# Existence and Uniqueness Results for Nonlinear Differential-Algebraic Equations

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## 1 Introduction

Differential-algebraic equations (DAEs) are frequently identified as implicit equations

$$F(t, x, x') = 0 \tag{1}$$

for which $x'$ cannot be expressed explicitly as a function of $t$ and $x$. Such DAEs arise in many areas of science and engineering. In particular, the motions of a constraint mechanical system and electronic circuits may be modelled using equations of type (1). In recent years the literature on the numerical solution of (1) has grown rapidly [BCP89], [GM86], [HLR89], [LP86]. However, up to now, existence theories for nonlinear DAEs are available for a few selected classes of systems only.

It is well–known that the solution behaviour of DAEs may differ significantly from that of standard ordinary differential equations (ODEs). However, as long as we assume the solutions of a DAE to be some smooth path in the space of dependent variables, we expect a DAE to induce a dynamical system (flow) [AMR83] on a suitable submanifold of that space. But this connection between DAEs and dynamical systems is certainly not obvious. The aim of this paper is to outline how differential–geometric approaches can provide a natural tool for analyzing general DAEs of type (1).

In particular, it has been shown in [RR90], [Rei90], [Rhe84] that rather general autonomous DAEs induce smooth vector fields on suitable domains. Therefore we introduced in [Rei90] the notion of a regular DAE as a DAE to which a vector field uniquely corresponds. The concept of regularity leads not only to general existence and uniqueness results but also opens up the possibility for simplifying the computational approaches by means of appropriate transformation techniques [PR89], [Rei90].

In the present paper we want to elaborate the concept of regularity for general non-autonomous DAEs of type (1). After surveying the necessary background material, we state a formal definition for the regularity of a DAE and derive an existence and uniqueness result for regular DAEs. In Sect. 3 and 4 regular DAEs are characterized in terms of the constraint manifolds and the derivative arrays

[BCP89] of a DAE. As, in the literature, DAEs are categorized by many other concepts like index [BCP89], [Gea88], [Gri90], [LP86], [Mar90] and solvability [BCP89], we begin a comparison between the various techniques in the subsequent sections. Finally, we state a technique for the reduction of higher index problems to lower index ones that does not introduce any additional degrees of freedom and integral invariants, respectively. To distinguish reduction techniques with this property from arbitrary reductions, we call them index transformations as suggested in [Gea88]. Additionally, we apply our results to linear time–varying DAEs and give a detailed discussion of a simple electronic circuit which appears to illustrate our concepts.

## 2 Differential-Geometric Background

In the section we collect some basic material needed throughout the remainder of this presentation. For further details we refer to standard texts on differential geometry such as [AMR83] and [Lan62].

We begin with some standard terminology. If $M$ is a differentiable manifold, then $TM$ denotes the *tangent bundle* of $M$ and $T_xM$ the *tangent space* of $M$ at $x \in M$. Throughout this paper we consider only manifolds that are imbedded in $\mathbb{R}^n$; that is, *sub-manifolds* of $\mathbb{R}^n$. In accordance with this, we can always assume that the tangent space $T_xM$ has been identified with a $d$-dimensional linear subspace of $\mathbb{R}^n$, where $d = \dim(M)$.

Now, let $M$ be a differentiable sub-manifold of $\mathbb{R} \times \mathbb{R}^n$, let $\mathrm{pr}_1 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be the projection onto the first component in $\mathbb{R} \times \mathbb{R}^n$, let $J$ be an open set of $\mathbb{R}$ with $\mathrm{pr}_1(M) = J$, and let $p : M \to J$ be the restriction of $\mathrm{pr}_1$ to $M$; i. e. $p = \mathrm{pr}_1|M$. Then the triple $(M, p, J)$ is a *sub-bundle* of $\mathbb{R} \times \mathbb{R}^n$ if $\mathrm{pr}_1(T_{(t,x)}M) = \mathbb{R}$ for all $(t, x) \in M$. For any sub-bundle $(M, p, J)$ the manifolds $M(t) \subset \mathbb{R}^n$ defined by

$$\{t\} \times M(t) = (M \cap \{t\} \times \mathbb{R}^n) \qquad (t \in J)$$

are called the *fibres* of $M$ at $t$.

