Collocation for transferable differential-algebraic equations

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1 Introduction

In the present paper collocation methods for the determination of approximation solutions of a certain class of implicit ordinary differential equations (ODE's)

$$f(u'(t), u(t), t) = 0, \qquad t \in [a, b] \ , \tag{1.1}$$

with respect to the boundary condition

$$g(u(a), u(b)) = 0 \tag{1.2}$$

are developed.

If the Jacobian $f'_y(y, x, t) := \frac{\partial f}{\partial y}(y, x, t)$ exists and is regular for all (y, x, t) to be considered, then (1.1) is at least locally equivalent to an explicit ODE

$$u'(t) = \bar{f}(u(t), t) \qquad t \in [a, b] \ , \tag{1.3}$$

if (1.1) is solvable. Hence, all methods known for ODE's can be applied, but if $f'_y(y, x, t)$ is singular for all (y, x, t), to be considered, then a transformation into (1.3) is not possible.

Now let the null-space of $f'_y(y, x, t)$ be independent of y and x, i. e. $\ker(f'_y(y, x, t)) =: N(t)$, and letthis null-space be smooth. The smoothness implies the existence of a continuously differentiable matrix function $Q : [a, b] \rightarrow \mathbb{R}^m$, such that $Q(t)$ is a projection from $\mathbb{R}^m$ onto $N(t)$ for all $t \in [a, b]$. We call equation (1.1) with these properties a differential-algebraic equation (DAE). In [GM86] the denotation of DAE's is used in a more general form, but the definition given above includes the most important cases. A motivation of the notation DAE is given by the property that

$$f(u'(t), u(t), t) = f(P(t)u'(t), u(t), t) \ , \tag{1.4}$$

where $P(t) = I - Q(t)$. Hence, only the P-components of $u'(t)$ occur in (1.1). The Q-components are uniquely determined by algebraic conditions.

In the following let us suppose additionally that the null-space of $f'_y(y, x, t)$ is independent of t, i. e. $N(t) \equiv N$. This condition ensures a constant partition of a solution of (1.1), (1.2) into P- and Q-components, respectively. In spite of this restriction the class of DAE's is sufficient for the mathematical description of many natural and technical processes.

A simpler form of (1.1), the semiexplicit DAE

$$\begin{aligned} v'(t) &= h_1(v(t), w(t), t) \\ 0 &= h_2(v(t), w(t), t) \ , \end{aligned} \tag{1.5}$$

which is treated for instance in [HLR89], fulfils these properties, obviously. If now $\frac{\partial h_2}{\partial w}(v(t), w(t), t)$ is regular, (1.5) can be at least locally transformed into the state variable system

$$\begin{aligned} v'(t) &= \varphi(v(t), t) \\ w(t) &= \psi(v(t), t) \ . \end{aligned} \tag{1.6}$$

Nevertheless, for transforming (1.1) into (1.6) the bijectivity of the matrix function

$$G(y, x, t) := f'_y(y, x, t) + f'_x(y, x, t)Q$$

has to be required for all (y, x, t), to be considered, where $f'_x(y, x, t) := \frac{\partial f}{\partial x}(y, x, t)$.

DAE's with these properties are called *transferable* ([GM86]). Now it does not seem to be useful to transform (1.1) into (1.6) and then to apply a well–known method to (1.6), because these transformations are numerically very expensive in general. The aim is to modify these methods in such a way that they can be applied to (1.1) directly. From the literature, some applications of methods, which have been developed for explicit ODE's are known. For instance, Runge-Kutta methods ([GM86], [HLR89]), multistep methods ([Gea71], [GP84], [GM86]), difference methods ([GM86]) and shooting methods ([Lam]) are treated. Collocation methods for transferable DAE's are examined in ([Asc87]), too.

Collocation methods applied to explicit ODE's are investigated in [Asc86], [Bad88], [BA87], [dBS73], [Rus74], [RC72], [ACR81] for instance. There, it is required that the examining class of differential equations has solutions in a certain function space. For the approximation of a solution it is required that this solution belongs to an appropriate choosen subspace of the solution space, the collocation space. Then the collocation approximation is uniquely determined by the conditions, that it fulfils the differential equation in a set of points, the collocation points and the boundary conditions. Hence, a collocation method is completely determined by the choice of a collocation space and a set of collocation points. The most usual choice of collocation spaces are the spaces of piecewise polynomials as they are used in [dBS73], [Rus74] and [RC72]. Because collocation methods have been successfully applied to explicit ODE's, similar results are expected from the application to DAE's.

For instance, in [Asc87] U. Ascher proposes a generalization of COLNEW ([BA87]) for transferable DAE's with time dependent null-space. For this purpose a linear DAE is transformed into an equivalent system of explicit ODE's and pure algebraic equations. Then, the convergence results can be easily derived from the collocation theory for explicit ODE's. However, this idea does not yield satisfactory methods for nonlinear DAE's, because the numerical transformation may be very expensive. He gives an alternative by approximating all solution components by piecewise polynomials of the same degree and the same smoothness, but this proposal leads to weak instabilities.

In the present paper the possibility of a time–independent partition into differential and algebraic components is given by the assumption that the null-space of $f'_y(y,x,t)$ is constant. So, differential and algebraic components can be approximated by piecewise polynomials of different degree and different smoothness. For the class of BVP's of transferable DAE's described above, collocation methods are developed by the suitable choice of collocation points and collocation spaces (Chap. 2).

For the collocation, spaces of piecewise polynomials are chosen. The collocation methods arise from an interpolation method for the algebraic components and a collocation method which is known from ODE's for the differential components.

For constructing realizations of the collocation methods, basis representations of approximation solutions are used. The algebraic components are represented by a Lagrange basis. For representing of the differential components a Runge-Kutta monomial basis and a generalized Hermite basis, respectively, are proposed and discussed in Sect. 2.3. With the aid of these basis representations realizations of the collocation methods are then constructed in Chap. 3. In Chap. 4 properties of feasibility as stability, superconvergence for the differential components at the grid points and an increase of the condition number of the system as $0(h^{-1})$, where h is the maximum distance of two grid points, are given. Finally, a solution algorithm is proposed.

2 The Problem and Its Discretization

2.1 The Specification of the Problem

Summarizing the assumptions discussed in the introduction, the present paper is concerned with the nonlinear implicit DAE

$$f(u'(t), u(t), t) = 0 \qquad t \in [a,b] \ , \tag{2.1}$$

subject to the boundary conditions

$$g(u(a), u(b)) = 0 \ . \tag{2.2}$$

The functions f and g are supposed to have the following properties:

(Vf) 1. Let $V \subseteq \mathbb{R}^m \times \mathbb{R}^m \times [a,b]$ be a region. $f : V \to \mathbb{R}^m$ is continuous and the partial derivatives with respect to the first components $f'_y(y,x,t) := \frac{\delta f}{\delta y}(y,x,t)$ and with respect to the second components $f'_x(y,x,t) := \frac{\delta f}{\delta x}(y,x,t)$ exist and are continuous on V.
2. The nullspace of $f'_x(y,x,t)$ is independent of y, x and t, such that $ker(f'_y(y,x,t)) =: N$ and, moreover, $\dim N = m - r (r \le m, r \in \mathbb{N})$.

3. The function $G(y,x,t) := f'_y(y,x,t) + f'_x(y,x,t)Q$ is non-singular on V, where Q is a projection onto N.

Let $P = I - Q, g'_a(x(a),x(b)) = \frac{\partial g}{\partial x(a)}(x(a),x(b))$ and $g'_b(x(a),x(b)) := \frac{\partial g}{\partial x(b)}(x(a),x(b))$.

