backslash\mathcal{K}}[\mu_{\mathcal{K}} \cdot (w^s)_{\mathcal{E}_{\mathcal{K}}}]|\varphi\rangle = \int_{\kappa(U)} dx u_{\mathcal{N}}^s \mu_{\mathcal{K}} \sum_{\mathcal{J} \subseteq \mathcal{N}\backslash\mathcal{K}} (-1)^{|\mathcal{J}|} \nu_{\mathcal{J}} \delta_{\mathcal{K}\cup\mathcal{J}}[f^s\varphi] \quad (7.4)$$

in detail. Note that $f_{g//\mathcal{J}}$ depends only on the variables x_e associated to edges of $E(t_g \setminus t_{h_1 \cup \ldots \cup h_k})$ with $\{h_1, \ldots, h_k\} = \max \mathcal{J}_{<g}$ (not on the marked element $x_g^{i_g}$ though!). This is exactly the set of coordinates on which the maps ν_g depend. Therefore divergences corresponding to elements $g//\mathcal{J}$ are also renormalized by the subtraction points ν_g associated to g.

To make the following calculations more readable we simplify the notation: For $\mathcal{K} \subseteq \mathcal{N}$ write $\tilde{g}$ for the $\mathcal{K}$-contracted graph $g//\mathcal{K}$. Let $\mathcal{K} = \{g_1, \ldots, g_n\}$ (if $G \in \mathcal{K}$ assume $g_n = G$) and define the subsets $\mathcal{H}_i \subseteq \mathcal{N}$ by

$$\mathcal{H}_i := \{h \mid \tilde{h} \sqsubset \tilde{g}_i\} \text{ for } i = 1, \ldots, n.$$

We want to show that the integral

$$\int_{\kappa(U)} dx\, u_{\mathcal{N}}^s \prod_{i=1}^n (\nu'_{g_i} - \nu_{g_i}) \sum_{\mathcal{J} \subseteq \mathcal{N}\backslash\mathcal{K}} (-1)^{|\mathcal{J}|} \nu_{\mathcal{J}} \prod_{\gamma \in \mathcal{K}\cup\mathcal{J}\cup\{G\}} f_{\tilde{\gamma}//\mathcal{J}}^s\, \delta_{\mathcal{K}\cup\mathcal{J}}[\varphi]$$

factorizes into a product of integrals according to (7.1) and (7.2). To see this split the sum into two parts, the first one summing over subsets $\mathcal{I} \subseteq \mathcal{N} \setminus \mathcal{K}$ that contain an element of $\mathcal{H}_1$, i.e. $\mathcal{I} \cap \mathcal{H}_1 \neq \emptyset$, the second one over subsets $\mathcal{J} \subseteq \mathcal{N} \setminus \mathcal{K}$ with $\mathcal{J} \cap \mathcal{H}_1 = \emptyset$. The first sum can then be written as

$$\sum_{\substack{\mathcal{J} \subseteq \mathcal{N}\backslash\mathcal{K} \\ \mathcal{J} \cap \mathcal{H}_1 = \emptyset}} (-1)^{|\mathcal{J}|} \sum_{\emptyset \neq \mathcal{L} \subseteq \mathcal{H}_1} (-1)^{|\mathcal{L}|} \nu_{\mathcal{J}} \nu_{\mathcal{L}} \delta_{\mathcal{K}\cup\mathcal{J}\cup\mathcal{L}}[f^s] \delta_{\mathcal{K}\cup\mathcal{J}\cup\mathcal{L}}[\varphi].$$

Now

$$\delta_{\mathcal{K}\cup\mathcal{J}\cup\mathcal{L}}[\varphi] = \delta_{\mathcal{K}\cup\mathcal{J}}[\varphi],$$

because all $g \in \mathcal{L}$ satisfy $\tilde{g} \sqsubset \tilde{g}_1$ and from this follows $g \leq g_1$ for the partial order on $\mathcal{N}$. Therefore all $g \in \mathcal{L}$ are scaled by $x_{g_1}^{i_{g_1}}$ which is set to zero by

$\delta_{\mathcal{K}}$. Again, since all elements of $\mathcal{L}$ are smaller than g_1,

$$\delta_{\mathcal{K}\cup\mathcal{J}\cup\mathcal{L}}[f^s] = f_{\tilde{G}//\mathcal{J}\cup\mathcal{L}} \prod_{\gamma\in(\mathcal{K}\cup\mathcal{J}\cup\mathcal{L})\backslash\{G\}} f_{\tilde{\gamma}//\mathcal{J}\cup\mathcal{L}}$$

$$= f_{\tilde{G}//\mathcal{J}} \prod_{\gamma\in\mathcal{K}\backslash\{G\}} f_{\tilde{\gamma}//\mathcal{J}\cup\mathcal{L}} \prod_{\xi\in\mathcal{J}\backslash\{G\}} f_{\tilde{\xi}//\mathcal{J}\cup\mathcal{L}} \prod_{\eta\in\mathcal{L}\backslash\{G\}} f_{\tilde{\eta}//\mathcal{J}\cup\mathcal{L}}$$

$$= f_{\tilde{G}//\mathcal{J}}\, f_{\tilde{g}_1//\mathcal{L}} \prod_{\gamma\in\mathcal{K}\backslash\{g_1,G\}} f_{\tilde{\gamma}//\mathcal{J}} \prod_{\xi\in\mathcal{J}\backslash\{G\}} f_{\tilde{\xi}//\mathcal{J}} \prod_{\eta\in\mathcal{L}\backslash\{G\}} f_{\tilde{\eta}//\mathcal{L}}.$$

In the last line we have used that $\tilde{g}_1$ is immune to contraction by elements lying outside of $\mathcal{H}_1$. Thus, the factor

$$\sum_{\emptyset\neq\mathcal{L}\subseteq\mathcal{H}_1} (-1)^{|\mathcal{L}|} \nu_{\mathcal{L}}\, f_{\tilde{g}_1//\mathcal{L}} \prod_{\eta\in\mathcal{L}\backslash\{G\}} f^s_{\tilde{\eta}//\mathcal{L}}$$

can be pulled out of the first sum.

