

Multiple polylogarithms and linearly reducible Feynman graphs

Christian Bogner and Martin Lüders

ABSTRACT. We review an approach for the computation of Feynman integrals by use of multiple polylogarithms, with an emphasis on the related criterion of linear reducibility of the graph. We show that the set of graphs which satisfies the linear reducibility with respect to both Symanzik polynomials is closed under taking minors. As a step towards a classification of Feynman integrals, we discuss the concept of critical minors and exhibit an example at three-loops with four on-shell legs.

1. Introduction

In recent years we witnessed rapid progress in the development of techniques for the computation of higher order corrections in perturbative quantum field theory. While other talks at this conference cover progress in the computation of entire amplitudes, our talk refers to the 'classical' approach of computing the amplitude by its Feynman graphs, which is inevitable when meeting the needs of present collider experiments. In this field of research, it has shown to be fruitful to discuss Feynman integrals in their own right, without restrictions to a particular quantum field theory.

Computations of higher order corrections to observables often start from the consideration of hundreds or thousands of Feynman integrals with tensor structure, and proceed via effective standard procedures to reduce the problem, possibly to a relatively small number of scalar integrals. At higher loop-orders, the evaluation of the latter remains to be the hard part of the problem. There is no algorithm which would succeed in the analytical computation of every Feynman integral. However, there is a variety of powerful methods which have been useful for a wide range of relevant cases, such as the Mellin-Barnes approach (see [9, 59, 58, 55]), the expansion of hypergeometric functions [47, 38, 39], differential equation methods [42, 52, 33, 48, 49], difference equations [43, 56, 57, 46] or position-space methods [28] (also see [37]). In this talk we focus on the approach of iteratively integrating out Feynman parameters by use of multiple polylogarithms.

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In order to choose an appropriate strategy for the computation of a given Feynman integral, it would be desirable in general, to know in advance, which are the classes of functions and numbers the integral may evaluate to. As a slightly more refined question of this type we may ask: Which scalar Feynman integrals can be expressed by multiple polylogarithms and multiple zeta values, and for which integrals do we need a wider range of functions and numbers? In the past few years, questions of this type turned out to define a fruitful common field of research for quantum field theorists and algebraic geometers alike. While the physicist's interest in these questions is given by the desire to compute specific integrals or to learn about the 'number content' of a given quantum field theory, the mathematician arrives at the same question from a different direction. In a very general context, Feynman integrals can be viewed as period integrals, and the question of evaluating to multiple zeta values is related to the question whether an underlying motive is mixed Tate over \mathbb{Z} (see [12, 20, 21, 24, 2, 3, 4]).

A definite classification of Feynman graphs with respect to the above questions is missing. However, for vacuum and two-point graphs important progress was made by considering the first Symanzik polynomial, given by the Feynman parametric representation of a Feynman integral. Even though many vacuum-type Feynman integrals evaluate to multiple zeta values [17, 18], this is not the case in general. A first vacuum graph whose period has to belong to a set of numbers beyond multiple zeta values was exhibited in a recent article by Brown and Schnetz [24] (also see [22]). When allowing the Feynman integrals to depend on kinematical invariants and particle masses, we can ask for graphs where multiple polylogarithms are not sufficient to express the result. Here the first cases show up at much lower loop-order, such as in the case of massive sunrise graphs and related graphs with a cut through three massive edges (see e.g. [6, 7]).

In this talk we review a criterion on graphs which is related to the above questions and show that if a graph satisfies the criterion, its minors do so as well. In graph theory such a minor monotony is an important and desirable feature. In section 2 we begin with a brief reminder on scalar Feynman integrals, their two Symanzik polynomials and the approach of integrating out Feynman parameters by use of multiple polylogarithms. In section 3 we briefly review the criterion of linear reducibility of a graph, which is used to decide whether a given integral can be computed by use of the method. If this is the case, the functions and numbers in all intermediate steps and in the result will not exceed combinations of multiple polylogarithms and their values at rational points. In this way the criterion and the corresponding algorithm are useful tools for addressing the above questions. In the case of integrals only involving the first Symanzik polynomial, the criterion was extensively studied in [20, 21]. As the iterated integration over Feynman parameters can be expected to be useful in the case of integrals depending on kinematical invariants and particle masses as well, we intend to extend the discussion to the second Symanzik polynomial. In section 4 we consider linear reducibility with respect to both Symanzik polynomials and show that the set of linearly reducible graphs is closed under taking minors. This property is useful for a classification, as it allows us to characterize families of reducible graphs by a small number of graphs not belonging to the family. In a case study we exhibit such a 'forbidden minor' at the level of massless three-loop graphs with four on-shell legs. Section 5 contains our conclusions.

2. Multiple Polylogarithms and Feynman Integrals

In this section we recall some general facts about Feynman integrals, Symanzik polynomials and a method to compute period integrals by use of multiple polylogarithms. Let us begin with a generic Feynman graph G with n edges, loop-number (i.e. first Betti number) $L \geq 1$ and with r external half-edges (or 'legs'). We label each edge e_i by an integration variable α_i (Feynman parameter), an integer ν_i (exponent of the Feynman propagator), a real or complex variable m_i (particle mass). Each leg is labelled by a vector p_j (external momentum).

To this labelled graph G we associate the scalar Feynman integral in Dimensional Regularization:

$$(2.1) \quad I_G = \frac{\Gamma(\nu - LD/2)}{\prod_{j=1}^n \Gamma(\nu_j)} \int_{\alpha_j \geq 0} \delta\left(1 - \sum_{i=1}^n \alpha_i\right) \left(\prod_{j=1}^n d\alpha_j \alpha_j^{\nu_j-1}\right) \frac{\mathcal{U}_G^{\nu-(L+1)D/2}}{\mathcal{F}_G^{\nu-LD/2}}$$

where $\nu = \sum_{i=1}^n \nu_i$. (We omit to write a trivial prefactor by which the integral becomes independent of the physical mass-scale.) The Feynman integral I_G and the function \mathcal{F}_G depend on the particle masses and on certain kinematical invariants, which are quadratic functions of the external momenta. The functions \mathcal{U}_G and \mathcal{F}_G are the first and second Symanzik polynomial of the graph. A definition is given below. Usually a Feynman integral is associated to a Feynman graph by Feynman rules in momentum or position space, and we refer to the literature [40, 50] for the standard computation leading from there to the Feynman parametric representation given in eq. 2.1.

