On the Existence and Approximate Solution of Discontinuous Differential-Algebraic Systems

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We consider differential-algebraic systems where discontinuities w. r. t. the state variable do arive. Among the great variety of problems of this type, we have control systems with discontinuous feedback from different fields of application, e. g., robotics, aircraft control or chemical engineering, mechanical systems with dry friction, electric circuits with switch elements. One way to model such systems is to replace the discontinuous characteristic of a system component by a multivalued one (cf. e. g., Chua et al. (1987) [CDK87], Cook (1986) [Coo86]). As an example, the force of dry friction $h(\dot{x}) = P \operatorname{sgn} \dot{x}$ will be replaced by $g(\dot{x}) = \begin{cases} P \operatorname{sgn} \dot{x}, & \dot{x} \neq 0, \\ [-P, P], & \dot{x} = 0, \end{cases}$ if $\dot{x}$ is the velocity and $p > 0$ a constant. Filippov (1960) [Fil60] used this type of set-valuled regularization to define a solution of a differential equation with discontinuous right-hand side by the solution of an associated differential inclusion with a correspondingly regularized set-valued right-hand side.

We generalize this definition in the first section to differential-algebraic systems $Ax' + Bx = h(t, x)$ with time-dependent matrices A and B, A singular, and h discontinuous in both variables, where we will distinguish between a differentiable and a nondifferentiable solution part as proposed by Griepentrog&März (1986) [GM86] for smooth differential-algebraic equations. In Sect. 2 we investigate the existence of a solution of the related differential-algebraic inclusion (DAI) $Ax' + Bx \in g(t, x)$, where g is an upper semicontinuous multifunction with compact and convex values in R^n. Assuming that $(A + BQ)$ is nonsingular and Q a projection onto the nullspace of A, we show that the DAI can be transformed into a semistate system, and hence the differentiable solution part has to satisfy an associated differential inclusion. In general, there is no solution of the associated differential inclusion. However, we will show that a generalized solution exists, and we give conditions where a generalized solution is a solution.

The third section is devoted to a convergence analysis of multistep and Runge-Kutta methods for the approximate solution of DAI. It is based on a discretization theoretical approach to the convergence problem for differential inclusions (cf. Niepage (1987) [Nie87b], Niepage&Wendt (1987) [NW87]) which was applied to the convergence and existence problem for DAI with constant matrices (cf. Niepage (1987-88,89) [Nie88],[Nie89]). We will prove here the same convergence result for the time-dependent case and under a weaker assumption on the coefficients in the approximation schemes.

1 Filippov-Solutions of Discontinuous Systems

We consider differential-algebraic systems

$$A(t)x'(t) + B(t)x(t) = h(t, x(t)) \tag{1.1}$$

on a real interval $I = [a, b]$ with time-dependent real $n \times n$-matrices $A, B : I \to L(R^n)$ where $A(t)$ is singular on I and $h : I \times R^n \to R^n$ a discontinuous function w. r. t. both variables, in general. Let $Q(t)$ be an arbitrary but fixedly chosen projection onto the nullspace $\ker(A(t))$ of $A(t)$, $P(t) = E - Q(t)$, $t \in I$, where E is the identity matrix. We assume

(A1) A, B are continuous and Q is continuously differentiable on I.

As in the case of differential equations with discontinuous right-hand side, we define a solution of (1.1) by means of an associated inclusion replacing h by its multivalued Filippov-regularization

$$g(t, x) = \bigcap_{\varepsilon > 0} \bigcap_{\mu(N)=0} \operatorname{conv}\{h(s, y) | (s, y) \in B_\varepsilon(t, x) \backslash N\}, (t, x) \in I \times R^n \ . \tag{1.2}$$

Here conv$A$ denotes the closed convex hull of a subset $A \subset R^n$, $B_\varepsilon(z)$ a closed ε-ball around z and μ the Lebesgue-measure in R^n.

Definition 1. A measurable function $x : I \to R^n$ is called a *(Filippov-) solution of the differential-algebraic equation (1.1)* if Px is absolutely continuous on I and

$$A((Px)' - P'x) + Bx \in g(t, x) \tag{1.3}$$

holds for almost every (a. e.) $t \in I$.

Hence, we have

$$A(t)x'(t) + B(t)x(t) \in g(t, x(t)), \quad a.e.\ t \in I \ , \tag{1.4}$$

for each absolutely continuous solution x of (1.1) and vice versa. Consequently, if A is regular on I, each solution is a solution in the sense of Filippov for the discontinuous differential equation (1.1) (cf. [Fil60]). On the other hand, it becomes evident from the properties of the Filippov-regularization g of h collected below that the given definition of a solution is exactly the one used by Griepentrog/März (1986) [GM86] for differential-algebraic systems (1.1) with continuous h.

We summarize now some basic definitions and results needed for the characterization of the Filippov-function and the subsequent investigations. Let $c(R^n) = \{A \subset R^n | A \text{ nonvoid, compact}\}, \operatorname{co}(R^n) = \{A \in c(R^n) | A \text{ convex}\}$. For sets $A, B \in c(R^n)$, the semidestance of A to B is given by $d(A, B) = \max_{x \in A} \min_{y \in B} |x - y|$, where $|\cdot|$ denotes the Euclidean norm in R^n. We need the following properties of the semidistance.

Lemma 2. *Let $A, B, C, D \in c(R^n)$, $\lambda \in R, T, S$ real $n \times n$-matrices with norm $\|T\| = (\sum_{i,j=1}^{n} t_{ij}^2)^{1/2}$ if $T = (t_{i,j})_{n,n}$. Then we have:*

(i) $d(A, B) = 0$ *iff* $A \subseteq B$,
(ii) $(A, B) \leq d(A, C) + d(C, B)$,
(iii) $d(\lambda A, \lambda B) = |\lambda| d(A, B)$,
(iv) $d(A + B, C + D) \leq d(A, C) + d(B, D)$,
(v) $d(convA, convB) \leq d(A, B)$,
(vi) $d(TA, TB) \leq \|T\| d(A, B)$,
(vii) $d(TA, SA) \leq \|T - S\| d(A, 0)$ *where* 0 *is the zero in* R^n.