Let $f : J \times U \to \mathbb{R}^n$ be a given mapping of the class $C^r$, $r > 0$, on the open set $J \times U \subset \mathbb{R} \times \mathbb{R}^n$. Then we denote by $D_t f(t, x)$ and $D_x f(t, x)$ the partial derivatives of $f$ with respect to the first and second argument, respectively. Now most classical examples of differentiable sub–bundles of $\mathbb{R} \times \mathbb{R}^n$ can be identified with the zeros of differentiable mappings $f : J \times U \to \mathbb{R}^n$. Specifically:

Let $f : J \times U \to \mathbb{R}^m$, $d = n + 1 - m > 0$, be a given mapping of the class $C^r$, $r > 0$, on the open set $J \times U \subset \mathbb{R} \times \mathbb{R}^n$ such that $0 \in f(J \times U)$ and, for all $(t, x) \in J \times U$, $\mathrm{im}[D_x f(t, x)] = \mathbb{R}^m$. Then the triple $(M, p, J)$ with

$$M = \{(t, x) \in J \times U : f(t, x) = 0\}$$

is a $C^r$-sub-bundle of $\mathbb{R} \times \mathbb{R}^n$. The fibres of the bundle are given by

$$M(t) = \{x \in U : f(t, x) = 0\} \qquad (t \in J) \ ,$$

and the tangent space of $M$ at $(t,x)$ is given by

$$T_{(t,x)}M = \{(s,p) \in \mathbb{R} \times \mathbb{R}^n : D_t f(t,x)s + D_x f(t,x)p = 0\} \ .$$

The following definition is motivated by the fact that the variable $t$ in (1) satisfies the differential equation $t' = 1$. Thus, for any sub-bundle $(M,p,J)$ of $\mathbb{R} \times \mathbb{R}^n$, we define the sets $S_{(t,x)}M$ and $SM$ by

$$S_{(t,x)}M = \{p \in \mathbb{R}^n : (1,p) \in T_{(t,x)}M\}$$

and

$$SM = \bigcup_{(t,x)\in M} \{(t,x)\} \times S_{(t,x)}M \ ,$$

which we call the *restricted tangent space* of $M$ at $(t,x)$ and *restricted tangent bundle* of $M$, respectively.

## 3 Regular DAEs

From now on we study DAEs of the type

$$F(t,x,x') = 0 \tag{2}$$

where $F : J \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping of the class $C^r$, $r > 0$, on the open set $J \times U \times \mathbb{R}^n \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. By a *solution of a DAE* (2) we mean a mapping $x : I \to \mathbb{R}^n$ of the class $C^1$ on the open interval $I \subset J$ such that, for all $t \in I$,

$$F(t,x(t),x'(t)) = 0$$

where $x'(t) = D_t x(t)$.

In [Rhe84], Rheinboldt has suggested the idea to consider DAEs of the type

$$F(x,x') = 0 \tag{3}$$

as an implicit description of autonomous vector fields on manifolds. Inspired by this, we proposed the notion of regularity for DAEs of the type (3) in [Rei90]. Here we apply this concept to the non-autonomous case (2). To start with, we give a definition for the notion of a non-autonomous vector field and its solutions.

**Definition 1.** Let $(M,p,J)$ be a $C^r$-sub-bundle, $r > 0$, of $\mathbb{R} \times \mathbb{R}^n$. Then a mapping $v : M \to \mathbb{R}^n$ of the class $C^k$, $k \geq 0$, is called a (*non-autonomous*) *vector field* of the class $C^k$ on $M$ if $v(t,x) \in S_{(t,x)}M$ for all $(t,x) \in M$.

By a *solution of a vector field* $v : M \to \mathbb{R}^n$ we mean a differentiable mapping $x : I \to \mathbb{R}^n$ on the open interval $I \subset J$ such that $x(t) \in M(t)$ and

$$x'(t) = v(t,x(t))$$

for all $t \in I$. □

*Remark.* In several papers the sub-bundle $(M, p, J)$ is assumed to be trivial; i. e. $M = J \times Q$, where $Q$ is a sub-manifold of $\mathbb{R}^n$. However, the subsequent discussion will reveal that we need, indeed, the more general definition of a non-autonomous vector field on a non-trivial sub-bundle $(M, p, J)$. Furthermore, the manifold $M$ cannot be regarded as a linear space by globally coordinating it in general; i. e., the manifold $M$ cannot be described by only one chart in general. □

*Remark.* One can associate with a given vector field $v : M \to \mathbb{R}^n$ the non-autonomous ODE

$$x' = v(t, x) \qquad ((t, x) \in M)$$

on the manifold $M$. Thus the notions of a vector field and an ODE on a manifold are synonymous in case the manifold $M$ has been embedded in $\mathbb{R}^n$. □

We can now introduce the notion of regularity for DAEs of the type (2).

