Oscillations in Differential-Algebraic Equations

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1 Introduction

The computation of oscillations is an important problem in fields of electrical networks, biological models and motions in mechanical systems. Very often the description of such problems is possible by differential-algebraic equations (dae's) of the structure

$$f(x'(t), x(t), t) = 0, \qquad t \in [0, T] \tag{1.1}$$

with periodic boundary condition.

The calculation of nonautonomous oscillation does not contain additional difficulties with respect to the general boundary value case. The period T is known and if the equation of first variation of (1.1) has the trivial solution only, the transferable nonautonomous periodic boundary problem is well-posed. However, our experience of the ODE-case for autonomous oscillation make us consider this problem carefully.

In the autonomous oscillation case we discover the same problems we are facing in the ODE-case.

- The period T of the oscillation is unknown.
- With $x_\star$, also $x_\star(\cdot + \eta)$ solves (1.1).
- The equation of first variation of (1.1) has nontrivial solutions.

In this paper we describe how to overcome these difficulties . More accurately, we consider the general dae

$$f((Px)'(t) - P'(t)x(t), x(t), t) = 0, \qquad t \in [0, T] \tag{1.2}$$

with a periodic boundary condition

$$P(0)x(0) - P(T)x(T) = 0 \ . \tag{1.3}$$

(For the definition of the projector P see Sect. 2, Assumption (A1)). We define the operator

$$Fx := \begin{pmatrix} f((Px)' - P'x, x, \cdot) \\ P(0)x(0) - P(T)x(T) \end{pmatrix}. \tag{1.4}$$

Then (1.2), (1.3) can be written in operator form

$$Fx = 0 \ , \tag{1.5}$$

where $F : \mathcal{C} \to Y$ and $\mathcal{C} := \{x | x \in C([0,T], R^n), Px \in C^1([0,T], R^n)\}$, $Y := C([0,T], R^n) \times R^n$. The solution of (1.5) in the nonautonomous case represents a well-posed problem (see Sect. 3) if the equation of first variation of (1.5)

$$F'(x_\star)z = 0 \tag{1.6}$$

has the trivial solution only. Based on the experience of the case of ordinary differential equations [HNL81], [Shi77], we overcome the difficulties in the autonomous case by extending the operator F (Sect. 4). Special considerations are necessary for periodic solutions in higher index dae's. This is interesting in mechanical motions, which have index three in general. Here, regularization or index reduction methods form a different structure of the periodic boundary condition (1.3). In Sect. 5 we investigate the index 3 case via index reduction methods.

Some examples illustrate the problems in the nonautonomous, autonomous and higher index case. The numerical solutions were calculated by a multiple shooting method [Lam] (Sect. 6).

2 Well-Posedness of General Boundary Value Problems

The consideration of autonomous oscillation requires the investigation of a general boundary value problem (bvp). First we investigate the linear problem

$$A(t)x'(t) + B(t)x(t) = q(t), \qquad t \in [a,b] \tag{2.1}$$

$$l(x) = b \tag{2.2}$$

($A, B \in C([a,b], L(R^n)), l : C([a,b], R^n) \to R^n$ linear, bounded).

(A1) *Denote by* $N(t) := \ker(A(t))$ *and* $Q(t)$ *a projection onto* $N(t)$*, let* Q *be smooth and* $P(t) := I - Q(t)$. We assume that (2.1) is transferable, i. e. *the matrix*

(A2)

$$G(t) := A(t) + B(t)Q(t)$$

is nonsingular for all $t \in [a,b]$.

The general solution of (2.1) is given by

$$x(t) = X(t,a)c + x_q(t) \ , \tag{2.3}$$

where $X(\cdot, a)$ denotes the fundamental matrix of the homogeneous dae (2.1) and x_q denotes the solution of the inhomogeneous dae (2.1) satisfying the initial condition

$$P(a)x_q(a) = 0 \tag{2.4}$$

(see [GM86]).

We introduce the 'general' shooting matrix

$$S := l(X(\cdot, a)) \ . \tag{2.5}$$

By $l(X(\cdot, a))$ we denote the matrix whose column vectors are $l(X_i(\cdot, a))$ ($i = 1, \ldots, n$), where $X_i(t, a)$ are column vectors of the matrix $X(t, a)$.

Using the representation

$$X(t, s) = P_s(t)Y(t, s)P(s) \tag{2.6}$$

($P_s := I - QG^{-1}B$ - the canonical projector, $Y(t, s)$ solves the system $Y' = (P'P_s - PG^{-1}B)Y, \quad Y(s, s) = I$, see [GM86])
and the linearity of l

$$l(X(\cdot, a)) = l(X(\cdot, a)P(a)) = l(X(\cdot, a))P(a) \tag{2.7}$$

holds. In general,

$$\mathrm{im}(S) \subseteq M := \mathrm{im}(l), \quad N(a) \subseteq \ker(S) \tag{2.8}$$

is valid.