Proof. For (i)-(v), cf. [Bri70]. (vi) follows directly from the definition. To see (vii), let $u, v \in A$. Then $|Tu - Sv| \leq |Tu - Su| + |Su - Sv| \leq \|T - S\||u| + \|S\||u - v|$. Hence, $\min_{v \in A} |Tu - Sv| \leq \|T - S\||u|$, which yields $d(TA, SA) \leq \|T - S\| \max_{u \in A} |u| = \|T - S\| d(A, 0)$. □

A multifunction $f : K \subset R^m \to c(R^n)$ is called upper semicontinuous (u. s. c.) if, for arbitrarily fixed $x \in K$ and each $\varepsilon > 0$, there is a $\delta > 0$ such that $f(y) \subseteq B_\varepsilon(f(x))$ for all $y \in B_\delta(x)$. In other words, f is u. s. c. on K iff $d(f(x_i), f(x)) \to 0$ follows from $x_i \to x$ $(i \in N)$, $x, x_i \in K$, $i \in N$. Hence, each u. s. c. multifunction has a closed graph. Conversely, if f has a closed graph and is bounded, i. e., $d(f(x), 0) \leq m$, $x \in K$, then f is u. s. c. f is said to be cointinuous, if it is a continuous mapping from K into the metric space $(c(R^n), h)$ with the Hausdorff-metric $h(A, B) = \max(d(A, B), d(B, A))$, $A, B \in c(R^n)$.

For the Filippov-function g replacing the discontinuous right-hand side h in (1.1), we have (cf. [Fil60]).

Proposition 3. *Let $h : I \times R^n \to R^n$ be measurable and bounded. Then the Filippov-regularization g defined by (1.2) is u. s. c. and bounded. If h is continuous in $z \in I \times R^n$, we have $g(z) = h(z)$.*

We remark that the given definition of the Filippov-function is a slight restriction of the original one used by Filippov for the qualitative investigation of discontinuous ordinary differential equations. However, in most cases of practical importance, e. g. mechanical systems with dry friction or control systems with discontinuous control, it is equivalent to it (for details, cf. [Fil85]).

2 Transferability and Existence of Solutions

In what follows, we consider, more general, a differential-algebraic inclusion (DAI)

$$\begin{aligned} A(t)((P(t)x(t))' - P'(t)x(t)) + B(t)x(t) \in g(t, x(t)), \text{ a. e. } t \in I \ , \\ P(a)x(a) = P(a)\alpha, \quad \alpha \in R^n \ . \end{aligned} \tag{2.1}$$

Suppose the assumptions made in the previous section on the matrices A, B, P are satisfied and $g : I \times R^n \to c(R^n)$, e. g., the Filippov-regularization of the discontinuous right-hand side h of the differential-algebraic equation (1.1).

Definition 4. A measurable function $x : I \to R^n$ for which Px is absolutely continuous and (2.1) is valid is called a *solution of the DAI (2.1)*.

We will investigate the existence of Filippov-solutions of discontinuous equations (1.1) in the more general setting of a DAI (2.1). To this end, we show that (2.1) can be transformed into a multivalued semistate system similar to the case of smooth differential-algebraic equations if we assume

(A2) $G(t) = A(t) + B(t)Q(t)$ is regular on I.

Then (2.1) is equivalent to

$$P(Px)' - PP'x + G^{-1}Bx \in G^{-1}g(t,x),$$

i. e.,

$$P(Px)' - P'Qx + G^{-1}BPx + G^{-1}BQx = s \tag{2.2}$$
$$s \in G^{-1}g(t,x) \tag{2.3}$$

since $G^{-1}A = P$ and $PP' = P'Q$. Multiply (2.2) by P and Q and notice $GQ = BQ$, we obtain its representation by two equations

$$P(Px)' - P'Qx + PG^{-1}BPx = Ps \tag{2.2a}$$
$$QG^{-1}BPx + Qx = Qs \ . \tag{2.2b}$$

If we use (2.2b) to replace Qx in (2.2a) and (2.3), we obtain

$$P(Px)' + P(E+P')G^{-1}BPx = P(E+P')s \ ,$$

hence

$$(Px)' - P'(Px) + P(E+P')G^{-1}B(Px) = P(E+P')s$$

and

$$s \in G^{-1}g(t, Px - QG^{-1}BPx + Qs) \ .$$

Definition 5. We say the pair (u,s) is a *solution of the multivalued semistate system*

$$u' - P'u + P(E+P')G^{-1}Bu = P(E+P')s \tag{2.4a}$$
$$s \in G^{-1}g(t,(E-QG^{-1}B)u + Qs) \tag{2.4b}$$
$$u(a) = P(a)\alpha, \alpha \in R^n \tag{2.4c}$$

if $u : I \to R^n$ is absolutely continuous, $s : I \to R^n$ is measurable, and (2.4a), (2.4b) holds for a. e. $t \in I$ together with (2.4c).

Lemma 6. *Let (A1), (A2) be satisfied. If (u,s) is a solution of (2.4), then $x = (E-QG^{-1}B)u+Qs$ is a solution of the DAI (2.1). Conversely, if x is a solution of (2.1), then $u = Px$ and s given by (2.2) solve (2.4).*

The proof follows from the considerations made above and from the fact that $u = Pu$ on I for a solution (u, s) of (2.4) since $(Qu)' = Q'u + Qu' = -P'u + QP'u = -P'u + P'Pu = -P'(Qu)$ and $(Qu)(a) = 0$, i. e., $Qu = 0$ on I. The realated components of (2.1) and (2.4) are absolutely continuous resp. measurable because of (A1) and (A2), which yields, in particular, that $G^{-1} : I \to L(R^n)$ is continuous.

