# Relations among multiple zeta values

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## 1 Single zeta values

**Theorem 1.1** (Euler). *For any $n \in \mathbb{N}$, the even zeta values are*

$$\zeta(2n) = \frac{(-1)^{n-1} B_{2n} (2\pi)^{2n}}{2(2n)!} = -\frac{(2\pi i)^{2n} B_{2n}}{2(2n)!} \in \mathbb{Q} \cdot \pi^{2n} = \mathbb{Q} \cdot \zeta^n(2) \tag{1.1}$$

*for the* Bernoulli numbers $B_n$ *defined by*

$$\frac{x}{e^x - 1} = \sum_{0 \leq n} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} + \cdots$$

Hence, as rational multiples of powers of $\pi$ the even zetas are transcendental.

| $2n$ | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| $B_{2n}$ | $\frac{1}{6}$ | $-\frac{1}{30}$ | $\frac{1}{42}$ | $-\frac{1}{30}$ |
| $\zeta(2n)$ | $\frac{\pi^2}{6}$ | $\frac{\pi^4}{90} = \frac{2}{5}\zeta^2(2)$ | $\frac{\pi^6}{945} = \frac{8}{35}\zeta^3(2)$ | $\frac{\pi^8}{9450} = \frac{24}{175}\zeta^4(2)$ |

For the odd zeta values we do not expect such relations at all:

**Conjecture 1.2.** *The elements $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.*

In particular, we expect all zeta values to be transcendental. But we only know very few results on irrationality:

**Theorem 1.3** ([1]). *$\zeta(3)$ is irrational[1].*

**Theorem 1.4** ([13]). *Infinitely many of the odd zetas $\zeta(3), \zeta(5), \zeta(7), \ldots$ are irrational. In fact, for any $\varepsilon > 0$ exists some $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$,*

$$\dim_{\mathbb{Q}} \operatorname{lin}_{\mathbb{Q}} \{1, \zeta(3), \zeta(5), \ldots, \zeta(2n+1)\} \geq \frac{1-\varepsilon}{1+\log 2} \log n. \tag{1.2}$$

**Theorem 1.5** ([18]). *At least one of $\{\zeta(5), \zeta(7), \zeta(9), \zeta(11)\}$ is irrational.*

**Theorem 1.6** ([19]). *For any odd $n \in \mathbb{N}$, at least one of $\{\zeta(n+2), \zeta(n+4), \ldots, \zeta(8n-1)\}$ is irrational.*

[1] For details on this proof see [12] and its recently updated version.

## 2 Multiple zeta values

**Definition 2.1** ([16])**.** *To $d \in \mathbb{N}$ integers $n_1, \ldots, n_d \in \mathbb{N}$ with $n_d > 1$, the* multiple zeta value

$$\zeta(n_1, \ldots, n_d) := \sum_{0<k_1<\ldots<k_d} \frac{1}{k_1^{n_1} \cdots k_d^{n_d}} \in \mathbb{R}_+ \tag{2.1}$$

*assigns a positive real number. We call $d$ its* depth *and $n_1 + \ldots + n_d$ its* weight*. Be aware of the also common reverse convention [9].*

Already Euler [5] new the case $d = 1$ of

**Theorem 2.2** (Sum theorem [7])**.** *For any depth $d \in \mathbb{N}$ and weight $w > 1$,*

$$\sum_{n_1+\ldots+n_d=w, n_d>1} \zeta(n_1, \ldots, n_d) = \zeta(w) \,. \tag{2.2}$$

**Example 2.3.** *In weight three, $\zeta(3) = \zeta(1,2)$ and $\zeta(4) = \zeta(1,3) + \zeta(2,2) = \zeta(1,1,2)$ in weight four.*

Symmetric MZVs are sums of products of single zeta values, as the stuffle identity supplies

**Theorem 2.4** ([9])**.** *Let $n_1, \ldots, n_d \geq 2$ then*

$$\sum_{\sigma \in S_d} \zeta\left(n_{\sigma(1)}, \ldots, n_{\sigma(d)}\right) = \sum_{\text{partitions } M \text{ of } \{1,\ldots,d\}} (-1)^{d-|M|} \prod_{P \in M} (|P|-1)! \cdot \zeta\left(\sum_{n \in P} n\right) . \tag{2.3}$$

**Corollary 2.5.** *Combining (2.3) with (1.1), we deduce*

$$E(2n, d) := \sum_{n_1+\ldots+n_d=n} \zeta(2n_1, \ldots, 2n_d) \in \mathbb{Q} \cdot \zeta(2n) \,. \tag{2.4}$$

*In fact their generating function is computed in [8] to*

$$F(t,s) := 1 + \sum_{1 \leq k \leq n} E(2n,k) t^n s^k = \frac{\sin\left(\pi\sqrt{t(1-s)}\right)}{\sqrt{1-s}\sin\left(\pi\sqrt{t}\right)}. \tag{2.5}$$