Theorem 1. *Let A1 and A2 hold.*
1. The homogeneous bvp (2.1), (2.2) has only the trivial solution iff

$$\ker(S) = N(a) \ .$$

2. The bvp (2.1), (2.2) is uniquely solvable on $\mathcal{C}$ for arbitrary right-hand sides $q \in C([a, b], R^n)$, $b \in \mathrm{im}(l)$ iff

$$\ker(S) = N(a) \tag{2.9a}$$
$$\mathrm{im}(S) = \mathrm{im}(l) \ . \tag{2.9b}$$

3. The bvp (2.1), (2.2) is well-posed iff the shooting matrix S satisfies condition (2.9).

Remark. Theorem 1 extends Theorem 1.2.25 in [GM86] for general bvp's.

Proof. 1.($\rightarrow$): The solution of (2.1), (2.2) with $q \equiv 0$, $b = 0$ is given by

$$x(\cdot) = X(\cdot, a)c, \quad l(x) = 0 \ .$$

It holds that

$$0 = l(x) = l(X(\cdot, a)c) = l(X(\cdot, a))c = Sc \ .$$

If $c \notin N(a)$, then x will be a nontrivial solution, hence $\ker(S) \subseteq N(a)$.
Taking into account (2.8) we derive $\ker(S) = N(a)$.

($\leftarrow$): From $Sc = 0$ ($c \subseteq N(a)$) it follows immediately that $x \equiv 0$.

2.($\rightarrow$): Let (2.1), (2.2) be uniquely solvable for arbitrary $q \in C([a,b], R^n)$, $b \in M$. The solution is of the form (2.3). According to 1. we immediately obtain $\ker(S) = N(a)$. (2.2) provides

$$Sc = b - l(x_q) =: d \tag{2.10}$$

with arbitrary $d \in M$. This means that $M \subset \mathrm{im}(S)$. Using (2.8), $M = \mathrm{im}(S)$ follows.

($\leftarrow$): (2.9) holds. Let $q \in C([a,b], R^n)$, $b \in M$. First we solve (2.1) with

$$P(a)(x(a) - x_a) = 0 \ .$$

Using (2.3) the solution is $x = X(\cdot, a)x_a + x_q$ and (2.10) provides

$$Sx_a = d \ . \tag{2.11}$$

Taking into account (2.9b) the general solution of (2.11) is

$$x_a = S^\dagger d + (I - S^\dagger S)c$$

for arbitrary c ($S^\dagger$ denotes the Moore-Penrose inverse). Using (2.6) and (2.9a) we have the only solution

$$x = X(\cdot, a)S^\dagger d + x_q \ ,$$

which fulfils the boundary condition.

3. Let $x \in \mathcal{C}$ be the solution of the boundary value problem for $q \in C([a,b], R^n)$, $b \in M$. Simultaneously, x satisfies the initial condition

$$P(a)(x(a) - S^\dagger d) = P(a)S^\dagger S(x(a) - S^\dagger d) = 0 \ .$$

Using Theorem 1.2.20 in [GM86] (well-posedness of initial value problems) we have

$$\begin{aligned} ||x|| &\leq K_0(||q||_\infty + ||P(a)S^\dagger d||) \\ ||x_q|| &\leq K_0||q||_\infty \end{aligned}$$

From (2.10) we derive the estimation

$$||P(a)S^\dagger d|| \leq ||P(a)S^\dagger|| \, ||b|| + ||P(a)S^\dagger l||_{C \rightarrow R^n} ||x_q||_\infty \ .$$

Altogether we have now

$$||x|| \leq K_1(||q||_\infty + ||b||)$$

with

$$K_1 = K_0 \max(1 + K_0||P(a)S^\dagger l||_{C \rightarrow R^n}, ||P(a)S^\dagger||).$$

Taking into consideration 2. the assertion is proved.

□

Remark. The special case of the two-point-boundary condition

$$l(x) := D_1 x(a) + D_2 x(b)$$

creates the same shooting matrix

$$S = l(X(\cdot, a)) = D_1 X(a, a) + D_2 X(b, a)$$

as considered in [GM86].

Now we are interested in the general nonlinear case

$$\begin{aligned} f(x'(t), x(t), t) = 0, \quad t \in [a, b] \\ g(x) = 0 \end{aligned} \tag{2.12}$$

(A1') *Let f be sufficiently smooth. The kernel of f'_y depends on t only* $(\ker(f'_y) = N(t))$*, and $Q(t)$ denotes a projection onto $N(t)$. Let Q be smooth and $P(t) := I - Q(t)$.*

(A2') *For transferability we assume that the matrix*

$$G(y, x, t) := f'_y(y, x, t) + f'_x(y, x, t)Q(t)$$

is nonsingular.

(A3') *Let g be sufficiently smooth and* $\mathrm{im}(g'_x) =: M$.

