On Quasilinear Index 2 Differential-Algebraic Equations

R. März

Humboldt-Universität zu Berlin, Fachbereich Mathematik,
PSF 1297, O-1086 Berlin, Germany

1 Introduction

This paper deals with autonomous differential algebraic equations (DAEs) of the form

$$Ax'(t) + g(x(t)) = 0 \ , \tag{1.1}$$

where the leading coefficient A is a constant singular $m \times m$ matrix, and $g : \mathcal{D} \subset \mathbb{R}^m \to \mathbb{R}^m$ is of class C^1 and C^2, respectively.

Type (1.1) equations arise in various applications, e. g. in describing the dynamics of circuits, constrained mechanical systems, chemical reactions subject to invariants etc. This is why great interest has been devoted to the analysis, geometry and numerical treatment of DAEs in recent years. With the present paper, we aim at a better understanding of so-called index 2 DAEs, their solvability and stability properties.

From the geometric viewpoint, an index 2 DAE should induce a smooth vector field on a certain manifold. However, even if it does so, this state space is unknown explicitly. In general, except for interesting case studies, the implicitly given state space is not available in practice. This is why we bestow on terms of (1.1) for practical criteria of solvability etc. in the present paper.

For DAEs (1.1) having index 1 (or, equivalently, being transferable) on some neighbourhood of $x_0 \in \mathcal{D}$ from the basic constraint manifold

$$M := \{z \in \mathcal{D} : g(z) \in \operatorname{im}(A)\} \ , \tag{1.2}$$

the solvability of initial value problems as well as the Lyapunov stability are discussed e. g. in [GM86], [Dol86], [Dol87]. While in the index 1 case the manifold M represents the state space at the same time, the state space for an index 2 equation becomes a lower dimensional manifold.
The existence and uniqueness theory presented in [RR90] applies also to index 2 DAEs (1.1). But this is a rather geometrically oriented theory. Further, it is emphasized that the C^2-regularity of $g(\cdot)$ turns out to be important for that theory.

Compared with this, our existence statements presented in Sect. 3 are based on the apparently more natural C^1-regularity of $g(\cdot)$ as well as on the understanding that a solution of (1.1) should be a C^1 function except for that component which belongs to $\ker(A)$, and which should be continuous actually. For uniqueness, the C^2 regularity of that part of $g(\cdot)$ which corresponds to the derivative free equations (constraints) within (1.1) seems to be essential. On the other hand, if the mentioned part of $g(\cdot)$ is C^2, then also the nullspace component of the solution is C^1. However, for perturbed equations $Ax'(t)+g(x(t)) = q(t)$, the natural regularity of the solution is again C^1 except for the nullspace component, which is continuous only.

It is well-known that a reduction of the index can be achieved by differentiating the constraints. This fact may become useful in theoretical investigations as well as practical computations (cf. e.g. [RR90], [BCP89], [Gri90]). However, one of the difficulties with reducing the index in this way is that the asymptotical behaviour of the solutions of the resulting system, (so-called underlying regular explicit differential equation) may differ strongly from the behaviour of the original DAE (cf. [FL]).

Example 1.

$$\begin{aligned} u' &= (\alpha(u) - v)\beta(u) + \gamma(u) \\ v' &= w \\ 0 &= v - \alpha(u) \ . \end{aligned} \tag{1.3}$$

Let α, β, γ be sufficiently smooth scalar functions. Obviously, this index 2 DAE (1.3) might be rewritten as

$$\begin{aligned} u' &= \gamma(u) \\ v &= \alpha(u) \\ w &= \alpha'(u)\gamma(u) \ . \end{aligned} \tag{1.4}$$

The equation $u' = \gamma(u)$ appears to be the inherent regular differential equation that is responsible for the dynamical behaviour of the whole system.
Differentiating the constraint in (1.3) twice and replacing the derivatives, leads to the "underlying" regular differential equation

$$\begin{aligned} u' &= (\alpha(u) - v)\beta(u) + \gamma(u) \\ v' &= w \\ w' &= \alpha''(u)\{(\alpha(u) - v)\beta(u) + \gamma(u)\}^2 - \alpha'(u)\beta(u)w + \\ &\quad +\alpha'(u)\{\alpha'(u)\beta(u) + (\alpha(u) - v)\beta'(u) + \\ &\quad +\gamma'(u)\}\{(\alpha(u) - v)\beta(u) + \gamma(u)\} \ . \end{aligned} \tag{1.5}$$

While the initial condition for (1.3) should be related to the component u only, say

$$u(t_0) = u_0 \ , \tag{1.6}$$

the system (1.5) requires initial conditions with respect to all components

$$u(t_0) = u_0, \quad v(t_0) = v_0, \quad w(t_0) = w_0 \ . \tag{1.7}$$

With the solution of (1.5), (1.7) we reobtain a solution of (1.3), (1.6) if it holds additionally that

$$v_0 = \alpha(u_0), \quad w_0 = \alpha'(u_0)\gamma(u_0) \ . \tag{1.8}$$

Now, let $\gamma(u_*) = 0$, $\gamma'(u_*) < 0$, so that $u_*, v_* := \alpha(u_*)$, $w_* = 0$ represent an asymptotically stable equilibrium of (1.3). However, whether $(u_*, v_*, w_*)^T$ is an asymptotically stable stationary solution of (1.5) depends on the eigenstructure of the matrix

$$\begin{bmatrix} \alpha'\beta + \gamma' & -\beta & 0 \\ 0 & 0 & 1 \\ \alpha'(\alpha'\beta + \gamma')^2 & -\alpha'\beta(\alpha'\beta + \gamma') & -\alpha'\beta \end{bmatrix}_{(u_*, v_*, w_*)} ,$$

the eigenvalues of which are

$$\lambda_1 = \gamma'(u_*), \quad \lambda_2 = 0, \quad \lambda_3 = 0 \ .$$

Since the nullspace of the matrix given by (1.9) has dimension one, we have to expect an unstable equilibrum.

When replacing equation $v' = w$ in (1.5) equivalently by

$$v' = \alpha'(u)\{(\alpha(u) - v)\beta(u) + \gamma(u)\} \ ,$$

the eigenvalues of the resulting matrix are even $\lambda_1 = \gamma'(u_*)$, $\lambda_2 = 0$, $\lambda_3 = -\alpha'(u_*)\beta(u_*)$.

The definition of Lyapunov stability given in the present paper reflects the behaviour of the inherent regular differential equation and the expectation from a rather geometrical viewpoint, respectively, but it is formulated in terms of (1.1), which are available also in practice. On this background, the stability statement presented in Sect. 4 generalizes the well-known Lyapunov Theorem on asymptotically stable stationary solutions in a quite natural way.

The main tool in the following investigations is an appropriate composition of classical ODE techniques and certain subspace/projector techniques.

The paper is organized in sections:

2. Fundamentals,
3. Local Solvability,
4. Lyapunov Stability.

2 Fundamentals

Let us start by considering the linear equation

$$Ax'(t) + Bx(t) = q(t) \tag{2.1}$$

with constant matrix coefficients $A, B \in L(\mathbb{R}^m)$, where A is singular. Let $q(\cdot)$ be continuous. It is well-known (e.g.[GM86]) that (2.1) has index 1 if and only if the matrix pencil $\{A, B\}$ is regular with index 1 or, equivalently, if the decomposition

$$\mathbb{R}^m = N \oplus S \tag{2.2}$$

holds, whereby we denote

$$N := \ker(A), \quad S := \{z \in \mathbb{R}^m : Bz \in \operatorname{im}(A)\} \ . \tag{2.3}$$

Choose $Q \in L(\mathbb{R}^m)$ to be a projector onto N, that is $Q^2 = Q$, $\operatorname{im}(Q) = N$. Denote $P := I - Q$.