Let

$$S(t,u) = \{s \in R^n | s \in G^{-1}g(t, Cu + Qs)\}, \quad (t,u) \in I \times R^n \tag{2.5}$$

where $C = E - QG^{-1}B$. Then the semistate system (2.4) can be written as

$$Lu(t) := u'(t) + K(t)u(t) = M(t)s(t) \tag{2.6a}$$

$$s(t) \in S(t, u(t)), \quad \text{a. e. } t \in I \tag{2.6b}$$

$$u(a) = P(a)\alpha$$

with $K = P(E + P')G^{-1}B - P'$, $M = P(E + P')$. If (u, s) is a solution of (2.4), then u is a solution of the corresponding initial value problem for the associated differential inclusion

$$Lu(t) \in T(t, u(t)), \quad \text{a. e. } t \in I \tag{2.7}$$

where $T(t,u) = M(t)S(t,u)$, $(t,u) \in I \times R^n$. Reversely, if u is an (absolutely continuous) solution of (2.7) and if there exists a measurable selection $s(t) \in S(t, u(t))$, a. e. $t \in I$, then (u, s) is a solution of (2.4).

For smooth differential-algebraic equations, Griepentrog&März (1986) [GM86] introduced the notion of transferability by a condition which permits the transformation into an equivalent solvable differential equation. More precisely, a system (1.1) where (A1) is satisfied, h is continuous and has a continuous partial derivative h'_x on $I \times R^n$ was defined to be transferable iff

$$[A(t) + (B(t) - h'_x(t,x))Q(t)] \text{ has a bounded inverse on } I \times R^n \ . \tag{2.8}$$

Then

$$(A + BQ)w + Bu = h(t, u + Qw) \tag{2.9}$$

has a unique solution $w = w(t,u) \in R^n$ for all $(t,u), w : I \times R^n \to R^n$ is continuous and $x = u + Qw$ is a solution of (2.1) (notice $g = h$) if u is a solution of

$$u' - P'u = P(E + P')w(t,u), \quad u(a) = P(a)\alpha \tag{2.10}$$

and vice versa. If we set $s = G^{-1}Bu + w$, then the unique solvability of (2.9) means that the set $S(t,u)$ consists of one element for each (t,u) and the differential inclusion (2.7) associated with the DAI (2.1) via the semistate system (2.6) is nothing else but (2.10).

We give a weaker definition of transferability that underlines the close relation of the DAI (2.1) and the differential inclusion (2.7).

Definition 7. We say that the DAI (2.1) is *transferable* (into a multivalued semistate system) if $S(t,u)$ is nonempty for all $(t,u) \in I \times R^n$.

Hence, transferability can be viewed as a minimal demand for the solvability of the associated differential inclusion (2.7). We assume for the right-hand sie g in (2.1)

(A3) $g : I \times R^n \to \mathrm{co}(R^n)$ is u. s. c. and bounded.

(A3) is satisfied for the Filippov-regularization of a discontinuous function by Proposition 3.

Theorem 8. *Let the assumptions (A1-3) be fulfilled. Then the DAI (2.1) is transferable and $s : I \times R^n \to c(R^n)$ is u. s. c. and bounded.*

Proof. Let $\varphi(t,x) = G^{-1}(t)g(t,x)$, $(t,x) \in I \times R^n$. Then $\varphi : I \times R^n \to \mathrm{co}(R^n)$ is bounded because of (A3), Lemma 2(vi) and the continuity of G^{-1}. Again by Lemma 2, we have, for $(t,x), (t',x') \in I \times R^n$,

$$
\begin{aligned}
&d(\varphi(t,x), \varphi(t',x')) \\
&\le d(G^{-1}(t)g(t,x), G^{-1}(t')g(t,x)) + d(G^{-1}(t')g(t,x), G^{-1}(t')g(t',x')) \\
&\|G^{-1}(t) - G^{-1}(t')\| d(g(t,x), 0) + \|G^{-1}(t')\| d(g(t,x), g(t',x')).
\end{aligned}
$$

Hence, φ is u. s. c. on $I \times R^n$ since g is so and G^{-1} continuous.

Let $(t,u) \in I \times R^n$ be arbitrarily fixed and $\psi(s) = \varphi(t, C(t)u + Q(t)s)$, $s \in R^n$. Then $\psi : R^n \to \mathrm{co}(R^n)$ is u. s. c. and there exists a compact convex subset of R^n which is mapped by ψ into itself because ψ is bounded. Consequently, ψ has a fixed point by Brouwer's fixed point theorem, i. e., $S(t,u)$ is nonvoid.

We show next that S has a closed graph. Let $s_i \in S(t_i, u_i)$, $\lim_{i\to\infty}(t_i, u_i, s_i) = (t,u,s)$. By Lemma 2,

$$
\begin{aligned}
d(s, \varphi(t, C(t)u + Q(t)s)) \le\ & d(s, s_i) + d(s_i, \varphi(t_i, S(t_i)u_i + Q(t_i)s_i) \\
& + d(\varphi(t_i, C(t_i)u_i + Q(t_i)s_i), \varphi(t, C(t)u + Q(t)s))
\end{aligned}
$$

from which we get $s \in S(t,u)$ since C and Q are continuous and φ is u. s. c. In particular, each set $S(t,u)$ is closed. Evidently, the multifunction S is bounded. Hence S is compact-valued and u. s. c. □

As an immediate consequence of Theorem 8 and assumption (A1), we obtain for the right-hand side $T = P(E + P')S$ of the associated differential inclusion (2.7)

Corollary 9. *If the assumptions (A1-3) are satisfied, then we have $T : I \times R^n \to c(R^n)$ is u. s. c. and bounded.*

In case of an ordinary differential equation (2.7), i. e. $T : I \times R^n \to R^n$ is a continuous function, there exists a solution for every initial value $P(a)\alpha$, $\alpha \in R^n$. For differential inclusions, the properties of T are not sufficient to guarantee the existence (– for a simple example, consider $u' \in T(u)$ on $I = [0,1]$

with $T(u) = \begin{cases} -1, & u > 0 \\ \{-1, 1\}, & u = 0 \\ 1, & u < 0 \end{cases}$ and $u(0) = 0$). But it is known that (2.7) has a solution for all initial values $P(a)\alpha$ if T is convex-valued (cf. [Fil60] or [AC84]).