Theorem 2. *Let (A1') - (A3') be valid, and $x_\star$ denotes a solution of (2.12). The bvp (2.12) is well-posed iff the shooting matrix*

$$S_\star := g'_x(X(\cdot, a)) \tag{2.13}$$

satisfies the conditions

$$\ker(S_\star) = N(a), \ \mathrm{im}(S_\star) = M \ , \tag{2.14}$$

where $X(\cdot, a)$ denotes the fundamental solution of (2.12).

Using Theorem 1 we prove the well-posedness of the nonlinear system (2.12) under the conditions (A1') - (A3') via linearization techniques, as it is done in [GM86].

3 Nonautonomous Oscillations

(P) *Let f be T-periodic,* i. e.

$$f(\cdot, \cdot, t) = f(\cdot, \cdot, t + T) \ .$$

This assumption implies that $N(t) \equiv N(t + T)$ and therefore

$$P(t) = P(t + T) \ .$$

The periodic boundary condition (1.3) has now the form

$$P(0)(x(0) - x(T)) = 0 \ . \tag{3.1}$$

Corollary 3. *Let the conditions (A1') - (A3'), (P) be valid, and let $x_\star$ denote a solution of (2.12). The equation of first variation (1.6) has only the trivial solution iff the matrix $(I - Y(T,0)P(0))$ is nonsingular.*

Proof. (1.6) has the representation

$$Az' + Bz = 0 \tag{3.2a}$$
$$P(0)(z(0) - z(T)) = 0 \tag{3.2b}$$

with $A := f'_y(x'_\star, x_\star, \cdot)$, $B := f'_x(x'_\star, x_\star, \cdot)$. The solution z of(3.2a) is given by

$$z(t) = X(t,0)z(0) = P_s(t)Y(t,0)P(0)z(0) \ .$$

This means that $P(t)z(t) = P(t)Y(t,0)P(0)z(0)$ and $Q(t)z(t) = Q(t)P_s(t)Y(t,0)P(0)z(0)$, consequently

$$z(t) \equiv 0 \Longleftrightarrow P(0)z(0) = 0 \ .$$

Consider $P(T)z(T) = P(0)z(0) \overset{(3.2b)}{=} P(0)Y(T,0)P(0)z(0)$ (recall that $PP_s = P$) and with $Q(T)Y(T,0)P(0) = 0$ (see [LM90]) we obtain

$$(I - Y(T,0)P(0))P(0)z(0) = 0 \ .$$

Now we have to investigate three cases:

a.) $I - Y(T,0)P(0)$ is nonsingular $\Longrightarrow P(0)z(0) = 0$.

b.)
$$I - Y(T,0)P(0)) = RQ(0) \tag{3.3}$$
and R represents an arbitrary matrix. Multiplying (3.3) by $Q(0)$ from the right we obtain $Q(0) = RQ(0)$ and, therefore, $I - Y(T,0)P(0) = Q(0)$. This implies $Y(T,0) = I$.

c.) $I - Y(T,0)P(0)) \equiv 0$, this means that $Y(T,0) = I$ and $P(0) = I$.

$Y(T,0) = I$ (case b. and c.) contradicts the assumption that (1.6) has the trivial solution only, because every function

$$z(t) = P_s(t)Y(t,0)P(0)c$$

solves (3.2) with arbitrary c.

$\square$

Theorem 4. *Let (A1') - (A3'), (P) be fulfilled and let (1.6) have the trivial solution only. Then (2.14) is valid.*

Proof. The shooting matrix has the special form

$$S_\star = P(0)(X(0,0) - X(T,0))$$

and the set M is given by $\mathrm{im}(g'_x) = \mathrm{im}(P(0)) =: M$. We use the structure of X and the periodicity of P_s and P.

$$\begin{aligned} S_\star &= P(0)(P_s(0)Y(0,0)P(0) - P_s(T)Y(T,0)P(0)) \\ &= (P(0) - P(0)Y(T,0)P(0)) \qquad (3.4) \\ &= P(0)(I - Y(T,0)P(0)) \ . \qquad (3.5) \end{aligned}$$

With $Q(T)Y(T,0)P(0) = 0$ and (3.4) we have

$$S_\star = (I - Y(T,0)P(0))P(0) \ . \qquad (3.6)$$

Using the Corollary we conclude with (3.5) and (3.6)

$$\begin{aligned} \mathrm{im}(S_\star) &= \mathrm{im}(P(0)) = M \\ \ker(S_\star) &= \ker(P(0)) = N(0) \ . \end{aligned}$$

□

Remark. Theorem 4 shows that the periodic boundary condition (3.1) for nonautonomous equations forms a well-posed boundary value problem.