It may be checked easily (cf.[GM86], p.198) that (2.2) becomes equivalent for

$$A_1 := A + BQ$$

to be nonsingular. In particular, in the index 1 case we may choose Q to project along S. Thus $P = I - Q$ projects onto S. By multiplying (2.1) by PA_1^{-1} resp. QA_1^{-1} we decouple the index 1 DAE (2.1) into the system

$$(Px)'(t) + PA_1^{-1}BPx(t) = PA_1^{-1}q(t) \tag{2.4a}$$

$$Qx(t) = QA_1^{-1}q(t) \ . \tag{2.4b}$$

Clearly, the inherent regular differential equation (or state space equation) is now given by (2.4a). Solving the IVPs

$$u'(t) + PA_1^{-1}BPu(t) = PA_1^{-1}q(t)$$
$$u(t_0) = u_0 \in S \ ,$$

with $x(\cdot) := u(\cdot) + QA_1^{-1}q(\cdot)$, we obtain all solutions of (2.1). Thereby, $u_0 \in S$ implies $u(t) \in S$, i. e. $Px(t) = u(t)$.

It should be mentioned that because of $AQ = 0$ equation (2.1) can be rewritten more precisely as

$$A(Px)'(t) + Bx(t) = q(t) \ , \tag{2.5}$$

which leads to the understanding of the solutions $x(\cdot)$ to be of class C, but with components $Px(\cdot)$ belonging to C^1. Denote the function space which the solutions have to belong to by

$$C_N^1 := \{x(\cdot) \in C : Px(\cdot) \in C^1\}.$$

The DAE (2.1) is called an *index 2 DAE* (cf. [Mar89]) if and only if $\{A, B\}$ forms a regular index 2 matrix pencil, or, equivalently, if

$$N_1 := \ker(A_1) \neq \{0\}, \quad \mathbb{R}^m = N_1 \oplus S_1 \tag{2.6}$$

where $S_1 := \{z \in \mathbb{R}^m : BPz \in \operatorname{im}(A_1)\}$. Since $N \cap S$ is no more trivial in the index 2 case, we are not allowed to choose the projector Q onto N along S.

Lemma 1. *Let $\{A, B\}$ be a regular index 2 matrix pencil. Then, projectors $Q, Q_1 \in L(\mathbb{R}^m)$ onto N resp. N_1 may be chosen in such a way that the relations*

$$Q = QP_1A_2^{-1}B$$
$$Q_1 = Q_1A_2^{-1}BP, \quad Q_1Q = 0 \ ,$$

are valid, where $P_1 := I - Q_1$, and the matrix $A_2 := A_1 + BPQ_1$ is nonsingular. Q_1 projects onto N_1 along S_1.

Proof. Let us start by denoting Q to be any projector onto N, $A_1 := A + BQ$, $A_2 := A_1 + B_1Q_1$, where $B_1 := BP$, and Q_1 denotes the projector onto N_1 along S_1. Due to [GM86], p.209, the relation

$$Q_1 = Q_1A_2^{-1}BP = Q_1A_2^{-1}B_1$$

is easily checked to be satisfied. Then we define $\hat{Q} := QP_1A_2^{-1}B$. $\hat{Q}$ projects onto N, too. Denote $\hat{A}_1 := A + B\hat{Q}$, $\hat{N}_1 := \ker(\hat{A}_1)$, $\hat{P} := I - \hat{Q}$, $\hat{S}_1 := \{z \in bbbr^m : B\hat{P}z \in \operatorname{im}(\hat{A}_1)\}$, $\hat{B}_1 := B\hat{P}$. Let $\hat{Q}_1$ be the projector onto $\hat{N}_1$ along $\hat{S}_1$. By elementary computations it may be shown that

$$\hat{S}_1 = S_1, \quad \hat{N}_1 = N_1$$

hold. Consequently, $\hat{Q}_1 = \hat{Q}_1\hat{A}_2^{-1}B\hat{P} = Q_1$. Furthermore, $\hat{Q}\hat{P}_1\hat{A}_2^{-1}B = QP_1A_2^{-1}BP_1\hat{A}_2^{-1}B = QP_1A_2^{-1}BP_1(I - QP_1A_2^{-1}BPP_1)A_2^{-1}B = QP_1A_2^{-1}B = \hat{Q}$, so that $\hat{Q}$ resp. $\hat{Q}_1$ appear to be the projectors announced in the assertion. □

By Lemma 1 we obtain a decoupling of (2.1) that is more or less similar to (2.4). Namely, multiplying (2.1) by $PP_1A_2^{-1}, QP_1A_2^{-1}$ and $Q_1A_2^{-1}$, respectively, we obtain

$$\begin{aligned} (PP_1x)'(t) + {} & PP_1A_2^{-1}BPP_1x(t) = PP_1A_2^{-1}q(t) && (2.7a)\\ -(QQ_1x)'(t) + {} & Qx(t) = QP_1A_2^{-1}q(t) && (2.7b)\\ & Q_1x(t) = Q_1A_2^{-1}q(t) \ . && (2.7c)\end{aligned}$$

Now, the inherent regular differential equation is given by (2.7a). Supposed q is continuous, but $Q_1A_2^{-1}q$ is of class C^1, system (2.7) completed by the initial condition

$$PP_1x(t_0) = u_0 \in \operatorname{im}(PP_1) \qquad (2.8)$$

becomes uniquely solvable on C_N^1. Furthermore, a fundamental solution matrix $X(\cdot)$ is uniquely determined by

$$AX'(t) + BX(t) = 0, \quad PP_1(X(t_0) - I) = 0 \ ,$$

and

$$\operatorname{im}(X(t)) = \operatorname{im}(PP_1) \qquad (2.9)$$

becomes the solution space. This gives rise to describing the second part of (2.6) more precisely as

$$\mathbb{R}^m = N_1 \oplus S_1, \quad S_1 = N \oplus \operatorname{im}(X(t)) \tag{2.10}$$

which makes the geometrical meaning of S_1 more transparent.

If $u \in C^1$ denotes the solution of the IVP

$$\begin{aligned} u' + PP_1A_2^{-1}BPP_1u &= PP_1A_2^{-1}q \\ u(t_0) = u_0 &\in \operatorname{im}(PP_1) \ , \end{aligned}$$

then $u(t) \equiv PP_1u(t)$, and

$$\begin{aligned} x &= PP_1x + PQ_1x + Qx \\ &= u + PQ_1A_2^{-1}q + QP_1A_2^{-1}q + (QQ_1A_1^{-1}q)' \end{aligned}$$

represents the solution of (2.7), (2.8). In this context we direct the attention also to the fact that for homogeneous systems (2.7) the components PQ_1x, Qx disappear.

Now let us consider the quasilinear equation

$$Ax'(t) + g(x(t)) = 0, \tag{2.11}$$

where $A \in L(\mathbb{R}^m)$ is again a singular matrix and $g : \mathcal{D} \subset \mathbb{R}^m \to \mathbb{R}^m$ is of class C^1. Let $Q \in L(\mathbb{R}^m)$ denote a projector onto $N := \ker(A)$, $P := I - Q$. Moreover, introduce

$$S(x) := \{z \in \mathbb{R}^m : g'(x)z \in \operatorname{im}A\}, \quad x \in \mathcal{D} \ . \tag{2.12}$$

Clearly, for $x \in M := \{y \in \mathcal{D} : g(y) \in \operatorname{im}(A)\}$, $S(x)$ represents the tangent space T_xM.

Recall that (2.11) is index 1 tractable (transferable) on $\mathcal{D}$ if

$$\mathbb{R}^m = N \oplus S(x), \quad x \in \mathcal{D} \ ,$$

is valid (cf. [GM86]). As in the linear case (cf.(2.4)), Px represents the state variable. On the contrary, index 2 DAEs are characterized by nontrivial intersections $N \cap S(x)$ as well as singular matrices $A_1(x) := A + g'(x)Q$.

