Linear Differential-Algebraic Equations

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1 Introduction

This paper deals with linear implicit differential equations

$$A(t)x'(t) + B(t)x(t) = f(t), \quad t \in I$$

where the matrix $A(t)$ shall be singular on the interval $I \subseteq \mathbb{R}$. Such equations are called *differential-algebraic equations* (DAEs). Furthermore, let A, B and f be continuous matrix and vector functions, respectively.

Analyzing and solving differential-algebraic equations the singularity of $A(t)$ causes some serious trouble which we do not know from ordinary differential equations (ODEs). This is due to the special structure of differential-algebraic equations, which is essentially more complex than that of regular ODEs (cf. “DAEs are not ODEs” by Petzold [Pet82]). Catchwords for such trouble are instability of numerical methods for DAEs, inconsistent initial values etc. For instance, solving an explicit ODE $x'(t) = g(x(t), t)$, an aribitrary point x_0 can be chosen as an initial point since for every point of the (x, t) plane there is a trajectory of the ODE passing this point. The situation for DAEs is essentially different: There we cannot choose an arbitrary initial point x_0, demanding the solution x evaluated in t_0 to be equal to x_0. Namely, $x(t_0)$ has to fulfil not only the “underlying ODE”, but the complete differential-algebraic equation.

Definition 1. An initial value x_0 is called *consistent* if there is a solution x of this equation with $x(t_0) = x_0$.

In recent years DAEs have been investigated by various authors, and many of them have introduced their own definitions and notations, e.g. “index”, “canonical form”, “solution”, “solvable” etc. This paper aims at comparing some of these concepts in order to realize which classes of linear DAEs can be treated by which methods and which cannot. We confine our considerations to DAEs that are given in (or that can be transformed into) so-called “normal forms”. Namely, when investigating linear differential-algebraic equations it is fairly common to consider them in “canonical normal forms”. There, the coefficient matrices have an especially simple block-diagonal form.

Chapter 2 contains the well-known Kronecker Canonical Form (KCF) by Gear/Petzold [GP84], while the following two chapters represent the corresponding concepts of Campbell [Cam83] and Clark [Cla88], respectively. Remark that we have to recall the KCF-definition in Sect. 5 again. This final section proves an important relation between the index concepts of [GP84] and [Mar87]. The latter concept will be explained in sect. 4. Generalizing two results of [Mar89a] and [Mar89b] sect. 5 shows that the "decoupling method" of [Mar87] and [Han90] is applicable to a larger class of linear normal DAEs than the "KCF reduction method" of [GP84].

2 The Kronecker Canonical Form for a Linear DAE, Global and Local Index

We consider linear differential-algebraic equations (DAEs)

$$A(t)x'(t) + B(t)x(t) = f(t), \quad t \in I \ , \tag{1}$$

where $A, B : I \to L(\mathbb{R}^m)$ are continuous matrix functions, and $f \in C(I, \mathbb{R}^m)$, and $A(t)$ is singular for all $t \in I$. Many authors have defined the *index* of a given DAE, which is considered to be an important characteristic of such an equation because it "measures the distance" between the DAE and an ordinary differential equation.

Gear and Petzold [GP84] investigated DAEs in *Kronecker Canonical Form* and defined the *global index* of a differential-algebraic equation:

Definition 2. * The differential-algebraic equation (1) with coefficients $A, B \in C(I, L(\mathbb{R}^m))$ is said to be in *Kronecker Canonical Form* (KCF) *with the global index* k if

$$A(t) = \begin{pmatrix} I_s & 0 \\ 0 & J \end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} W(t) & 0 \\ 0 & I_{m-s} \end{pmatrix} \text{ hold}$$

with $J^k = 0,\ J^l \neq 0 \quad \forall l < k$.

* Equation (1) is called a *DAE with the global index* k if there exist $E \in C(I, L(\mathbb{R}^m))$ and $F \in C^1(I, L(\mathbb{R}^m))$, $E(t)$ and $F(t)$ nonsingular for all $t \in I$, such that the transformed equation

$$\tilde{A}(t)y'(t) + \tilde{B}(t)y(t) = \tilde{f}(t)$$

(with $\tilde{A} = EAF$, $\tilde{B} = EBF + EAF'$ and $\tilde{f} = Ef$) is in KCF with the global index k.

In some papers (e. g. [Cam83],[Cla88]) great importance is attached to the so-called local index of the matrix pencil $(A(t), B(t))$.