4 Autonomous Oscillations

In autonomous equations (1.2) f does not depend on t explicitly. For $\ker(f'_y)$ this means that $N(t) \equiv N$, i. e. the projectors P and Q are constant. Let $x_\star$ be a T-periodic solution of

$$\begin{aligned} f((Px)'(t), x(t)) &= 0, \quad t \in [a,b] \\ P(x(0) - x(T)) &= 0, \end{aligned} \qquad (4.1)$$

then also $x_\star(. + \eta)$ represents a solution of (4.1) if we consider the periodic continuation of $x_\star$. By differentiation of (4.1) (if we consider the equivalent periodic boundary condition $P(x(t) - x(t+T)) = 0$) we obtain

$$\begin{aligned} Ax''_\star + Bx'_\star &= 0 \\ P(x'_\star(0) - x'_\star(T)) &= 0 \ , \end{aligned} \qquad (4.2)$$

where A and B are defined in (3.2). This implies that $F'(x_\star)z = 0$ has a nontrivial solution $z = x'_\star$. For the investigation of (4.1) we transform it into the interval $[0,1]$:

$$f(Px'(t)/T, x(t)) = 0, \quad t \in [0,1] \qquad (4.3a)$$

$$P(x(0) - x(1)) = 0. \qquad (4.3b)$$

To fix the phase of x we add a further differential equation

$$T' = 0 \tag{4.4}$$

and a boundary condition

$$l(x) = \beta \ , \tag{4.5}$$

where l is a Fréchet-differentiable functional and β a suitable number. Now we investigate the system (4.3) – (4.5). The first question concerns the transferability of (4.3a), (4.4).

Corollary 5. *(4.3a), (4.4) are transferable around $(x_\star, T_\star)$ if (4.3a) is transferable around $x_\star$.*

Proof. See [Lam91]. □

The second question askes for the validity of condition (2.14) for (4.3) – (4.5).

Theorem 6. *Let the assumptions (A1') - (A3') be fulfilled and let $x_\star$ denote a solution of (4.3). The equation $F'(x_\star)z = 0$ has only the nontrivial solution $z = x'_\star$. If $Px'_\star(0) \notin \mathrm{im}(S_\star)$ and $l'_x(x_\star)[x'_\star] \neq 0$, then the well-posedness condition (2.14) for the system (4.3) – (4.5) is fulfilled.*

Proof. We investigate the shooting matrix

$$\overline{S}_\star = \overline{g}'_{\overline{x}}(\overline{X}(\cdot, 0)) \tag{4.6}$$

where

$$\overline{g}(\overline{x}) := \begin{pmatrix} P(x(0) - x(1)) \\ l(x) \end{pmatrix} \tag{4.7}$$

$$\overline{x}(\cdot) := \begin{pmatrix} x(.) \\ T \end{pmatrix} ,$$

and $\overline{X}$ denotes the fundamental matrix of

$$\overline{F}\overline{x} := \begin{pmatrix} Fx \\ T' \end{pmatrix} = 0 \tag{4.8}$$

With the abbreviations $A := f'_y / T_\star, \quad B := f'_x$ we obtain

$$\overline{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} , \quad \overline{B} = \begin{pmatrix} B & -Ax'_\star / T_\star \\ 0 & 0 \end{pmatrix} .$$

With $\overline{X} := \begin{pmatrix} \psi & p \\ q^t & s \end{pmatrix}$ we have to solve

$$\begin{aligned} \overline{A}\overline{X}' + \overline{B}\overline{X} &= 0 \\ \overline{P}(\overline{X}(0,0) - I) &= 0 \ , \end{aligned} \tag{4.9}$$

where $\overline{P} := \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$. In detail, (4.9) reads

$$A\psi' + B\psi - Ax'_\star q^t/T_\star = 0 \tag{4.10a}$$
$$Ap' + Bp - Ax'_\star s/T_\star = 0 \tag{4.10b}$$
$$(q^t)' = 0 \tag{4.10c}$$
$$s' = 0 \tag{4.10d}$$

with initial conditions

$$\begin{aligned} P(\psi(0) - I) &= 0 \\ Pp(0) &= 0 \\ q^t(0) &= 0 \\ s(0) &= 0\ . \end{aligned}$$

We obtain $q^t \equiv 0$, $s \equiv 1$ and $\psi \equiv X$ immediately . With the solution representation of (4.10b) (see [LM90])

$$\begin{aligned} p(t) &= \int_0^t X(t,s)P(s)G^{-1}(s)Ax'_\star/T_\star ds \\ &= \int_0^t X(t,s)x'_\star(s)ds/T_\star \\ (x'_\star &\quad \text{is a solution of (1.6), i. e.} x'_\star(t) = X(t,s)x'_\star(s)) \\ p(t) &= tx'_\star(t)/T_\star, \end{aligned}$$

(4.6) has the structure

$$\overline{S}_\star = \begin{pmatrix} P(X(0,0) - X(1,0)) & Px'_\star(1)/T_\star \\ l'_x(x_\star)[X(\cdot,0)] & l'_x(x_\star)[x'_\star/T_\star] \end{pmatrix} \tag{4.11}$$

For the validity of (2.14) we have to show that
a.) $\ker(\overline{S}_\star) = \overline{N} = \ker(\overline{P})$ and
b.) $\operatorname{im}(\overline{S}_\star) = \operatorname{im}(\overline{g}'_\star) =: \overline{M}$.