(Vg) 1. Let $H \subseteq \mathbb{R}^m \times \mathbb{R}^m$ be a region with

$$\{(x_1,x_2) \in \mathbb{R}^m \times \mathbb{R}^m \mid (y_1,x_1,a),(y_2,x_2,b) \in V\} \subseteq H \ .$$

Let $M \subseteq \mathbb{R}^m$ be a subspace and $g : H \longrightarrow M$ is continuously differentiable.

2. $g(u(a),u(b)) = g(Pu(a),Pu(b))$.
3. $\operatorname{im}(g'_a(x_1,x_2), g'_b(x_1,x_2)) = M$ for all $(x_1,x_2) \in H$.
4. The boundary conditions are linearly independent.

The assumption 2) of (Vg) means that the boundary conditions yield conditions only for the P-components. This is a sensible restriction because in (2.1) only the P-components are differential components. The Q-components can be determined by purely algebraic equations. So it makes sense to define a space of continuous vector functions that are continuously differentiable in the P-components only.

Let the function space $\mathcal{C}$ be defined by

$$\mathcal{C} := \{u \in C([a,b],\mathbb{R}^m)/Pu \in C^1([a,b],\mathbb{R}^m)\}$$

$\mathcal{C}$ equipped with the norm

$$\|u\|_{\mathcal{C}} := \|u\|_\infty + \|Pu'\|_\infty := \sup_{t\in[a,b]} |u(t)| + \sup_{t\in[a,b]} |Pu'(t)|$$

is a Banach space.

Remark. 1. The problems (2.1),(2.2) with the properties (Vf) and (Vg) for f and g, respectively, are well-posed in the sense of Hadamard.

2. Let (Vf) and (Vg) be fulfilled for f and g, respectively. Then, the state variable system is equivalent to

$$\begin{aligned} v'(t) &= Pw(v(t),t) \\ u(t) &= v(t) + Qw(v(t),t), \qquad t \in [a,b] \ , \\ g(v(a),v(b)) &= 0 \end{aligned} \tag{2.3}$$

where $w(v(t),t) = w(t)$ is uniquely defined by $f(w(t), v(t) + Qw(t), t) = 0$. Furthermore, $v(t)Pu(t)$ and $w(v(t),t) = Pu'(t) + Qu(t)$. (The proof can be found in [GM86].)

3. The statements 1) and 2) can be shown with more general assumptions on g. The boundary conditions may depend on Q-components, too, but they must be given in such a form, that the locally unique solvability is secured.

2.2 The Collocation Method

The aim of this part is to formulate collocation methods for the class of BVP's of DAE's described in Sect. 2.1. The collocation methods are completely characterized by the collocation subspace and the set of collocation points. Because of good convergence results spaces of piecewise polynomials are chosen as in [Rus74],[RC72],[dBS73], too. For this let

$$\Pi_n := \{t_i \mid a = t_0 < t_1 < \ldots < t_n = b\}, \qquad n \in \mathbb{N} \tag{2.4}$$

be a sequence of quasiuniform grids on the interval $[a,b]$ and

$$h_i = t_i - t_{i-1},\ i = 1,\ldots,n$$
$$h := \max_{i=1,\ldots,n} h_i \le \lambda \min_{i=1,\ldots,n} h_i$$

with a constant $\lambda \in \mathbb{R}$. Let $P_l[\alpha,\beta]$ be the set of polynomials p with a degree of p not greater than l on the interval $[\alpha,\beta]$. Then,

$$P_{l,\Pi_n} := \prod_{i=1}^{n} P_l[t_{i-1},t_i]$$

describes the linear space of piecewise polynomials, where the product is understood as a topological product of metric spaces. Now a partition

$$\sigma_k := \{c_j / j = 1,\ldots,k; 0 \le c_1 < c_2 < \ldots < c_k \le 1\} \tag{2.5}$$

of $[0,1]$ is chosen. The set of collocation points Σ_{nk} is then determined by transforming the points of σ_k onto each subinterval $[t_{i-1},t_i]$, $i = 1,\ldots,n$, of the grid $\Pi_{n}i$

$$\Sigma_{nk} := \{t_{ij} / t_{ij} = t_{i-1} + c_j h_i, i = 1,\ldots,n; j = 1,\ldots,k\}\ .$$

For a complete determination of a collocation solution the collocation spaces have to be described more exactly. For this the special structure of transferable DAE's and the choice of the set of collocation points are taken into consideration. Let $U \subseteq \mathbb{R}^m$ and

$$C^l_{\pi_n}([a,b],U) := \begin{cases} [\prod_{i=1}^{n} C^l([t_{i-1},t_i],U)] \cap C^{l-1}([a,b],U), & \text{for } l \ge 1 \\ \prod_{i=1}^{n} C^l([t_{i-1},t_i],U), & \text{for } l = 0. \end{cases}$$

For $y \in C^l_{\pi_n}([a,b],U)$ let a norm be defined as follows

$$\| y \|_{l,\pi_n} := \sum_{j=1}^{l-1} \| y^{(j)} \|_\infty + \max_{i=1,\ldots,n} \max_{t \in [t_{j-1},t_i]} | y_i^{(l)}(t) |\ ,$$

where

$$y_i^{(l)}(t) := \begin{cases} \lim_{\varepsilon \to 0} y^{(l)}(t_{i-1}+\varepsilon) & \text{for } \ t = t_{i-1} \\ y^{(l)}(t) & \text{for } \ t \in (t_{i-1},t_i) \\ \lim_{\varepsilon \to 0} y^{(l)}(t_i - \varepsilon) & \text{for } \ t = t_i \end{cases}$$

and $\varepsilon > 0$.

Remark. This norm is chosen in such a way that, for $y \in C^l([a,b],U)$

$$\| y \|_{C^l} := \sum_{j=1}^{l} \| y^{(j)} \|_\infty = \| y \|_{l,\pi_n}$$

holds.

The equivalent system (2.3) makes the special structure of the transferable DAE clear. For the P-components we have a BVP of an explicit ordinary differential equation. So it seems to be reasonable to approximate the P-components of a solution u^* of (2.1), (2.2) by collocation methods, which are proposed for explicit ordinary differential equations. Since the Q-components are determined by algebraic equations only and, consequently, the differentiability is not necessary, an interpolation at the collocation points $t_{ij}, i = 1,\ldots,n, j = 1,\ldots,k$, will be proposed, such that polynomials of degree $k-1$ are uniquely determined on each interval $[t_{i-1}, t_i], i = 1,\ldots,n$.

The different smoothness properties of the P- and Q-components are taken into consideration for the construction of the approximation spaces. In dependence on the partition σ_k the following cases are distinguished:

I: $c_1 > 0; c_k < 1$

Since the differential equation (2.1) has to be fulfilled at the collocation points, the P-components of the approximation have to be at least piecewise continuously differentiable. Analogously, the Q-components have to be at least piecewise continuous. Therefore, let the collocation subspace be

$$X_n := C^1_{\pi_n}([a,b], P\mathbb{R}^m) \cap P_{k,\pi_n} \oplus C_{\pi_n}([a,b], Q\mathbb{R}^m) \cap P_{k-1,\pi_n} \quad . \tag{2.6}$$

In this case collocation methods are mostly considered in Gaussian points.

II: $c_1 = 0; c_k = 1$.