Definition [Mar89]. The DAE (2.11) is said to be *index 2 tractable* on $U \subset \mathcal{D}$ if $N_1(x) := \ker(A_1(x)) \neq \{0\}$, $N_1(x)$ has constant dimension, and the decomposition

$$\mathbb{R}^m = N_1(x) \oplus S_1(x), \quad x \in U \ , \tag{2.13}$$

is true, where

$$S_1(x) := \{z \in \mathbb{R}^m : g'(x)Pz \in \operatorname{im}(A_1(x))\} \ .$$

It should be mentioned that in [Mar89] the nullspace $N_1(x)$ is not required to have constant dimension. However, dimension changes of this nullspace are shown to give rise to bifurcations etc ([GM89]) even in linear DAEs with variable coefficients. For this reason we exclude those singularities now.

With index 2 tractability we aim at characterizing a further class of DAE so that properly formulated IVPs (e.g. with (2.8)) become uniquely solvable.
The following example makes clear that, on this background, index 2 tractability should be given on an open subset $U \subseteq \mathbb{R}^m$ of $\mathcal{D}$. We quote this DAE describing a nonlinear resistor circuit from [CD89].

Example 2. For

$$\begin{aligned} x_1' &= \alpha(x_3) \\ x_2' &= \beta(x_3) \\ x_3^3 + x_2 x_3 + x_1 &= 0 \end{aligned} \tag{2.14}$$

we compute

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g'(x) = \begin{pmatrix} 0 & 0 & -\alpha'(x_3) \\ 0 & 0 & -\beta'(x_3) \\ 1 & x_3 & x_2 + 3x_3^2 \end{pmatrix},$$

$$A_1(x) = \begin{pmatrix} 1 & 0 & -\alpha'(x_3) \\ 0 & 1 & -\beta'(x_3) \\ 0 & 0 & x_2 + 3x_3^2 \end{pmatrix},$$

$$\begin{aligned} N &= \{z \in \mathbb{R}^3 : z_1 = z_2 = 0\}, \qquad P = A \\ S(x) &= \{z \in \mathbb{R}^3 : z_1 + x_3 z_2 + (x_2 + 3x_3^2) z_3 = 0\} \\ N \cap S(x) &= \{z \in \mathbb{R}^3 : z_1 = z_2 = 0,\ (x_2 + 3x_3^2) z_3 = 0\} \ . \end{aligned}$$

Any solution must lie on the constrained manifold $M = \{y \in \mathbb{R}^3 : y_1 + y_2 y_3 + y_3^3 = 0\}$ (see Fig. 1 below). The points on the fold curves R and L belong to $H := \{y \in \mathbb{R}^3 : y_2 + 3y_3^2 = 0\}$.

Obviously, the DAE (2.14) is index 1 tractable on $\mathbb{R}^3 \setminus H$, and any point $x_0 \in M \setminus (H \cap M)$ becomes a consistent initial value. The IVPs for (2.14) completed by the initial conditions $x_1(t_0) = \tilde{x}_{1,0}$, $x_2(t_0) = \tilde{x}_{2,0}$, are uniquely solvable on C_N^1 supposed $P(\tilde{x}_0 - x_0)$ is sufficiently small.

However, on $M \cap H$ the situation may change considerably. $x \in M \cap H$ implies $A_1(x)$ to become singular. Formally, we calculate for those x

$$\begin{aligned} N_1(x) &= \{z \in \mathbb{R}^3 : z_1 - 2x_3 z_3 = 0,\ z_2 = z_3\} \\ S_1(x) &= \{z \in \mathbb{R}^3 : z_1 + x_3 z_2 = 0\} = S(x) \ . \end{aligned}$$

Hence, $z \in N_1(x) \cap S_1(x)$ leads to $z = 0$ provided that $x \in M \cap H, x \neq 0$. Thus, the DAE (2.14) appears to be index 2 tractable on $(M \cap H) \setminus \{0\}$, i.e. on the fold curves R and L, independently of the choice of $\alpha(\cdot)$, $\beta(\cdot)$. Note that $N_1(x)$ has constant dimension 1 there.

Fig. 1. Constraint manifold M

However, for

$$\alpha(x_3) = x_3^2, \qquad \beta(x_3) = -x_3$$

we easily check

$$x(t) = (x_{1,0} + tx_{3,0}^2, x_{2,0} - tx_{3,0}, x_{3,0})^T$$

to solve the DAE for any initial point $x_0 \in M$ at $t = 0$. Additionally, for $x_0 \in M \cap H$,

$$x(t) = \left(2(\frac{t}{6} + x_{3,0})^3, -3(\frac{t}{6} + x_{3,0})^2, \frac{t}{6} + x_{3,0}\right)^T$$

becomes also a solution that remains on the fold curve where it has started.

On the other hand, when choosing

$$\alpha(x_3) = \beta(x_3) = x_3 \ ,$$

typical trajectories on the upper surface ($x \in M, x_3 > 0$) cannot be continued when they reach the fold curve R because any continuation is constrained by M to move along the lower folded branch, which contradicts the direction imposed by $x_1' = x_2' = x_3 > 0$ (see Fig. 2). Those points Q are called impasse points. Here, both R and L are impasse curves (cf. [CD89]).

We might add the trivial case

$$\alpha(x_3) = \beta(x_3) = 0$$

to have unique solutions that are all stationary.

Now it seems to be clear that our example represents rather an index 1 DAE (on $\mathbb{R}^m \setminus H$) which degenerates on H. However, formally characteriying (2.14) to

Fig. 2. Impasse points

be index 2 tractable on $(M \cap H)\backslash\{0\} =: \tilde{H}$ would make no sense. We emphasize that in each neighbourhood (in $\mathbb{R}^3$ as well as in M) of $\tilde{H}$ there are points where the quality of the DAE becomes different to that onto $\tilde{H}$. Note that similar arguments are pointed out also in [RR90].

The following lemma describes a class of DAEs which are index 2 tractable always on open subsets of $\mathcal{D}$. Note that, for those DAEs, the unique solvability of properly formulated IVPs will be proved in Sect. 3. Clearly, in that case the solutions also do have extensions.

Lemma 2. *For given $x_0 \in \mathcal{D}$, $B := g'(x_0)$, let the matrix pencil $\{A, B\}$ be regular with index 2. Let Q, Q_1, A_2 be determined by Lemma 1, and let*

$$Q_1 A_2^{-1}\{g(z) - g(Pz)\} = 0, \qquad z \in U_0 \tag{2.15}$$

be satisfied for a neighbourhood $U_0 \subset \mathcal{D}$ of x_0. Then the DAE (2.11) is index 2 tractable on a neighbourhood $U \subset U_0$ of x_0.

Remark. Denote by $R \in L(\mathbb{R}^m)$ a projector onto $\mathrm{im}(A)$. Because of $Q_1 A_2^{-1} R = 0$, $Q_1 A_2^{-1} BQ = 0$, a sufficient condition for (2.15) is given by

$$(I - R)\{g(y) - g(Py) - BQy\} = 0, \qquad y \in U_0 \tag{2.16}$$

which means, roughly speaking, that the derivative free part $(I - R)g(x)$ within (2.11) may depend on the component Qx only in a linear way (cf. (2.18)).

Proof. First we form $A_1 := A + BQ$, $A_2 := A_1 + BPQ_1$ as well as projectors Q, Q_1 according to Lemma 1. Next we try to show that

$$A_1(x) := A + g'(x)Q = A_1 + (g'(x) - g'(x_0))Q$$

has a constant-dimensional nullspace on a certain neighbourhood $U_1 \subset U_0$ of x_0.