Let

$$f(t, u) = \operatorname{conv} S(t, u), \quad (t, u) \in I \times R^n \quad . \tag{2.11}$$

Theorem 10. *If (A1-3) hold, then $f : I \times R^n \to co(R^n)$ is u. s. c. and bounded, and the semistate system*

$$\begin{aligned} Lu(t) = M(t)s(t), \quad s(t) \in f(t, u(t)), \text{ a.e.} \quad t \in I \\ u(a) = P(a)\alpha, \;\; \alpha \in R^n \quad , \end{aligned} \tag{2.12}$$

related to (2.4) has a solution (u, s).

Proof. $f(t, u) \in \operatorname{co}(R^n), (t, u) \in I \times R^n$, since it is the closed convex hull of a compact set in R^n by Theorem 8. The remaining properties of the multifunction f are again a direct consequence of those established for S in Theorem 8 and of Lemma 2(v). The right-hand side of the associated differential inclusion

$$Lu \in Mf(t, u) \tag{2.13}$$

has the same properties, obviously. Thus, there is an absolutely continuous $u : I \to R^n$ for which (2.13) holds almost everywhere on I together with the given initial condition. The superposition $f(\cdot, u(\cdot)) : I \to \operatorname{co}(R^n)$ is u. s. c. since f is u. s. c. Hence, there exists a measurable function $s : I \to R^n$ such that $s(t) \in f(t, u(t))$ and $Lu(t) = M(t)s(t)$, a. e. $t \in I$, by a result of Himmelberg ([Him75], Th. 7.1). □

It is evident that the right-hand side of the differential inclusion (2.13) related to the semistate system (2.12) is given by $\operatorname{conv} T(t, u)$, $(t, u) \in I \times R^n$, the convexified right-hand side of the inclusion (2.7). The solutions of the convexified differential inclusions are usually called generalized solutions of the original one. We adapt this terminology to differential-algebraic systems (2.1).

Definition 11. Each solution (u, s) of the semistate system (2.12) is called a *generalized solution of the semistate system (2.4)* (resp. (2.6)). If (u, s) is a generalized solution of (2.4), then $x = (E - QG^{-1}B)u + Qs$ is said to be a *generalized solution of the equivalent DAI (2.1).*

Thus, if the assumptions (A1), (A2) are valid, each solution of (2.1) is also a generalized solution. Moreover, we have by Theorem 10

Corollary 12. *Let the assumptions (A1-3) be satisfied. Then the DAI (2.1) has a generalized solution.*

As a consequence, the differential-algebraic system (1.1) has a generalized (Filippov-) solution if (A1,2) is assumed and the right-hand side h is measurable and bounded.

We point out some cases where a generalized solution is a solution.

Corollary 13. *Assume (A1-3). If the algebraic inclusion (2.4b) has only one solution for each* $(t,u) \in I \times R^n$*, then each generalized solution* (u,s) *of (2.4) is a solution and* s *is continuous on* I.

Sufficient, for the unique solvability of the algebraic inclusion (2.4b) in case of smooth differential-algebraic equations (1.1), are the already discussed transferability condition (2.8) or

$$\|G^{-1}(t)h'_x(t,x)Q(t)\| \le \alpha < 1, \quad (t,x) \in I \times B_m(0)$$

as a sufficient condition for (2.8) where $|G^{-1}(t)h(t,x)| \le m$, $(t,x) \in I \times R^n$ by (A3) and $h = g$. In the more general case of a DAI (2.1), the algebraic inclusion (2.4b) has a unique solution if

$$H(s) = G^{-1}(t)g(t, C(t)u - Q(t)s) \text{ is monotonuous} \tag{2.14}$$

for arbitrarily fixed $(t,u) \in I \times R^n$ (cf. [Bre73], Prop. 2.1).

Corollary 14. *Assume (A1-3). Let* $g_{t,u}$*,* $g_{t,u}(s) = g(t, C(t)u + Q(t)s)$*, have a convex graph for arbitrarily fixed* $(t,u) \in I \times R^n$*. Then the DAI (2.1) has a solution.*

Obviously, $S(t,u)$ is convex for each (t,u) if $g_{t,u}$ has a convex graph. An important special case is a control system

$$\begin{aligned} u' &\in h_1(t,u) + A_1(t,u)v \\ v &\in h_2(t,u) \end{aligned} \quad , \tag{2.15}$$

with state variable u, control v, $h_1 : I \times R^r \to \mathrm{co}(R^r)$, $h_2 : I \times R^r \to \mathrm{co}(R^{n-r})$ u. s. c. and bounded and a continuous bounded $r\times(n-r)$–matrix A_1. Set $A = \begin{pmatrix} E & \\ & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & \\ & E \end{pmatrix}$, $g(t,x) = \{s = \binom{s_1}{s_2} \in R^r \times R^{n-r} | s_1 \in h_1(t,u) + A_1(t,u)s_2, s_2 \in h_2(t,u)\}$, $t \in I$, $x = \binom{u}{v} \in R^n$. Then (2.15) can be written as a DAI (2.1) where (A1-3) are fulfilled and each generalized solution is a solution since $g_{t,u}$ has a convex graph.

3 Multistep Approximations of Generalized Solutions

In this section, we will investigate the convergence of multistep methods for the approximate solution of the DAI (2.1) and discontinuous differential- algebraic equations (1.1) regularized in the sense of Filippov, in particular. We will formulate these methods in terms of the semistate system (2.12) used in the definition of a generalized solution, i. e.,

$$Lu := u' + Ku = Ms \tag{3.1a}$$

$$s \in f(t,u), \quad \text{a. e. } t \in I \tag{3.1b}$$

$$u(a) = P(a)\alpha, \ \alpha \in R^n \ , \tag{3.1c}$$

where the matrices $K = MG^{-1}B - P', M = P(E+P')$ are continuous on I by assumptions (A1,2), and $f(t,u) = \text{conv}\, S(t,u)$, for arbitrarily fixed $(t,u) \in I \times R^n$, is the closed convex hull of the solution set $S(t,u)$ of the algebraic inclusion

$$s \in G^{-1}(t)g(t, C(t)u + Q(t)s), \tag{3.2}$$

with $C = E - QG^{-1}B$ also continuous on I, occuring in the semistate systems (2.4) and (2.6), respectively.