The collocation conditions at the points $t_{ik} = t_{i-1} + c_k h_i = t_i + c_1 h_{i+1} = t_{i+1,1}$ give rise to additional conditions for smoothness at the grid points. So the P-components of the approximation have to be continuously differentiable on $[a,b]$ and the Q-components have to be continuous on $[a,b]$. Therefore, let the collocation subspaces be

$$X_n = C^1([a,b], P\mathbb{R}^m) \cap P_{k,\pi_n} \oplus C([a,b], Q\mathbb{R}^m) \cap P_{k-1,\pi_n} \quad . \tag{2.7}$$

This case includes the family of Lobatto-points.

Remark. A third case is possible: $c_1 = 0; c_k < 1$ (and $c_1 > 0; c_k = 1$, respectively). Partitions σ_k with that property are not suitable for the treatment of BVP's, because they yield unsymmetric methods.

Then the collocation method yields the following problem.

$$\begin{aligned} &\text{Solve} \\ &\qquad f(Pu_n'(t), u_n(t), t) = 0, \qquad t \in \Sigma_{nk} \quad , \\ &\qquad f(Pu_n(a), Pu_n(b)) = 0 \\ &\text{for} \quad u_n \in X_n \quad . \end{aligned} \tag{2.8}$$

2.3 Basis Representations of the Collocation Solution

For the representation and computation of the collocation solution bases for the collocation subspaces will be chosen. Special bases are discussed in [APR83], [BA87] and [dBS73]. In this section a Lagrange-basis, a Runge-Kutta monomial basis as it was used by the implementation of COLNEW [BA87] or in [Asc86], [AW83] and [Bad88], and a generalized Hermite-basis will be introduced. With the aid of the combination of these bases a representation of a collocation solution can be given.

Let $e_j \in P_{k-1}[0,1]$ be functions of the form

$$e_j(s) = \prod_{l-1, l=j}^{k} \frac{s - c_l}{c_j - c_l} \qquad j = 1, \ldots, k \ .$$

Then, the $e_j(s), j = 1, \ldots, k$ represent a Lagrange-basis of $P_{k-1}[0,1]$ (cf. [But87]). Moreover, let $\varphi_j \in P_k[0,1]$ be functions of the form

$$\varphi_j(s) = \int_0^s e_j(\tau) d\tau \qquad j = 1, \ldots, k \ .$$

Then, the $\varphi_j(s), j = 1, \ldots, k$ and $\varphi_0 \equiv 1$ represent a Runge-Kutta monomial basis of $P_k[0,1]$ [AW83].

Remark. The Runge-Kutta monomial basis was introduced as a basis of the collocation subspace $C^l_{\pi_n}([a,b], \mathbb{R}^m) \cap P_{k+l-1}$ for the appximation of explicit ordinary differential equations of l-th order ([APR83], [BA87]). In that case the monomials $t^{j-1}/(j-1)!$ for $1 \leq j \leq l$ are the first l basis functions on $P_{k+l-1}[0,1]$. The functions $\varphi_j, j = 1, \ldots, k$ are the polynomials of degree $k + l - 1$, which complete the basis.

Resulting from the structure of the approximation spaces X_n, the P-components of a collocation solution will be represented in the Runge-Kutta monomial basis and the Q-components of a collocation solution will be represented in the Lagrange basis. Then, a collocation solution is of the form

$$\begin{aligned} u_n(t) = P u_n(t_{i-1}) &+ h_i \sum_{j=1}^{k} \varphi_i \left(\frac{t - t_{i-1}}{h_i} \right) P u_n'(t_{ij}) \\ &+ \sum_{j=1}^{k} e_j \left(\frac{t - t_{i-1}}{h_i} \right) Q u_n(t_{ij}) \\ &\qquad \text{for} \quad t \in [t_{i-1}, t_i], i = 1, \ldots, n \ . \end{aligned} \tag{2.9}$$

For the following proposition of a basis representation let k be odd and σ_k be the set of k-step Lobatto-points. Then, $C_{(k+1)/2} = 1/2$. Now we consider functions $\eta_j \in P_k[0,1], j = 1, \ldots, k$, which are uniquely defined by

$$\eta_1'(c_j) = 0; j = 1, \ldots, k, j \neq (k+1)/2; \quad \eta_1(c_1) = 1; \quad \eta_1(c_k) = 0$$

$$\eta'_{k+1}(c_j)=0; j=1,\ldots,k; j\neq(k+1)/2; \quad \eta_{k+1}(c_1)=0;\ \eta_{k+1}(c_k)=1$$

$$\eta'_j(c_l)=\begin{cases}\delta_{j-1l} & j=2,\ldots,(k+1)/2,\ l=1,\ldots,k, l\neq(k+1)/2\\ \delta_{jl} & j=(k+3)/2,\ldots,k,\ l=1,\ldots,k, l\neq(k+1)/2\end{cases}$$

$$\eta_j(c_1)=\eta_j(c_k)=0, \qquad j=2,\ldots,k\ .$$

Then, $\eta_j, j=1,\ldots,k+1$ form a basis of the space $P_k[0,1]$. In further considerations this basis is called a *generalized Hermite basis.* An approximation solution u_n can be represented by means of the generalized Hermite basis for the P-components and the Lagrange basis for the Q-components as

$$\begin{aligned}u_n(t)&=\eta_1\left(\frac{t-t_{i-1}}{h_i}\right)Pu_n(t_{i-1})+\eta_{k+1}\left(\frac{t-t_{i-1}}{h_i}\right)Pu_n(t_i)+\\ &h_i\sum_{l=2}^{(k+1)/2}\eta_l\left(\frac{t-t_{i-1}}{h_i}\right)Pu'_n(t_{il-1})+h_i\sum_{l=(k+3)/2}^{k}\eta_l\left(\frac{t-t_{i-1}}{h_i}\right)Pu'_n(t_{il})+\\ &\sum_{l=1}^{k}e_l\left(\frac{t-t_{i-1}}{h_i}\right)Qu_n(t_{il}) \quad \text{for} \quad t\in[t_{i-1},t_i], i=1,\ldots,n\ . \end{aligned}\tag{2.10}$$

Remark. The representation (2.10) can be improved using (2.9). According to (2.9) the P-components of a collocation solution u_n have the form

$$Pu_n(t)=Pu_n(t_{i-1})+h_i\sum_{l=1}^{k}\varphi_l\left(\frac{t-t_{i-1}}{h_i}\right)Pu'_n(t_{il})\ . \tag{2.11}$$

If (2.11) is considered in $t_i, i=1,\ldots,n$, then we obtain

$$Pu_n(t_i)=Pu_n(t_{i-1})+h_i\sum_{l=1}^{k}\varphi_l(1)Pu'_n(t_{il}) \qquad i=1,\ldots,n\ .$$

If this equations is solved for $Pu'_n(t_{i(k+1)/2})$, it follows

$$\begin{aligned}&Pu'_n(t_{i(k+1)/2})=\\ &\left\{\frac{Pu_n(t_i)-Pu_n(t_{i-1})}{h_i}-h_i\sum_{\substack{l=1\\ l\neq(k+1)/2}}^{k}\varphi_l(1)Pu'_n(t_{il})\right\}/\varphi_{(k+1)/2}(1)\end{aligned}\tag{2.12}$$