**Example 2.6.** *$E(2n,2) = \sum_{k=1}^{n-1} \zeta(2n-2k, 2k) = \frac{3}{4}\zeta(2n)$ is already due to Euler [5]. Writing $\{2\}^n$ for the sequence of $n$ consecutive twos, note further*

$$E(2n,n) = \zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!}. \tag{2.6}$$

**Theorem 2.7** ([10])**.** MZV *with $n_1 = \ldots = n_{d-1} = 1$ are sums of products of single zetas, namely*

$$\sum_{n,m \geq 1} \zeta\left(\{1\}^{m-1}, n+1\right) s^n t^m = 1 - \frac{\Gamma(1-s)\Gamma(1-t)}{\Gamma(1-s-t)} = 1 - \exp\left(\sum_{n \geq 2} \zeta(n) \frac{t^n + s^n - (t+s)^n}{n}\right).$$

*Again, the case $m = 2$ was known to Euler [5] already.*

**Theorem 2.8** ([2])**.** *As was originally conjectured by D. Zagier, we have the identity*

$$\zeta(\{1,3\}^n) = \frac{2\pi^{4n}}{(4n+2)!}.$$

## 3 Shuffles, stuffles and regularization

**Definition 3.1** (Hoffman[2])**.** *The* Hoffman algebra $\mathfrak{h}$ *are the non-commutative polynomials*

$$\mathfrak{h} := \mathbb{Q}\langle x, y\rangle \tag{3.1}$$

*in two letters $x$ and $y$. It is graded by the* weight *(length of a word = number of letters) and* depth *(number of letters $y$). On the subspace*

$$\mathfrak{h}^0 := \mathbb{Q} \oplus y\mathfrak{h}x \tag{3.2}$$

*of* admissible words *words (beginning with $y$ and ending in $x$) we define the* period map *by*

$$\zeta : \mathfrak{h}^0 \longrightarrow \mathbb{R}, \qquad y_{n_1}\cdots y_{n_d} \mapsto \zeta(n_1, \ldots, n_d), \tag{3.3}$$

*where $y_n := yx^{n-1}$ for any $n \in \mathbb{N}$. This is extended linearly with $\zeta(1) = 1$ for the empty word. The* shuffle product $⧢$ *and* stuffle *(also called* harmonic*) product $\star$ are recursively defined by*

$$av⧢bw := a(v⧢bw)+b(av⧢w) \quad \text{and} \quad y_n v\star y_m w := y_n(v\star y_m w)+y_m(y_n v\star w)+y_{n+m}(v\star w) \tag{3.4}$$

*for any letters $a, b \in \{x, y\}$ and words $v, w \in \mathfrak{h}^0$. Both of these turn $\mathfrak{h}^0$ into a commutative, associative and free algebra.*

As was shown in Olivers' lectures, the sum representation (2.1) and the iterated integral representation provide

**Lemma 3.2** (Dobule-shuffle relations)**.** *$\zeta$ is an algebra morphism with respect to both products $⧢$ and $\star$:*

$$\zeta(v ⧢ w) = \zeta(v)\cdot\zeta(w) = \zeta(v \star w) \quad \text{for any} \quad v, w \in \mathfrak{h}^0. \tag{3.5}$$

**Example 3.3.** *From $y_2⧢y_2 = yx⧢yx = 4yyxx+2yxyx = 4y_1y_3+2y_2y_2$ and $y_2\star y_2 = 2y_2y_2+y_4$ we deduce $4\zeta(1,3) + 2\zeta(2,2) = \zeta^2(2) = 2\zeta(2,2) + \zeta(4)$ and therefore $\zeta(4) = 4\zeta(1,3)$.*

**Theorem 3.4** (Hoffman relation [9])**.** *For any $w \in \mathfrak{h}^0$, $w ⧢ y - w \star y \in \mathfrak{h}^0$ (does end in $x$) and*

$$\zeta(w ⧢ y - w \star y) = 0. \tag{3.6}$$

**Example 3.5.** *We find $y_2 \star y - y_2 ⧢ y = y_3 + y_1y_2 + y_2y - (2y_1y_2 + y_2y) = y_3 - y_1y_2 \in \mathfrak{h}^0$, thus $\zeta(3) = \zeta(1,2)$.*

**Conjecture 3.6.** *All algebraic relations over $\mathbb{Q}$ among* MZV *are consequences of the so-called* regularized double-shuffle relations *meaning* (3.5) *and* (3.6)*.*

## 4 Weight filtration

**Definition 4.1.** *Let $\mathcal{Z}_N := \operatorname{lin}_{\mathbb{Q}}\{\zeta(n_1, \ldots, n_d)\colon\ n_1 + \ldots + n_d = N\}$ denote the space of* MZV *of weight $N$ and $\mathcal{Z} := \sum_{N\geq 0} \mathcal{Z}_N$ their sum (the $\mathbb{Q}$-space spanned by all* MZV*).*

[2] In the literature the words are written in the reversed way: $y_n = x^{n-1}y$ and $\zeta(y_{n_d}\ldots y_{n_1}) := \zeta(n_d, \ldots, n_1)$. Here we used this order to be consistent with the first talk of Oliver Schnetz.