Definition 3. Let the matrix pencil $(A(t), B(t))$ be regular, i. e. let a $\lambda \in \mathbb{R}$ exist so that $\det(\lambda A(t) + B(t)) \neq 0$. Then $\mathrm{ind}(A(t), B(t)) := k(t) := \mathrm{index}((\lambda A(t) + B(t))^{-1} A(t))$ is called the *local index* of the matrix pencil $(A(t), B(t))$.

Note that $k(t)$ does not depend on the chosen value λ, as was shown e. g. in [GM86].

The following examples show that neither the regularity nor the local index of the local matrix pencil characterize the DAE (1) and the behaviour of its solutions (if they exist).

Example 1 (cf. [GM86], p.91). Let the coefficient matrices of (1) be

$$A(t) = \begin{pmatrix} -t & t^2 \\ -1 & t \end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Obviously, the local pencil $(A(t), B(t))$ is everywhere regular with the index 2, but the homogeneous initial value problem has the solution $x(t) = \gamma(t)\binom{t}{1} \neq 0$, where γ is an arbitrary smooth scalar function with $\gamma(0) = 0$.

Example 2 (cf. [GM86], p.91). Consider equation (1) with

$$A(t) = \begin{pmatrix} 0 & 0 \\ 1 & -t \end{pmatrix} \quad \text{and} \quad B(t) = \begin{pmatrix} 1 & -t \\ 0 & 0 \end{pmatrix}.$$

The matrix pencil $(A(t), B(t))$ is singular (which means by definition that it is not regular) since

$$\det(\lambda A(t) + B(t)) = \det \begin{pmatrix} 1 & -t \\ \lambda & -\lambda t \end{pmatrix} = 0 \ .$$

However, for each $f \in C^1(I)$ and for each consistent initial condition $x(t_0) = x^0$ the DAE (1) has exactly one solution x:

$$x(t) = \begin{pmatrix} f_1(t) + t f_2(t) - t f_1'(t) \\ f_2(t) - f_1'(t) \end{pmatrix}.$$

Remark. The smooth transformations $E(t)$, $F(t)$ used in Definition 1 with $F \neq$ const. can change the local index of a pencil:

Example 3.

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Obviously, the matrix pencil (A, B) is regular with index 3. Let us transform it by means of

$$E(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F(t) = \begin{pmatrix} t & e^{-t} & 0 \\ 0 & 0 & t \\ 0 & 1/t & 0 \end{pmatrix}, \quad t \in (0, T] \ .$$

Then we obtain

$$\tilde{A}(t) = \begin{pmatrix} 0 & 0 & 0 \\ t & e^{-t} & 0 \\ 0 & 0 & t \end{pmatrix} \quad \tilde{B}(t) = \begin{pmatrix} t & e^{-t} & 0 \\ 1 & -e^{-t} & t \\ 0 & 1/t & 1 \end{pmatrix},$$

and we calculate $\operatorname{ind}(\tilde{A}(t), \tilde{B}(t)) = 2$.

If $F(t) \neq$ const., then, by smooth transformations, the regularity of a pencil $(A(t), B(t))$ can get lost, or a singular local pencil can become regular with a certain index:

Example 4.

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with } t \in (0, T] \ .$$

The corresponding matrix pencil is singular for all t, but applying the transformation of Definition 1 with

$$E(t) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F(t) := \begin{pmatrix} t & 1/t \\ 0 & t \end{pmatrix}$$

we obtain the pencil

$$(\tilde{A}(t), \tilde{B}(t)) = \left(\begin{pmatrix} 0 & 0 \\ t & 1/t \end{pmatrix}, \begin{pmatrix} 2t & 2/t \\ 1 & -1/t^2 \end{pmatrix} \right) \ ,$$

which is regular with index 2.

Another canonical form was introduced by Campbell [Cam83]. We want to outline his main ideas and statements in the next section.

3 The Standard Canonical Form of a DAE

Campbell considered linear DAEs (1) and assumed the matrix functions $A, B : I \to L(\mathbb{R}^m)$ as well as the vector function $f : I \to \mathbb{R}^m$ to be sufficiently smooth. The solution of (1) should be smooth (i. e. continuously differentiable) and satisfy (1).