Inserting (2.12) into (2.11) yields

$$\begin{aligned}Pu_n(t)&=Pu_n(t_{i-1})+h_i\sum_{\substack{l=1\\ l\neq(k+1)/2}}^{k}\varphi_l\left(\frac{t-t_{i-1}}{h_i}\right)Pu'_n(t_{il})+\\ &h_i\varphi_{(k+1)/2}\left(\frac{t-t_{i-1}}{h_i}\right)\left\{\frac{Pu_n(t_i)-Pu_n(t_{i-1})}{h_i}-\right.\end{aligned}$$

$$
\left. - h_i \sum_{\substack{l=1\\ l\neq(k+1)/2}}^{k} \varphi_l(1) P u_n'(t_{il}) \right\} / \varphi_{(k+1)/2}(1)
$$
$$
= \left\{ 1 - \varphi_{(k+1)/2}\left(\frac{t-t_{i-1}}{h_i}\right) / \varphi_{(k+1)/2}(1) \right\} P u_n(t_{i-1}) +
$$
$$
\left\{ \varphi_{(k+1)/2}\left(\frac{t-ti-1}{h_i}\right) / \varphi_{(k+1)/2}(1) \right\} P u_n(t_i) +
$$
$$
h_i \sum_{\substack{l=1\\ l\neq(k+1)/2)}}^{k} \left\{ \varphi_l\left(\frac{t-t_{i-1}}{h_i}\right) - \left[\varphi_{(k+1)/2}\left(\frac{t-t_{i-1}}{h_i}\right) / \varphi_{k+1)/2}(1) \right] \varphi_l(1) \right\} P u_n'(t_{il}) \ . \tag{2.13}
$$

This transformation corresponds to the elimination of the derivative of the P-components of the approximation solution at the points $t_{(k+1)/2}, i = 1, \dots, n$. Moreover, this transformation generates a representation of the generalized Hermite-basis with the aid of the Runge-Kutta monomial basis:

$$
\eta_j\left(\frac{t-t_{i-1}}{h_i}\right) = \left\{ \begin{array}{l} \varphi_{j-1}\left(\frac{t-t_{i-1}}{h_i}\right) - \left[\varphi_{(k+1]/2}\left(\frac{t-t_{i-1}}{h_i}\right) / \varphi_{(k+1)/2}(1)\right] \varphi_{j-1}(1) \\ \varphi_{j-1}\left(\frac{t-t_{i-1}}{h_i}\right) - \left[\varphi_{(k+1)/2}\left(\frac{t-t_{i-1}}{h_i}\right) / \varphi_{(k+1)/2}(1)\right] \varphi_j(1) \end{array} \right.
$$
$$
\eta_1\left(\frac{t-t_{i-1}}{h_i}\right) = 1 - \varphi_{(k+1)/2}\left(\frac{t-t_{i-1}}{h_i}\right) / \varphi_{(k+1)/2}(1)
$$
$$
\eta_{k+1}\left(\frac{t-t_{i-1}}{h_i}\right) = \varphi_{(k+1)/2}\left(\frac{t-t_{i-1}}{h_i}\right) / \varphi_{(k+1)/2}(1) \ .
$$

The convergence of the collocation method given in (2.8) can be shown for the two basis representations (2.9) and (2.10). The proof is deduced with the aid of statements of discretization theory ([Vai76], [Vai78]). The idea of the proof can be found in ([Deg]), where the convergence is shown for the collocation method (2.8) with the basis representation (2.10) and $k = 3$.

3 Construction of Realizations of the Collocation Methods

3.1 Realizations Using the Basis Representation (2.9)

In this section the construction of a collocation scheme for case I is discussed in detail. Since in case II the idea is similar, we will refer to differences only in some remarks.

For simplification let

$$
\begin{array}{ll} a_{jl} = \varphi_l\left(\frac{t_{ij}-t_{i-1}}{h_i}\right) & , l = 1, \dots, k, j = 1, \dots, k \ , \\ b_l = \varphi_l(1) & , l = 1, \dots, k \ . \end{array} \tag{3.1}
$$

Now, by inserting (2.9) into (2.8) the following equation system for computing the coefficients $Pu_n'(t_{ij}), Qu_n(t_{ij}), i = 1,\ldots,n, j = 1,\ldots,k$, and $Pu_n(t_i), i = 0,\ldots,n$, is obtained

$$
\begin{aligned}
& f(Pu_n'(t_{ij}), Pu_n(t_{i-1}) + h_i \sum_{l=1}^{k} a_{jl} Pu_n'(t_{il}) + Qu_n(t_{ij}), t_{ij}) = 0 \\
& \qquad\qquad\qquad\qquad i = 1,\ldots,n, \quad j = 1,\ldots,k \\
& \qquad\qquad\qquad\qquad \text{(collocation conditions)} \\
& Pu_n(t_i) = Pu_n(t_{i-1}) + h_i \sum_{l=1}^{k} b_l Pu_n'(t_{il}), \quad i = 1,\ldots,n \\
& \qquad\qquad\qquad\qquad \text{(conditions of continuity)} \\
& g(Pu_n(t_0), Pu_n(t_n)) = 0 \\
& \qquad\qquad\qquad\qquad \text{(boundary conditions)} \ .
\end{aligned}
\tag{3.2}
$$

Since the continuity conditions occur only for the P-components, first of all an underdetermined equation system is obtained. To overcome this underdetermination some additional suitable conditions are considered, namely

$$Q\bar{x}_i = 0 \qquad , i = 0,\ldots,n \quad \text{with} \quad \bar{x}_i \in \mathbb{R}^m \ . \tag{3.3}$$

Then the solution vector x is given in the form

$$x = \begin{pmatrix} Px_0 + Q\bar{x}_0 \\ Px_{11}' + Qx_{11} \\ Px_{12}' + Qx_{12} \\ \vdots \\ Px_{1k}' + Qx_{1k} \\ Px_1 + Q\bar{x}_1 \\ \vdots \\ Px_n + Q\bar{x}_n \end{pmatrix} , \tag{3.4}$$

where $x_i, \bar{x}_i \in \mathbb{R}^m, i = 0,\ldots,n, x_{ij}' \in \mathbb{R}^m, i = 1,\ldots,n, j = 1,\ldots,k$ and

$$
\begin{aligned}
Qx_{ij} &= Qu_n(t_{ij}), & i &= 1,\ldots,n, j = 1,\ldots,k \ , \\
Px_{ij}' &= Pu_n'(t_{ij}), & i &= 1,\ldots,n, j = 1,\ldots,k \ , \\
Px_i &= Pu_n(t_i), & i &= 0,\ldots,n
\end{aligned}
$$

are the coefficients in the basis representation (2.9). Consequently, (3.2) and (3.3) yield the system

$$f(Px_{ij}', Px_{i-1} + h_i \sum_{l=1}^{k} a_{jl} Px_{il}' + Qx_{ij}, t_{ij}) = 0 \quad i = 1,\ldots,n, \ j = 1,\ldots,k$$

$$Q\bar{x}_i + Px_i - Px_{i-1} - h_i \sum_{l=1}^{k} b_l Px_{il}' = 0 \quad i = 1,\ldots,n$$

$$\begin{pmatrix} g(Px_0, Px_n) \\ \mathcal{O} \end{pmatrix} + TQ\bar{x}_0 = 0 \tag{3.5}$$

where $\mathcal{O} = (0, \ldots, 0)^T \in \mathbb{R}^{m-r}$. For a combination of the boundary conditions and $Q\bar{x}_0 = 0$ in such a way that m linearly independent conditions are obtained, a linear function $T \in L(\mathbb{R}^m)$ mapping N bijectively onto $M^\perp$ is applied to $Q\bar{x}_0$ (cf. [GM86]).