Eq. 2.1 defines a very general class of integrals which deserves our attention for several reasons. Firstly, the class contains the Feynman integrals of scalar quantum field theory such as ϕ^3 - or ϕ^4 -theory. Secondly, any Feynman integral with a tensor-structure, arising from a physical quantum field theory, can in principle be expressed in terms of scalar integrals of the above class [56, 57]. Thirdly, as we allow the ν_j to take arbitrary integer values, there are well-known identities between these scalar integrals which can be used for efficient reduction procedures [29]. As a consequence, integrals of the above class appear in a wide range of physical set-ups and their evaluation is the bottleneck of many computational problems in particle physics.

The parameter D can either be fixed to the integer space-time dimension or, as the integral is very often ill-defined in the desired dimension, one may consider I_G in Dimensional Regularisation where D is a complex variable. Then, in order to separate the pole-terms and obtain finite contributions in four-dimensional Minkowski space, one usually attempts to compute the coefficients of a Laurent-expansion

$$I_G = \sum_{j=j_0}^{\infty} c_j \epsilon^j,$$

with $D = 4 - 2\epsilon$, to a desired order. Even though the computation of the functions c_j can be very difficult, we can make a general statement about them. It is shown in [15] that if for an arbitrary Feynman graph we evaluate any function c_j at algebraic values of the squared particle masses m_i^2 and kinematical invariants s_i , where all $m_i^2 \geq 0$ and all $s_i \leq 0$, we obtain a *period* according to the definition of Kontsevich

and Zagier [41]. For the special case where the Feynman integral takes the form

$$(2.2) \quad \mathcal{P}_G = \int_{\alpha_j \geq 0} \delta \left(1 - \sum_{i=1}^n \alpha_i \right) \left(\prod_{j=1}^n d\alpha_j \right) \frac{1}{\mathcal{U}_G^{D/2}}$$

this statement was already proven in [8]. It seems that Feynman integrals in fact evaluate to a restricted subset of periods and it is an important challenge to understand which one this is.

Let us now recall the definition of the Symanzik polynomials of a Feynman graph G . The first Symanzik polynomial is defined as

$$\mathcal{U}_G = \sum_T \prod_{e_i \notin T} \alpha_i,$$

where the sum is over all spanning trees of the graph G . The second Symanzik polynomial is defined as

$$\mathcal{F}_G = \mathcal{F}_{0,G} + \mathcal{U}_G \sum_{i=1}^n \alpha_i m_i^2$$

with

$$\mathcal{F}_{0,G} = \sum_{(T_1, T_2)} \left(\prod_{e_i \notin (T_1, T_2)} \alpha_i \right) s_{(T_1, T_2)}.$$

Here the sum runs through all spanning two-forests (T_1, T_2) of G , where T_1 and T_2 denote the connected components of the forest.

In order to define the kinematical invariants $s_{(T_1, T_2)}$, we introduce an arbitrary orientation on G . We firstly say that each external momentum p_j is incoming at the vertex at the corresponding leg. We furthermore label each oriented edge by a momentum-vector q_i . If the edge e_i is oriented from vertex v_j to v_k then q_i is said to be incoming at v_k and $-q_i$ is incoming at v_j . Momentum-conservation on G is reflected in our labels by the condition that the sum of all external momenta p_j is zero, and at each vertex, the sum of all incoming momenta is zero. By these conditions, except for L momenta, each of the q_i can be expressed as a linear combination of external momenta. The kinematical invariants are defined as

$$s_{(T_1, T_2)} = \left(\sum_{e_j \notin (T_1, T_2)} \pm q_j \right)^2$$

where the sign of q_j is fixed by the condition that we sum over the momenta incoming at the component T_2 . Note that by momentum conservation, the $s_{(T_1, T_2)}$ are functions of the external momenta.

As an alternative to the above construction by spanning trees and forests, there are several ways to obtain both Symanzik polynomials from determinants of certain matrices [16, 13, 51, 21]. To demonstrate such a derivation, let us label each edge e_i by an auxiliary variable y_i . Each vertex v_i is labelled by

$$u_i = \begin{cases} z_j & \text{if a leg with incoming momentum } p_j \text{ is attached,} \\ 0 & \text{if no leg is attached.} \end{cases}$$

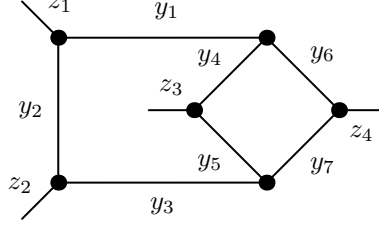


FIGURE 1. The non-planar double-box

For a Feynman graph with vertices v_1, \dots, v_m we consider an $m \times m$ matrix M whose entries are:

$$M_{ij} = \begin{cases} u_i + \sum y_k \text{ for } i = j, e_k \text{ attached to } v_i \text{ at exactly one end,} \\ -\sum y_k \text{ for } i \neq j, e_k \text{ connecting } v_i \text{ and } v_j. \end{cases}$$

We compute the determinant

$$\mathcal{V}(y_1, \dots, y_n, z_1, \dots, z_r) = \det(M)$$

and consider the function

$$\mathcal{W}(\alpha_1, \dots, \alpha_n, z_1, \dots, z_r) = \mathcal{V}(\alpha_1^{-1}, \dots, \alpha_n^{-1}, z_1, \dots, z_r) \prod_{i=1}^n \alpha_i$$

which is a polynomial in the α - and z -variables. Note that M depends on a chosen ordering on the vertices but \mathcal{W} does not.