Lemma 1 yields

$$\begin{aligned} Q_1 A_2^{-1} BQ &= Q_1 A_2^{-1}(A + BQ + BPQ_1)Q = Q_1 Q = 0 \\ Q_1 A_2^{-1} A_1 &= Q_1 A_2^{-1} A_2 P_1 = 0 \ . \end{aligned}$$

Taking (2.15) into account we derive

$$Q_1 A_2^{-1} g'(y) = Q_1 A_2^{-1} g'(y) P \quad \text{for } y \in U_0 \ ,$$

hence

$$Q_1 A_2^{-1} g'(x) Q = Q_1 A_2^{-1} g'(x) PQ = 0 \text{ for } x \in U_0 \ .$$

Since Q_1 projects onto $\ker(A_1)$ along $S_1 = \{z \in \mathrm{I\!R}^m \ : BPz \in \ \mathrm{im}(A_1)\}$, the relation $Q_1 A_2^{-1} A_1(x) = 0$ implies that $\mathrm{im}(A_2^{-1} A_1(x)) \subseteq S_1$ holds for all $x \in U_0$. Consequently,

$$\mathrm{rank}(A_2^{-1} A_1(x)) \leq \mu := \dim(S_1) = \mathrm{rank}(P_1)$$

for $x \in U$. Then, because of $A_2^{-1} A_1(x_0) = A_2^{-1} A_1 = P_1$, we conclude that $A_2^{-1} A_1(x)$ has constant rank μ on a neighbourhood $U_1 \subset U_0$ of x_0. This means that in fact

$$\dim(\ker(A_1(x)) = m - \mu, \quad \text{for } x \in U_1 \ .$$

$A_1(x)$ depends continuously on x since g is assumed to be C^1. Therefore, on U_1, also the orthogonal projector $Q_1(x)$ onto $N_1(x)$ is continuous. Hence $A_2(x) := A_1(x) + g'(x)PQ_1(x)$ depends also continuously on x. Further, $A_2(x_0) = A_1 + BPQ_1(x_0)$ is nonsingular simultaneously with A_2. Since A_2 is nonsingular, $A_2(x)$ is so too for x from a neighbourhood $U \subset U_1$ of x_0. □

We finish this section discussing condition (2.15) in some detail. (2.15) is easily shown to be equivalent to

$$(I - R)\{g(y) - g(Py)\} \in \ \mathrm{im}((I - R)BQ), \quad y \in U_0 \ . \tag{2.17}$$

Return shortly to our example (2.14). For $x_0 \in H$ we obtain with $R = P = A$ that

$$(I - R)BQ = 0 \ ,$$

but $(I - R)\{g(y) - g(Py)\} = (0, 0, y_3^3 + y_2 y_3)^T$, and it is not surprising that condition (2.15) does not hold in this case.

Furthermore, if the derivative free part of (2.11) $(I - R)g(x)$ has the special form

$$(I - R)g(y) = \varphi(Py) + (I - R)BQy, \ y \in U_0 \ , \tag{2.18}$$

i. e. $(I - R)g(y)$ depends linearly on Qy, then (2.17) is satisfied.

The so-called semi-explicit DAEs

$$\begin{aligned} u' + h(u,v) &= 0 \\ k(u,v) &= 0 \end{aligned} \tag{2.19}$$

form a large practically important class of DAEs. The circuits discussed in [CD89], inclusively example (2.14), belong to that class, too. For (2.19), condition (2.17) (resp. (2.15)) reads

$$\begin{gathered} k(u,v) - k(u,0) \in \operatorname{im}(k_v'(u_0,v_0)) \\ \text{on a neighbourhood of } (u_0,v_0). \end{gathered} \tag{2.20}$$

In particular, semi-discrete versions of Navier-Stokes systems for incompressible fluids (e.g. [Hun87], [BCP89]) lead to

$$k(u,v) \equiv Ku + c \ ,$$

where K and c are a constant matrix and a constant vector, respectively, so that (2.20) holds trivially.

On the other hand, all Hessenberg-form DAEs (e.g.[BCP89]) obviously have the property (2.19), since then $k(u,v)$ is supposed to be independent of v at all.

3 Local Solvability

Consider the quasilinear DAE

$$Ax'(t) + g(x(t)) = 0 \ , \tag{3.1}$$

where $g : \mathcal{D} \to \mathbb{R}^m$ is of class C^1. Given $x_0 \in M := \{z \in \mathcal{D} : g(y) \in \operatorname{im}(A)\}$ we form $B := g'(x_0)$ as well as $A_1 := A + BQ$, $A_2 := A_1 + BPQ_1$, Q, Q_1 according to Lemma 1. Of course, thereby the matrix pencil $\{A, B\}$ is assumed to be regular with index 2 again. First we formulate the main solvability assertion we are trying to prove in this section.

Theorem 3. *Let $g : \mathcal{D} \to \mathbb{R}^m$ be of class C^1, $x_0 \in M$ given, and let $\{A, g'(x_0)\}$ be a regular index 2 pencil. Additionally, let the conditions*

$$QA_2^{-1} g(x_0) = 0 \tag{3.2}$$

$$Q_1 A_2^{-1}\{g(y) - g(Py)\} = 0, \ y \in B(x_0, \rho) \tag{3.3}$$

be valid.

Then there is a $\tau > 0$ so that all IVPs for (3.1) completed by the initial condition

$$PP_1 x(t_0) = PP_1 x^o, \ x^o \in \mathbb{R}^m \ , \tag{3.4}$$

$$\textit{where}\ |PP_1(x^o - x_0)| \le \tau \ , \tag{3.5}$$

become solvable on C_N^1.

Remark. 1. Recall that by (3.3) the DAE (3.1) is guaranteed to be index 2 tractable on a neighbourhood of x_0.

2. C_N^1 solutions have continuously differentiable P-components but only continuous Q-components.
3. Condition (3.2) means that $x_0 \in M$ belongs to the implicitly given state space of our DAE. More precisely, if $x_*(\cdot) : I \to \mathbb{R}^m$ solves the DAE (3.1) and it holds that $PP_1x_*(t_0) = PP_1x_0$ (what corresponds to $\tau = 0$ in (3.4), (3.5)), then also
$$x_*(t_0) = x_0$$
becomes true. This fact becomes more transparent in the following explanations.

To gain an idea on how to prove solvability we decompose the linear part of (3.1) in the same way as we generated system (2.7). Thereby, to simplify the expressions we suppose $x_0 = 0$. On the contrary, as usually we might transform the DAE (3.1) into a new one

$$A\tilde{x}'(t) + \tilde{g}(\tilde{x}(t)) = 0$$

by $\tilde{x} = x - x_0$, $\tilde{g}(y) := g(y + x_0)$ in order to obtain this situation.

Rewrite (3.1) as

$$Ax'(t) + Bx(t) + h(x(t)) + g(0) = 0 \ , \tag{3.6}$$

where $h(y) := g(y) - g(0) - g'(0)y$. Obviously, h is of class C^1, and $h(0) = 0$, $h'(0) = 0$. Multiplying (3.6) by $PP_1A_2^{-1}$, $QP_1A_2^{-1}$ and $Q_1A_2^{-1}$, respectively, we decompose this DAE into the system (cf.(2.7))

$$(PP_1x)' + PP_1A_2^{-1}BPP_1x + PP_1A_2^{-1}(h(x) + g(0)) = 0 \tag{3.7a}$$
$$-(QQ_1x)' + Qx + QP_1A_2^{-1}(h(x) + g(0)) = 0 \tag{3.7b}$$
$$Q_1x \quad + Q_1A_2^{-1}h(Px) \quad = 0 \tag{3.7c}$$

The third equation (3.7c) appears in this special form since (3.3) together with $Q_1A_2^{-1}g(0) = 0$, $Q_1A_2^{-1}BQ = 0$ imply $Q_1A_2^{-1}(h(y) + g(0)) = Q_1A_2^{-1}h(Py), y \in B(x_0, \rho)$.