a.) Consider

$$\overline{S}_\star \begin{pmatrix} z \\ \alpha \end{pmatrix} = 0 \tag{4.12}$$

or the first equation of (4.12) in detail

$$P(X(0,0) - X(1,0))z + \alpha P x'_\star(1)/T_\star = 0$$

With $Px'_\star(1) = Px'_\star(0)$ and $Px'_\star(0) \notin \operatorname{im}(S_\star)$ it follows that $\alpha = 0$. Now we choose $z = Q\tilde{z} + \tau P x'_\star(0) \in \ker(S_\star) = N \oplus \{Px'_\star(0)\}$ and consider the second equation of (4.12).

$$\begin{aligned} 0 &= l'_x(x_\star)[X(\cdot,0)(Q\tilde{z} + \tau P x'_\star(0))] \\ &= \tau l'_x(x_\star)[X(\cdot,0)x'_\star(0)] \\ &= \tau l'_x(x_\star)[x'_\star]\ . \end{aligned}$$

Using $l'_x(x_\star)x'_\star \neq 0$ we have $\tau = 0$ and $\ker(\overline{S}_\star) = \overline{N}$.

b.) $\mathrm{im}(\overline{g}'_{\overline{x}}) = \left\{ \begin{pmatrix} x \\ v \end{pmatrix} : x \in \mathrm{im}(P), v \in R \right\}$

We consider the equation $\begin{pmatrix} x \\ v \end{pmatrix} = \overline{S}_\star \begin{pmatrix} z \\ \alpha \end{pmatrix}$ or, in detail,

$$\begin{aligned} x &= P(X(0,0) - X(1,0))z + \alpha P x'_\star(0)/T_\star \\ v &= l'_x(x_\star)[X(\cdot,0)z + \alpha x'_\star(\cdot)/T_\star] \ . \end{aligned}$$

Using the representation $z = \beta P x'_\star(0)/T_\star + y$ $(\beta \in R\ ,\ \langle y, P x'_\star(0)\rangle_{R^n} = 0)$ we have

$$\begin{aligned} x &= P(X(0,0) - X(1,0))y + \alpha P x'_\star(0)/T_\star \\ v &= l'_x(x_\star)[X(\cdot,0)y + (\alpha+\beta) P x'_\star(0)/T_\star] \end{aligned}$$

and the possibility to represent each value $x \in \mathrm{im}(P)$ and $v \in R$ by variation of y, α and/or β. This implies

$$\mathrm{im}(\overline{S}_\star) = \overline{M} \ .$$

$\square$

Remark. 1. The assumption $Px'_\star(0) \notin \mathrm{im}(S_\star)$ is of the same structure as in the case. Here we require that $x'_\star(0)$ represents a simple eigenvector to the eigenvalue 0 of $S_\star$ (see [HNL81]).The condition $l'_x(x_\star)x'_\star \neq 0$ corresponds to the ODE-case.
2. The selection of the functional l is closely connected with the method of solution of (4.3) – (4.5). The functional $l(x) := x_{i_0}(0) - d$ with suitable i_0 and d is the so-called "phase normalization" or, if Fourier approximation is used, the choice

$$l(x) = \begin{cases} a_{i_0 j_0} - d \text{ or} \\ b_{i_0 j_0} - d \end{cases}$$

is successful for suitable i_0, j_0 and d (see [HNL81], [Shi77, Zhe88]). Further functionals we find in [Wil81].

5 Higher index problems

In this chapter we restrict ourselves to a class of dae's which are closely connected with mechanical systems . We consider the system

$$x' = f_1(x, v, t), \tag{5.1a}$$
$$v' = f_2(x, v, \lambda, t), \quad t \in [0,1] \tag{5.1b}$$
$$0 = f_3(x, t) \tag{5.1c}$$

$(x(t), v(t) \in R^{n_1}, \lambda(t) \in R^{n_2})$ and for (5.1) having index 3 we assume that $\frac{\partial f_3}{\partial x}\frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial \lambda}$ is nonsingular. We are interested in periodical solutions of (5.1).

The numerical solution of higher index problems requires the transformation of (5.1) to an index 1 problem. Every differentiation of the restriction (5.1c) reduces the index of (5.1) by one (see [GP84]).

$$\frac{d}{dt} f_3(x(t), t) = 0 \tag{5.2a}$$

$$\frac{d^2}{dt^2} f_3(x(t), t) = 0 \ . \tag{5.2b}$$

This means that (5.1a), (5.1b), (5.2b) form an index 1 system. However, the equations (5.1c) and (5.2a) are also valid for all t, especially for $t = 0$ and $t = 1$.

The equivalence of the conditions (5.1c) and (5.2b) is only given by taking into consideration additional necessary boundary conditions

$$H(x(0), v(0), 0) := \begin{cases} f_3(x(0), 0) = 0 \\ \frac{d}{dt} f_3(x(0), 0) = 0 \end{cases} . \tag{5.3}$$

Remark. The easiest choice of t is $t = 0$, but for numerical reasons also other points are possible.