We consider multistep methods, for simplicity, on equidistant grids $I_i : t_j = a + hj,\ j = 0, 1, \cdots, s, t_s = b,\ i \in N$, with stepsize $h \to 0\ (i \in N)$ and constant coefficients,

$$h^{-1}\sum_{k=0}^{m} a_k u_{j+k} + \sum_{k=0}^{m} a'_k K_{j+k} u_{j+k} = \sum_{k=0}^{m} b_k M_{j+k} s_{j+k},\ j = 0, \ldots, s-m \quad \text{(3.3a)}$$

where the initial approximations $u_o, \cdots, u_{m-1}$ result from a starting procedure

$$\begin{aligned} h^{-1}(u_j - u_{j-1}) + \sum_{k=0}^{j} a'_{jk} K_k u_k &= \sum_{k=0}^{j} b_{jk} M_k s_k,\ j = 1, \cdots, m-0, \\ u &= P(a)\alpha_i \ , \end{aligned} \quad \text{(3.3b)}$$

and

$$s_j \in f(t_j, u_j),\ j = 0, \cdots, s \ .$$

Let $K_j = K(t_j), M_j = M(t_j),\ j = 0, \cdots, s$, and the real coefficients $a'_{jk}, b_{jk}, k = 0, \cdots, j,\ j = 1, \cdots, m-1,\ a_k, a'_k, b_k, k = 0 \cdots, m$ be chosen independently on the grids $I_i,\ i \in N$.

Lemma 15. *Let (A1-3) be satisfied. Assume $a_m \neq 0$. Then, to each stepsize $h \leq h_0$, there is a solution $u = (u_j) \in (R^n)^{s+1}, s = (s_j) \in (R^n)^{s+1}$ of the multistep method (3.3).*

Proof. Let $j \in \{0, \cdots, s-m\}$ be arbitrarily chosen. We get from (3.3)

$$(a_m E + h a'_m K_{j+m}) u_{j+m} \in c + h b_m M_{j+m} f(t_{j+m}, u_{j+m}) \tag{3.4}$$

where

$$c = -\sum_{k=0}^{m-1} (a_k u_{j+k} + h a'_k K_{j+k} u_{j+k} - h b_k M_{j+k} s_{j+k})$$

and $(u_{j+k}, s_{j+k}),\ k = 0, \cdots, m-1$ are known from previous steps. The matrices K_j are equibounded for all j and all stepsizes h since K is continuous by (A1,2). Hence, there exists an h_0 such that $T_j = a_m E + h a'_m K_{j+m}$ has an inverse if $0 < h \leq h_0$ because of $a_m \neq 0$.

We can write (3.4) now as a fixed point problem for u_{j+m},

$$u \in Fu, \quad Fu = T_j^{-1}c + h b_m T_j^{-1} M_{j+m} f(t_{j+m}, u), u \in R^n \ .$$

Obviously, $F : R^n \to \mathrm{co}(R^n)$ since f has this property. We get by Lemma 2,

$$\begin{aligned} d(Fu', Fu) &\leq d(hb_m T_j^{-1} M_{j+m} f(t_{j+m}, u'), hb_m T_j^{-1} M_{j+m} f(t_{j+m}, u)) \\ &\leq h|b_m| |T_j^{-1} M_{j+m}| d(f(t_{j+m}, u'), f(t_{j+m}, u)) \ \text{ and} \\ d(Fu, 0) &\leq |T_j^{-1} c| + h|b_m| |T_j^{-1} M_{j+m}| d(f(t_{j+m}, u), 0), u, u' \in R^n \ . \end{aligned}$$

Hence, F is u. s. c. and bounded by Theorem 10. In particular, there is a convex compact set mapped by F into itself. Thus u_{j+m} exists by Brouwer's fixed point theorem as a solution of (3.4). Therefore (u_{j+m}, s_{j+m}) is a solution of (3.3) for arbitrary $s_{j+m} \in f(t_{j+m} u_{j+m})$. The existence of $u_j, s_j, j = 1, \cdots, m-1$ can be shown in the same way. □

We will prove the convergence of the multistep method to a generalized solution by a discretization theoretical approach. In particular, we rely on results for discrete approximations of Hilbert spaces. Since f is bounded, we have for each solution (u, s) of (3.1) that s is from the Sobolev-space $H^0 = H^0(I, R^n)$ of quadratically summable functions and u from the Sobolev-space $H^1 = H^1(I, R^n)$ of quadratically summable functions with quadratically summable generalized derivative.

Using the same notations for the operators induced by the matrices K and M, (3.1a) can be viewed as an operator equation in H^0 where

$$L = D + K \tag{3.5}$$

with the differential operator

$$D : H^1 \to H^0, Du = u', K : H^1 \to H^0 \ ,$$

i. e.

$$L : H^1 \to H^0 \ \text{ and } \ M : H^0 \to H^0 \ .$$

Analogously, (3.1b) defines an operator inclusion

$$s \in Fu \tag{3.6}$$

in a space of multifunctions

$$MH^0 = \{u : I \to \mathrm{co}(R^n) | u \text{ measurable}, \int_I d^2(u(t), 0) dt < \infty\}$$

where $F : H^1 \to MH^0, (Fu)(t) = f(t, u(t)),\ t \in I$ and ϵ is defined by $s(t) \in (Fu)(t)$, a. e. $t \in I$. Moreover, by $d_0(u, v) = (\int_I d^2(u(t), v(t)) dt)^{1/2}$ a semidistance is defined for $u, v \in MH^0$ with basic properties corresponding to those of d compiled in Lemma 2(i)-(iv). The space of multifunctions
$MC = \{u : I \to \mathrm{co}(R^n) | u \text{ continuous}\}$ is dense in MH^0 w. r. t. the metric h_0, $h_0(u, v) = \max(d_0(u, v), d_0(v, u))$, $u, v \in MH^0$ (for details cf. [Nie87b]). Notice that H^0 can be considered as a subspace of MH^0 and $|u - v|_0 = d_0(u, v), u, v \in H^0$ if $|\cdot|_0$ is the usual norm in H^0.