Thinking of a decomposition $x = PP_1x + PQ_1x + Qx$, $PP_1x =: u$, $Q_1x =: v$, $Qx =: w$, we are inspired by (3.7c) to apply the Implicit Function Theorem to

$$F(v, u) := v + Q_1A_2^{-1}h(u + Pv) = 0 \ . \tag{3.8}$$

Clearly, F is of class C^1, $F(0,0) = I$, $F_v'(0,0) = 0$, so that there is a C^1 function

$$f : B(0, \rho_1) \subset \mathbb{R}^m \to \mathbb{R}^m$$

with

$$F(f(u), u) = 0, \ u \in B(0, \rho_1), f(0) = 0, \quad f'(0) = 0, \text{ as well as } f(u) \equiv Q_1f(u).$$

Supposed $x(\cdot) : J \to \mathbb{R}^m$, $t_0 \in J$, $x(t_0) = x_0 = 0$ solves our DAE (3.1), for the components $v(\cdot) = Q_1x(\cdot)$, $u(\cdot) = PP_1x(\cdot)$, $w(\cdot) = Qx(\cdot)$, the relations

$$\begin{aligned} v(t) &= f(u(t)) \\ v'(t) &= f'(u(t))u'(t) \qquad\qquad (3.9)\\ &= -f'(u(t))\{PP_1A_2^{-1}Bu(t) + PP_1A_2^{-1}(h(u(t) + Pv(t) + w(t)) + \\ &\quad +g(0))\} \\ v'(t_0) &= 0 \ , \end{aligned}$$

$$w(t) = Qv'(t) - QP_1A_2^{-1}(h(u(t) + Pv(t) + w(t)) + g(0)) \ . \qquad (3.10)$$

are valid for t belonging to certain interval $J_0 \subset J$, $J_0 \ni t_0$.
Now, $w(t_0) = 0$; $u(t_0) = 0$, $v(t_0) = 0$, $v'(t_0) = 0$ imply that $QP_1A_2^{-1}g(0) = 0$ has to be true, i.e.

$$QA_2^{-1}g(0) = QQ_1A_2^{-1}g(0) = 0 \ .$$

Thus, condition (3.2) appears to be necessary for the consistency of the initial value x_0.

Proof of Theorem 3. Continue to assume $x_0 = 0$. By (3.7b), (3.9), (3.10) we are led to apply the Implicit Function Theorem once more to

$$\begin{aligned} K(w,u) &:= w + QP_1A_2^{-1}(h(u + Pf(u) + w) + g(0)) + \qquad (3.11)\\ &+Qf'(u)\{PP_1A_2^{-1}Bu + PP_1A_2^{-1}(h(u + Pf(u) + w) + g(0))\} = 0 \ . \end{aligned}$$

The consistency conditions $x_0 \in M$ and (3.2) yield $QP_1A_2^{-1}g(0) = 0$, hence $K(0,0) = 0$. The function K is continuous with a continuous partial derivative K_w'. We have $K_w'(0,0) = I$. Thus (3.11) describes a continuous function

$$k : B(0, \rho_2) \subset \mathbb{R}^m \to \mathbb{R}^m$$

with

$$K(k(u), u) = 0, u \in B(0, \rho_2), k(0) = 0, k(u) \equiv Qk(u).$$

Next we solve the regular IVP

$$\begin{aligned} &u' + PP_1A_2^{-1}Bu + PP_1A_2^{-1}(h(u + Pf(u) + k(u)) + g(0)) = 0 \qquad (3.12)\\ &u(t_0) = PP_1x^0. \end{aligned}$$

For sufficiently small PP_1x^0, this IVP has a C^1 solution $u(\cdot) : J \to \mathbb{R}^m$. Multiplying (3.12) by the projector $I - PP_1$ we show that $((I - PP_1)u)'(t) = 0$ holds. Because of $u(t_0) \in \operatorname{im}(PP_1)$ this leads to the identity $u(t) \equiv PP_1u(t)$, i.e. the IVP solution remains in $\operatorname{im}(PP_1)$ if it starts there.

Define the continuous function $x(\cdot) : J \to \mathbb{R}^m$ by $x(t) := u(t) + Pv(t) + w(t)$, $v(t) := f(u(t))$, $w(t) := k(u(t))$. We have $v(t) = Q_1v(t), w(t) = Qw(t)$ due to our construction. Since $u(\cdot), v(\cdot)$ are C^1 functions, $Px(\cdot) = u(\cdot) + Pv(\cdot)$ is also C^1, hence $x(\cdot)$ belongs to C_N^1.

Finally, it is easy to check that $x(\cdot)$ solves the IVP (3.1), (3.4) in fact. □

Remark. The inherent regular differential equation (3.12) of (3.1) may have more than one solution passing through PP_1x^0 in general. But if $k(\cdot)$ is also continuously differentiable, then there is only one such solution, and also its component $Qx(\cdot) = k(u(\cdot))$ becomes continuously differentiable.

Theorem 4. *Let the assumptions of Theorem 3 be satisfied. Additionally, let the function*

$$Q_1A_2^{-1}g(\cdot) \text{ be of class } C^2 \ . \tag{3.13}$$

Then the IVPs (3.1), (3.4), (3.5) are uniquely solvable, and the solutions belong to class C^1.

Proof. It remains to show that $k(\cdot)$ is of class C^1. For this purpose we compute (cf.(3.7c))

$$Q_1A_2^{-1}h(y) = Q_1A_2^{-1}g(y) - Q_1A_2^{-1}BPy \ .$$

This makes clear that $F(u,v)$ defined in (3.8) becomes a C^2 function, hence the implicitly given $f(\cdot)$ belongs to that class, too. As a consequence, $K(w,u)$ given by (3.11) appears to be continuously differentiable, hence also $k(\cdot)$ is C^1. □

It should be pointed out that the C^2 regularity condition (3.13) applies to the derivative free part of (3.1) only.

Finally, given a solution $x(\cdot) : J \rightarrow \mathbb{R}^m$ we ask for its extension. For given $t_1 > t_0$, $t_1 \in J$, the consistency condition $x_1 := x(t_1) \in M$ holds trivially. Our construction in Theorem 3 provides also an index 2 pencil $\{A, g'(x_1)\}$. Thus, the second consistency condition

$$QA_2(x_1)^{-1}g(x_1) = 0$$

is also satisfied. Note that $B(\cdot), Q_1(\cdot), A_1(\cdot), A_2(\cdot)$ should be evaluated at x_1 now. It remains to check (3.3) resp. its equivalent formulations at the new point x_1. Clearly, if $(I-R)g'(y)Q$ is independent of y, then Theorem 3 may immediately be applied to the new initial value x_1. Note that the two new functions F and K defined by (3.8) resp. (3.11) depend on x_1, hence also the implicitly given functions $f(\cdot), k(\cdot)$ as well as their domains of definition do so. Possibly, the extended solution leaves the chart where it has started, and enters a new one.

Of course, on the other hand it is also possible to extend the solution on the original chart (with fixed x_0) as long as the solution does not leave the domains of $f(\cdot), k(\cdot)$ and $h(\cdot)$, respectively.

A further very interesting question is how perturbations influence the solution of the IVP (3.1), (3.4). In particular, the behaviour of numerical methods is closely related to this problem. Consider the perturbed equation

$$Ax'(t) + g(x(t)) = q(t) \tag{3.14}$$

completed by the initial condition

$$PP_1x(t_0) = PP_1x_0 \ . \tag{3.15}$$

Recall that Theorem 4 provides a unique solution $x_*(\cdot) : J \to \mathbb{R}^m$ of the unperturbed problem (3.1), (3.15), and it holds that

$$x_*(t_0) = x_0 \ .$$

Theorem 5. *Let all assumptions of Theorem 4 be satisfied.*

Then, for sufficiently small $\sigma > 0$, the IVP (3.14), (3.15) becomes uniquely solvable with a solution belonging to C^1_N supposed $q(\cdot)$ is of class C, $p(\cdot) := Q_1 A_2^{-1} q(\cdot)$ is of class C^1 and

$$\|q\|_\infty + \|(Q_1 A_2^{-1} q)'\|_\infty \le \sigma \ .$$

Moreover, it holds that

$$\|x - x_*\|_\infty + \|(Px)'H - (Px_*)'\|_\infty \le \gamma\{\|q\|_\infty + \|(Q_1 A_2^{-1} q)'\|_\infty\} \ . \tag{3.16}$$

Proof. Assume again $x_0 = 0$. Decouple (3.14) in the same way as we proceed with (3.1) to obtain (3.7). This gives the background of the following investigations.