Remark. 1. The equation system (3.5) and the solution vector x (3.4) are of dimension $((k+1)n+1)m =: l_1$, but in case II, (3.4) and (3.5) can be reduced to the dimension $(kn+2)m =: l_2$. Then $a_{1l} = 0$, $a_{kl} = b_l, l = 1, \ldots, k$ and $x_{i1} = x_{i-1k} = x_{i-1}, i = 1, \ldots, n$. So the following reduced system is obtained

$$\begin{aligned} f(Px_i', Px_i + Qx_i, t_i) &= 0 \qquad i = 0, \ldots, n \\ f(Px_{ij}', Px_{i-1} + h_i \sum_{l=1}^{k} a_{jl} Px_{il}' + Qx_{ij}, t_{ij}) &= 0 \qquad i = 1, \ldots, n \ , \\ & \qquad\quad j = 2, \ldots, k-1 \\ Q\bar{x}_i + Px_i - Px_{i-1} - h_i \sum_{l=1}^{k} b_l Px_{il}' &= 0 \qquad i = 1, \ldots, n \end{aligned}$$

$$\begin{pmatrix} g(Px_0, Px_n) \\ \mathcal{O} \end{pmatrix} + TQ\bar{x}_0 = 0 \ . \tag{3.6}$$

2. If, in (3.5), the approximation method for the P-components is considered, then the equivalence to a k-step implicit Runge-Kutta method can be shown, as it was done for instance in [Wei74] for explicit ordinary differential equations. The parameters described in (3.1) correspond to the parameters of the equivalent implicit Runge-Kutta method.

3.2 Realizations Using the Basis Representation (2.10)

Inserting (2.10) into (2.8) the following system for computing of the coefficients $Pu_n(t_i), i = 0, \ldots, n, Pu_n'(t_{ij}), i = 1, \ldots, n, j = 1, \ldots, k, j \neq (k+1)/2$ is obtained

$$\begin{aligned} f(Pu_n'(t_i), u_n(t_i), t_i) &= 0, \quad i = 0, \ldots, n \ , \\ f(Pu_n'(t_{ij}), L(Pu_n)(t_{ij}) + Qu_n(t_{ij}), t_{ij}) &= 0, \quad i = 1, \ldots, n \ , \\ & \qquad j = 2, \ldots, k-1, \quad j \neq (k+1)/2 \ , \\ f(L'(Pu_n)(t_{i(k+1)/2}), L(Pu_n)(t_{i(k+1)/2}) &+ Qu_n(t_{i(k+1)/2}), t_{i(k+1)/2}) = 0 \ , \\ & i = 1, \ldots, n \\ & \qquad \text{(collocation conditions)} \\ g(Pu_n(t_0), Pu_n(t_n)) &= 0 \\ & \qquad \text{(boundary conditions)} \end{aligned} \tag{3.7}$$

where

$$
\begin{aligned}
L(Pu_n)(t_{ij}) = \eta_1 \left(\frac{t_{ij}-t_{i-1}}{h_i}\right) Pu_n(t_{i-1}) + \eta_{k+1} \left(\frac{t_{ij}-t_{i-1}}{h_i}\right) Pu_n(t_i) \\
+h_i \sum_{l=2}^{(k-1)/2} \eta_l \left(\frac{t_{ij}-t_{i-1}}{h_i}\right) Pu_n'(t_{il-1}) + \\
h_i \sum_{l=(k+3)/2}^{k} \eta_l \left(\frac{t_{ij}-t_{i-1}}{h_i}\right) Pu_n'(t_{il}), \qquad i = 1, \ldots, n \ , \\
j = 1, \ldots, k \ .
\end{aligned}
$$

Here the continuity conditions are not necessary, because they are implicitly included in the special basis. Since the boundary conditions yield only r linearly independent conditions an underdetermined system is obtained again. This is why the condition $Q\bar{x} = 0$ with $\bar{x} \in \mathbb{R}^m$ is added. Then the solution vector x is given in the form

$$
x = \begin{pmatrix}
x_0 \\
Px_0' + Q\bar{x} \\
Px_{12}' + Qx_{12} \\
Px_{13}' + Qx_{13} \\
\vdots \\
Px_{1(k-1)/2}' + Qx_{1(k-1)/2} \\
Px_{1(k+3)/2}' + Qx_{1(k+3)/2} \\
\vdots \\
Px_{1k-1}' + Qx_{1k-1} \\
x_1 \\
Px_1' + Qx_{1(k+1)/2} \\
\vdots \\
x_n \\
Px_n' + Qx_{n(k+1)/2}
\end{pmatrix} , \tag{3.8}
$$

where $x_i \in \mathbb{R}^m, i = 0, \ldots, n, x_i' \in \mathbb{R}^m, i = 0, \ldots, n, x_{ij}' \in \mathbb{R}^m, i = 1, \ldots, n,$ $j = 2, \ldots, k-1, j = (k+1)/2, x_{ij} \in \mathbb{R}^m, i = 1, \ldots, n, j = 2, \ldots, k-1$, and

$$
\begin{aligned}
x_i &= u_n(t_i) \quad i = 0, \ldots, n \\
Px_i' &= Pu_n'(t_i) \quad i = 0, \ldots, n \\
Px_{ij}' &= Pu_n'(t_{ij}) \quad i = 1, \ldots, n, j = 2, \ldots, k-1, j \neq (k+1)/2 \ , \\
Qx_{ij} &= Qu_n(t_{ij}) \quad i = 1, \ldots, n, j = 2, \ldots, k-1 \ ,
\end{aligned}
$$

are the coefficients in the representation (2.10) of the collocation solution u_n. Finally, the following collocation scheme is obtained

$$
f(Px_i', x_i, t_i) = 0 \qquad i = 0, \ldots, n \ ,
$$

$$
f(Px_{ij}', L(Px)(t_{ij}) + Qx_{ij}, t_{ij}) = 0, \ i = 1, \ldots, n, j = 2, \ldots, k-1, j \neq (k+1)/2 \ ,
$$

$$f(L'(Px)/t_{i(k+1)/2}), L(Px)(t_{i(k+1)/2}) + Qx_{i(k+1)/2}, t_{i(k+1)/2}) = 0 \; i = 1, \dots, n \;,$$

$$\begin{pmatrix} g(Px_0, Px_n) \\ \mathcal{O} \end{pmatrix} + TQ\bar{x} = 0 \tag{3.9}$$

where $\mathcal{O} = (0, \dots, 0)^T \in \mathbb{R}^{m-r}$ and $T \in L(\mathbb{R}^m)$ mapping N bijectively onto $M^{\perp}$.

Remark. 1. The advantage of the use of the basis representation (2.10) is, that the dimension of the system (3.9) is only $l_3 := ((k-1)n + 2)m$, i.e. it has been reduced by nm equations in comparison to (3.6) by the transformation (2.13).
2. For $k = 3$ this method in detail discussed in [Deg].

4 Feasibility and Convergence of the Collocation Schemes

In the following the properties of feasibility and convergence are shown only for the collocation scheme (3.5), because the idea is similar for the collocation schemes (3.6) and (3.9). Now, the proof of stability and consistency of the scheme (3.5) uses an idea of H.B. Keller and A.B. White ([KW75]). Firstly, consistency and stability are proved for linear IVP's. Hence, an argument, by now standard, provides the stability of the scheme applied to linear BVP's. Then, commuting linearization and discretization provides the desired results.

Let (2.1), (2.2) be described by an operator equation

$$Fu = 0$$

with $F : X \longrightarrow Y$ and $x := C^1([a,b], P\mathbb{R}^m) \oplus C([a,b], Q\mathbb{R}^m)$ and $Y = C([a,b], \mathbb{R}^m) \times \mathbb{R}^r$. X and Y considered with

$$\begin{aligned} \|x\|_X &:= \|Px\|_{1,\pi_n} + \|Qx\|_{0,\pi_n} && \text{for all } x \in X \text{ and} \\ \|(y,d)^T\|_y &:= \|y\|_{0,\pi_n} + |d| && \text{for all } (y,d)^T \in Y \;, \end{aligned}$$

respectively, are Banach-spaces.