According to the grids $I_i^0 = I_i \backslash \{t_0\}, i \in N$, a discrete metric approximation $(MH^0, (MH_i^0))$ can be defined now in the sense of Stummel [Stu70] by means of operators of pointwise restriction (r_i^0), $(r_i^0 u)_j = u(t_j), j = 1, \cdots, s, u \in MC$. Hence, $MH_i^0 = \{u = (u_j) | u_j \in \text{co}(R^n), j = 1, \cdots, s\}$ where $d_{0,i}(u, v) = (\sum_{j=1}^{s} h d^2(u_j, v_j))^{1/2}$ is a semidistance with the above mentioned basic properties and $|u - v|_{0,i} = d_{0,i}(u, v)$, $u, v \in H_i^0$ if $(H^0(H_i^0))$ is the discrete approximation of the subspace H^0 of MH^0 given in this way. Denote by $(H^0, (\bar{H}_i^0))$, $(H^1, (H_i^1)), (C, (C_i))$ the corresponding discrete approximations of H^0, H^1 and the space of continuous function $C = C(I, R^n)$ w. r. t. the grids I_i.

The discrete equations (3.3a) with the starting procedure can be written as

$$L_i u := L_{i,1} u + L_{i,2} u = L_{i,3} s \quad , \tag{3.7}$$

with $L_{i,1} = P_{i,1} D_i, D_i : H_i^1 \to H_i^0, (D_i u)_j = h^{-1}(u_j - u_{j-1})$, $P_{i,1} : H_i^0 \to H_i^0$

$$(P_{i,1} y)_j = \begin{cases} y_j, & j = 1, \cdots, m-1 \\ \sum_{k=1}^{m} \alpha_k y_{j-m+k}, & j = m, \cdots, s \end{cases}$$

where

$$a_k = \begin{cases} -\alpha_1, & k = 0 \\ \alpha_k - \alpha_{k+1}, & k = 1, \ldots, m-1 \\ \alpha_m, & k = m \end{cases} \tag{3.8}$$

and

$$L_{i,2} = P_{i,2} K_i, K_i : H_i^1 \to \bar{H}_i^0, (K_i u)_j = K_j u_j, P_{i,2} : \bar{H}_i^0 \to H_i^0$$

$$(P_{i,2} y)_j = \begin{cases} \sum_{k=0}^{j} a'_{jk} y_k, & j = 1, \cdots, m-1 \\ \sum_{k=0}^{m} a'_k y_{j-m+k}, & j = m, \cdots, s \end{cases}$$

and $L_{i,3} = P_{i,3} M_i$ with $M_i : \bar{H}_i^0 \to \bar{H}_i^0$, $P_{i,3} : \bar{H}_i^0 \to H_i^0$ defined analogously to $K_i, P_{i,2}$.

The discrete inclusions (3.3b) are in the space MH_i^0

$$s \in F_i u \tag{3.9}$$

where $F_i : H_i^1 \to MH_i^0, (F_i u)_j = f(t_j, u_j)$, i. e., ϵ in (3.9) means $s_j \in (F_i u)_j$, $j = 1, \cdots, s$.

We assume on the multistep approximations in (3.3a):

(A4) Let the difference approximations satisfy: $a_m \neq 0$,

(i) $\sum_{k=0}^{m} a_k = 0, \sum_{k=0}^{m} k a_k = 1$ (– consistency),

(ii) except for the simple root $\lambda = 1$, the characteristic polynomial $\sum_{k=0}^{m} a_k \lambda^k$ has roots only inside the unit circle (– strong stability).

(iii) The remaining approximations are consistent, $\sum_{k=0}^{m} a'_k = 1, \quad \sum_{k=0}^{m} b_k = 1.$

Suppose that the initial values converge,

(A5) $P(a)(\alpha_i - \alpha) \to 0 \quad (i \in N)$.

Then the corresponding operators from (3.7) satisfy

Lemma 16. *Let (A4,5) be valid. Then we have for the sequence* $(P_{i,1})$

(i) $P_{i,1} \rightharpoonup i_d$ *(- identity) in* H^0*, i. e.,* $y_i \rightharpoonup y$ $(i \in N)$ *in* $(H^0, (H_i^0))$ *implies* $P_{i,1} y_i \rightharpoonup y$ $(i \in N)$ *in* $(H^0, (H_i^0))$,
(ii) $(P_{i,1})$ *is inversely stable, i. e., there is a constant* $k_1 > 0$ *and* $i_1 \in N$ *such that*

$$k_1 |y|_{0,i} \leq |P_{i,1} y|_{0,i}, \quad y \in H_i^0, \quad i \geq i_1 \ . \tag{3.10}$$

Moreover,

(iii) $P_{i,2} \to i_d$ *in* H^0,
(iv) $u_i \rightharpoonup u$ $(i \in N)$ *in* $(H^1, (H_i^1))$ *implies* $L_{i,2} u_i \to Ku$ $(i \in N)$ *in* $(H^0, (H_i^0))$,
(v) there is a constant $k > 0$ *and* $i_0 \in N$ *such that*

$$k|u|_{1,i} \leq |L_i u|_{0,i} + |u_0|, \quad u \in H_i^1, \quad i \geq i_0 \ ; \tag{3.11}$$

(vi) $P_{i,3} \rightharpoonup i_d$ *in* H^0.