Now we investigate transferable dae's which are closely related with such transformed higher index dae's. We consider the system

$$x_1'(t) = \phi_1(x_1(t), x_2(t), x_3(t)) \tag{5.4a}$$

$$x_2'(t) = \phi_2(x_1(t), x_2(t), x_3(t)) \tag{5.4b}$$

$$0 = \phi_3(x_1(t), x_2(t), x_3(t)) \tag{5.4c}$$

$(x_1(t) \in R^{n_{11}}, x_2(t) \in R^{n_{12}}, n_{11} + n_{12} = 2n_1)$ and the boundary conditions are given as

$$H(x_1(0), x_2(0), 0) = 0 \tag{5.5a}$$

$$x_2(0) - x_2(1) = 0 \tag{5.5b}$$

with $H : R^{n_{11}} \times R^{n_{12}} \times [0, 1] \to R^{n_{11}}$. For H let the following assumption be valid **(H)**:

1. There exists a function $\tilde{H}$ with

$$x_1(0) = \tilde{H}(x_2(0), 0)$$
$$x_1(1) = \tilde{H}(x_2(1), 1) \ .$$

2. $\tilde{H}$ is a 1–periodic function

$$\tilde{H}(\cdot, t) = \tilde{H}(\cdot, t + 1) \ .$$

Theorem 7. *Let (A1') - (A3') be valid for (5.4), (5.5). The equation of first variation of (5.4), (5.5) has the trivial solution only and the boundary condition (5.5) fulfils condition (H). Then the solution of (5.4), (5.5) is periodic and well-posed.*

Proof. 1.Periodicity: We have to show that the condition

$$P\left(\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} - \begin{pmatrix} x_1(1) \\ x_2(1) \\ x_3(1) \end{pmatrix}\right) = 0$$

is valid. This means that

$$x_1(0) - x_1(1) = 0 \tag{5.6a}$$
$$x_2(0) - x_2(1) = 0 \ . \tag{5.6b}$$

(5.6b) is identical with (5.5b). Using condition (H) we have

$$\begin{aligned} x_1(0) &= \tilde{H}(x_2(0), 0) \stackrel{(H)}{=} \tilde{H}(x_2(0), 1) \stackrel{(5.5b)}{=} \tilde{H}(x_2(1), 1) \\ &= x_1(1) \ . \end{aligned}$$

2. The well-posedness requires

$$\operatorname{im}(S_\star) = \operatorname{im}(g'_x), \quad \ker(S_\star) = \ker(P) = N \ .$$

$S_\star$ is given by

$$S_\star = g'_x(X(\cdot, 0)) = \begin{pmatrix} H'_{x_1} & H'_{x_2} & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} X(0,0) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{pmatrix} X(1,0) \ .$$

Using the structure of X, P and

$$Y(1,0) =: \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix}$$

it follows that

$$S_\star = \begin{pmatrix} H'_{x_1} & H'_{x_2} & 0 \\ -Y_{21} & I - Y_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} P \ . \tag{5.7}$$

However, the trivial solvability of (5.4), (5.5) implies that the matrix

$$\begin{pmatrix} H'_{x_1} & H'_{x_2} \\ -Y_{21} & I - Y_{22} \end{pmatrix} \tag{5.8}$$

becomes nonsingular and, therefore, $\ker(S_\star) = N$. With (5.7) and the nonsingularity of (5.8) also

$$\operatorname{im}(S_\star) = \operatorname{im}(P) = \operatorname{im}(g'_x)$$

is valid. □

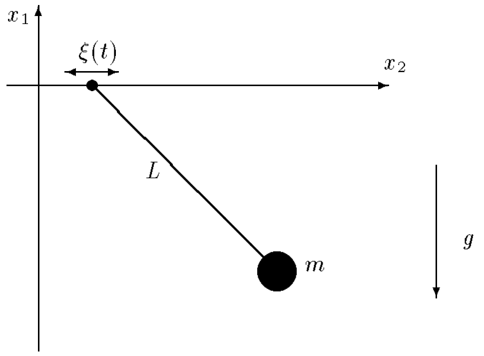

Fig. 1. Pendulum with moved origin

For a better understanding of condition (H) we consider a mechanical example: the rigid pendulum with moved origin.

$$\begin{aligned} \dot{x}_1 &= v_1 && (5.9a)\\ M\,\dot{v}_1 &= -g + 2x_1\lambda && (5.9b)\\ \dot{x}_2 &= v_2 && (5.9c)\\ M\,\dot{v}_2 &= 2(x_2-\xi)\lambda && (5.9d)\\ x_1^2 + (x_2-\xi)^2 &= L^2\ . && (5.9e) \end{aligned}$$