In the second sum over the subsets $\mathcal{J} \subseteq \mathcal{N}\backslash\mathcal{K}$ with $\mathcal{J}\cap\mathcal{H}_1 = \emptyset$ the factor $f^s_{\tilde{g}_1}$ appears in every summand because $\tilde{g}_1$ is not affected by $\delta_{\mathcal{J}}$. Recall that $\delta_{\mathcal{K}}[\varphi]$ depends only on the coordinates $\{x_e\}_{e\in E(t//\mathcal{K})}$ to conclude that

$$\int_{\kappa(U)} dx\, u^s_{\mathcal{N}} \prod_{i=1}^n (\nu'_{g_i} - \nu_{g_i}) \sum_{\mathcal{J}\subseteq\mathcal{N}\backslash\mathcal{K}} (-1)^{|\mathcal{J}|} \nu_{\mathcal{J}}\, \delta_{\mathcal{K}\cup\mathcal{J}}[f^s]\, \delta_{\mathcal{K}\cup\mathcal{J}}[\varphi]$$

$$= \int_{V_1} dx\, u^s_{g_1} u^s_{\mathcal{H}_1} (\nu'_{g_1} - \nu_{g_1}) \left(\sum_{\emptyset\neq\mathcal{L}\subseteq\mathcal{H}_1} (-1)^{|\mathcal{L}|} \nu_{\mathcal{L}} \prod_{\eta\in\mathcal{L}\backslash\{G\}} f^s_{\tilde{\eta}//\mathcal{L}}\, f^s_{\tilde{g}_1//\mathcal{L}} + f^s_{\tilde{g}_1} \right)$$

$$\times \int_{V_2} dx\, u^s_{\mathcal{N}\backslash(\mathcal{H}_1\cup\{g_1\})} \prod_{i=2}^n (\nu'_{g_i} - \nu_{g_i}) \sum_{\substack{\mathcal{J}\subseteq\mathcal{N}\backslash\mathcal{K}\\\mathcal{J}\cap\mathcal{H}_1=\emptyset}} (-1)^{|\mathcal{J}|} \nu_{\mathcal{J}}\, \delta_{\mathcal{K}\backslash\{g_1\}\cup\mathcal{J}}[f^s]\, \delta_{\mathcal{K}\cup\mathcal{J}}[\varphi]$$

$$= \langle R_{\nu}[\tilde{w}_{g_1//\mathcal{K}}]|\nu'_{g_1}\rangle \int \cdots .$$

Here we have changed the domain of integration from $\kappa(U)$ to $V_1 \times V_2$ with V_i constructed as follows: Pick a linear extension of the partial order on $\mathcal{K} = \{g_1,\ldots,g_n\}$ and let $g_1 = \tilde{g}_1$ be the minimal element (the proof works also without this assumption, but this simplifies it considerably). Define $X^{\tilde{g}_1}$ as in Equation (7.3) and $X^{G'}$ similarly for $G' := G/\tilde{g}_1$. Recall the wonderful construction from Definition 4.8 and set for every g in the minimal or maximal building set $\mathcal{B}$

$$Z^1_g := Z_g \cap (X^{\tilde{g}_1} \times \{0\}) \text{ and } Z^2_g := Z_g \cap (\{0\} \times X^{G'}).$$

Define $V_1 := X^{\tilde{g}_1} \setminus \cup_{\gamma\in\mathcal{B}} Z^1_\gamma$ and $V_2 := X^{G'} \setminus \cup_{\gamma\in\mathcal{B}} Z^2_\gamma$. Then V_1 is a local chart domain for the wonderful model for $\tilde{g}_1$ with respect to the nested set

$\mathcal{H}_1 \cup \{g_1\}$ and adapted basis $B|_{E(t_{\tilde{g}_1})}$. The same holds for G' with respect to $\mathcal{N} \setminus (\mathcal{H}_1 \cup \{g_1\})$ and $B|_{E(t_{G/\tilde{g}_1})}$. For the original chart on Y we have

$$\kappa(U) = X^G \setminus (\cup_{\gamma \in \mathcal{B}} Z_\gamma) \subseteq V_1 \times V_2,$$

and the difference is an union of linear subspaces, i.e a set of measure zero. Moreover, the integrand is finite because all divergences associated to the elements of $\mathcal{K}$ get "damped" by $\mu_{\mathcal{K}}$, the remaining divergences coming from elements of $\mathcal{N} \setminus \mathcal{K}$ are renormalized and $\psi \in \mathcal{D}(\beta(U_{\mathcal{N},B}))$ vanishes in a neighborhood of all Z_γ, which covers the divergences of $\mathcal{B} \setminus \mathcal{N}$. Thus, changing the domain is justified and by Fubini's theorem the integral factorizes into the desired product. The last equality holds because of

$$(\nu'_{g_1} - \nu_{g_1}) \left(\sum_{\emptyset \neq \mathcal{L} \subseteq \mathcal{H}_1} (-1)^{|\mathcal{L}|} \nu_{\mathcal{L}} \prod_{\eta \in \mathcal{L}} f^s_{\tilde{\eta}//\mathcal{L}} \; f^s_{\tilde{g}_1//\mathcal{L}} + f^s_{\tilde{g}_1} \right)$$

$$= f^s_{\tilde{g}_1} \nu'_{g_1} - \nu_{g_1} (f^s_{\tilde{g}_1} \nu'_{g_1})|_{x^{i_{g_1}}_{g_1} = 0} + \sum_{\emptyset \neq \mathcal{L} \subseteq \mathcal{H}_1} (-1)^{|\mathcal{L}|} \nu_{\mathcal{L}} \prod_{\eta \in \mathcal{L} \cup \{g_1\}} f^s_{\tilde{\eta}//\mathcal{L}} \nu'_{g_1}$$

$$- \sum_{\emptyset \neq \mathcal{L} \subseteq \mathcal{H}_1} (-1)^{|\mathcal{L}|} \nu_{\mathcal{L}} \nu_{g_1} (\prod_{\eta \in \mathcal{L} \cup \{g_1\}} f^s_{\tilde{\eta}//\mathcal{L}} \nu'_{g_1})|_{x^{i_{g_1}}_{g_1} = 0}$$

$$= f^s_{\tilde{g}_1} \nu'_{g_1} + \sum_{\emptyset \neq \mathcal{L} \subseteq \mathcal{H}_1 \cup \{g_1\}} (-1)^{|\mathcal{L}|} \nu_{\mathcal{L}} \delta_{\mathcal{L}} [f^s_{\tilde{g}_1} \nu'_{g_1}].$$

A technical detail: If g_1 had another divergent subgraph $h \in \mathcal{B} \setminus \mathcal{N}$, the renormalization by R_ν would not take care of this and the integral would still diverge. But in this case all variables $\{x_e\}_{e \in E(t_h)}$ are set to zero by $\delta_{\mathcal{K}}$. Then the whole summand associated to $\mathcal{K}$ in Formula (7.1) vanishes because $\delta_{\mathcal{K}}[\varphi] = \delta_{\mathcal{K}}[\psi \circ \rho] = 0$ since $\text{supp}(\psi) \subseteq \kappa(U)$ is disjoint from $\{x_e = 0 \mid e \in E(t_h)\}$.