Proof. To see (i), we remark first that (A4)(i) implies $\sum_{k=1}^{m} \alpha_k = 1$ by (3.8). $P_{i,1} \rightharpoonup i_d$ is equivalent to $P_{i,1}^* \to i_d$ for the sequence $(P_{i,1}^*)$ of adjoint operators (cf. [Stu70]). Obviously, $(P_{i,1}^*)$ is stable since we have constant coefficients $\alpha_k, k = 1, \ldots, m,, i \in N$. It is also easy to check that $(P_{i,1}^*)$ is consistent with i_d in H^0 on the dense subspace $C^\infty(I, R^n)$. Hence, (i) is valid. In the same way, we get $P_{i,3}^* \to i_d$, i.e., (vi) as well as (iii) from (A4) (iii).

In case of difference approximations $P_{i,1}$ with constant coefficients (3.10) is equivalent to (A4)(ii) (cf. [Gri70], [Gri83]). Hence, (ii) is valid.

We show (iv). From $u_i \rightharpoonup u$ in $(H^1, (H_i^1))$, it follows that $u_i \to u$ in $(C, (C_i))$. Hence, $K_i u_i \to Ku$ in $(C, (C_i))$ since the operators are defined by pointwise restriction on I_i of the continuous function $K(\cdot)$. Thus, $K_i u_i \to Ku$ in $(H^0, (\bar{H}_i^0))$ and $L_{i,2} u_i \to Ku$ in $(H^0, (H_i^0))$ by (iii). Therefore, $(L_{i,2})$ is discretely weakly convergent,

$$L_{i,2} \rightharpoonup K \quad (i \in N) \ ,$$

and discretely weakly compact, i. e., $u_i \rightharpoonup 0$ in $(H^1, (H_i^1))$ implies $L_{i,2} u_i \to 0$ in $(H^0, (H_i^0))$. These properties guarantee together with (i), (ii), (A5) that (v) is satisfied (cf. [Nie87a], Theorem (1)). □

We are now in position to show the convergence of discrete solutions to a generalized solution of the DAI (2.1).

Theorem 17. *Let assumptions (A1-5) be satisfied. Then there is an* $i_0 \in N$ *such that each discrete problem (3.3) has a solution* (u_i, s_i) *for* $i \geq i_0$*, and for a subsequence* $N' \subset N$*, we have*

$$u_i \rightharpoonup u \; (i \in N') \; in \; (H^1, (H_i^1)), \quad s_i \rightharpoonup s \; (i \in N') \; in \; (H^0, (H_i^0))$$

where (u, s) *is a solution of (3.1).*

Proof. (u_i, s_i), $i \geq i_0 = i(h_0)$ exists by Lemma 15, where $|s_i|'_{0,i} = \left(\sum_{j=0}^{s} h|s_j|^2 \right)^{1/2}$ $\leq k'$, $i \geq i_0$, by (3.3b) and (A3). Thus, there is a subsequence $N' \subset N$ and $s \in H^0$ such that

$$s_i \rightharpoonup s \quad (i \in N') \text{ in } (H^0, (\bar{H}_i^0)) \tag{3.12}$$

since bounded sequences in discrete approximations of separable Hilbert-spaces are discretely weakly compact (cf. [Stu70]). Hence, in the right- hand side of (3.7), $M_i s_i \rightharpoonup Ms$ in $(H^0, (\bar{H}_i^0))$ since $(M_i s_i)_j = M(t_j) s_j$, $j = 0, \cdots, s$, where the matrix $M(\cdot)$ is continuous on I, and we get

$$L_{i,3} s_i \rightharpoonup Ms \quad (i \in N') \text{ in } (H^0, (H_i^0)) \tag{3.13}$$

by Lemma 16(vi). Therefore, $(L_{i,3} s_i)$ is bounded and the sequence $(L_i u_i)_{i \in N'}$ is also bounded due to (3.7). Then it follows by the stability inequality (3.11) and (A5) that $(u_i)_{i \in N'}$ is bounded, hence discretely weakly compact in $(H^1, (H_i^1))$. Consequently, there exist a $u \in H^1$ and a subsequence of N', again denoted by N', such that

$$u_i \rightharpoonup u \quad (i \in N') \text{ in } (H^1, (H_i^1)) \; . \tag{3.14}$$

It is easy to see now that we have $D_i u_i \rightharpoonup Du$ $(i \in N')$ in $(H^0, (H_i^0))$ for the difference operators, which gives $L_{i,1} u_i \rightharpoonup Du$ by Lemma 16(i). Hence,

$$L_i u_i \rightharpoonup Lu \quad (i \in N') \text{ in } (H^0, (H_i^0)) \; . \tag{3.15}$$

Thus, $Lu = Ms$ by (3.12) and (3.7), i. e., (u, s) is a solution of (3.1a). Moreover, the initial condition (3.1c) is satisfied by $u_i \to u$ $(i \in N')$ in $(C, (C_i))$ and (A5).

It remains to show that (u, s) is a solution of (3.1b) resp. (3.9). Due to a theorem of Zaremba (1936) [Zar36], there are continuous multifunctions g_k : $I \times R^n \to \text{co}(R^n)$,

$$f(t, x) \subset g_{k+1}(t, x) \subset g_k(t, x), \quad k \in N \tag{3.16}$$

$$d(g_k(t, x), f(t, x)) \to 0 \quad (k \in N), \quad (t, x) \in I \times R^n \; . \tag{3.17}$$

For $u \in C$ and $G^k u = g_k(\cdot, u(\cdot))$, we get

$$d_0(Fu, G^k u) = 0, \quad k \in N \quad \text{and } d_0(G^k u, Fu) \to 0 \quad (k \in N)$$

by a theorem of Lebesgue from (3.16) and (3.17), respectively, where d_0 is the semidistance in MH^0.