Instead of (3.8) we use now

$$\tilde{F}(v, u, q) := v + Q_1 A_2^{-1} h(u + Pv) - Q_1 A_2^{-1} q \ , \tag{3.17}$$

which leads to an implicitly given C^2 function $\tilde{f} : B(0, \rho_1) \subset \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ with

$$\tilde{F}(\tilde{f}(u, q), u, q) = 0$$

$$\tilde{f}(0,0) = 0, \quad \tilde{f}'_u(0,0) = 0, \quad \tilde{f}'_q(0,0) = Q_1 A_2^{-1}, \quad \tilde{f}(u, 0) \equiv f(u) \ .$$

Further, instead of $k(u)$ (cf. (3.11) we construct the C^1 function $\tilde{k}(u, q, p)$ via the Implicit Function Theorem applied to

$$\begin{aligned} &\tilde{K}(w, u, q, p) := w + QP_1 A_2^{-1}(h(u + P\tilde{f}(u, q) + w) + g(0) - q) \\ &\quad + Q\tilde{f}'_u(u, q)\{-PP_1 A_2^{-1} Bu - PP_1 A_2^{-1}(h(u + P\tilde{f}(u, q) + w) + g(0) - q)\} \\ &\quad + Q\tilde{f}'_q(u, q)p \\ &\tilde{K}(0,0,0,0) = 0, \quad \tilde{K}'_w(0,0,0,0) = I \ . \end{aligned}$$

Note that $\tilde{k}(u, 0, 0) \equiv k(u)$. Now, we construct the solution of the IVP (3.14), (3.15) for a given perturbation $q(\cdot)$ described above. First we solve the regular IVP

$$\begin{aligned} &u' + PP_1 A_2^{-1} Bu + PP_1 A_2^{-1}\{h(u + P\tilde{f}(u, q) + \tilde{k}(u, q, (Q_1 A_2^{-1} q)') + \\ &\quad + g(0) - q\} = 0 \\ &u(t_0) = 0 \ . \end{aligned} \tag{3.18}$$

Then we define

$$\begin{aligned} v(t) &\equiv \tilde{f}(u(t), q(t)), \\ w(t) &\equiv \tilde{k}(u(t), q(t), (Q_1 A_2^{-1} q)'(t)) \end{aligned}$$

and compose the C_N^1 function $x(\cdot) := u(\cdot) + Pv(\cdot) + w(\cdot)$. It is easy to check that $x(\cdot)$ satisfies the system (cf. (3.7))

$$u' + PP_1A_2^{-1}Bu + PP_1A_2^{-1}(h(x) + g(0) - q) = 0 \tag{3.19a}$$

$$-(Qv)' + w + QP_1A_2^{-1}(h(x) + g(0) - q) = 0 \tag{3.19b}$$

$$v + Q_1A_2^{-1}(h(Px) - q) = 0\ , \tag{3.19c}$$

which is the perturbed DAE (3.14). Trivially, $PP_1x(t_0) = PP_1u(t_0) = 0$ is valid, i. e. $x(\cdot)$ is the unique solution of (3.14), (3.15) in fact.

Since the implicitly given functions $\tilde{f}, \tilde{k}$ are Lipschitz, we have

$$\|v - v_*\|_\infty \le L_{\tilde{f}}(\|u - u_*\|_\infty + \|q\|_\infty) \tag{3.20}$$

$$\|w - w_*\|_\infty \le L_{\tilde{k}}(\|u - u_*\|_\infty + \|(Q_1A_2^{-1}q)'\|_\infty + \|q\|_\infty)\ , \tag{3.21}$$

where $v_* := Q_1x_*$, $w_* := Qx_*$, $u_* := PP_1x_*$. By Peano's Lemma, (3.18) implies immediately

$$\|u - u_*\|_\infty \le L_0(\|q\|_\infty + \|(Q_1A_2^{-1}q)'\|_\infty)\ , \tag{3.22}$$

further

$$\|u' - u_*'\|_\infty \le L_1(\|q\|_\infty + \|(Q_1A_2^{-1}q)'\|_\infty)\ . \tag{3.23}$$

Taking into account that

$$v'(t) = \tilde{f}_u'(u(t), q(t))u'(t) + \tilde{f}_q'(u(t), q(t))(Q_1A_2^{-1}q)'(t)$$

holds, we derive

$$\|v' - v_*'\|_\infty \le L_2(\|q\|_\infty + \|(Q_1A_2^{-1}q)'\|_\infty)\ . \tag{3.24}$$

Finally because of $Px - Px_* = u + Pv - u_* - Pv_*$, (3.20)-(3.24) yield (3.16). □

Remark. 1. It should be stressed that the perturbation $q(\cdot)$ may cause the solution $x(\cdot)$ to slip out of the state space manifold implicitly given by (3.1). At the initial point, too, we have

$$x(t_0) = x_*(t_0) = x_0 \tag{3.25}$$

only in case we suppose that

$$Q_1A_2^{-1}q(t_0) = 0,\ (Q_1A_2^{-1}q)'(t_0) = 0, \quad QP_1A_2^{-1}q(t_0) = 0$$

is true. However, in general (3.15) does not imply (3.25).

2. The inequality (3.16) makes clear that at least for the class of DAEs characterized by Theorem 4, index 2 tractability yields the DAE (3.1) to have the so-called perturbation index 2 (cf. [HLR89]). However, (3.16) reflects the nature of the DAE (3.1) more precisely than the inequality

$$\|x - x_*\|_\infty \le \tilde{\gamma}\{\|q\|_\infty + \|q'\|_\infty\}\ ,$$

which is used to define the perturbation index 2.

3. If the perturbation q is of class C^1 and, additionally, $Q_1 A_2^{-1} q$ is from C^2, then the solution $x(\cdot)$ belongs to C^1. Hence, while C^1 regularity seems to be quite natural for the solution of the autonomous DAE (3.1) (cf. Theorem 4), the conditions for the solution of the perturbed equation also to be of class C^1 are rather spurious.
4. The index 2 DAEs arising as semi-discrete versions of the Navier-Stokes equations for incompressible fluids (cf.[Hun87],[BCP89]) have the very special form

$$\begin{aligned} y'(t) + \mathcal{S}(y(t)) + Dz(t) &= 0 \\ G(y(t) - r(t)) &= 0 \end{aligned} , \tag{3.26}$$

G, D constants matrices, GD nonsingular. Here we compute

$$PP_1 A_2^{-1} h(x) \equiv PP_1 A_2^{-1} h(Px) \tag{3.27}$$

since the nullspace component $z(\cdot)$ appears only linearly in (3.26). Note that we have here

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad Q_1 = \begin{pmatrix} H & 0 \\ -(GD)^{-1}G & 0 \end{pmatrix}, \quad PP_1 = \begin{pmatrix} I - H & 0 \\ 0 & 0 \end{pmatrix},$$

whereby $H := D(GD)^{-1}G$ projects onto $\mathrm{im}(D)$ along $\ker(G)$. Due to (3.27) the component w disappears in (3.19a). In the consequence we obtain a decoupling of (3.16) into

$$\begin{aligned} &\|Px - Px_*\|_{C^1} \le \gamma_1 \|q\|_\infty \\ &\|Qx - Qx_*\|_\infty \le \gamma_2(\|q\|_\infty + \|(Q_1 A_2^{-1} q)'\|_\infty) \ . \end{aligned}$$

4 Lyapunov Stability

On the background of the above considerations the following definition seems to reflect the geometrical meaning of Lyapunov stability properly, but at the same time it seems to be also practically quite useful, since there is no need to evaluate the implicitly given state space explicitly.