The remaining integral is of the same structure as the one we started with, so we can repeat the process for $g_2 \in \mathcal{K}$ (notice how this relies heavily on $\mathcal{K} \subseteq \mathcal{N}$ being nested and the stability of t under contractions). After a finite number of steps we obtain a product of renormalized densities, each factor representing an element of $\mathcal{K}$, possibly times a last remaining factor,

$$\int_{V_2} dz \, u^s_{\mathcal{N} \setminus (\cup_{i=1}^n \mathcal{H}_i \cup \{g_i\})} \sum_{\substack{\mathcal{J} \subseteq \mathcal{N} \setminus \mathcal{K} \\ \mathcal{J} \cap (\cup_i \mathcal{H}_i) = \emptyset}} (-1)^{|\mathcal{J}|} \nu_{\mathcal{J}} \, f^s_{\tilde{G}//\mathcal{J}} \prod_{\gamma \in \mathcal{J} \setminus \{G\}} f^s_{\tilde{\gamma}//\mathcal{J}} \, \delta_{\mathcal{K} \cup \mathcal{J}}[\varphi].$$

If G lies in $\mathcal{K}$, then $\delta_{\mathcal{K}}[\varphi] = \psi(0)$ is constant and the procedure ends before this last step since the $\mathcal{H}_i$ cover $\mathcal{N}$. If G is not in $\mathcal{K}$, then $G//\mathcal{K}$ could have remaining divergences, given by the elements of $\mathcal{N} \setminus (\mathcal{K} \cup \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_n)$, and therefore this last integral is just the renormalized expression for $\tilde{w}^s_{G//\mathcal{K}}$,

applied to the test function $\delta_{\mathcal{K}}[\varphi]$,

$$\sum_{\substack{\mathcal{J}\subseteq\mathcal{N}\setminus\mathcal{K} \\ \mathcal{J}\cap(\cup_i\mathcal{H}_i)=\emptyset}} (-1)^{|\mathcal{J}|}\nu_{\mathcal{J}}\ f^s_{\tilde{G}//\mathcal{J}}\prod_{\gamma\in\mathcal{J}\setminus\{G\}} f^s_{\tilde{\gamma}//\mathcal{J}}\ \delta_{\mathcal{K}\cup\mathcal{J}}[\varphi]$$

$$=\sum_{\mathcal{J}\subseteq\mathcal{N}\setminus(\mathcal{K}\cup\mathcal{H}_1\cup...\cup\mathcal{H}_n)} (-1)^{|\mathcal{J}|}\nu_{\mathcal{J}}\ \delta_{\mathcal{J}}[f^s_{\tilde{G}//\mathcal{K}}\delta_{\mathcal{K}}[\varphi]]$$

$$=\langle R_\nu[\tilde{w}^s_{G//\mathcal{K}}]|\delta_{\mathcal{K}}[\varphi]\rangle.$$

Last but not least, there remains one technical detail to take care of: The exponents in u^s_g do not match the ones provided by the definition of $\tilde{w}^s_{G/\mathcal{K}}$ and $\tilde{w}^s_{g//\mathcal{K}}$. This would not happen if we had defined subgraphs $g\subseteq G$ as given by there edge set $E(g)\subseteq E(G)$ but with $V(g)=V(G)$ (we chose not to do so because in the formulation presented here, X is spanned by variables associated to edges of an adapted spanning tree, not by the elements of V' like in [BBK10]). However, we can also just rescale the complex regularization parameter $s=1-\frac{d_{\tilde{g}}}{d_g}(1-\tilde{s})$ without affecting the whole construction to obtain the correct exponent in $u^s_{\tilde{g}}$. On the other hand, this discrepancy does not show up in the limit $s\to 1$ which we are allowed to take because this proof shows that every term in (7.1) and (7.2) is well-defined at $s=1$. Putting everything together we arrive at the desired formula. $\qquad\square$

Now we consider minimal subtraction. It will turn out to work in exactly the same manner as the case above. This is already clear if we think of R_1 as a non-smooth version of R_ν by making the (forbidden) substitution $\nu_g(x_g)=\theta(1-|x^{i_g}_g|)$.

Let $\mathcal{N}$ be a nested set for the building set $I(\mathcal{D})$ or $\mathcal{D}$. Locally in U the minimal subtraction operator R_1 is given by

$$\tilde{w}^s(x)=\prod_{\gamma\in\mathcal{N}} u^s_\gamma(x^{i\gamma}_\gamma)f^s(x)|dx| \overset{R_1}{\longmapsto} \prod_{\gamma\in\mathcal{N}}\left(u^s_\gamma(x^{i\gamma}_\gamma)\right)_{\heartsuit}f^s(x)|dx|,$$

where $(\dots)_{\heartsuit}$ denotes the regular part

$$\langle\left(u^s_g\right)_{\heartsuit}|\varphi\rangle:=\int dx\ |x|^{-1+d_g(1-s)}\big(\varphi(x)-\theta(1-|x|)\varphi(0)\big)$$

and the factor f^s is treated as test function. If instead $\theta^c(x):=\theta(c-|x|)$ with $c>0$ is used as cutoff, the principal part of the Laurent expansion does not change while the regular part $(u^s_g)_{\heartsuit_c}$ gets an additional δ-term

$$2\sum_{k\geq 0}\frac{d^k_g\log^{k+1}(c)}{k+1!}(s-1)^k\delta,$$

as predicted by Theorem 3.7. Therefore the whole Laurent series for u_g^s is given by

$$\langle u_g^s | \varphi \rangle = \frac{2}{d_g} \langle \delta | \varphi \rangle (1-s)^{-1} + \sum_{k \geq 0} \langle \frac{1}{k!} \left(|x|^{-1} \log^k(|x|) \right)_{\heartsuit_c} | \varphi \rangle (1-s)^k$$

$$= \frac{2}{d_g} \langle \delta | \varphi \rangle (1-s)^{-1} + \sum_{k \geq 0} \langle \frac{1}{k!} \left(|x|^{-1} \log^k(|x|) \right)_{\heartsuit} | \varphi \rangle (1-s)^k$$

$$+ \sum_{k \geq 0} \frac{1}{k+1!} d_g^k \log^{k+1}(c) \langle \delta | \varphi \rangle (1-s)^k.$$