Definition 4. * The DAE (1) is called *solvable in* $t_0 \in I$ if there is a $\lambda \ (= \lambda(t_0)) \in \mathbb{R}$ such that the matrix $\lambda A(t_0) + B(t_0)$ is nonsingular. (This means that the local matrix pencil is supposed to be regular.)
* The number $\operatorname{ind}(A(t_0), B(t_0)) := k(t_0) := \operatorname{index}((\lambda A(t_0) + B(t_0))^{-1} A(t_0))$ is called *the local index* of this matrix pencil.

In [Cam83], Definition 3.1, Campbell's normal form as well as the "safe transferability" into this form are defined:

Definition 5. * The *Standard Canonical Form of size r (SCF-r)* of DAE (1) is

$$\begin{pmatrix} G_{11} & & & & \\ G_{21} & 0 & & 0 & \\ G_{31} & G_{32} & 0 & & \\ \vdots & & \ddots & \ddots & \\ G_{r1} & \dots & \dots & G_{r,r-1} & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ \vdots \\ x_r' \end{pmatrix} =$$

$$\begin{pmatrix} H_{11} & 0 & \\ \vdots & \ddots & \\ H_{r1} & \dots & H_{rr} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} + \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} . \tag{2}$$

* The DAE (1) is called to be *safely transformable* into SCF if a scaling with $E(t)$ and a change of variables $x =: Fy$ or $x =: \Phi y$ (where $E(t)$ and F are nonsingular matrices and Φ is a scalar function not vanishing on I) yield a DAE of the structure (2).

Campbell also proved a necessary and sufficient criterion for the save transformability into SCF:

Theorem 6. *Let the local index of (1) satisfy the relation $k(t) \leq k\ \forall t \in I$. Then (1) is safely transformable into SCF-k if and only if* $\ker G^k = \text{const.}$, *and there is a chain $\{N_i\}_{i=1,\ldots,k}$ of constant subspaces N_i with $N_i \supset N_{i+1}$, $G_\lambda N_i \subseteq N_{i+1}$ $\forall i = 0,\ldots,k-1$ where $N_0 = \ker G_\lambda^k$ and $N_k = 0$.*

In case of safe transformability the following assertions concerning initial values hold:

Theorem 7 (cf. [Cam83], Th. 4.1). *Let (1) be solvable and safely transformable into SCF.*

* *For each $f \in C^r(I)$, there exist solutions of (1), which are uniquely determined by consistent initial values.*
* *Let M be an arbitrary complement of* $\ker G_\lambda^k$ *in $\mathbb{R}^m$ and $x_0 = x_{01} + x_{02}$ with $x_{01} \in M$, $x_{02} \in \ker G_\lambda^k$. Then, for each x_{01}, there is exactly one x_{02} such that x_0 is a consistent initial value, i. e. there is a solution x with $x(t_0) = x_0$.*

Remark. 1. The Examples 1 – 4 show that regularity as well as the local index of the local pencil $(A(t_0), B(t_0))$ are not characteristics of a differential-algebraic equation.
2. If (1) is a differential-algebraic equation in SCF-r, then its global or local index is not necessarily equal to r.

4 Modified Standard Canonical Form and Hessenberg Form

Another concept of considering DAEs in normal forms is that of Clark, who investigated a generalization of the SCF as well as equations in Hessenberg form in his paper [Cla88].

Definition 8. The DAE is called *solvable* if, for each sufficiently smooth function f, there is at least one solution x of (1), all solutions are continuously differentiable on I, and a solution is uniquely determined by its value $x(t_0)$.

The investigation is confined to regular DAEs (1):

Definition 9. * An equation (1) is called *regular* in I if the pencil (A, B) is regular, i. e., for all $t \in I$ there is a $\lambda_t \in \mathbb{R}$ such that $\lambda_t A(t) + B(t)$ is invertible.

* The DAE is called a *strictly regular* one if λ does not depend on t.
* Again, the *local index* of (1) is defined to be index $A_\lambda(t)$ where $A_\lambda(t) := (\lambda A(t) + B(t))^{-1} A(t)$ while $\max_{t\in I}(\operatorname{ind} A_\lambda(t))$ is defined to be the *global index* of (1).