**Definition 2.** Let a DAE of type (2) be given. Then we call this DAE a *regular DAE* if there is a unique sub-bundle $(M, p, J)$ of $\mathbb{R} \times \mathbb{R}^n$ and a unique vector field $v : M \to \mathbb{R}^n$ on $M$ such that a differentiable mapping $x : I \to \mathbb{R}^n$ is a solution of the vector field $v$ if and only if $x$ is a solution of the given DAE.

The manifold $M$ is then called the *configuration space* and the vector field $v$ the *corresponding vector field* of the DAE. □

Clearly, the above definition allows for the formulation of a solution theory for regular DAEs by 'translating' related results for vector fields to regular DAEs. For example, one can prove the following existence and uniqueness result:

**Theorem 3.** *Let (2) be a regular DAE, let $M$ be the configuration space, and let $v$ be the corresponding vector field of this DAE. We assume the vector field $v$ to be of the class $C^k$, $k > 0$. Then, for any $(t_0, x_0) \in M$ there exists a solution $x : I \to \mathbb{R}^n$ of the DAE on an open interval $I$ containing $t_0$ with $x(t_0) = x_0$. Moreover, any solution is a mapping of the class $C^k$ and any two solutions $x_1$ and $x_2$ with $x_1(t_0) = x_2(t_0)$ are equal on the intersection of their domain.* □

**Proof:** In [AMR83] the above result is stated for vector fields and consequently, because of the notion of regularity, this result carries over to regular DAEs. □

Naturally, the existence and uniqueness of solutions has been addressed in several other papers too [BCP89], [RR90], [Rhe84]. For example, the following definition is given in [BCP89]:

**Definition 4.** A DAE (2) is called *solvable* on a subset $\Gamma \times \Omega$ of $\mathbb{R} \times \mathbb{R}^n$ if there is a mapping $\mu : \Gamma \times \Omega^o \to \Omega$ on an open set $\Gamma \times \Omega^o$ of $\mathbb{R} \times \mathbb{R}^k$ such that

(i) if $x : I \to \mathbb{R}^n$ is a solution of the DAE whose graph lies in $\Gamma \times \Omega$, then $x(t) = \mu(t; y_0)$ for all $t \in I$ and some $y_0 \in \Omega^o$,
(ii) the graph of $\mu$ is a $(k + 1)$-dimensional sub-manifold of $\mathbb{R} \times \mathbb{R}^n$. □

Clearly, regularity implies the solvability of a DAE in case the corresponding vector field is differentiable; i. e., we obtain:

**Proposition 5.** *Let (2) be a regular DAE, let $M$ be the configuration space, and let $v$ be the corresponding vector field of the DAE. Furthermore, we assume the vector field $v$ to be of the class $C^1$. Then this DAE is solvable on a sufficiently small neighbourhood of $(t, x)$ on $M$ for any $(t, x) \in M$.* □

*Remark.* Let $(M, p, J)$ be a sub-bundle of $\mathbb{R} \times \mathbb{R}^n$. By a *dynamical system* (or flow) [AMR83] on a set $W$ we mean a mapping $\phi : \Gamma \times W \to \mathbb{R}^n$ such that

(i) $W$ is an open set of $M$ and $(W, p, \Gamma)$ is a sub-bundle of $\mathbb{R} \times \mathbb{R}^n$,
(ii) $\phi(t_2 + t_1; t_0, x_0) = \phi(t_2; \phi(t_1; t_0, x_0))$ and $\phi(t; t, x) = x$ as long as the mappings are defined.

Now Theorem 3 implies that, in case the corresponding vector field $v$ is differentiable, the solutions $x : I \to \mathbb{R}^n$ of (2) define a dynamical system $\phi$ in a neighbourhood $W$ of an arbitrary point $(t_0, x_0) \in M$ in $M$; i. e.

$$\phi(t; t_1, x_1) = x(t)$$

for all $(t_1, x_1) \in W$, $t \in I$. Furthermore, let $\phi : \Gamma \times W \to \mathbb{R}^n$ be a dynamical system defined by the solutions of (2), then (2) is solvable on a subset $\Gamma \times \Omega$ of $\mathbb{R} \times \mathbb{R}^n$ with $(M \cap \Gamma \times \Omega) = W$. For that reason let $W(t_0)$ be the fibre of the sub-bundle $(W, p, \Gamma)$ at $t_0 \in \Gamma$ and let $\rho : W(t_0) \to \Omega^o \subset \mathbb{R}^k$ be a chart of the fibre $W(t_0) \subset \mathbb{R}^n$. Then we can define the mapping $\mu : \Gamma \times \Omega^o \to \Omega$ of Definition 4 by

$$\mu(t; y_0) = \phi(t; t_0, x_0)$$

where $y_0 = \rho(x_0)$.