(5.9) represents an index 3 dae. Under the assumption $\xi \in C^2$ we obtain by differentiating (5.9e)

$$\begin{aligned} x_1v_1 + (x_2-\xi)(v_2-\dot{\xi}) = 0 && (5.10a)\\ v_1^2 + (v_2-\dot{\xi})^2 + \frac{-x_1g + 2\lambda(x_1^2+(x_2-\xi)^2)}{M} - (x_2-\xi)\,\ddot{\xi} = 0\ . && (5.10b) \end{aligned}$$

If we choose the boundary condition

$$\begin{aligned} x_1^2(0) + (x_2(0)-\xi(0))^2 - L^2 = 0 && (5.11a)\\ x_1(0)v_1(0) + (x_2(0)-\xi(0))(v_2(0)-\dot{\xi}\,(0)) = 0 && (5.11b)\\ x_1(0) - x_1(1) = 0 && (5.11c)\\ v_2(0) - v_2(1) = 0\ , && (5.11d) \end{aligned}$$

condition (H) is fulfilled. (5.9e) and (5.10a) are valid for $t \in [0,1]$, i. e. also

$$x_1^2(1) + (x_2(1) - \xi(1))^2 - L^2 = 0$$
$$x_1(1)v_1(1) + (x_2(1) - \xi(1))(v_2(1) - \dot{\xi}(1)) = 0$$

and

$$x_2 = \sqrt{L^2 - x_1^2} + \xi$$
$$v_1 = -(x_2 - \xi)(v_2 - \dot{\xi})/x_1$$

are valid for $t = 0, 1$ if $x_1(0) \neq 0$.

Remark. 1. The validity of condition (5.11a), (5.11b) on the whole t-interval turns out to be useful for assumption (H). Such a condition is also fulfilled by using physical constants like energy, impulse as boundary conditions.
2. For autonomous DAE's the results of Sect. 5 are also applicable when taking into consideration the special constellation in this case (see the preceding section and Example 4 below).

6 Numerical Examples

For the solution of the examples we use a multiple shooting method whose first version is presented in [Lam91]. As integration method we use the BDF-code IVPDAE and the nonlinear equations are solved by the code NLSOLV. Both codes have been developed by the author.

EXAMPLE 1 *The first example describes a two-stage amplifier with periodic input. This problem has been communicated to P.Rentrop by K.Glashoff and H.-J.Oberle. It is documented as IVP in [RRS89] .*

$$\begin{aligned}
c_5(y_2' - y_1') &= y_1/R_9 \\
c_5(y_1' - y_2') &= (y_2 - u_b)/R_8 + \alpha f(y_4 - y_3) \\
-c_4 y_3' &= y_3/R_7 - f(y_4 - y_3) \\
c_3(y_5' - y_4') &= y_4/R_5 + (y_4 - u_b)/R_6 + (1-\alpha) f(y_4 - y_3) \\
c_3(y_4' - y_5') &= (y_5 - u_b)/R_4 + \alpha f(y_7 - y_6) \\
-c_2 y_6' &= y_6/R_3 - f(y_7 - y_6) \\
c_1(y_8' - y_7') &= y_7/R_1 + f(y_7 - y_6) \\
c_1(y_7' - y_8') &= (y_8 - u_e)/R_0
\end{aligned}$$

where

$$f(y) := \beta(exp(y/u_f) - 1) \ ,$$

$c_i = i \cdot 10^{-6}$ $\quad i = 1, \ldots, 5,\ R_0 = 1000,\ R_i = 9000,\ i = 1, \ldots, 9$
$u_b = 6,\quad u_e = 0.1 sin(200\pi t),\quad u_f = 0.026,\ \alpha = 0.99,\ \beta = 10^{-6}$

EXAMPLE 2 *The second example which is solved by Fourier methods in [Zhe88] represents an electronic oscillator.*

$$\begin{aligned} c_1(u_2'/T - u_3'/T) &= (u_3 - u_1)/R_1 \\ (u_3 - u_1)/R_1 &= u_1/R_2 + c_2 u_1'/T \\ u_2 &= f(u_1) \\ T' &= 0 \end{aligned}$$

$$f(u) := arctan(\Lambda u)$$

$c_1 = 1,\ R_1 = 1,\ \Lambda = 28.$
We choose the boundary condition

$$\begin{aligned} u_1(0) &= u_1(1) \\ u_3(0) &= u_3(1) \end{aligned}$$

and the functional l as

$$l(u) := u_1(0) - 0.1 \ .$$

EXAMPLE 3 *We consider the rigid pendulum with moved origin (4.9)(a)-(d), (4.10)(b) with boundary conditions (4.11). We choose the parameters* $M = 1,\ g = 9.81,\ L = 1,$

$$\xi(t) = sin(\frac{2\pi}{3.5}t) \ .$$

EXAMPLE 4 *The determination of the period of the rigid pendulum we investigate in example 4. The describing equations are*

$$\begin{aligned} x_1'/T &= v_1 \\ M v_1'/T &= -g + 2x_1\lambda \\ x_2'/T &= v_2 \\ M v_2'/T &= 2x_2\lambda \\ T' &= 0 \\ v_1^2 + v_2^2 - \frac{x_1 g}{M} + 2\lambda(x_1^2 + x_2^2) &= 0 \ . \end{aligned}$$