Definition 6. Let (3.1) be index-2-tractable on a neighbourhood $U \subset \mathcal{D}$ of the trajectory of a given solution $x_*(\cdot) : [t_0, \infty) \to \mathbb{R}^m$ from class C^1_N.
$x_*(\cdot)$ is said to be stable in Lyapunov's sense if there is a $\tau > 0$, and moreover there exists a $\delta(\varepsilon) > 0$ to each $\varepsilon > 0$ such that

1. all IVPs
$$Ax'(t) + g(x(t)) = 0, \quad PP_1 x(t_0) = PP_1 x^0 \ , \tag{4.1}$$
where $|PP_1(x^0 - x_*(t_0))| \le \tau$, and P_1 is determined by the choice $x_0 := x_*(t_0)$, have unique solutions of class C^1_N, which are defined on $[t_0, \infty)$,
2. $|PP_1(x(t_0) - x_*(t_0))| \le \delta(\varepsilon)$ implies $|x(t) - x_*(t)| \le \varepsilon$ for all $t \ge t_0$.

Moreover, $x_*(\cdot)$ is called asymptotically stable if it is stable, and for sufficiently small $\tau_0 \le \tau$, $|PP_1(x(t_0) - x_*(t_0))| \le \tau_0$ implies

$$|x(t) - x_*(t)| \to 0 \quad (t \to \infty).$$

Turn shortly to the homogeneous constant coefficient equation $Ax' + Bx = 0$. Its decoupled form (2.7) shows that the solution components $Q_1x(t)$, $Qx(t)$ vanish identically, i.e.

$$x(t) \equiv PP_1x(t) \ ,$$

and only (2.7a) is responsible for the stability behaviour.

Lemma 7. *Given a regular index 2 matrix pencil $\{A, B\}$. Let Q, Q_1 be given by Lemma 1, and*

$$\begin{array}{c} M := -PP_1A_2^{-1}B \\ \mu := \operatorname{rank}(PP_1) \end{array} \ . \tag{4.2}$$

Then grad $\det(\lambda A + B) = \mu$, *i. e. $\{A, B\}$ has μ eigenvalues, say $\lambda_1, \dots, \lambda_\mu$. Moreover, $\lambda_1, \dots, \lambda_\mu$ belong also to the spectrum of M, where the corresponding eigenvectors lie within $im(PP_1)$. The remaining eigenvalues of M are zero, the corresponding eigenvectors span* $\ker(PP_1)$.

Proof. Note that $M = MPP_1 = PP_1M$. Suppose $(\lambda A + B)w = 0$, $w \ne 0$. Multiplying by $PP_1A_2^{-1}$, $QP_1A_2^{-1}$, $Q_aA_2^{-1}$ we decouple into

$$\begin{aligned} \lambda PP_1w - Mw &= 0 \\ -\lambda QQ_1w + Qw &= 0 \\ Q_1w &= 0 \ , \end{aligned}$$

hence $w = P_1w = Pw = PP_1w, Mw = \lambda w$.

On the other hand, suppose

$$Mz = \lambda z, \quad z \ne 0 \ .$$

If $\lambda \ne 0$, then $M = PP_1M$ yields $z = PP_1z$ immediately. In this case we have

$$\begin{aligned} \lambda PP_1z &= -PP_1A_2^{-1}BPP_1z \\ &= -P_1A_2^{-1}BPP_1z \qquad \text{since } QP_1A_2^{-1}BP = 0 \ , \end{aligned}$$

furthermore,

$$\lambda PP_1z = -A_2^{-1}BPP_1z \qquad Q_1A_2^{-1}BPP_1 = 0 \ .$$

Finally, $-BPP_1z = \lambda A_2PP_1z = \lambda APP_1z$, i.e. $(\lambda A + B)PP_1z = 0, \quad PP_1z \ne 0$.

If $\lambda = 0$ but $PP_1z \ne 0$, we compute as above

$$0 = Mz = -PP_1A_2^{-1}BPP_1z, \ -BPP_1z = 0 \ ,$$

hence PP_1z is an eigenvector of $\{A, B\}$ which corresponds to the eigenvalue $\lambda = 0$.

Moreover, $\ker(PP_1) \subset \ker(M)$, and $z \in \ker(PP_1)$, $z \in \operatorname{im}(M) \subseteq \operatorname{im}(PP_1)$ imply $z = 0$, hence the related eigenstructure of M is simple. □

Lemma 8. *Given matrices $M, \Pi \in L(\mathbb{R}^m)$, $\Pi^2 = \Pi$, $M = \Pi M = M\Pi$, $\mathrm{rank}\Pi =: \mu$. Let M have μ nontrivial eigenvalues $\lambda_1, \dots, \lambda_\mu$, and let them all have negative real parts.*

Then there is a constant $\beta > 0$ as well as a regular matrix $C \in L(\mathbb{C}^m)$so that the inequality

$$Re\langle Mz, z\rangle_C \le -\beta |z|_C^2 \text{ for } z \in \ im(\Pi) \tag{4.3}$$

becomes valid.

Remark. The scalar product is defined by $\langle u, v\rangle_C := \langle C^{-1}u, C^{-1}v\rangle_2$, $|u|_C^2 := \langle u, u\rangle_C$.

Proof. Let $T^{-1}MT =: J$ be the Jordan normal form of M, $J = \mathrm{diag}(N, 0)$, $N \in L(\mathbb{C}^\mu)$,

$$N = \begin{bmatrix} \lambda_1 & * & & \\ & \ddots & \ddots & \\ & & & * \\ & & & \lambda_\mu \end{bmatrix},$$

where $*$ represent 1 or 0 according to the Jordan structure of M. Denote $C := TD_\varepsilon$, $D_\varepsilon := \mathrm{diag}(1, \varepsilon, \dots, \varepsilon^{m-1})$, $\tilde{M} := C^{-1}MC$, $\tilde{\Pi} := C^{-1}\Pi C$. More precisely, we have $\tilde{M} = \mathrm{diag}(N_\varepsilon, 0)$, $\tilde{\Pi} = \mathrm{diag}(I, 0)$,

$$N_\varepsilon = \begin{bmatrix} \lambda_1 & *_\varepsilon & & \\ & \ddots & \ddots & \\ & & & *_\varepsilon \\ & & & \lambda_\mu \end{bmatrix},$$

where $*_\varepsilon$ is ε or 0. For $z = \Pi z$ denote $C^{-1}\Pi z =: \binom{r}{0}$, $r \in \mathbb{C}^\mu$.
Now, it follows that

$$\begin{aligned} &\langle Mz, z\rangle_C = \langle C^{-1}Mz, C^{-1}z\rangle_2 = \\ &= \langle C^{-1}MCC^{-1}\Pi z, C^{-1}\Pi z\langle_2 \Big|_{\mathbb{C}^m} = \langle N_\varepsilon r, r\rangle_2 \Big|_{\rangle^\mu} \end{aligned}$$

for $z \in \mathrm{im}(\Pi)$. Choose $\beta > 0$ so that $\mathrm{Re}\lambda_j < -2\beta$, $j = 1, \dots, \mu$. Since $\frac{1}{2}(N_\varepsilon + N_\varepsilon^*)$ tends to $\mathrm{diag}(\mathrm{Re}\lambda_1, \dots, \mathrm{Re}\lambda_\mu)$ for $\varepsilon \to 0$, we are able to fix an $\varepsilon > 0$ (and therefore the matrix C) in such a way that

$$\mu_2(N_\varepsilon) = \lambda_{\max}(\frac{1}{2}(N_\varepsilon + N_\varepsilon^*)) < -\beta \ .$$

$\mu_2(\cdot)$ denotes the so-called logarithmic norm (with respect to the Euclidean norm, cf. [DV85]). Finally,

$$\begin{aligned} \mathrm{Re}\langle Mz, z\rangle_C &= \mathrm{Re}\langle N_\varepsilon r, r\rangle_2 \Big|_{\mathbb{C}^\mu} \le \mu_2(N_\varepsilon)|r|_2^2 \Big|_{\mathbb{C}^\mu} \\ &= \mu_2(N_\varepsilon)|z|_C^2 \ . \end{aligned}$$

□

It should be mentioned that Lemma 8 slightly modifies well-known facts (e. g. [DV85]) in order to make them appropriate for investigating what is called Lyapunov stability on the subspace $\mathrm{im}(\Pi)$ (cf. [GM86],p.74). It will be pointed out that in the case of index-2-tractable DAEs Lyapunov stability for the DAE itself means the same as Lyapunov stability on $\mathrm{im}(PP_1)$ for the inherent explicit regular differential equation.