Remark. 1. The definition of solvability does not comply with the common use of this term, which means that there exist solutions.
2. The denotations "solvability" in the sense of Campbell [Cam83] and of Clark [Cla88] do not correspond (cf. Examples 1 and 2).
3. The definition of the global index in [Cla88] does not seem to be very useful since the global index of a DAE in the sense of Gear/Petzold [GP84] can be smaller than all local indices of this DAE, even if the local pencil remains regular:

Example 5 (cf. [GM86], p. 103).

$$A(t) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & t \\ 0 & 1 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; t \in [-1,1] =: I \; .$$

The corresponding matrix pencil is strictly regular because

$$\det(\lambda A(t) + B(t)) = \det \begin{pmatrix} 1 & -\lambda t & 0 \\ \lambda & 1 & \lambda t \\ 0 & \lambda & 1 \end{pmatrix} = 1 \quad \forall t \in I \;,$$

and we compute the index of

$$A_\lambda(t) = \begin{pmatrix} \lambda t & -t & \lambda t^2 \\ 1 & 0 & t \\ -\lambda & 1 & -\lambda t \end{pmatrix} \quad \text{to be equal to 3 for all } t \in I \; .$$

Consequently, the pencil is (strictly) regular with index 3. However, on the other hand, one can show that the global index in the sense of Gear and Petzold [GP84] equals 2.

A generalization of Campbell's SCF-r is the *Modified Standard Canonical Form (MSCF)* defined in [Cam83] as follows:

Definition 10. The DAE (1) is in *MSCF-r* if it is given in the form

$$E(t)x_1'(t) + F(t)x_1(t) = f_1(t)$$

$$\begin{pmatrix} 0 & & & \\ N_{21} & 0 & & \\ N_{31} & N_{32} & 0 & \mathbf{0} \\ \vdots & & \ddots & \ddots \\ N_{r-1,1} & \dots & \dots N_{r-1,r-2} & 0 \end{pmatrix} x_2'(t) + x_2(t) = f_2(t) + G(t)x_1'(t) + H(t)x_1(t) \;,$$

where the first equation is a regular differential equation or a solvable index-1 system.

A modified version of Theorem 1 is given in [Cla88], Th. 2.1:

Theorem 11. *Let $A_\lambda := (\lambda A + B)^{-1}A$ have the index $k(t) \le r$ for all $t \in I$, $N_0 := \ker A_\lambda^r$. Then the DAE (1) is safely transformable into MSCF-r if and only if*

(i) core-rank$A_\lambda := \operatorname{rank}(A_\lambda A_\lambda^D) = \text{const.}$ on I and
(ii) there is a chain $\{N_i\}_{i=1,\dots,r}$ of constant subspaces N_i where $N_r = 0$, $N_i \supset N_{i+1}$ and $A_\lambda N_i \subseteq N_{i+1}$ $\forall i = 0,\dots,r-1$.

Besides the MSCF, Clark investigates another structural form for linear semi-explicit differential-algebraic equations

$$\begin{aligned} x_1'(t) &= \bar{B}_{11}(t)x_1(t) + \bar{B}_{12}(t)x_2(t) + f_1(t) \\ 0 &= \bar{B}_{21}(t)x_1(t) + \bar{B}_{22}(t)x_2(t) + f_2(t) \end{aligned} \quad . \tag{3}$$

Supposed $\operatorname{rank}\bar{B}_{22}(t) = \text{const.}$, one can consider such an equation with $\bar{B}_{22} = 0$ without loss of generality, cf. [Cla88]. Note that often DAEs are given in such a special form as will be considered in the following.

Definition 12. Equation (3) (resp. the matrix triple $(\bar{B}_{11}, \bar{B}_{12}, \bar{B}_{21})$) is called to be in *Hessenberg form of size r (H_r)* if the three coefficient matrices have the form

$$\bar{B}_{11}(t) = \begin{pmatrix} B_{11}(t) & \dots & \dots & B_{1,r-1}(t) \\ B_{21}(t) & B_{22}(t) & & \vdots \\ 0 & B_{32}(t) & \ddots & \\ \vdots & \ddots & & \\ 0 & \dots\dots & B_{r-1,r-2}(t) & B_{r-1,r-1}(t) \end{pmatrix}$$

$\bar{B}_{12}(t) = (B_{1r}(t), 0, \dots, 0)^T$, $\bar{B}_{21}(t) = (0, \dots 0, B_{r,r-1})$, and the product: $\pi_r(t) = B_{r-1,r-2}(t)\dots B_{21}(t)B_{1r}(t)$ is nonsingular on I.

Theorem 13. *Let the matrix functions $B_{i,j}$ be r times continuously differentiable on I $\forall i,j = 1,\dots,r-1$. Then the DAE (2) with H_r structure is solvable with the local and global index r if and only if π_r is invertible for all $t \in I$.*

Theorem 14 ([Cla88], Th. 3.3). *Let $(\bar{B}_{11}, \bar{B}_{12}, \bar{B}_{21})$ be a Hessenberg triple. Then the matrix pencil (A, B) of the DAE (3) is strictly regular on I.*

Remark. Not all linear differential-algebraic equations having the global index 2 in the sense of [GP84] are in H_2 form or safely transformable into this special form.