The boundary conditions are:

- *Periodicity*

$$v_2(0) = v_2(1) \ ,$$

- *initial conditions of index reduction*

$$\begin{aligned} x_1^2(0) + x_2^2(0) - L^2 &= 0 \\ x_1(0)v_1(0) + x_2(0)v_2(0) &= 0 \ , \end{aligned}$$

- *energy of the system*

$$\frac{M}{2}(v_1^2(0) + v_2^2(0)) + (L + x_1(0))g - 4.5 = 0 \ .$$

We select the funtional

$$l := v_1(0) - 1.3$$

and choose the same parameters as in example 3.

Numerical results:
Example 1: $n = 8,\ \mathrm{rank}(f'_y) = 5,\ t \in [0, 10^{-2}]$

$$\begin{aligned} \text{relative accuracy of integration (eps)} &= 10^{-6} \\ \text{accuracy of defect of shooting equation (epsdef)} &= 5 \cdot 10^{-4} \\ \text{number of shooting points (m)} &= 4. \end{aligned}$$

Table 1. Start values of Example 1

t=	0	0	0	0025	0.005	0	0075
x_i							
	1	27	-3	16	-0.413	1	32
	4	74	0	283	2.9	4	68
	2	76	2	82	3.11	2	96
	2	71	3	0	3.26	2	95
	2	92	3	18	3.48	3	20
	2	85	2	91	2.82	2	76
	3	01	3	06	2.96	2	91
	-0	0044	0	0835	0.00156	-0	084

Table 2. Newton iteration of Example 1

iteration number (it)	defect of shooting equation (defect)		damping parameter (λ)	
0	0	7299		
1	0	3395	0	5
2	0	08754	0	5
3	0	01937	0	5
4	0	00792	0	5
5	0	00489	0	5
6	0	00189	1	0
7	0	000045 $\leq$ epsdef	1	0

Fig. 2. Solution of Example 1

Example 2: $n = 4,\ \mathrm{rank}(f'_y) = 3,\ t \in [0,1],\ \mathrm{eps} = 10^{-6},\ \mathrm{epsdef} = 5 \cdot 10^{-6},\ m = 1\ .$

Table 3. Start values (solution for $\Lambda = 4$) and results of Example 2

t= 0.0		
	Start values	solution after 7 iterations
x_i		
	0.1	0.1
	0.3805	1.22777239
	-0.05002	0.0821154202
	6.50824	11.3962306

Table 4. Newton iteration of Example 2

it	defect	λ
x_i		
0	1.475	1.0
1	0.9892	1.0
2	0.7294	1.0
3	0.06762	1.0
4	0.02299	1.0
5	0.02075	1.0
6	0.001415	1.0
7	0.0000768	1.0

Fig. 3. Solution of Example 2

Example 3: $n = 5,\ \mathrm{rank}(f_y') = 4,\ t \in [0,1],\ \mathrm{eps} = 10^{-7},\ \mathrm{epsdef} = 5,\ \cdot 10^{-3},\ m = 4.$ The sensitivity of this example is documented by the estimated condition number of $4.6 \cdot 10^3$.

Example 4: $n = 6,\ \mathrm{rank}(f_y') = 5,\ t \in [0,1],\ \mathrm{eps} = 10^{-6},\ \mathrm{epsdef} = 5 \cdot 10^{-6},\ m = 1.$

Table 5. Start values of Example 3

t=	0.0	0.25	0.5	0.75
x_i				
	-1.0	0.735	-1.0	0.735
	0.0	0.0	0.0	0.0
	0.0	0.318	0.0	-0.318
	-4.0	0.0	4.0	0.0
	-21.0	4.66	-21.0	4.66

Table 6. Newton iteration of Example 3

it	defect	λ
x_i		
0	0.2358	1.0
1	0.0266	1.0
2	0.0038	1.0

Fig. 4. Motion of the pendulum of Example 3

Table 7. Start values and results of Example 4

t= 0.0	
Start values	solution after 12 iterations
x_i	
-0.66	-0.720300157
1.3	1.3
0.75	0.693662515
1.1	1.34992188
-4.74	-5.28921680
2.13	2.13887233

Table 8. Newton iteration of Example 4

it	defect	λ
x_i		
0	3.289D-01	
1	1.557D-01	1.0
2	5.267D-02	1.0
3	2.145D-02	1.0
4	8.630D-03	1.0
5	3.559D-03	1.0
6	1.455D-03	1.0
7	6.044D-04	1.0
8	2.539D-04	1.0
9	9.631D-05	1.0
10	7.320D-05	1.0
11	8.561D-06	1.0
12	3.428D-07	1.0

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