Recall (cf. (2.7)) $u'(t) = Mu(t)$, $M := -PP_1A_2^{-1}B$ to be that inherent regular equation within $Ax'(t) + Bx(t) = 0$. As we pointed out before, we have $x(t) \equiv PP_1x(t) \equiv u(t)$ for trivial reasons here. Let the eigenvalues of the pencil $\{A, B\}$ have negative real parts. Then the nontrivial eigenvalues of M have them, too. Additionally, M has the eigenvalue zero, but it is related to $\ker(PP_1)$.

Choosing
$$W(u(t)) := |PP_1u(t)|_C^2 \tag{4.4}$$
we may estimate due to Lemma 8
$$\begin{aligned}\frac{d}{dt}W(u(t)) &= 2\mathrm{Re}\langle C^{-1}MCC^{-1}PP_1u(t), C^{-1}PP_1u(t)\rangle > \\ &\leq -2\beta|PP_1u(t)|_C^2 = -2\beta W(u(t))\end{aligned}$$
hence
$$W(u(t)) \leq e^{-2\beta(t-t_0)}W(u(t_0)) \ .$$
This means that the trivial solution of $u' = Mu$ is asymtotically stable on $\mathrm{im}(PP_1)$. On the other hand, since $x(t) \equiv u(t)$ $(u(t_0) \in \mathrm{im}(PP_1))$, the trivial solution of the DAE $Ax' + Bx = 0$ appears to be asymptotically stable.

In the following we are concerned with nonlinear equations
$$Ax'(t) + g(x(t)) = 0 \tag{4.5}$$
and try to generalize Lyapunov's Theorem in an appropriate way.

Theorem 9. *Let $g : \mathcal{D} \to \mathbb{R}^m$ be of class C^2, $x_0 \in \mathcal{D}$ be a stationary solution of (4.5), i.e. $g(x_0) = 0$. Let the pencil $\{A, B\}$, $B := g'(x_0)$, be regular with index 2, and let all its eigenvalues have negative real parts. Moreover, let condition (3.2) be valid.*

Then x_0 is an asymptotically stable equilibrium of (4.5).

Proof. For transparency, assume again $x_0 = 0$. Since $g(x_0) = 0$, the consistency conditions of Theorem 3 are satisfied obviously. The IVPs (4.5), $PP_1x(t_0) = PP_1x^0$ have local solutions $x(\cdot) : J \to \mathbb{R}^m$ of class C^1 (cf. Theorem 4) supposed $|PP_1(x^0 - x_0)| \leq \tau$, τ sufficiently small. Thereby, the component $PP_1x(\cdot) =: u(\cdot)$ satisfies the regular IVP (cf.(3.12))
$$u' = Mu - PP_1A_2^{-1}h(u + Pf(u) + k(u)) \tag{4.6}$$
$$u(t_0) = PP_1x^0, \quad |PP_1x^0| \leq \tau \ , \tag{4.7}$$
where $M := -PP_1A_2^{-1}B$. By Lemma 7, M has the structure demanded in Lemma 8, hence the scalar product $< .,. >_C$ described there is available. Recall from Theorem 4 that $h(\cdot), f(\cdot)$ are C^2 but $k(\cdot)$ is C^1, thus $\tilde{h}(\cdot)$, $\tilde{h}(u) :=$

$-PP_1A_1^{-1}h(u + Pf(u) + k(u))$, is a C^1 function. We have $|f(u)| \leq \gamma_1|u|^2$, $|k(u)| \leq \gamma_2|u|$, $|h(x)| \leq \gamma_3|x|^2$, which leads to

$$|\tilde{h}(u)| \leq |PP_1A_2^{-1}|\gamma_3|u + Pf(u) + k(u)|^2 \leq \gamma_4|u|^2$$

for $u \in B(0, \rho_2)$. Next, by usual arguments we extend the solution of (4.6), (4.7) to the infinite interval. Namely, due to Lemma 8 we have for $W(u(t)) := |PP_1u(t)|_C^2 = |u(t)|_C^2$ that

$$\begin{aligned}
&\frac{d}{dt}W(u(t)) = \frac{d}{dt} < u(t), u(t) >_C = 2\mathrm{Re} < C^{-1}u'(t), C^{-1}u(t) >_2 \\
&= 2\mathrm{Re} < C^{-1}MCC^{-1}u(t), C^{-1}u(t) >_2 + 2\mathrm{Re} < C^{-1}\tilde{h}(u(t), C^{-1}u(t) >_2 \\
&\leq -\beta|u|_C^2 + 2\|C\|\|C^{-1}\|\gamma_4|u||u|_C^2 \\
&\leq (-\beta + 2\gamma_4\|C\|C^{-1}\||u|)W(u(t))
\end{aligned}$$

Making $B(0, \rho_2)$ and τ, respectively, smaller, we obtain $-\beta + 2\gamma_4\|C\|\|C^{-1}\||u| \leq -\beta_0$ with $\beta_0 > 0$, consequently

$$W(u(t)) \leq e^{-\beta_0(t-t_0)}W(u(t_0)) \ . \tag{4.8}$$

Now it is obvious that the solution $u(\cdot)$ of (4.6), (4.7) is defined on $[t_0, \infty)$, and that $u(t) \equiv PP_1u(t)$. Moreover, the trivial solution $u_*(t) \equiv 0$ of (4.6) is asymptotically stable on $\mathrm{im}(PP_1)$ in Lyapunovs sense (cf. [GM86], §1.25). Following the construction described in the proof of Theorem 3 we know

$$x(t) = u(t) + Pv(t) + w(t) \ ,$$

$v(t) := f(u(t))$, $w(t) := k(u(t))$ to solve the DAE (4.5). Clearly, also $v(\cdot), w(\cdot)$ may be defined on $[t_0, \infty)$. Further, $u(t) \to 0 (t \to \infty)$, implies that $v(t) \to 0$, $w(t) \to 0$ because of $f(0) = 0$, $k(0) = 0$. □

Once more we characterize the asymptotical stability on $\mathrm{im}(PP_1)$ for (4.6) to be the main point of our considerations.

Finally, let us shortly return to our example (1.3). In this special simple case we compute with $x = (u, v, w)^T$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = g'(x_*) = \begin{pmatrix} \alpha'\beta + \gamma' & -\beta & 0 \\ 0 & 0 & 1 \\ -\alpha' & 1 & 0 \end{pmatrix}_{(u_*, v_*, w_*)},$$

hence $\det(\lambda A + B) = -\lambda + \gamma'(u_*)$, that is, $\lambda_1 = \gamma'(u_*) < 0$ is the only eigenvalue of the pencil $\{A, B\}$. By Theorem 9, the equilibrium $(u_*, v_*, w_*)^T$ is asymptotically stable.

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