Let us assume that at least two legs are attached to the graph, i.e. $r \geq 2$. We expand \mathcal{W} as $\mathcal{W} = \mathcal{W}^{(1)} + \mathcal{W}^{(2)} + \dots + \mathcal{W}^{(r)}$ where $\mathcal{W}^{(k)}$ is homogeneous of degree k in the z -variables. We can directly read off the first Symanzik polynomial from the first term in this expansion, as it satisfies

$$\mathcal{W}^{(1)}(\alpha_1, \dots, \alpha_n, z_1, \dots, z_r) = \mathcal{U}_G(\alpha_1, \dots, \alpha_n) \sum_{i=1}^r z_i.$$

The massless second Symanzik polynomial \mathcal{F}_{0G} is directly obtained from $\mathcal{W}^{(2)}$. By construction, $\mathcal{W}^{(2)}$ is homogeneous of degree 2 in the z -variables. We replace each product $z_i z_j$ in $\mathcal{W}^{(2)}$ by the scalar-product of the corresponding external momentum vectors $p_i \cdot p_j$. By momentum-conservation, $\sum_{i=1}^r p_i = 0$, we express each of the scalar products by the functions $s_{(T_1, T_2)}$. As result we obtain \mathcal{F}_{0G} [16].

As an example let us compute the two Symanzik polynomials of the massless non-planar double-box, shown in figure 1 with auxiliary y - and z -variables. For

this graph and a chosen ordering on the vertices we have

$$M = \begin{pmatrix} M_{11} & -y_2 & 0 & 0 & -y_1 & 0 \\ -y_2 & M_{22} & -y_3 & 0 & 0 & 0 \\ 0 & -y_3 & M_{33} & -y_7 & 0 & -y_5 \\ 0 & 0 & -y_7 & M_{44} & -y_6 & 0 \\ -y_1 & 0 & 0 & -y_6 & M_{55} & -y_4 \\ 0 & 0 & -y_5 & 0 & -y_4 & M_{66} \end{pmatrix},$$

where $M_{11} = y_1 + y_2 + z_1$, $M_{22} = y_2 + y_3 + z_2$, $M_{33} = y_3 + y_5 + y_7$, $M_{44} = y_6 + y_7 + z_4$, $M_{55} = y_1 + y_4 + y_6$, $M_{66} = y_4 + y_5 + z_3$. Proceeding in the described way we compute

$$\begin{aligned} \mathcal{U}_G &= (z_1 + z_2 + z_3 + z_4)^{-1} \mathcal{W}^{(1)} \\ &= (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) + (\alpha_4 + \alpha_5)(\alpha_6 + \alpha_7), \\ \mathcal{F}_{0G} &= \mathcal{W}^{(2)}|_{z_i z_j = p_i p_j, \sum_{i=1}^4 p_i = 0} \\ &= -p_1^2 \alpha_2 (\alpha_1 (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) + \alpha_4 \alpha_6) \\ &\quad - p_2^2 \alpha_2 (\alpha_3 (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) + \alpha_5 \alpha_7) \\ &\quad - p_3^2 (\alpha_4 \alpha_5 (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7) + \alpha_3 \alpha_4 \alpha_7 + \alpha_1 \alpha_5 \alpha_6) \\ &\quad - p_4^2 (\alpha_6 \alpha_7 (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) + \alpha_1 \alpha_4 \alpha_7 + \alpha_3 \alpha_5 \alpha_6) \\ &\quad - (p_1 + p_2)^2 (\alpha_1 \alpha_3 (\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7) + \alpha_1 \alpha_5 \alpha_7 + \alpha_3 \alpha_4 \alpha_6) \\ &\quad - (p_1 + p_3)^2 \alpha_2 \alpha_5 \alpha_6 - (p_2 + p_3)^2 \alpha_2 \alpha_4 \alpha_7. \end{aligned}$$

It is often sufficient to consider the Feynman integral after setting some of its legs on-shell, which means that the corresponding external momenta are fixed by setting their square to a squared particle mass. In our example we may assume massless particles and set $p_i^2 = 0$ for all $i = 1, \dots, 4$. The corresponding Feynman integral was evaluated in dimensional regularization by classical polylogarithms in reference [58]. We will return to Symanzik polynomials of graphs with four on-shell legs in section 4.

Let us now turn to iterated integrals. Let k be the field of either the real or the complex numbers and M a smooth manifold over k . We consider a piecewise smooth path on M , given by a map $\gamma : [0, 1] \rightarrow M$, and some smooth differential 1-forms $\omega_1, \dots, \omega_n$ on M . The iterated integral of these 1-forms along the path γ is defined by

$$\int_{\gamma} \omega_n \dots \omega_1 = \int_{0 \leq t_1 \leq \dots \leq t_n \leq 1} f_n(t_n) dt_n \dots f_1(t_1) dt_1,$$

where $f_i(t) dt = \gamma^*(\omega_i)$ is the pull-back of ω_i to $[0, 1]$. With the term iterated integral we will more generally refer to k -linear combinations of such integrals.

We will consider classes of iterated integrals which define the same function for any two homotopic paths. Such integrals are called *homotopy invariant*. They are well-defined functions of variables given by the end-point of γ . In such iterated integrals the differential forms and the order in which we integrate over them have to satisfy a property known as the *integrability condition*. The condition is best formulated on tensor products of 1-forms over some field $K \subseteq k$, which we denote by $[\omega_1 | \dots | \omega_m]$. Let D denote a K -linear map from tensor products of smooth 1-forms on M to tensor products of all forms on M , given by

$$D([\omega_1 | \dots | \omega_m]) = \sum_{i=1}^m [\omega_1 | \dots | \omega_{i-1} | d\omega_i | \omega_{i+1} | \dots | \omega_m] + \sum_{i=1}^{m-1} [\omega_1 | \dots | \omega_{i-1} | \omega_i \wedge \omega_{i+1} | \dots | \omega_m].$$