Remark. For a constant nullspace N of $f'_y(y, x, t)$ we have $\mathcal{C} = C^1([a,b], P\mathbb{R}^m) \oplus C([a,b], Q, \mathbb{R}^m)$.

Moreover, (3.5) shall be considered as an operator equation in suitable normed spaces. For this, the finite dimensional spaces $\mathcal{C}_n := \mathbb{R}^{l_1}$ and $C_n := \mathbb{R}^{l_1}$ as well as restriction operators $r_n : X \longrightarrow \mathcal{C}_n$ and $q_n : Y \longrightarrow C_n$ can be defined. $\mathcal{C}_n$ and C_n are considered with the norms

$$\begin{aligned} |x|_{e_n} := \max_{i=0,\dots,n} |Px_i + Q\bar{x}_i| + \max_{\substack{i=1,\dots \\ j=1,\dots,k}} |Px'_{ij} + Qx_{ij}| + \\ \max_{\substack{i=1,\dots,n \\ j=1,\dots,k}} |Px_{i-1} + h_i \sum_{l=1}^{k} a_{jl} Px'_{il}| \;, \end{aligned}$$

where $x \in \mathcal{C}_n$ is of the form (3.4) and

$$|x|_{C_n} := \max_{i=1,\ldots,(k+1)n+1} |x_i|$$

with $x_i \in \mathbb{R}^m, i = 1, \ldots, (k+1)n+1$. Then (3.5) can be formulated as an operator equation

$$G_n x_n = 0 \tag{4.1}$$

with $G_n : \mathcal{C}_n \longrightarrow C_n$. Moreover, let restriction operators r_n and q_n be defined as $r_n : X \longrightarrow \mathcal{C}_n$ with

$$r_n u = \begin{pmatrix} Pu(t_0) \\ Pu'(t_{11}) + Qu(t_{11}) \\ Pu'(t_{12}) + Qu(t_{12}) \\ \vdots \\ Pu'(t_{1k}) + Qu(t_{1k}) \\ Pu(t_1) \\ \vdots \\ Pu(t_n) \end{pmatrix} \tag{4.2}$$

and $q_n : Y \longrightarrow C_n$ with

$$q_n \begin{pmatrix} f \\ d \end{pmatrix} = \begin{pmatrix} f(t_{11}) \\ f(t_{12}) \\ \vdots \\ f(t_{1k}) \\ 0 \\ f(t_{21}) \\ \vdots \\ f(t_{nk}) \\ 0 \\ d \end{pmatrix} . \tag{4.3}$$

Additionally, norms for the matrices $E \in L(\mathbb{R}^{l_1})$ are introduced

$$|E|_{C_n} := \max_{|z|_{C_n}=1} |Ez|_{C_n} \ ;$$
$$\|E\|_{C_n} := \max_{|z|_{C_n}=1} |Ez|_{C_n} \ ;$$
$$|E|_{\mathcal{C}_n} := \max_{|z|_{C_n}=1} |Ez|_{\mathcal{C}_n} \ .$$

Let $S_{\mathcal{C}_n}(z,\rho) := \{w \in \mathcal{C}_n \,|\, |w-z|_{\mathcal{C}_n} \le \rho\}$.

Lemma 1. *Let $u^* \in X$ be a solution of the BVP (2.1), (2.2), the assumptions (Vf) and (Vg) be fulfilled for f and g, respectively, and let $f'_y(y,x,t)$,*

$f'_x(y,x,t)$,$g'_a(x(a),x(b))$ and $g'_b(x(a),x(b))$ be Lipschitzian on V and H, respectively. Then there exists a $\rho_0>0$ such that G_n is defined on $S_{\mathcal{C}_n}(r_n u^*,\rho_0)$, it is continuously differentiable and the derivatives satisfy the condition

$$\|G'_n(z)-G'_n(\tilde z)\|_{C_n} \le L|z-\tilde z|_{\mathcal{C}_n}$$

for arbitrary grids Π_n with (2.4), $z,\tilde z\in S_{\mathcal{C}_n}(r_n u^*,\rho_0)$ with a Lipschitz-constant L independent of Π_n.

Proof. G_n is defined on $S_{\mathcal{C}_n}(r_n u^*,\rho_0)$ and continuously differentiable there because of the assumptions (Vf) and (Vg). It has only to be shown that G'_n is Lipschitzian. Using the assumptions of smoothness of f and g this statement follows by simple estimations. □

Now, a formulation for the stability of the collocation scheme is given.

Definition 2. Let $u^*\in X$ be a solution of (2.1), (2.2) and the assumptions (Vf) and (Vg) be fulfilled for f and g, respectively. Then, the collocation scheme $G_n x_n=0$ for (2.1), (2.2) is *stable* on Π_n with $h\le h_{\max}$ if there exist constants $S>0$ and $\rho>0$, such that for arbitrary Π_n with $h\le h_{\max}$, $x^1,x^2\in S_{\mathcal{C}_n}(r_n u^*,\rho)$

$$|x^1-x^2|_{\mathcal{C}_n}\le S|G_n x^1-G_n x^2|_{C_n} \;.$$

Sufficient conditions for the stability of the collocation scheme (4.1) are given in the following theorem.

Theorem 3. *Let the assumptions of Lemma 1 be satisfied. Let an $h_{\max}>0$ exist such that, for all Π_n with $h\le h_{\max}$, $G'_n(r_n u^*)$ is bijective and $|(G'_n(r_n u^*))^{-1}|_{\mathcal{C}_n}\le S^*$ holds with a constant $S^*>0$ independent of Π_n. Then the collocation scheme (4.1) is stable for (2.1), (2.2) on Π_n with $h\le h_{\max}$.*

Proof. This statement about the stability of a collocation scheme is a special formulation of a theorem given by H.B. Keller in [Kel73] and can be shown analogously. □

In the following G_n is shown to satisfy the assumptions of Theorem 3. First the stability of this scheme for an IVP we show, which is obtained by linearization of (2.1), (2.2) in a solution $u^*\in X$. For this, consider

$$G_n^{0\,\prime}(r_n u^*)z=\begin{cases} A_{ij}Pz'_{ij}+B_{ij}(Pz_{i-1}+h_i\sum\limits_{l=1}^{k}a_{jl}Pz'_{il})+B_{ij}Qz_{ij} \\ \qquad i=1,\dots,n,\ j=1,\dots,k \\ Q\bar z_i+Pz_i-Pz_{i-1}-h_i\sum\limits_{l=1}^{k}b_lPz'_{il} \\ \qquad i=1,\dots,n \\ T_1Pz_0+TQ\bar z_0 \end{cases},$$

where $T_1 \in L(\mathbb{R}^m)$ is a linear function mapping $N^\perp$ bijectively onto M. T_1 exists because of the assumptions (Vf) and (Vg). Moreover, let

$$A_{ij} = f_y' P u^{*'}(t_{ij}, (Pu^*(t_{i-1}) + h_i \sum_{l=1}^{k} a_{jl} P u^{*'}(t_{il}) + Qu^*(t_{ij}), t_{ij})$$
$$i = 1, \ldots, n, j = 1, \ldots, k$$
$$B_{ij} = f_x'(P u^{*'}(t_{ij}), Pu^*(t_{i-1}) + h_i \sum_{l=1}^{k} a_{jl} P u^{*'}(t_{il}) + Qu^*(t_{ij}), t_{ij})$$
$$i = 1, \ldots, n, j = 1, \ldots, k \ .$$

Theorem 4. *Let u^* be a solution of (2.1), (2.2). Let f and g fulfil (Vf) and (Vg), respectively. Then ${G_n^0}'(r_n u^*)$ is nonsingular for all Π_n with $h \leq h_{\max}$ and there is a constant C independent of Π_n such that*

$$|({G_n^0}'(r_n u^*))^{-1}|_{\mathcal{C}_n} \leq C \ .$$

Proof. Here the stability of ${G_n^0}'(r_n u^*)$ on Π_n with $h \leq h_{\max}$ has to be shown. For this purpose let ${G_n^0}'(r_n u^*)z = v$. Now, $|z|_{\mathcal{C}_n} \leq C|v|_{C_n}$ for arbitrary $z \in \mathcal{C}_n, v \in C_n, \Pi_n$ with $h \leq h_{max}$ and h_{max} sufficiently small has to be shown.