Write $\mathcal{N} = \{g_1, \ldots, g_n\}$ and for $k \in \{1, \ldots, n\}$ let $x_k^{i_k}$ denote the associated marked element. Write $\hat{x}$ for the collection of all other coordinates. For a test function $\varphi \in \mathcal{D}(\kappa(U))$ set

$$\phi_s(x_1^{i_1}, \ldots, x_n^{i_n}) := \int d\hat{x} \, f^s(x) \varphi(x).$$

By expanding the successive application of the regular parts $(u_g^s)_{\heartsuit}$ and re-ordering the sum we see that R_1 is expressed by a formula similar to the one for subtraction at fixed conditions:

$$\langle R_1^{\mathcal{N}}[\tilde{w}^s] | \varphi \rangle = \langle (u_{g_1}^s)_{\heartsuit} | \langle (u_{g_2}^s)_{\heartsuit} | \ldots \langle (u_{g_n}^s)_{\heartsuit} | \phi_s \rangle \ldots \rangle \rangle$$

$$= \langle (u_{g_1}^s)_{\heartsuit} | \langle (u_{g_2}^s)_{\heartsuit} | \ldots \int dx_n^{i_n} |x_n^{i_n}|^{-1+d_n(s-1)}$$

$$\times \left(\phi_s(x_1^{i_1}, \ldots, x_n^{i_n}) - \theta^1(x_n^{i_n}) \phi_s(x_1^{i_1}, \ldots, x_{n-1}^{i_{n-1}}, 0) \right) \rangle \ldots \rangle$$

$$= \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} \int dx \prod_{j=1}^{n} u_{g_j}^s(x_j^{i_j}) \prod_{\gamma \in \mathcal{K}} \theta_\gamma(x_\gamma^{i_\gamma}) \delta_{\mathcal{K}}[f^s \varphi](x).$$

We do the same for the regular parts obtained by cutting off at c,

$$\langle R_c^{\mathcal{N}}[\tilde{w}^s] | \varphi \rangle = \sum_{\mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} \int dx \prod_{j=1}^{n} u_{g_j}^s(x_j^{i_j}) \prod_{\gamma \in \mathcal{K}} \theta_\gamma^c(x_\gamma^{i_\gamma}) \delta_{\mathcal{K}}[f^s \varphi](x),$$

and write $\theta^{c'}(x) = \theta^c(x) + \vartheta^{c,c'}(x)$ with

$$\vartheta^{c,c'}(x) = \begin{cases} 1 & \text{for } c < |x| < c', \\ 0 & \text{else.} \end{cases}$$

Then we can express the difference between two minimal subtraction operators $R_{c'}^{\mathcal{N}}$ and $R_c^{\mathcal{N}}$, applied to $\tilde{w}^s$, by the density

$$\varphi \longmapsto \sum_{\emptyset \neq \mathcal{K} \subset \mathcal{N}} (-1)^{|\mathcal{K}|} \int dx \, u_{\mathcal{N}}^s \Theta_{\mathcal{K}}^{c',c} \delta_{\mathcal{K}}[f^s \varphi].$$

Here $\Theta_{\mathcal{K}}^{c',c} := \prod_{\gamma \in \mathcal{K}} \theta^{c'}(x_\gamma^{i_\gamma}) - \prod_{\gamma \in \mathcal{K}} \theta^c(x_\gamma^{i_\gamma})$ is a "multidimensional cutoff", supported on

$$\{x \in \mathbb{R}^{|\mathcal{K}|} \mid c < |x_i| < c' \text{ for all } i \in \{1, \dots, |\mathcal{K}|\}\}.$$

Expanding $\Theta_{\mathcal{K}}^{c,c'}$ and reordering the sum, we have

$$\begin{aligned}
\Theta_{\mathcal{K}}^{c',c} &= \prod_{\gamma \in \mathcal{K}} (\theta_\gamma^c + \vartheta_\gamma^{c,c'}) - \prod_{\gamma \in \mathcal{K}} \theta_\gamma^c \\
&= \sum_{\emptyset \neq \mathcal{J} \subseteq \mathcal{K}} (\prod_{\gamma \in \mathcal{J}} \vartheta_\gamma^{c,c'})(\prod_{\eta \in \mathcal{K} \setminus \mathcal{J}} \theta_\eta^c) \\
&= \sum_{\emptyset \neq \mathcal{J} \subseteq \mathcal{K}} \vartheta_{\mathcal{J}}^{c,c'} \theta_{\mathcal{K} \setminus \mathcal{J}}^c.
\end{aligned}$$

Putting everything together, we find that a change of the renormalization point c is expressed by a sum of densities supported on components of the exceptional divisor given by subsets $\mathcal{K} \subseteq \mathcal{N}$,

$$\langle (R_{c'}^{\mathcal{N}} - R_c^{\mathcal{N}})[\tilde{w}^s] | \varphi \rangle = \sum_{\emptyset \neq \mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} \sum_{\emptyset \neq \mathcal{J} \subseteq \mathcal{K}} \int dx \, u_{\mathcal{N}}^s \vartheta_{\mathcal{J}}^{c,c'} \theta_{\mathcal{K} \setminus \mathcal{J}}^c \delta_{\mathcal{K}}[f^s \varphi].$$

We can repeat the argumentation from the case of subtraction at fixed conditions to arrive at the formulae of Proposition 7.1 and Theorem 7.3:

$$\langle (R_{c'}^{\mathcal{N}} - R_c^{\mathcal{N}})[\tilde{w}^s] | \varphi \rangle = \sum_{\emptyset \neq \mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} \langle R_c^{\mathcal{N} \setminus \mathcal{K}}[\vartheta_{\mathcal{K}}^{c,c'} \cdot (\tilde{w}^s)_{\mathcal{E}_{\mathcal{K}}}] | \varphi \rangle,$$

$$\langle (R_{c'}^{\mathcal{N}} - R_c^{\mathcal{N}})[\tilde{w}^s] | \varphi \rangle = \sum_{\emptyset \neq \mathcal{K} \subseteq \mathcal{N}} (-1)^{|\mathcal{K}|} c_{\mathcal{K}} \langle R_c[\tilde{w}_{G/\mathcal{K}}^s] | \delta_{\mathcal{K}}[\varphi] \rangle.$$

Viewing the maps $\theta_\gamma^{c'}$ as test functions, the constants $c_{\mathcal{K}}$ are exactly the same as in (7.2). They are given by densities of $\mathcal{K}$-contracted graphs, evaluated at their respective renormalization points, $c_\gamma = \prod_{\gamma \in \mathcal{K}} \langle R_c[\tilde{w}_{\gamma//\mathcal{K}}^s] | \theta_\gamma^{c'} \rangle$.