A K -linear combination of tensor products $\xi = \sum_{l=0}^m c_{i_1, \dots, i_l} [\omega_{i_1} | \dots | \omega_{i_l}]$, $c_{i_1, \dots, i_l} \in K$, is called an *integrable word* if it satisfies the equation

$$D\xi = 0.$$

Let Ω be a finite set of smooth 1-forms and let $B_m(\Omega)$ denote the vector space of integrable words of length m with 1-forms in Ω . Now we return from words to integrals by consider the integration map on integrable words:

$$(2.3) \quad \sum_{l=0}^m \sum_{i_1, \dots, i_l} c_{i_1, \dots, i_l} [\omega_{i_1} | \dots | \omega_{i_l}] \mapsto \sum_{l=0}^m \sum_{i_1, \dots, i_l} c_{i_1, \dots, i_l} \int_{\gamma} \omega_{i_1} \dots \omega_{i_l}.$$

A fundamental theorem of Chen [27] states that this map is an isomorphism from $B_m(\Omega)$ to the set of homotopy invariant iterated integrals in 1-forms in Ω of length less or equal to m , if Ω satisfies further conditions which we do not specify here.

In the following we fix $K = \mathbb{Q}$ and discuss two sets of 1-forms for which the theorem applies. For a coordinate t_1 on an open subset of \mathbb{C} we firstly consider the set of closed 1-forms

$$\Omega_n^{\text{Hyp}} = \left\{ \frac{dt_1}{t_1}, \frac{dt_1}{t_1 - 1}, \frac{t_2 dt_1}{t_1 t_2 - 1}, \dots, \frac{(\prod_{i=2}^n t_i) dt_1}{\prod_{i=1}^n t_i - 1} \right\}.$$

As a trivial consequence of $dt_1 \wedge dt_1 = 0$, any tensor product of 1-forms in Ω_n^{Hyp} is an integrable word. By applying the integration map eq. 2.3 to these words, we obtain the class of hyperlogarithms [45]. In particle physics it is very common to use sub-classes of hyperlogarithms. As an example, we may consider Ω_2^{Hyp} and fix the constant $t_2 = -1$. To physicists, the iterated integrals obtained from this restriction are well known as harmonic polylogarithms [53] and suffice for the evaluation of many Feynman integrals.

We want to focus on a class of functions of severable variables, obtained from another set of 1-forms, where now all the t_1, \dots, t_n are considered to be coordinates in an open subset of \mathbb{C}^n :

$$\Omega_n^{\text{MPL}} = \left\{ \frac{dt_1}{t_1}, \dots, \frac{dt_n}{t_n}, \frac{d\left(\prod_{a \leq i \leq b} t_i\right)}{\prod_{a \leq i \leq b} t_i - 1} \text{ where } 1 \leq a \leq b \leq n \right\}.$$

For this set the integrability condition is not trivial and there are words for which it is not satisfied. The homotopy invariant iterated integrals which we obtain via the integration map from the integrable words in Ω_n^{MPL} form the vector space $\mathcal{B}(\Omega_n)$ of multiple polylogarithms in n variables. We use the notation $\mathcal{B}_m(\Omega_n)$ for the vector space of such functions obtained from integrable words of length $\leq m$. There is an explicit map [14] to construct all integrable words in Ω_n^{MPL} , closely related to the 'symbol' in [34, 36, 32].

The functions in $\mathcal{B}(\Omega_n)$ were extensively studied in reference [19]. We just want to recall a few statements which are relevant for the following considerations. Firstly, the multiple polylogarithms of Goncharov [35], frequently used in the physics literature, are contained in this class. As we want to use the elements of $\mathcal{B}(\Omega_n)$ in an iterative integration procedure, it is important for us to know their primitives and limits. It is proven in [19] that $\mathcal{B}(\Omega_n)$ is closed under taking primitives. Furthermore if we take the limits of elements of $\mathcal{B}(\Omega_n)$ at t_n equal to 0 and

1, we obtain \mathcal{Z} -linear combinations of elements in $\mathcal{B}(\Omega_{n-1})$, where \mathcal{Z} denotes the \mathbb{Q} -vector space of multiple zeta values.

Now let us consider definite integrals of the form

$$I = \int_0^1 dt_n \frac{\beta(\{g_i\})}{f}$$

where f is a polynomial and $\beta(\{g_i\}) \in \mathcal{B}(\Omega_n)$ is a multiple polylogarithm whose arguments are some irreducible polynomials g_i . Let us call f and the g_i the *critical polynomials* of the integrand. If f and the g_i are linear in t_n we can evaluate the above integral and from the mentioned properties it is clear that the result will be a \mathcal{Z} -linear combination of elements in $\mathcal{B}_m(\Omega_{n-1})$. If the result can be again expressed by functions of the form of the above generic integrand and the critical polynomials are linear in t_{n-1} then we can continue and integrate over this variable from 0 to 1, and so on.

Such an iterative procedure can be used to compute Feynman integrals. For recent examples in the physics literature, partly relying on different parametrizations, we refer to [30, 31, 1, 26, 5]. Aiming at such a computation one has to express the Feynman integral by a finite parametric integral such that the integrand can be written in the above form, and if after each integration step, the critical polynomials are linear in at least one of the remaining parameters. The method was introduced systematically in [20] and demonstrated for certain Feynman parametric integrals of the type of eq. 2.2, coming from primitive logarithmically divergent vacuum Feynman graphs. However, the approach is not restricted to such graphs. Reference [23] presents a method to express Feynman integrals with UV sub-divergences by finite parametric integrals to which the approach may apply. The treatment of graphs with infrared divergences is not excluded in principle, but we are missing a canonical method to express IR-divergent integrals by finite ones. In principle, the method of sector decomposition [11] allows us to write down the coefficients of a Laurent expansion for a dimensionally regularized, infrared divergent integral in terms of finite integrals over Feynman parameters, however, the polynomials in these integrals usually become very complicated. In view of the above approach one would ideally wish for a method, where the critical polynomials in the finite integrals could be obtained from the Symanzik polynomials in a rather simple way.