Let

$$W_{ij} = Pz_{ij}' + Qz_{ij}$$

$$U_{ij} = Pz_{ij} \text{with } Pz_{ij} = Pz_{i-1} + h_i \sum_{l=1}^{k} a_{jl} P z_{il}' \quad , i = 1, \ldots, n, j = 1, \ldots, k \ ,$$

$$u_{i-1} = Pu_{i-1} + Q\bar{u}_{i-1} = Pz_{i-1} + Q\bar{z}_{i-1}, \quad i = 1, \ldots, n \ .$$

Then ${G_n^0}'(r_n u^*)z = v$, implies

$$A_{ij}W_{ij} + B_{ij}(U_{ij} + QW_{ij}) \ = \ v_{ij}, \qquad i = 1, \ldots, n, j = 1, \ldots, k \tag{4.4}$$
$$U_{ij} - h_i \sum_{l=1}^{k} a_{jl} P W_{il} = \ Pu_{i-1}, \qquad i = 1 \ldots, n, j = 1, \ldots, k \tag{4.5}$$
$$u_i - Pu_{i-1} - h_i \sum_{l=1}^{k} b_l P W_{il} \ = \ v_i, \qquad i = 1, \ldots, n \tag{4.6}$$
$$T_1 P u_0 + TQ\bar{u}_0 \ = \ v_0 \tag{4.7}$$

Because of $A_{ij} + B_{ij}Q = G_{ij}$, it follows that (4.4) is equivalent to

$$G_{ij}W_{ij} + B_{ij}U_{ij} = v_{ij} \ .$$

Then the functional matrix of the equation system (4.4), (4.5) is given by

$$\mathcal{G}_i = \begin{pmatrix} \operatorname{diag}(G_{i1}, G_{i2}, \ldots, G_{ik}) & \operatorname{diag}(B_{i1}, B_{i2}, \ldots, B_{ik}) \\ h_i(A \otimes P) & I, \end{pmatrix}$$

where

$$A \otimes P = \begin{pmatrix} a_{11}P & \cdots & a_{1k}P \\ \vdots & & \vdots \\ a_{k1}P & \cdots & a_{kk}P \end{pmatrix} .$$

Since $G^{-1}(y,x,t)$ exists and $G^{-1}(y,x,t)$ and $f'_y(y,x,t)$ are bounded on V due to the assumptions, the matrix

$$H = \{\text{diag}(G_{i1}, G_{i2}, \ldots, G_{ik}) - h_i \text{ diag}(B_{i1}, B_{i2}, \ldots, B_{ik})(A \otimes P)\}$$

has a globally bounded inverse for $0 \leq h \leq h_{\max}$ with sufficiently small h_{max}. However, then

$$\mathcal{G}_i^{-1} = \begin{pmatrix} H^{-1} & -h_i \text{diag}(B_{i1}, B_{i2}, \ldots, B_{ik}) \\ h_i(A \otimes P)H^{-1} & I + h_i(A \otimes P)H^{-1}\text{diag}(B_{i1}, B_{i2}, \ldots, B_{ik}) \end{pmatrix}$$

is also bounded from Hadamard's Theorem ([OR77]) we derive

$$|W_{ij}| \leq \gamma\{|v_{ij}| + |Pu_{i-1}|\} \quad i = 1, \ldots, n, j = 1, \ldots, k \ , \tag{4.8}$$
$$|U_{ij}| \leq \gamma\{|v_{ij}| + |Pu_{i-1}|\} \quad i = 1, \ldots, n, j = 1, \ldots, k \ . \tag{4.9}$$

Hence it follows from (4.6) that

$$\begin{aligned} |u_i| &\leq |Pu_{i-1}| + h_i \left(\sum_{l=1}^{k} |b_l| \right) \|P\| \gamma\{|v_i l| + |Pu_{i-1}|\} + |v_i| \\ &\leq (1 + \alpha h_i)|Pu_{i-1}| + (1 + \alpha h_i)|v|_{C_n} \ , \end{aligned} \tag{4.10}$$

where $\alpha = \gamma \left(\sum_{l=1}^{k} |b_l| \right) \|P\|$.

Let $\alpha_i = 1 + \alpha h_i$. Then, successively inserting

$$Pu_i = Pu_{i-1} + h_i \sum_{l=1}^{k} b_l PW_{il} + Pv_i$$

into (4.10)

$$|u_i| \leq \prod_{j=1}^{i} \alpha_j |Pu_0| + \prod_{j=1}^{i} \alpha_j |v|_{C_n}, \quad i = 1, \ldots, n \ .$$

Due to the assumption that the grid Π_n is quasiuniform, we have $\min_{i=1,\ldots,n} h_i \geq h/\lambda$ and

$$\prod_{j=1}^{i} \alpha_j \leq \exp(\alpha\lambda(t_i - t_0)) \ .$$

Finally,

$$|u_i| \leq \exp(\alpha\lambda(t_i - t_0))(|Pu_0| + |v|_{C_n}), \quad i = 1, \ldots, n \ .$$

With (4.7) it follows that

$$|Pu_0| \leq |P(u_0 + Q\bar{u}_0| \leq \|P\||Pu_0 + Q\bar{u}_0| =$$

$$\|P\||T^+TPu_0 + Q\bar{u}_0| \leq \|P\|\|T^+\||T_1Pu_0 + TQ$$

and, analogously, $|Q\bar{u}_0| \leq C_2|v|_{C_n}$ with constants C_1 and C_2. So we have the estimation

$$|u_i| \leq S|v|_{C_n} \qquad i = 0, \dots, n \ ,$$

with a constant S. Inserting this inequality into (4.8) and (4.9) finally yields

$$|z|_{\mathcal{C}_n} \leq \tilde{S}|v|_{C_n}$$

with a constant $\tilde{S}$ independent of Π_n. □

Now, using statements of discretization theory ([Kel73],[Vai76])

$$|(G_n'(r_nu^*))^{-1}|_{\mathcal{C}_n} \leq S \tag{4.11}$$

can be shown for a constant S with the aid of the representation

$$G_n'(r_nu^*) = G_n^{o'}(r_nu^*) + (G_n'(r_nu^*) - G_n^{o'}(r_nu^*)) \ .$$

The following theorem properties which are important for the feasibility of this method summarizes.

Theorem 5. *Let* $u^* \in C_{\Pi_n}^{k+2}([a,b], P\mathbb{R}^m) \oplus C_{\Pi_n}^{k+1}([a,b], Q\mathbb{R}^m)$ *be an isolated solution of (2.1), (2.2) and let the assumptions of Lemma 1 be fulfilled. Let* $f \in C^2(S_{\mathbb{R}^m}(Pu^{*'}(t), \delta) \times S_{\mathbb{R}^m}(u^*(t), \delta) \times [a,b], \mathbb{R}^m)$ *and* $g \in C^2(S_{\mathbb{R}^m}(Pu^*(a), \delta) \times S_{\mathbb{R}^m}(Pu^*(b)\delta), \mathbb{R}^r)$. *Then*

(i) the condition number of the system (4.1) is bounded
(ii) G_n *is stable on* Π_n
(iii) the local error $\tau_n := G_nr_nu^* = 0(h^{k+1})$
(iv) the global error $e_n := r_nu^* - x_n^* = 0(h^{k+1})$, *where* x_n^* *is a solution of* $G_nx_n = 0$.