Eventually we would like to apply the formulae presented here not on distributions given by single graphs but on the formal sum of all graphs expressing a given interaction (an *amplitude*). The study of the behaviour of amplitudes under a change of renormalization points allows in the best cases even to make non-perturbative statements; the main idea is that physical observables do not depend on the choices made in fixing a renormalization scheme. This leads to a differential equation, the *renormalization group equation* (cf. [Col84]). In the language of the renormalization Hopf algebra this translates to (combinatorial) *Dyson-Schwinger equations* (cf. [Kre13]).

So far we have not used any differential methods, but to explore these objects within the wonderful framework it seems that subtraction at fixed conditions is then the way to go. This is reserved for future work.

Chapter 8

Back to Physics

This chapter connects the geometric method of extending distributions presented in this thesis to physics. We show that renormalization on wonderful models satisfies the *locality principle* of Epstein and Glaser [EG73]. After that we finish with an outlook of how to relate our approach to the method of Epstein-Glaser, i.e. to the renormalization of amplitudes, and how Hopf algebras can be utilized to describe the wonderful renormalization process.

8.1 Connection to the Epstein-Glaser method

The Epstein-Glaser locality principle is the position space analog of locality of counterterms. It decides whether a given theory is renormalizable, i.e. if adding counterterms to renormalize the Lagrangian keep its form invariant.

In [BBK10] this principle is formulated in a version for single graphs.

Definition 8.1 (Locality principle)**.** Let G be a connected graph. Let $\mathscr{R}$ denote a renormalization operator. $\mathscr{R}$ satisfies the *locality principle* of Epstein and Glaser if

$$\mathscr{R}[v_G] = \mathscr{R}[v_g]\mathscr{R}[v_h]v_{G\backslash(g\cup h)} \text{ on } X^G \backslash X_s^{G\backslash(g\cup h)} \tag{8.1}$$

holds for all disjoint pairs g, h of connected and divergent subgraphs of G.

This is to be understood in the sense of distributions. For all test functions $\varphi \in \mathcal{D}(X^G)$ with support disjoint from $X_s^{G\backslash(g\cup h)}$ the renormalization of v_G is already determined by the renormalized distributions v_g and v_h. Note that v_g and v_h depend on disjoint sets of variables and $v_{G\backslash(g\cup h)}$ is regular on $X^G \backslash X_s^{G\backslash(g\cup h)}$. Therefore the product on the right hand side of Equation (8.1) is well-defined.

Another way to formulate the locality principle is that for causal disconnected regions (g and h are disjoint) the renormalized distribution is given by "lower order" (i.e. subgraph-) distributions. Recall that in the recursive

procedure of Epstein and Glaser (Section 2.3) this is one of the main ingredients in the construction; it allows to recursively construct the n-th order term T^n in the formal series for the S-matrix up to the small diagonal in M^n.

Theorem 8.2. *Let $\mathscr{R}$ be given by minimal subtraction or subtraction at fixed conditions on the minimal wonderful model for the divergent arrangement of a connected and at most logarithmic graph.*

In both cases $\mathscr{R}$ satisfies the locality principle (8.1).

Proof. We follow the lines of [BBK10] but correct the proof by adding some essential details missing there.

Let Y_g, Y_h and Y denote minimal wonderful models for g, h and G. For $G^* := G \setminus (g \cup h)$ define $X^- := X^{G^*}$ as in (7.3). In the language of wonderful models the theorem states that $Y' = Y_g \times Y_h \times X^-$ is a (minimal) wonderful model for the divergent arrangement of the graph $g \cup h$ in $S := \operatorname{supp}(\varphi) \subseteq X$.

Let $\mathcal{B}$ denote the minimal building set in the divergent arrangement. The proof is based on two claims: Every $\mathcal{B}(g \cup h)$-nested set is given by a disjoint union $\mathcal{N}_g \,\dot\cup\, \mathcal{N}_h$ of $\mathcal{B}(g)$- and $\mathcal{B}(h)$-nested sets (one of them possibly empty). Secondly, $Y \setminus \beta^{-1}(X^-)$ is covered by the open sets $U_{\mathcal{N},B}$ where $\mathcal{N}$ is a disjoint union of $\mathcal{N}_g$ and $\mathcal{N}_h$ as above, both non-empty.

Proof of first claim: Since g and h are disjoint, Lemma 5.17 implies $\mathcal{B}(g \cup h) = \mathcal{B}(g) \cup \mathcal{B}(h)$. This shows that if $\mathcal{N}_g$ and $\mathcal{N}_h$ are nested with respect to $\mathcal{B}(g)$ and $\mathcal{B}(h)$, then $\mathcal{N}_g \cup \mathcal{N}_h$ is $\mathcal{B}(g \cup h)$-nested. On the other hand, every subset of a nested set is nested itself. With $\mathcal{B}(g \cup h) = \mathcal{B}(g) \cup \mathcal{B}(h)$ the claim follows.

Proof of second claim: If γ is an element of $\mathcal{B}(G) \setminus \mathcal{B}(g \cup h)$, then it must contain an edge e in $E(t_\gamma \setminus t_g)$ for t an adapted spanning tree. From

$$A_\gamma^\perp = \bigcup_{e' \in E(t_\gamma)} A_e^\perp \subseteq A_e^\perp$$

and $e \in E(G^*)$ it follows that

$$\mathcal{E}_\gamma = \beta^{-1}(A_\gamma^\perp) \subseteq \beta^{-1}(X^-).$$

Now consider the open sets $U_{\mathcal{N},B} \subseteq Y$ where $\mathcal{N} = \mathcal{N}_g \,\dot\cup\, \mathcal{N}_h$ is $I(g \cup h)$-nested and B marked accordingly. We need to show that local charts of this type cover every $\mathcal{E}_\gamma$ for $\gamma \in \mathcal{B}(g \cup h)$ and that every $x \in X \setminus X^-$ is the preimage of some $y \in U_{\mathcal{N},B}$ under the map β: W.l.o.g. assume $\gamma \subseteq g$ and pick $\eta \in \mathcal{B}(h)$. Then $\mathcal{N} := \{\gamma, \eta\}$ is $\mathcal{B}(G)$-nested. Let B be marked accordingly and let $\underline{\hat{x}}_\gamma$ and $\underline{\hat{x}}_\eta$ denote the collection of coordinates $\{x_e\}_{e \in E(t_\gamma)}$ and $\{x_e\}_{e \in E(t_\eta)}$