For the following discussion let us assume, that in some way we have already been able to express a given Feynman integral by finite integrals of the type I and that the critical polynomials are the Symanzik polynomials of the graph. We focus on the criterion, that after each integration over a Feynman parameter, the new critical polynomials have to be linear in a next Feynman parameter. The reduction algorithm to be reviewed in section 3 allows us to study this criterion as it computes for each integration step a set in which the critical polynomials are contained.

As a further motivation of the following discussion, let us have a glance at two well-known Feynman graphs in view of the mentioned criterion. For the massless two-loop graph of figure 2 (a) it was proven by use of the Mellin-Barnes approach and expansions by nested sums that each coefficient of the ϵ -expansion is a combination of multiple zeta values [10]. Reference [20] confirmed this statement for this two-loop graph and several higher-loop graphs by relating them to integrals of the type of eq. 2.2 whose integrands satisfy the criterion.

The case of the equal-mass two-loop sunrise graph, shown in figure 2 (b), is very different. The desired coefficients in the ϵ -expansion can be derived from the

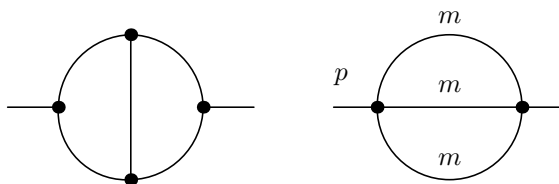


FIGURE 2. (a) Massless two-loop graph, (b) Equal-mass sunrise graph

$D = 2$ -dimensional version of the Feynman integral,

$$I_{\text{sunrise}} = \int_{\alpha_j \geq 0} d\alpha_1 d\alpha_2 d\alpha_3 \delta \left(1 - \sum_{i=1}^3 \alpha_i \right) \frac{1}{\mathcal{F}_G},$$

with the second Symanzik polynomial

$$\mathcal{F}_G = -p^2 \alpha_1 \alpha_2 \alpha_3 + m^2 (\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_1 \alpha_3) (\alpha_1 + \alpha_2 + \alpha_3),$$

playing the role of the critical polynomial. As \mathcal{F}_G is not linear in any of the Feynman parameters, integral I_{sunrise} fails the criterion. It is in perfect match with this simple observation, that the known result of the sunrise integral involves elliptic integrals [44]. In general, it is possible that the polynomials of an integrand fail the criterion and still the integral can be expressed by multiple polylogarithms. However, the criterion may provide a useful first classification and give a hint where to look for integrals, which exceed the class of multiple polylogarithms. We also want to mention reference [25], where different criteria are used to search for such integrals.

3. Linear Reducibility

Let us briefly review the polynomial reduction algorithm of [20]. Let $S = \{f_1, \dots, f_N\}$ be a set of polynomials in the variables $\alpha_1, \dots, \alpha_n$ with rational coefficients.

(1) If there is an index $1 \leq r_1 \leq n$ such that all polynomials in S are linear in α_{r_1} we can write

$$f_i = g_i \alpha_{r_1} + h_i \text{ for all } 1 \leq i \leq N,$$

where $g_i = \frac{\partial f_i}{\partial \alpha_{r_1}}$ and $h_i = f_i|_{\alpha_{r_1}=0}$. We define

$$(3.1) \quad S'_{[r_1]} = \left\{ (g_i)_{1 \leq i \leq N}, (h_i)_{1 \leq i \leq N}, (h_i g_j - g_i h_j)_{1 \leq i < j \leq N} \right\}$$

and furthermore we define $S_{[r_1]}$ to be the set of irreducible polynomials in $S'_{[r_1]}$. In $S_{[r_1]}$ we neglect all constants and monomials.

(2) If there is a $1 \leq r_2 \leq n$ such that all polynomials in $S_{(r_1)}$ are linear in α_{r_2} , we repeat the above step, now with $S_{(r_1)}$ and α_{r_2} in the roles of S and α_{r_1} , and obtain a new set of polynomials which we call $S_{[r_1](r_2)}$. Then, assuming that

starting from S the above steps can be done first for α_{r_2} and then for α_{r_1} , we compute $S_{[r_2](r_1)}$ and take the intersection of both sets:

$$S_{[r_1, r_2]} = S_{[r_1](r_2)} \cap S_{[r_2](r_1)}.$$

Whenever we speak of intersections here and in the following, we mean the common zero loci, such that if a polynomial appears in two sets with a different constant prefactor, it nevertheless belongs to the intersection. Then we choose a next variable in which all polynomials of $S_{[r_1, r_2]}$ are linear and continue in the same way. At each iteration we apply step (1) and take the intersection

$$S_{[r_1, r_2, \dots, r_k]} = \bigcap_{1 \leq i \leq k} S_{[r_1, \dots, \hat{r}_i, \dots, r_k](r_i)}.$$

If a set $S_{[r_1, \dots, \hat{r}_i, \dots, r_k]}$ contains a polynomial which is non-linear in α_{r_i} , the set $S_{[r_1, \dots, \hat{r}_i, \dots, r_k](r_i)}$ is undefined and omitted in the intersection. If this happens for all $1 \leq i \leq k$ the set $S_{[r_1, r_2, \dots, r_k]}$ is undefined and the algorithm stops. Unless this situation occurs, we obtain for each sequence of variables $(\alpha_{r_1}, \dots, \alpha_{r_k})$, $k \leq n$, a sequence of sets $S_{(r_1)}$, $S_{[r_1, r_2]}$, \dots , $S_{[r_1, r_2, \dots, r_k]}$.

DEFINITION 3.1. We say that S is *Fubini reducible* (or *linearly reducible*) if there is an ordering of all n variables $(\alpha_{r_1}, \dots, \alpha_{r_n})$ such that every polynomial in $S_{[r_1, r_2, \dots, r_k]}$ is linear in $\alpha_{r_{k+1}}$ for all $1 \leq k < n$.