From the previous examples, we know that $\zeta(3) = \zeta(1,3)$, so $\mathcal{Z}_3 = \mathbb{Q}\cdot\zeta(3)$ is one-dimensional.

$$\zeta(4) = \zeta(1,1,2) = 4\zeta(1,3) = \frac{4}{3}\zeta(2,2) = \frac{2}{5}\zeta^2(2)$$

shows that the four MZV of weight four span only a one-dimensional space $\mathcal{Z}_4 = \mathbb{Q}\cdot\zeta(4)$ as well. Recall the following table of Oliver's talk obtained by using all available relations:

| weight $N$ | conjectured basis of $\mathcal{Z}_N$ | $d_N$ | MZV of weight $N$ |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | | 0 | 0 |
| 2 | $\zeta(2)$ | 1 | 1 |
| 3 | $\zeta(3)$ | 1 | 2 |
| 4 | $\zeta(2)^2$ | 1 | 4 |
| 5 | $\zeta(5), \zeta(2)\zeta(3)$ | 2 | 8 |
| 6 | $\zeta(2)^3, \zeta(3)^2$ | 2 | 16 |
| 7 | $\zeta(7), \zeta(2)\zeta(5), \zeta(2)^2\zeta(3)$ | 3 | 32 |
| 8 | $\zeta(2)^4, \zeta(2)\zeta(3)^2, \zeta(3)\zeta(5), \zeta(3,5)$ | 4 | 64 |

**Conjecture 4.2** (D. Zagier, [16])**.** *$\mathcal{Z}$ is graded by the weight with Hilbert-Poincaré series*

$$\sum_{k\geq 0} d_k t^k = \frac{1}{1-t^2-t^3}. \tag{4.1}$$

*Equivalently, the following two statements hold:*

1. *All relations are homogeneous in weight: $\mathcal{Z} = \bigoplus_{N\geq 0}\mathcal{Z}_N$*
2. $\dim_{\mathbb{Q}}\mathcal{Z}_N = d_N$ *where* $d_0 = d_2 = 1$, $d_1 = 0$ *and then* $d_N = d_{N-2} + d_{N-3}$.

Note that this is a very strong claim as it implies conjecture 1.2 and thus transcendence of all odd zeta values. However, the results of F. Brown on *motivic multiple zeta values* (which will be featured in his upcoming lectures) imply

**Theorem 4.3** ([4])**.** *The* Hoffman-*elements span $\mathcal{Z}$ in each weight, that is*

$$\mathcal{Z}_N = \operatorname{lin}_{\mathbb{Q}}\{\zeta(n_1,\ldots,n_r)\colon\ n_1,\ldots,n_r\in\{2,3\} \quad \textit{and} \quad n_1+\ldots+n_r = N\}. \tag{4.2}$$

*In particular this implies (as was also proved independently in [14])*

$$\dim_{\mathbb{Q}}\mathcal{Z}_N \leq d_n. \tag{4.3}$$

In fact the results of [4] prove the existence of a surjective algebra morphism

$$\phi:\quad \mathbb{Q}[f_2]\otimes_{\mathbb{Q}}(\mathbb{Q}\langle f_3, f_5, f_7,\ldots\rangle, ⧢) \longrightarrow \mathcal{Z} \tag{4.4}$$

which preserves the weight filtrations. In this picture, conjecture 4.2 is equivalent to $\phi$ being an isomorphism.

## 5 Depth filtration

**Definition 5.1.** *Let* $\mathcal{Z}_N^{(d)} := \mathrm{lin}_{\mathbb{Q}} \{\zeta(n_1, \ldots, n_r)\colon\ n_1 + \ldots + n_r = N \quad \text{and} \quad r \leq d\}$ *denote the span of* MZV *of weight* $N$ *and depth* $\leq d$.