where the marked elements $x_\gamma^{i_\gamma}$ and $x_\eta^{i_\eta}$ are set to 1. The map $\rho_{\mathcal{N},B}$ scales $\hat{\underline{x}}_\gamma$ by $x_\gamma^{i_\gamma}$, $\hat{\underline{x}}_\eta$ by $x_\eta^{i_\eta}$ and leaves all other coordinates unaltered - it does not "mix" coordinates because g and h are disjoint. Recall that $\rho_{\mathcal{N},B}$ is the essential part in the definition of the chart

$$\kappa_{\mathcal{N},B}^{-1} : X \setminus \bigcup_{\xi \in \mathcal{B}(G)} Z_\xi \longrightarrow U_{\mathcal{N},B} \subseteq Y,$$

$$x \longmapsto \left(x_\gamma^{i_\gamma} \hat{\underline{x}}_\gamma, \ldots, x_\eta^{i_\eta} \hat{\underline{x}}_\eta; [\hat{\underline{x}}_\gamma], \ldots, [\hat{\underline{x}}_\eta] \right).$$

If ξ is in $\mathcal{B}(G) \setminus \mathcal{B}(g \cup h)$, the Z_ξ are given by $\{x_e = 0 \mid e \in E(t_\xi)\}$ which is a subset of X^-. Similarly, for $\xi \in \mathcal{B}(g \cup h)$ with either $\xi < \gamma$ or $\xi < \eta$ we have $Z_\xi = \{x_e = 0 \mid e \in E(t_\xi)\}$. If $\xi \in \mathcal{B}(g \cup h)$ and either $\xi > \gamma$ or $\xi > \eta$, then $Z_\xi = \{x_\gamma^{i_\gamma} = 0, x_e = 0 \mid e \in E(t_\xi \setminus t_\gamma)\}$ or with γ replaced by η. Finally, $Z_\gamma = Z_\eta = \emptyset$. From this description it is clear that as the marking of B varies, the sets $U_{\mathcal{N},B}$ cover $\mathcal{E}_\gamma$ (and $\mathcal{E}_\eta$ as well). Additionally, as we vary over $\mathcal{B}(g \cup h)$-nested sets of the form $\mathcal{N}_g \dot{\cup} \mathcal{N}_h$ and markings of B, we find preimages $y \in Y$ in $U_{\mathcal{N},B}$ for every $x \in X \setminus X^-$ by solving a system of linear equations (after fixing the marked elements the system is trivial; if the marked elements are required to be zero, we switch to another chart).

The previous discussion shows also that for such nested sets $\mathcal{N} = \mathcal{N}_g \dot{\cup} \mathcal{N}_h$ and marked bases $B = B_g \dot{\cup} B_h$ we have

$$U_{\mathcal{N},B} = U_{\mathcal{N}_g,B_g} \times U_{\mathcal{N}_h,B_h} \times \left(X^{G^*} \setminus \bigcup_{\gamma \in \mathcal{B}(G) \setminus \mathcal{B}(g \cup h)} Z_\gamma \right),$$

where $U_{\mathcal{N}_g,B_g}$ and $U_{\mathcal{N}_h,B_h}$ are chart domains for Y_g and Y_h. A similar decomposition holds for the charts

$$\kappa_{\mathcal{N},B} = \kappa_{\mathcal{N}_g,B_g} \times \kappa_{\mathcal{N}_h,B_h} \times \mathrm{id}$$

and for the blowdown β, locally given by $\rho_{\mathcal{N},B} = \rho_{\mathcal{N}_g,B_g} \times \rho_{\mathcal{N}_h,B_h} \times \mathrm{id}$.

Thus, outside of $\beta^{-1}(X^-)$ both models Y and Y' look locally the same and we can find a partition of unity

$$\chi_{\mathcal{N},B} = \chi_{\mathcal{N}_g,B_g} \times \chi_{\mathcal{N}_h,B_h} \times \mathrm{id},$$

subordinate to the sets $U_{\mathcal{N},B} \cap \beta^{-1}(S)$ where $\mathcal{N} = \mathcal{N}_g \dot{\cup} \mathcal{N}_h$ and $B = B_g \dot{\cup} B_h$. With the notation introduced in the proof of Theorem 7.3 we have $f = $