The linear reducibility of the set of critical polynomials of an integrand as in eq. 2 is a criterion for the integral to be computable by the above approach. The criterion is sufficient but not necessary. The sets $S_{[r_1, r_2]}$ contain the critical polynomials of the integrand after the first k integrations, but might as well contain spurious polynomials which drop out in the integration procedure. A more refined reduction algorithm presented in reference [21] omits such cases, but not necessarily all of them. Furthermore the occurrence of a quadratic polynomial does not always forbid us to continue with the computation.

By applying the above algorithm to first Symanzik polynomials, it was shown in [20] that several vacuum Feynman integrals can be computed and evaluate to combinations of multiple zeta values. Moreover the same is true for coefficients of a dimensional series expansion of Feynman integrals with two legs, obtained from cutting one edge in one of these vacuum graphs, as in the case of figure 2 (a). The linear reducibility of first Symanzik polynomials is extensively studied in terms of Dodgson polynomials in [21]. It is shown that the first five iterations of the reduction succeed for any first Symanzik polynomial. Moreover, a first Symanzik polynomial is reducible, if its graph has vertex width less or equal three. (This is the class of graphs which decompose into two connected components after removing three vertices or less.) These results explain, why one has to go up to complicated graphs at high loop orders to find first examples for vacuum-type Feynman integrals which exceed the set of multiple zeta values [24, 22].

In the following, we want to consider the above algorithm applied to both Symanzik polynomials. In order to include \mathcal{F} which may depend on particle masses and external momenta, we slightly extend the above formulation of step (1), allowing f_i to be polynomials whose coefficients are rational numbers or algebraic functions of additional parameters s_1, \dots, s_m . The rest of the algorithm is not affected by this change.

It will be useful to consider polynomial reduction in coordinates for products of \mathbb{P}^1 . Let $P(\alpha_1, \dots, \alpha_n)$ be a polynomial in n Feynman parameters and consider

an associated function of new variables $x_1, \dots, x_n, y_1, \dots, y_n$ defined by

$$\bar{P}(x_1, \dots, x_n, y_1, \dots, y_n) = \left(\prod_{i=1}^n y_i^{d_i(P)} \right) P \left(\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n} \right),$$

where $d_i(P)$ denotes the maximal degree of P in α_i for all $1 \leq i \leq n$. We obtain our original polynomial back by setting

$$(3.2) \quad \bar{P}(\alpha_1, \dots, \alpha_n, 1, \dots, 1) = P(\alpha_1, \dots, \alpha_n).$$

The function \bar{P} is by definition a polynomial in $x_1, \dots, x_n, y_1, \dots, y_n$ and it is linear in x_k if and only if P is linear in α_k .

Let us consider a set of polynomials $S = \{P_1, \dots, P_N\}$ which are all linear in α_k . We can write

$$P_i = \alpha_k \frac{\partial}{\partial \alpha_k} P_i + P_i|_{\alpha_k=0}.$$

Consider the corresponding set of polynomials $\bar{S} = \{\bar{P}_1, \dots, \bar{P}_N\}$, obtained from S by changing to the projective coordinates. Each of these polynomials is linear in x_k and we can write

$$\bar{P}_i = x_k \bar{P}_i|_{x_k=1, y_k=0} + y_k \bar{P}_i|_{x_k=0, y_k=1}.$$

Let us use the convention that in a Fubini reduction with respect to the x -variables, we neglect the above prefactor y_k in the sense that in eq. 3.1 the terms are given by $g_i = \bar{P}_i|_{x_k=1, y_k=0}$ and $h_i = \bar{P}_i|_{x_k=0, y_k=1}$, which does not affect the linear reducibility. Step (1) of the reduction algorithm applied to \bar{S} with respect to x_k then gives the set $\bar{S}'_{(k)}$ consisting of the irreducible factors of

$$(3.3) \quad \bar{S}'_{(k)} = \left\{ \left(\bar{P}_i|_{x_k=1, y_k=0} \right)_{1 \leq i \leq N}, \left(\bar{P}_i|_{x_k=0, y_k=1} \right)_{1 \leq i \leq N}, \right. \\ \left. \left(\bar{P}_i|_{x_k=0, y_k=1} \cdot \bar{P}_j|_{x_k=1, y_k=0} - \bar{P}_i|_{x_k=1, y_k=0} \cdot \bar{P}_j|_{x_k=0, y_k=1} \right)_{1 \leq i < j \leq N} \right\}.$$

Considering a Fubini reduction in x -variables instead of α -variables, we convince ourselves that the factorizations into irreducible polynomials are not affected by the change of variables. Indeed if P factorizes as

$$P = f_1 \cdot f_2$$

into two polynomials f_1, f_2 then \bar{P} factorizes as

$$\bar{f}_1 \cdot \bar{f}_2 = \left(\prod_{i=1}^n y_i^{d_i(f_1)} \right) f_1 \left(\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n} \right) \cdot \left(\prod_{i=1}^n y_i^{d_i(f_2)} \right) f_2 \left(\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n} \right) = \bar{P},$$

because of $d_i(f_1) + d_i(f_2) = d_i(P)$ for all $1 \leq i \leq n$. As a consequence we obtain:

LEMMA 3.2. *P is linearly reducible with respect to the variables $\alpha_1, \dots, \alpha_n$ if and only if \bar{P} is linearly reducible with respect to the variables x_1, \dots, x_n .*

Now let us choose some coordinate $x_l, 1 \leq l \leq n$, and define two new sets of polynomials by restrictions $x_l = 0, y_l = 1$ and $x_l = 1, y_l = 0$ respectively:

$$\begin{aligned} \bar{S}^{(l, 0, 1)} &= \bar{S}|_{x_l=0, y_l=1}, \\ \bar{S}^{(l, 1, 0)} &= \bar{S}|_{x_l=1, y_l=0}. \end{aligned}$$

To these sets we apply one reduction step with respect to x_k . At first let us assume $l \neq k$. Step (1) of the algorithm gives

$$\begin{aligned}\bar{S}_{(k)}^{(l,0,1)} &= \text{irreducible factors of } \bar{S}'_{(k)}|_{x_l=0, y_l=1}, \\ \bar{S}_{(k)}^{(l,1,0)} &= \text{irreducible factors of } \bar{S}'_{(k)}|_{x_l=1, y_l=0}.\end{aligned}$$