Proof. (i) Analogously as in Lemma 1 $\|G_n'(r_nu^*)\|_{C_n} \leq C$ is obtained with a constant C independent of Π_n. The boundedness of the condition number of the system (4.2) follows with (4.11).
(ii) The stability of G_n follows from (4.11) and Theorem 4.
(iii) By means of Taylor expansions we obtain the statement about the local error $\tau_n = 0(h^{k+1})$.
(iv) The estimation for the global error follows obviously from (ii) and (iii).

□

Remark. 1. Since, for implementation, the error computation bases on the C_n-norm, it is useful to obtain an estimation in that norm. Obviously, for arbitrary $x \in \mathbb{R}^{l_1}$ we have $|x|_{C_n} \leq |x|_{\mathcal{C}_n}$. Hence, $|r_nu^* - x_n^*|_{C_n} = 0(h^{k+1})$ and with (4.11)

$$|(G_n'(r_nu^*))^{-1}|_{C_n} \leq |(G_n'(r_nu^*))^{-1}|_{\mathcal{C}_n} \leq S \ .$$

Moreover, we derive $|G_n'(r_nu^*)|_{C_n} = 0(h^{-1})$ and the condition number of system (4.1) as an operator from C_n into C_n increases as $0(h^{-1})$.

2. Under consideration of a appropriately chosen $\mathcal{C}_n$-norm, Theorem 5 can be shown for the systems (3.6) and (3.9), too.

Theorem 6. *Let the assumptions of Theorem 5 be fulfilled. Then the nonlinear operator equation $G_n x_n = 0$ can be solved by means of the Newton method for each fixed Π_n with $h \leq h_{\max}$. With the initial guess $x_n^{(0)}$ from $S_{C_n}(r_n u^*, \rho)$ the Newton method is feasible and converges quadratically.*

Proof. This statement follows from the above remark and Theorem 5.12 in [Sch79]. □

4.1 Statements About Superconvergence

The convergence of collocation methods for explicit ordinary differential equations is considered in a series of papers, cf. [dBS73], [Rus74],[Asc86], [GS66]. In these papers it is shown that, if the points of the partition σ_k satisfy the orthogonality conditions

$$\int_0^1 \Phi(t) \prod_{l=1}^{k} (t - c_l) dt = 0 \quad \text{for all}\, \Phi \in P_{\bar{n}}[0,1]$$

with $0 \leq \bar{n} \leq k, \bar{n} \in \mathbb{N}$, then the collocation approximation u_n^* converges to the exact solution of the problem as

$$|u^*(t_i) - u_n^*(t_i)| = 0(h^{k+\bar{n}}) \quad \text{for}\, t \in \Pi_n$$

whenever the grid Π_n is sufficiently fine and the solution u^* is sufficiently smooth, i.e. $u^* \in C_{\Pi_n}^{k+1}([a,b], \mathbb{R}^m) \cap C([a,b], \mathbb{R}^m)$.

Now we aim at this transfering these convergence results onto DAE's. Here, the order $p \geq k+1$ of superconvergence can be expected only for the P-components. This property can be shown with the aid of the state variable system (2.3). Moreover let

$$\begin{aligned} A = (a_{jl})_{j=1,\ldots,k, l=1,\ldots,k} &= \left(\varphi_l \left(\frac{t_{ij} - t_{i-1}}{h_i} \right) \right)_{l=1,\ldots,k, j=1,\ldots,k} \\ &= (\varphi_l(c_j))_{l=1,\ldots,k, j=1,\ldots,k} \quad . \end{aligned}$$

Theorem 7. *Let the assumptions (Vf) and (Vg) be fulfilled for f and g, respectively. Let*

$$\begin{aligned} v'(t) &= Pw(v(t),t) \\ u(t) &= v(t) + Qw(v(t),t), \qquad t \in [a,b] \ , \\ g(v(a), v(b)) &= 0 \end{aligned} \tag{4.12}$$

be the equivalent state variable system of (2.1), (2.2), where $w = w(v(t),t)$ is defined by $f(w, v + Qw, t) = 0$. Then, applying the collocation method (3.5) to (2.1), (2.2) yields exactly the same solution as the application to (4.12) if A is non-singular.

Proof. The proof is carried out analogously to the proof in [GM86]. □

Remark. In the case I the matrix A is always nonsingular. In the case II the matrix A is of the form

$$A = \begin{pmatrix} 0 & 0 \\ \bar{a} & \bar{A} \end{pmatrix} ,$$

where $\bar{a}$ is a $(k-1)$-vector and $\bar{A}$ is a non-singular $(k-1) \times (k-1)$-matrix ([AW83]). In this case the proof is similar. □

Theorem 8. *For f and g of (2.1), (2.2) let the assumptions (Vf) and (Vg) be fulfilled. Let $u^* \in C^{p+1}([a,b], P\mathrm{I\!R}^m) \oplus C([a,b], Q\mathrm{I\!R}^m)$ be an isolated solution of (2.1), (2.2). The second partial derivatives of f and g with respect to the first and second component exist and are continuous. The points of the partition σ_k satisfy the orthogonality condition*

$$\int_0^1 \Phi(t) \prod_{l=1}^{k} (t - c_l) dt = 0 \quad \textit{for all}\, \Phi \in P_{p-k}[0,1] \ .$$

Then, for sufficiently small h, the following error estimation at the gridpoints is obtained

$$|Pu^*(t_i) - Pu_n^*(t_i)| = 0(h^p) \ ,$$

where u_n^* is the collocation solution of (2.1), (2.2).

Proof. We have the equivalence of (2.1), (2.2) to the state variable system (4.12). Because of the continuity of the second partial derivatives of f, there exist continuous second partial derivatives of w. With the state variable system (4.12) an explicit ordinary differential equation for the P-components is obtained. Hence, the statements about superconvergence for explicit ordinary differential equations can be transferred, such that

$$|v^*(t_i) - v_n^*(t_i)| = 0(h^p), \qquad i = 0, \dots, n \ ,$$

where v^* is a solution and v_n^* is a collocation solution of (4.12) (cf. for instance Theorem 13 in [Asc86]). Because of the equivalence of the collocation methods for (2.1), (2.2) and (4.12) given in Theorem 7, we have

$$|Pu^*(t_i) - Pu_n^*(t_i)| = 0(h^p) \qquad i = 0, \dots, n \ ,$$

where u^* is a solution and u_n^* a collocation solution of (2.1), (2.2). □

Remark. In Gaussian and Lobatto points the order of superconvergence $p = 2k$ and $p = 2k-2$ is obtained for the P-components of the collocation solution at the grid points.

5 Algorithm for computing a collocation solution

For the computation of collocation solutions the following algorithm, which consists of three steps, is proposed.

1. Determination of a solution of the nonlinear collocation system (3.5), ((3.6), (3.9), respectively).
2. Computation of an estimation of the global error.
3. Selection of a new grid, if necessary.

Basing on statements given by R.D. Skeel ([Ske82]) an estimation of the global error using the first step of deferred corrections can be developed. For grid selection an algorithm given by M. Lentini and V. Pereyra ([LP77], [PS75]) can be generalized to this class of BVP's for DAE's.

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