5 Decoupling the DAE

Due to [Mar87] and [Han90] we introduce a method which can transform the DAE (1) into a special form consisting of an explicit ODE and a number of equations without deriatives. Let the matrix $A(t) \in L(\mathbb{R}^m)$ be singular but have constant rank on I.

Definition 15. Such equations are called *normal* DAEs.

Furthermore we suppose that the matrix function A has a *smooth nullspace*, i. e. that there exists a function $Q \in C^1(I, L(\mathbb{R}^m))$ where $Q(t)$ represents a projector from $\mathbb{R}^m$ onto $\ker A(t) =: N(t)$ for all $t \in I$. Denote the complementary projector by $P(t)$. Then $AP = A$ holds and we can understand the given DAE as

$$A(t)(Px)'(t) + (B - AP')(t)x(t) = f(t) \tag{4}$$

Now it seems reasonable to require solutions of this reformulated DAE to belong to the function space

$$\mathcal{C} := \{x \in C(I, \mathbb{R}^m) : Px \in C^1(I, \mathbb{R}^m)\} \ .$$

We define

$$\mathcal{A}x := A(Px)' + (B - AP')x \quad \text{for } x \in \mathcal{C} \ ,$$

hence $\mathcal{A}$ maps from $\mathcal{C}$ into $C(I, \mathbb{R}^m)$.

We are interested in the problem whether the DAE (4) has a finite dimensional solution space or, equivalently, whether an initial value problem arising from (4) by adding a certain number of appropriate initial conditions has a unique solution. Obviously, this is a matter of fact if and only if $\dim(\ker \mathcal{A}) < \infty$ holds.

Definition 16. The differential-algebraic equation (4) is called *tractable* if the map $\mathcal{A}$ has a finite dimensional nullspace.

Clearly, differential-algebraic equations having a global index are tractable. März [Mar87] introduced another index definition, the *index of tractability*. Originating from the coefficients of the DAE (4) we construct a matrix chain in the following way:

$$\begin{aligned}
&A_0(t) := A(t)\\
&B_0(t) := B(t) - A(t)P'(t)\\
&A_i(t) := A_{i-1}(t) + B_{i-1}(t)Q_{i-1}(t)\\
&B_i(t) := B_{i-1}(t)P_{i-1}(t) - A_i(t)(PP_1 \dots P_i)'(t)(PP_1 \dots P_{i-1})(t) \ (i = 1, 2, \dots) \ .
\end{aligned}$$

There $Q_i(t)$ denotes a projector from $\mathbb{R}^m$ onto $\ker A_i(t)$, and $P_i(t) := I - Q_i(t)$. Furthermore, let the projectors be chosen in such a way that

$$Q_j(t)Q_i(t) = 0$$

holds for all $t \in I$ and for $j > i$.

Definition 17. The DAE (4) is called *tractable with index k* if the matrix $A_k(t)$ is nonsingular on I whereas all previous matrices $A_0(t), \ldots, A_{k-1}(t)$ are singular for all $t \in [t_0, T]$.

Theorem 18. *Tractability with index k remains invariant under scalings of (4) by a matrix function* $E \in C(I, L(I\!R^m))$ *and transformations* $x =: Fy$ *with* $F \in C^1(I, L(I\!R^m))$, *where* $E(t)$ *and* $F(t)$ *are nonsingular on* I.

This assertion can be proved inductively, cf. [Han90]. Using a result of [GM89] one can show that an equation in KCF with the global index k is tractable with this index:

Theorem 19. *Let the matrices* $A(t) = \begin{pmatrix} I_s & 0 \\ 0 & J \end{pmatrix}$ *and* $B(t) = \begin{pmatrix} W(t) & 0 \\ 0 & I_{m-s} \end{pmatrix}$ *be given and let* J *be a constant, nilpotent matrix of the order* k. *Let the matrices* $A_0, A_1, \ldots$ *be calculated according to the matrix chain above. Then all matrices* $A_0, \ldots, A_{k-1}$ *are singular while* A_k *is nonsingular on* I.

Then the following result can be obtained immediately:

Theorem 20. *Let the DAE (4) have the global index k. Then this equation is also tractable with this index.*

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