$f_g f_h f_{G^*}$. Then in every such $U_{\mathcal{N},B}$

$$
\begin{aligned}
\langle R_\nu[\tilde{w}_G]|\psi\rangle &= \sum_{\mathcal{K}\subseteq\mathcal{N}}(-1)^{|\mathcal{K}|}\langle \nu_{\mathcal{K}}\cdot(\tilde{w}_G)_{\mathcal{E}_{\mathcal{K}}}|\psi\rangle \\
&= \sum_{\emptyset\neq\mathcal{K}_g\subseteq\mathcal{N}_g}(-1)^{|\mathcal{K}_g|}\int dx\, u_{\mathcal{N}}\nu_{\mathcal{K}_g}\delta_{\mathcal{K}_g}[f\psi] \\
&\quad + \sum_{\emptyset\neq\mathcal{K}_h\subseteq\mathcal{N}_h}(-1)^{|\mathcal{K}_h|}\int dx\, u_{\mathcal{N}}\nu_{\mathcal{K}_h}\delta_{\mathcal{K}_h}[f\psi] \\
&\quad + \sum_{\mathcal{K}_g\dot{\cup}\mathcal{K}_h\subseteq\mathcal{N}}(-1)^{|\mathcal{K}_g|+|\mathcal{K}_h|}\int dx\, u_{\mathcal{N}}\nu_{\mathcal{K}_g}\nu_{\mathcal{K}_h}\delta_{\mathcal{K}_g\dot{\cup}\mathcal{K}_h}[f\psi] \\
&= \sum_{\emptyset\neq\mathcal{K}_g\subseteq\mathcal{N}_g}(-1)^{|\mathcal{K}_g|}\int dx\, u_{\mathcal{N}}\nu_{\mathcal{K}_g}\delta_{\mathcal{K}_g}[f_g]\Big(f_h\delta_{\mathcal{K}_g}[f_{G^*}]\delta_{\mathcal{K}_g}[\psi] \\
&\quad + \sum_{\emptyset\neq\mathcal{K}_h\subseteq\mathcal{N}_h}(-1)^{|\mathcal{K}_h|}\nu_{\mathcal{K}_h}\delta_{\mathcal{K}_h}[f_h]\delta_{\mathcal{K}_g\dot{\cup}\mathcal{K}_h}[f_{G^*}]\delta_{\mathcal{K}_g\dot{\cup}\mathcal{K}_h}[\psi]\Big) \\
&\quad + \sum_{\mathcal{K}_h\subseteq\mathcal{N}_h}(-1)^{|\mathcal{K}_h|}\int dx\, u_{\mathcal{N}}\nu_{\mathcal{K}_h}\delta_{\mathcal{K}_h}[f_h]\delta_{\mathcal{K}_h}[f_{G^*}]\, f_g\,\delta_{\mathcal{K}_h}[\psi] \\
&= \sum_{\substack{\mathcal{K}_g\subseteq\mathcal{N}_g \\ \mathcal{K}_h\subseteq\mathcal{N}_h}}(-1)^{|\mathcal{K}_g|+|\mathcal{K}_h|}\int dx\, u_{\mathcal{N}}\nu_{\mathcal{K}_g}\nu_{\mathcal{K}_h}\delta_{\mathcal{K}_g}[f_g]\delta_{\mathcal{K}_h}[f_h] \\
&\quad \times \delta_{\mathcal{K}_g\dot{\cup}\mathcal{K}_h}[f_{G^*}]\delta_{\mathcal{K}_g\dot{\cup}\mathcal{K}_h}[\psi] \\
&= \big\langle R_\nu[\tilde{w}_g]\otimes R_\nu[\tilde{w}_h] \mid \langle \tilde{w}_{G^*}|\psi\rangle\big\rangle.
\end{aligned}
$$

Applying this to $\psi = (\chi_{\mathcal{N},B}\circ\kappa_{\mathcal{N},B}^{-1})\beta^*\varphi$ and summing over all nested sets and corresponding markings shows (8.1). The case of minimal subtraction works in the same way (cf. the discussion in Chapter 7). This finishes the proof. $\qquad\square$

To connect the graph by graph method presented in this thesis with the Epstein-Glaser construction we need to renormalize the sum of all graphs with a fixed vertex order. Thus, we need a space that serves as a universal wonderful model for all at most logarithmic graphs on n vertices. There are two obvious candidates, the minimal and the maximal wonderful models of the graph lattice $\mathcal{G}$ for the complete graph K_n. Since every divergent subgraph of a graph is saturated, the set $\mathcal{G}(K_n)$ contains all possible divergences of such a graph. In other words, these two models are universal in the sense that for every graph G on n vertices there exist canonical proper projections

$$p_{\max}^G : Y_{\mathcal{G}(K_n)} \longrightarrow Y_{\mathcal{D}(G)}, \tag{8.2}$$

$$p_{\min}^G : Y_{I(\mathcal{G}(K_n))} \longrightarrow Y_{I(\mathcal{D}(G))}. \tag{8.3}$$

This follows from Definition 4.3. The theorem above suggests to focus on minimal building sets. Let Y denote $Y_{I(\mathcal{G}(K_n))}$ and G be a connected and at most logarithmic graph on n vertices. The idea is to compose the projection $p_{\min}^G$ with the blowdown β of the wonderful model $Y_{I(\mathcal{D}(G))}$ and consider the pullback $\tilde{w}_G$ of v_G under this map. In the additional charts that cover the components of $\mathcal{E} \subseteq Y$ not corresponding to elements of $I(\mathcal{D}(G))$ we set the local densities $\tilde{w}_G$ to zero. Then we proceed as before to obtain a renormalized density on Y. A detailed description is left for future work, but we make one further observation that highlights the connection between wonderful renormalization and the Epstein-Glaser method. Recall that the wonderful model $Y_{I(\mathcal{G}(K_n))}$ is equivalent to the Fulton-MacPherson compactification of the configuration space $F_n(M)$, for which the structure of $I(\mathcal{G}(K_n))$-nested sets is encoded by rooted trees [FM94]. As shown in [BK04], Epstein-Glaser renormalization can also be formulated in terms of rooted trees. On the other hand, the Hopf algebra of rooted trees H_{rt} satisfies an universal property in the category of renormalization Hopf algebras [Kre13], as does the Fulton-MacPherson compactification in the category of (minimal) wonderful models (Equation (8.2))!

8.2   Connection to renormalization Hopf algebras

As shown in [BBK10], the renormalization Hopf algebra of Feynman graphs is encoded in the stratification of the exceptional divisor $\mathcal{E}$ of a wonderful model associated to a graph G. We sketch the arguments and finish with a discussion of a Hopf algebraic formulation of wonderful renormalization.

Let H be the free algebra on the vector space spanned by (isomorphism classes) of connected, divergent (at most logarithmic) graphs. The multiplication on H is given by disjoint union, the empty graph being the unit element. In [BK08] it is shown that H endowed with a coproduct Δ given by

$$\Delta(G) := \sum_{\gamma \in \mathcal{D}} \gamma \otimes G//\gamma$$

is indeed a Hopf algebra. To cope with the case of minimal building sets, i.e. irreducible graphs, we can mod out by the ideal I generated by all irreducible decompositions as defined in Chapter 5.

On $\tilde{H} := H/I$ it is the antipode $S : \tilde{H} \to \tilde{H}$ that disassembles G into parts determined by its irreducible divergent subgraphs and prepares so the renormalization process. In terms of contraction relative to nested sets S is given by

$$S(G) = \sum_{\mathcal{N}} (-1)^{|\mathcal{N}|} \prod_{\gamma \in \mathcal{N}} \gamma//\mathcal{N}$$

where the sum is over $I(\mathcal{D}(G))$-nested sets.

This is the starting point of Hopf algebraic renormalization. The goal is then to formulate the whole wonderful renormalization process in terms of the convolution product of a twisted antipode with Feynman rules, similar to renormalization in momentum or parametric space. This is not straightforward due to the local formulation of the renormalization operators, but motivated by another, more direct approach.