Let $\varepsilon > 0$ be arbitrarily chosen. Then there is a multifunction $g = g_k$ such that

$$d_0(Gu, Fu) \leq \varepsilon \ , \tag{3.18}$$

$Gu = g(\cdot, u(\cdot))$, $u \in C$. Define $G_i : H_i^1 \to MH_i^0$, $(G_i u)_j = g(t_j, u_j)$, $j = 1, \ldots, s$. Then $s_i \in G_i u_i$, $i \in N'$. by (3.1b) and (3.16). With operators $p_i : MH_i^0 \to MH^0$ of piecewise constant extension, $(p_i y)(t) = y_j$, $t_j \leq t < t_{j+1}$, $j = 1, \ldots, s-1$, we obtain

$$p_i s_i \in p_i G_i u_i, \quad i \in N' \tag{3.19}$$

and $p_i s_i \rightharpoonup s$ $(i \in N')$ in H^0 from (3.12). By a theorem of Mazur, there exists a sequence of convex compinations $z_k = \sum\limits_{i \geq k} c_i^k p_i s_i$, $\sum c_i^k = 1, c_i^k \geq 0$, $i \in N'$, $k \in N$, (at most finitely many $c_i^k > 0$ for fixed k) such that

$$z_k \to s \quad (k \in N) \text{ in } H^0 \ , \tag{3.20}$$

and

$$z_k \in \sum_{i \geq k} c_i^k G_i u_i, \quad k \in N \ , \tag{3.21}$$

by (3.19). From (3.18), (3.20), (3.21), we get by Lemma 2

$$\begin{aligned} d_0(s, Fu) \leq d_0(s, z_k) + d_0(z_k, \textstyle\sum c_i^k p_i G_i u_i) + \\ d_0(\textstyle\sum c_i^k p_i G_i u_i, Gu) + d_0(Gu, Fu) \\ \leq 2\varepsilon + d_0(\textstyle\sum c_i^k p_i G_i u_i, Gu) \\ \leq 2\varepsilon + \textstyle\sum c_i^k d_0(p_i G_i u_i, Gu), \quad k \geq k_0(\varepsilon) \end{aligned} \tag{3.22}$$

since Gu is a convex-valued multifunction, hence $Gu = \sum c_i^k Gu$. Finally,

$$\begin{aligned} d_{0,i}(G_i u_i, r_i(Gu)) &= \left(\sum_{j=1}^{s} h d^2(g(t_j, u_j), g(t_j, u(t_j))) \right)^{1/2} \\ &\leq \left(h \sum_{j=1}^{s} h^2(g(t_j, u_j), g(t_j, u(t_j))) \right)^{1/2} \\ &\leq \max_{j=1,\ldots,s} h(g(t_j, u_j), g(t_j, u(t_j)))(b-a)^{1/2} \to 0 \ (i \in N') \end{aligned}$$

since g is continuous and $u_i \to u$ $(i \in N')$ in $(C, (C_i))$. Therefore, there is a $k(\varepsilon) \geq k_0(\varepsilon)$ such that $d_0(p_i G_i u_i, Gu) \leq \varepsilon$, $i \geq k(\varepsilon)$, i. e., by (3.22), $s \in Fu$ since ε was arbitrarily chosen. □

We can get the same result for multistep methods with varible coefficients on nonequidistant grids, analogously. Corresponding conditions on coefficients and stepsizes can be found in [NW87], [Nie87a],[Nie88] where they were deduced for such general multistep schemes applied to initial and boundary value problems for differential inclusions and the DAI (2.1) with constant matrices A, B. In this connection, we mention that the nonnegativity assumption $b_k \geq 0$, $k = 0, \ldots, m$, used there in the right-hand side approximation, is superfluous.

If we have m initial values $u_0, \cdots, u_{m-1}$, not necessarily obtained by the above starting procedure, we have to demand additionally

$$|u_j - u_{j-1}| \geq k \cdot h, \quad j = 1, \cdots, m-1$$

with a constant k independent of stepsize h to guarantee a stability inequality (3.11).

The formulation (3.3) of the multistep method is more appropriate for the investigation of the convergence than for practical use. To determine the component (u_j, s_j) of a discrete solution, we have to solve a multivalued fixed point problem (3.4) for u_j, in general, where $f(t, u)$ is the closed convex hull of the solution set of the algebraic inclusion (3.2). In case of explicit methods, $b_m = b_{jj} = 0$, $j = 1, \cdots, m-1$, u_j can be computed explicitly from the previous $u_{j-1}, \cdots, u_{j-m}$, $s_{j-1}, \cdots, s_{j-m}$ and it is enough to solve

$$s \in G^{-1}(t_j) g(t_j, C(t_j) u_j + Q(t_j) s) \tag{3.23}$$

to obtain $s_j \in f(t_j, u_j)$.

If we apply an explicit method to a control system (2.15), (3.23) becomes simply a selection problem:

$$\begin{aligned} s_{2j} &\in h_2(t_j, u_j) \\ s_{1,j} &\in A_1(t_j, u_j) s_{2,j} + h_1(t_j, u_j) \\ s_j &= \begin{pmatrix} s_{1,j} \\ s_{2,,j} \end{pmatrix} . \end{aligned}$$

If the algebraic inclusion (3.2) has only one solution for each $(t, u) \in I \times R^n$, we get the following stronger convergence result.

Corollary 18. *Let (A1-5) be valid. Assume that (3.2) is uniquely solvable for every $(t, u) \in I \times R^n$. Then we have*

$$s_i \rightarrow s \quad (i \in N') \textit{ in } (C, (C_i))$$

for each sequence of discrete solutions (u_i, s_i) with $u_i \rightharpoonup u$ $(i \in N')$ in $(H^1, (H_i^1))$, and $x = (E - QG^{-1}B)u + Qs$ is a solution of DAI (2.1).

The proof is a simple consequence of the fact that $f(t, u) = S(t, u) : I \times R^n \rightarrow R^n$ is continuous by Theorem 8 and $u_i \rightarrow u$ $(i \in N')$ in $(C, (C_i))$.

Moreover, if the associated differential equation (2.7) has also only one solution, then each sequence of discrete solutions converges to a solution of the DAI.

Finally, we remark that it is possible to get the same convergence results for multivalued analogues of Runge-Kutta methods if we assume again (A1-5) and observe that such methods can be described by a scheme (3.3), where $f(t_j, u_j)$ in (3.3b) is replaced by conv $f(B_{\varepsilon_i}(t_j, u_j))$, $j = 0, \cdots, s$, $i \in N$ and $\varepsilon_i \rightarrow 0$ $(i \in N)$ (for more details, cf. [Nie90]).

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