The stuffle relation involves MZV of different depths, so in contrast to the weight, the depth can only be a filtration on $\mathcal{Z}$. For example recall the example

$$\zeta(\{2\}^n) = \zeta(\underbrace{2, \ldots, 2}_{n \text{ twos}}) = \frac{\pi^{2n}}{(2n+1)!} \in \mathbb{Q} \cdot \zeta(2n) = \mathcal{Z}_{2n}^{(1)} \tag{5.1}$$

that is of depth one (not $n$). Similarly note

**Theorem 5.2** ([17])**.** *Setting* $H(n) := \zeta(\{2\}^n)$ *and* $H(a,b) := \zeta\left(\{2\}^a, 3, \{2\}^b\right)$,

$$H(a,b) = 2 \sum_{r=1}^{a+b+1} (-1)^r \left[ \binom{2r}{2a+2} - \left(1 - \frac{1}{2^{2r}}\right) \binom{2r}{2b+1} \right] H(a+b-r+1)\zeta(2r+1)\,. \tag{5.2}$$

In particular it implies that $H(a,b) \in \mathcal{Z}_{2a+2b+3}^{(2)}$ is of depth at most two, even though it is originally a MZV of high depth $a+b+1$.

**Conjecture 5.3** (D. Broadhurst and D. Kreimer, [3])**.** *$\mathcal{Z}$ is graded by the weight and the depth filtration has dimensions*

$$d_{n,k} = \dim_{\mathbb{Q}} \left( \mathcal{Z}_n^{(k)} / \mathcal{Z}_n^{(k-1)} \right)$$

*given by the generating series*

$$\sum_{n,k} d_{n,k} x^k y^n = \frac{1 + \mathbb{E}y}{1 - \mathbb{O}y + \mathbb{S}y^2(1-y^2)} \tag{5.3}$$

*where* $\mathbb{E} := \frac{x^2}{1-x^2} = x^2 + x^4 + x^6 + \ldots$ *and* $\mathbb{O} := \frac{x^3}{1-x^2} = x^3 + x^5 + x^7 + \ldots$ *count the even and odd zetas in each weight, while* $\mathbb{S} := \frac{x^{12}}{(1-x^4)(1-x^6)}$ *is the generating series of the dimensions of the spaces of cusp forms of* $\mathrm{SL}_2(\mathbb{Z})$ *in each weight.*

Note that this is a refinement of conjecture 4.2 (set $y = 1$ and use $d_n = \sum_k d_{n,k}$). Expanding (5.3) to the first orders in the depth $y$, observe

$$\sum_{n,k} d_{n,k} = 1 + \underbrace{(\mathbb{E} + \mathbb{O})y}_{\text{single zetas}} + (\underbrace{\mathbb{E}\mathbb{O}}_{\text{odd weight}} + \underbrace{\mathbb{O}^2 - \mathbb{S}}_{\text{even weight}}) + (\cdots)y^3 + \ldots \tag{5.4}$$

In particular we see in depth two, that the products $\mathbb{E}\mathbb{O}$ of even and odd single zeta values are the only generators in odd weights. This is known more generally as

**Theorem 5.4** ([11, 15])**.** *Every* $\zeta(n_1, \ldots, n_d)$ *with weight* $n = n_1 + \ldots + n_d$ *and depth* $d \not\equiv n \mod 2$ *of different parity is a* $\mathbb{Q}$*-linear combination of products of* MZV *of depth smaller than* $d$.

**Example 5.5.**

$$\begin{aligned}\zeta(4,2,2) = {} & \zeta(4)\,\zeta(2,2) + \zeta(2)\left[4\zeta(4,2) + 6\zeta(3,3) + 7\zeta(2,4) + 8\zeta(1,5)\right] \\ & - 8\zeta(6,2) - 10\zeta(5,3) - \frac{33}{2}\zeta(4,4) - 12\zeta(3,5) - \frac{15}{2}\zeta(2,6)\end{aligned}$$

In particular, every depth-two $\zeta(n_1, n_2)$ with odd $n_1 + n_2$ is a sum of products of single zetas. An explicit formula is given as

**Theorem 5.6** (Proposition 7 in [17]). *For $m \geq 1$, $n \geq 2$ of odd weight $k = m + n = 2K + 1$,*

$$\zeta(m,n) = (-1)^m \sum_{s=0}^{K-1} \left[ \binom{k-2s-1}{m-1} + \binom{k-2s-1}{n-1} - \delta_{n,2s} + (-1)^m \delta_{s,0} \right] \zeta(2s)\,\zeta(k-2s)\,.$$

On the other hand, for even weights $N$, the contribution $\mathbb{O}^2$ to (5.4) counts the $\zeta(n,m)$ with odd entries $n, m \geq 3$ and $n + m = N$. However, these are in general not independent. The first relation (up to $\mathbb{Q} \cdot \zeta(N) = \mathcal{Z}_N^{(1)}$) appears at weight $N = 12$:

$$28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12)\,. \tag{5.5}$$

The origin of these *exotic* relations in depth two correspond [6] to period polynomials for cusp forms of $\mathrm{SL}_2(\mathbb{Z})$ and are counted by $\mathbb{S}$. These connections might be enlightened by the upcoming lectures of José I. Burgos.

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