The combinatorial character of Zimmermann's forest formula is a first hint at a Hopf algebra structure underlying renormalization. The locally defined wonderful renormalization operators resemble the classical formula for subtracting divergences only in certain charts. To connect with the forest formula and translate it into Hopf algebraic terms we could use the following idea. It is based on the fact that if a graph has only subdivergences that are nested into each other, then local subtractions resemble the forest formula correctly. Working modulo primitive elements of H, or $\tilde{H}$, every graph can be written as a sum of graphs that behaves like an element with purely nested subdivergences [BK08]. This shows that in principle wonderful renormalization fits into the Hopf algebraic framework. Of course, it is worthwhile to establish the connection on a more abstract level using geometrical methods.

Once this is achieved, the whole world of renormalization Hopf algebras can be explored and used in the position space setting.

CUTKOSKY RULEZ!

Bibliography

[AG08]   A. Aizenbud and D. Gourevitch. Schwartz functions on Nash manifolds. *Intern. Math. Research Notices*, 2008.

[AS94]   S. Axelrod and I. M. Singer. Chern-simons perturbation theory II. *J. Diff. Geom.*, 39, 1994.

[Ati70]   M. Atiyah. Resolution of singularities and division of distributions. *Comm. Pure Appl. Math.*, XXIII, 1970.

[BBK10] C. Bergbauer, R. Brunetti, and D. Kreimer. Renormalization and resolution of singularities. *arXiv:0908.0633*, 2010.

[BD13]   F. Brown and D. Doryn. Framings for graph hypersurfaces. *arXiv:1301.3056v1*, 2013.

[BEK06] S. Bloch, H. Esnault, and D. Kreimer. On motives associated to graph polynomials. *Commun. Math. Phys.*, 267, 2006.

[BF00]   R. Brunetti and K. Fredenhagen. Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds. *Commun. Math. Phys.*, 208, 2000.

[Bir67]   G. Birkhoff. *Lattice Theory*. Colloquium Publications Vol. XXV. American Mathematical Society, 1967.

[BK04]   C. Bergbauer and D. Kreimer. The hopf algebra of rooted trees in epstein-glaser renormalization. *Ann. Henri Poincaré*, 6, 2004.

[BK08]   S. Bloch and D. Kreimer. Mixed hodge structures and renormalization in physics. *Commun. Numb. Theory and Phys.*, 2(4), 2008.

[Bro10]   F. Brown. On the periods of some feynman integrals. *arXiv:0910.0114v2*, 2010.

[BS80]   N. N. Bogolyubov and D. V. Shirkov. *Introduction to the Theory of Quantized Fields*. Wiley & Sons, third edition, 1980.

[CK01]   A. Connes and D. Kreimer. Renormalization in quantum field theory and the riemann-hilbert problem II: the β-funcion, diffeomorphisms and the renormalization group. *Commun. Math. Phys.*, 216, 2001.

[Col84]   J. Collins. *Renormalization: An Introduction to Renormalization, the Renormalization Group and the Operator Product Expansion.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1984.

[CP95]   C. De Concini and C. Procesi. Wonderful models of subspace arrangements. *Selecta Math. (N.S.)*, 1(3), 1995.

[EFG07]   K. Ebrahimi-Fard and L. Guo. Rota-baxter algebras in renormalization of perturbative quantum field theory. *Fields Inst. Commun.*, 50, 2007.

[EG73]   H. Epstein and V. Glaser. The role of locality in perturbation theory. *Ann. Poincare Phys. Theor.*, A19, 1973.

[Fei05]   E. M. Feichtner. De concini-procesi wonderful arrangement models: A discrete geometer's point of view. *Combinatorial and Computational Geometry*, 52, 2005.

[FM94]   W. Fulton and R. MacPherson. A compactification of configuration spaces. *Ann. Math.*, 139, 1994.

[GH94]   P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* Wiley Classics Library. John Wiley & Sons, 1994.

[GM87]   M. Goresky and R. MacPherson. *Stratified Morse Theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 14. Springer, New York, 1987.

[GS64]   I. M. Gelfand and G. E. Shilov. *Generalized Functions I.* Academic Press, New York, 1964.

[Haa55]   R. Haag. On quantum field theories. *Matematisk-fysiske Meddelelser*, 29(12), 1955.

[Hir64]   H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero: I,II. *Ann. of Math.*, 79(1,2), 1964.

[Hö90]   L. Hörmander. *The Analysis of Linear Partial Differential Operators I, Distribution Theory and Fourier Analysis.* Classics in Mathematics. Springer, Berlin, second edition, 1990.

[Kol07]   J. Kollár. *Lectures on Resolution of Singularities.* Annals of Mathematics Studies, 166. Princeton University Press, 2007.

[Kon94]   M. Kontsevich. *Feynman diagrams and low-dimensional topology, in: First European Congress of Math. (Paris 1992), Vol. II.* Progress in Mathematics, 120. Birkhäuser, 1994.

[Kre13]   D. Kreimer. Renormalization & renormalization group. *Unpublished lecture notes by L. Klaczynski*, 2013.

[Li09]   L. Li. Wonderful compactification of an arrangement of subvarieties. *Michigan Math. J.*, 58(2), 2009.

[Nic07]   L. I. Nicolaescu. *Lectures on the Geometry of Manifolds.* World Scientific Publishing Company, second edition, 2007.

[Reb12]   A. Rebenko. *Theory of Interacting Quantum Fields.* Studies in Mathematics. De Gruyter, 2012.

[Sch66]   L. Schwartz. *Théorie des distributions.* Hermann, new edn. edition, 1966.

[Sch10]   O. Schnetz. Quantum periods: A census of ϕ^4 transcendentals. *Commun. Numb. Theory and Phys.*, 4(1), 2010.

[Sta97]   R. P. Stanley. *Enumerative Combinatorics I.* Cambridge University Press, Cambridge, second edition, 1997.

[SW00]   R.F. Streater and A.S. Wightman. *PCT, Spin and Statistics, and All That.* Landmarks in Physics. Princeton University Press, 2000.

[Uly02]   A. P. Ulyanov. Polydiagonal compactification of configuration spaces. *J. Algebraic Geom.*, 11(1), 2002.

[Wei96]   S. Weinberg. *The Quantum Theory of Fields I.* Cambridge University Press, second edition, 1996.

[Wig59]   E. Wigner. *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra.* Academic Press, 1959.

[Zim69]   W. Zimmermann. Convergence of bogoliubov's method of renormalization in momentum space. *Comm. Math. Phys.*, 15(3), 1969.

Selbstständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation selbstständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Marko Berghoff   Berlin, 07.10.2014