If a polynomial \bar{P} factorizes as $\bar{P} = f_1 \cdot f_2$ then furthermore

$$\begin{aligned}\bar{P}|_{x_l=0, y_l=1} &= f_1|_{x_l=0, y_l=1} \cdot f_2|_{x_l=0, y_l=1}, \\ \bar{P}|_{x_l=1, y_l=0} &= f_1|_{x_l=1, y_l=0} \cdot f_2|_{x_l=1, y_l=0},\end{aligned}$$

and the maximal degrees with respect to another variable x_i satisfy $d_i(f_j) \geq d_i(f_j|_{x_l=0, y_l=1})$ and $d_i(f_j) \geq d_i(f_j|_{x_l=1, y_l=0})$ for $j = 1, 2$ and for all $i = 1, \dots, n$. This means that for each polynomial $f \in \bar{S}_{(k)}$ there is at most one polynomial $f' \in \bar{S}_{(k)}^{(l,0,1)}$, and $d_i(f) \geq d_i(f')$ for all $i = 1, \dots, n$. The same is true for $\bar{S}_{(k)}^{(l,1,0)}$.

Lastly we observe that in the case of $l = k$, the irreducible factors of $\bar{S}_{(k)}^{(l,0,1)}$ and $\bar{S}_{(k)}^{(l,1,0)}$ are already contained in $\bar{S}_{(k)}$. This proves the following lemma:

LEMMA 3.3. *If \bar{S} is linearly reducible with the ordering $(x_{r_1}, \dots, x_{r_n})$ then for any $1 \leq l \leq n$ the sets $\bar{S}|_{x_l=0, y_l=1}$ and $\bar{S}|_{x_l=1, y_l=0}$ are linearly reducible with $(x_{r_1}, \dots, \hat{x}_l, \dots, x_{r_n})$.*

In combination with the previous lemma we obtain:

LEMMA 3.4. *Let $S = \{P_1, \dots, P_N\}$ be a set of polynomials which is linearly reducible with $(\alpha_{r_1}, \dots, \alpha_{r_n})$ and whose members are linear in α_l . Then the sets $S^l = \left\{ \frac{\partial P_1}{\partial \alpha_l}, \dots, \frac{\partial P_N}{\partial \alpha_l} \right\}$ and $S_l = \{P_1|_{\alpha_l=0}, \dots, P_N|_{\alpha_l=0}\}$ are linearly reducible with $(\alpha_{r_1}, \dots, \hat{\alpha}_l, \dots, \alpha_{r_n})$.*

4. Towards a Classification by Critical Minors

Let G be a graph and E_G its set of edges. For $e \in G$ we denote by $G \setminus e$ the graph obtained from G by *deletion* of e . Furthermore we write $G // e$ for the graph obtained from G by *contraction* of e . This is the graph where the end-points of e are identified and then e is removed. For any distinct edges $e_1, e_2 \in E_G$ the operations of deleting (or contracting) e_1 and deleting (or contracting) e_2 commute. Therefore we can more generally write $\gamma = G \setminus D // C$ with distinct $D, C \subset E_G$, for the unique graph obtained from G by deleting all edges in D and contracting all edges in C . Any such γ is called a *minor* of G .

If G is connected and there is an edge such that $G \setminus e$ is disconnected then e is called a *bridge*. When speaking of Feynman graphs, we may ignore disconnected graphs, so we introduce the convention that the corresponding Symanzik are zero:

$$\mathcal{U}_{G \setminus e} = 0, \mathcal{F}_{0G \setminus e} = 0 \text{ if } e \text{ is a bridge.}$$

Let us furthermore call $e_t \in E_G$ a *tadpole* if it is attached to the same vertex at both ends. In this case we have a factorization in the corresponding Feynman parameter:

$$\begin{aligned}\mathcal{U}_G &= \mathcal{U}_{G \setminus e_t} \alpha_t, \\ \mathcal{F}_{0G} &= \mathcal{F}_{0G \setminus e_t} \alpha_t.\end{aligned}$$

For any e which is not a tadpole we have the well-known deletion/contraction identities:

$$\begin{aligned}\mathcal{U}_G &= \mathcal{U}_{G \setminus e} \alpha_e + \mathcal{U}_{G // e}, \\ \mathcal{F}_G &= \mathcal{F}_{G \setminus e} \alpha_e + \mathcal{F}_{G // e}.\end{aligned}$$

Now we consider a set of arbitrary particle masses m_1, \dots, m_n distributed over the edges of G , with the restriction that at least one edge e_k is massless, $m_k = 0$. Then the second Symanzik polynomial

$$\mathcal{F}_G = \mathcal{F}_{0G} + \mathcal{U}_G \cdot \sum_{i \neq k} \alpha_i m_i^2$$

is linear in α_k and the above relations are extended by

$$\begin{aligned}\mathcal{F}_G &= \mathcal{F}_{G \setminus e_k} \alpha_k \text{ if } e_k \text{ is a tadpole,} \\ \mathcal{F}_G &= \mathcal{F}_{G \setminus e_k} \alpha_k + \mathcal{F}_{G // e_k} \text{ if } e_k \text{ is not a tadpole.}\end{aligned}$$

Equivalently, if e_k is a massless edge of G then

$$(4.1) \quad \frac{\partial}{\partial \alpha_k} \mathcal{U}_G = \mathcal{U}_{G \setminus e_k}, \quad \mathcal{U}_G|_{\alpha_k=0} = \mathcal{U}_{G // e_k},$$

$$(4.2) \quad \frac{\partial}{\partial \alpha_k} \mathcal{F}_G = \mathcal{F}_{G \setminus e_k}, \quad \mathcal{F}_G|_{\alpha_k=0} = \mathcal{F}_{G // e_k}.$$

In the following let us call a Feynman graph G linearly reducible, if $\{\mathcal{U}_G, \mathcal{F}_G\}$ is linearly reducible. We arrive at the main statement of these notes:

THEOREM 4.1. *If G is a linearly reducible Feynman graph then any minor of G is linearly reducible as well.*

PROOF. Let G be an arbitrary Feynman graph, e_j any of its edges and $\tilde{G} = G|_{m_j=0}$ the graph obtained from G by setting the mass m_j associated to e_j equal to zero. Assume that G is linearly reducible. Then G is linearly reducible for any value of m_j and therefore \tilde{G} is linearly reducible as well. As e_j is massless in \tilde{G} we can apply equations 4.1 and 4.2 and obtain

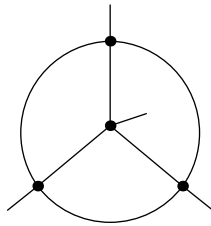
$$\begin{aligned}S_{\tilde{G} \setminus e_j} &= \left\{ \mathcal{U}_{\tilde{G} \setminus e_j}, \mathcal{F}_{\tilde{G} \setminus e_j} \right\} = \left\{ \frac{\partial}{\partial \alpha_j} \mathcal{U}_{\tilde{G}}, \frac{\partial}{\partial \alpha_j} \mathcal{F}_{\tilde{G}} \right\}, \\ S_{\tilde{G} // e_j} &= \left\{ \mathcal{U}_{G // e_j}, \mathcal{F}_{G // e_j} \right\} = \left\{ \mathcal{U}_{\tilde{G}}|_{\alpha_j=0}, \mathcal{F}_{\tilde{G}}|_{\alpha_j=0} \right\}.\end{aligned}$$

By lemma 3.4 it follows from the linear reducibility of \tilde{G} that $\tilde{G} \setminus e_j$ and $\tilde{G} // e_j$ are linearly reducible. G and \tilde{G} have the same minors with respect to deleting or contracting e_j :

$$G \setminus e_j = \tilde{G} \setminus e_j, \quad G // e_j = \tilde{G} // e_j$$

and therefore $S_{G \setminus e_j} = S_{\tilde{G} \setminus e_j}$ and $S_{G // e_j} = S_{\tilde{G} // e_j}$. By induction, this proves the theorem. \square

A set of graphs \mathcal{G} is called *minor closed* if for all $G \in \mathcal{G}$ every minor of G belongs to \mathcal{G} as well. Let \mathcal{H} be any set of graphs and let $\mathcal{G}_{\mathcal{H}}$ be the set of all graphs which do not have a minor in the set \mathcal{H} . Then the set $\mathcal{G}_{\mathcal{H}}$ is minor closed, and the graphs in \mathcal{H} are called *forbidden minors* of $\mathcal{G}_{\mathcal{H}}$. A theorem of Robertson and Seymour [54] states that any minor closed set of graphs can be defined in such a way by a finite set of forbidden minors.

FIGURE 3. K_4 with four legs

A graph H is called a *critical minor* of a set of graphs \mathcal{G} if the minors of H belong to \mathcal{G} but H does not. By removing a graph from \mathcal{H} which is a minor of another graph in \mathcal{H} , we do not change the set $\mathcal{G}_{\mathcal{H}}$. Therefore, to define a minor closed set $\mathcal{G}_{\mathcal{H}}$, it is sufficient to let \mathcal{H} consist only of critical minors. A well-known example for a characterization by critical minors is given by the set of planar graphs. Due to a theorem of Wagner [60], the set of planar graphs is the set with forbidden minors $\{K_{3,3}, K_5\}$.

In general we have to be careful when adapting notions from pure graph theory to the study of Feynman graphs, which are equipped with labels and special properties. However, theorem 4.1 suggests to attempt an analogous characterization of the set of linearly reducible Feynman graphs by critical minors, i. e. by a set of Feynman graphs which are not linearly reducible but have linearly reducible minors. We conclude these notes with a first step towards such a characterization for Feynman graphs with a dependence on external momenta. Let Λ be the set of massless Feynman graphs with four on-shell legs, attached to four distinct vertices. The on-shell condition restricts the four external momenta to satisfy $p_i^2 = 0$ such that the dependence on the external momenta can be expressed by two Mandelstam variables. By use of a Maple-implementation of the Fubini algorithm, we find linear reductions for all two-loop graphs of this class. At three loops there are several graphs in this class for which our program fails to find a linear reduction. Most of these graphs have the graph of figure 3 as a minor. This is the complete four-vertex graph K_4 where we attached an on-shell leg at each vertex. From direct observation of the two Symanzik polynomials one can see that the graph is not linearly reducible. As on the other hand all its minors are linearly reducible, the graph plays the role of a critical minor for the set of all linearly reducible graphs. Our case study suggests that only a few further critical minors are needed to distinguish all linearly reducible graphs of Λ at three loops. In this way, a large class of graphs can be separated into linearly reducible and irreducible members by knowing only a small number of critical minors.

5. Conclusions

In this talk we reviewed the criterion of linear reducibility of Symanzik polynomials, which can be used to decide whether a corresponding Feynman integral can be computed by iteratively introducing hyperlogarithms or multiple polylogarithms. Brown showed in [21] that with respect to the first Symanzik polynomial, the set

of linearly reducible graphs is closed under taking minors and that therefore a characterization of this set by forbidden, critical minors is possible. We extended this line of argument to the case of graphs with masses and kinematical invariants, by showing, that minor closedness is true for both Symanzik polynomials. We exhibit a first critical minor with a non-trivial dependence on kinematical invariants.

We expect a classification with respect to the criterion of linear reducibility to be useful for the more difficult question of which Feynman integrals evaluate to multiple polylogarithms. Due to the simplicity of its derivation and the property of minor closedness, such a systematic classification of a large class of Feynman graphs appears feasible.

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INSTITUTE OF PHYSICS, HUMBOLDT UNIVERSITÄT ZU BERLIN, UNTER DEN LINDEN 6, 10099 BERLIN, GERMANY

E-mail address: bogner@math.hu-berlin.de, lueders@physik.hu-berlin.de