On the other hand, let (2) be solvable on a subset $\Gamma \times \Omega$, then a dynamical system $\phi$ can be defined on the graph of $\mu$ by

$$\phi(t; t_0, x_0) = \mu(t; y_0) \quad \text{with} \quad x_0 = \mu(t_0; y_0) \ .$$

□

## 4 Differential-Geometric Characterization of Regular DAEs

In this section we derive sufficient conditions for the regularity of a DAE. Furthermore, we state a technique by means of which one can obtain the configuration space $M$ and the corresponding vector field $v$ for a given regular DAE. To start with, we associate with any DAE the corresponding set as defined below.

**Definition 6.** Let a DAE of type (2) be given. Then we call the set

$$N = \{(t, x, p) \in J \times U \times \mathbb{R}^n : \quad F(t, x, p) = 0\}$$

the *corresponding set* of the DAE. □

Note that in Definition 6 the DAE (2) is considered as a nonlinear system of equations in the variables $t$, $x$, and $p$. For that reason the corresponding set $N$ of a DAE is a subset of $J \times U \times \mathbb{R}^n$. Clearly, a differentiable mapping $x : I \longrightarrow \mathbb{R}^n$ is a solution of the DAE if and only if $(t, x(t), x'(t)) \in N$ for all $t \in I$. Therefore, with respect to the solutions of a DAE, we can consider the corresponding set instead. This fact is explored further in the next observation.

**Observation 7.** Let $N$ be the corresponding set of a given DAE (2). We consider the set $M_1$ defined by

$$M_1 = \mathrm{pr}_{1,2}(N),$$

where $\mathrm{pr}_{1,2} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \times \mathbb{R}^n$ is the projection onto the first two components in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$.

The set $M_1$ is a subset of $\mathbb{R} \times \mathbb{R}^n$ and reflects the algebraic constraints on the solutions of the DAE. Let us assume that the triple $(M_1, p, J)$ is a differentiable sub-bundle of $\mathbb{R} \times \mathbb{R}^n$. Clearly, under this assumption, a differentiable mapping $x : I \longrightarrow \mathbb{R}^n$ is a solution of the given DAE if and only if

$$(t, x(t), x'(t)) \in (N \cap SM_1)$$

for all $t \in I$. In general, $(N \cap SM_1)$ will be a subset of $N$. Therefore, we consider the set

$$M_2 = \mathrm{pr}_{1,2}(N \cap SM_1) \ .$$

If the triple $(M_2, p, J)$ is now a differentiable sub-bundle of $\mathbb{R} \times \mathbb{R}^n$, then we conclude again that a differentiable mapping $x : I \longrightarrow \mathbb{R}^n$ is a solution of the DAE if and only if

$$(t, x(t), x'(t)) \in (N \cap SM_2)$$

for all $t \in I$. This process can be continued as long as the triples $(M_k, p, J)$ with

$$M_k = \mathrm{pr}_{1,2}(N \cap SM_{k-1})$$

are differentiable sub-bundles of $\mathbb{R} \times \mathbb{R}^n$ and may be stopped whenever $M_k = M_{k-1}$. □

This observation leads us, in a natural way, to the following definition:

**Definition 8.** Let $N$ be the corresponding set of a given DAE (2). We define a family $(M_k)_{k=0,\ldots,s}$ of sub-manifolds $M_k$ of $\mathbb{R} \times \mathbb{R}^n$ by the recursion:

(i) $M_0 = J \times U$
(ii) $M_{k+1} = \mathrm{pr}_{1,2}(N \cap SM_k) \quad (k = 0, \ldots, s-1)$,

where $s$ is the largest non-negative integer such that the triples $(M_k, p, J)$ are differentiable sub-bundles and $M_{s-1} \neq M_s$. In case $M_1 = J \times U$, we define $s = 0$.

We call the family $(M_k)$ the *family of constraint manifolds* and the integer $s$ the *degree* of the given DAE. □

*Remark.* In [Sza89], Szatkowski has proposed a similar definition for autonomous DAEs, independent of us.

In [RR90], [Rhe84] Rheinboldt has described a procedure for obtaining a family of manifolds as the solutions of an overdetermined system of equations. In the next section we show that this family is identical with our family of constraint manifolds. Therefore, our definition can be considered as a differential-geometric interpretation of the procedure given in [RR90], [Rhe84].

In [Rei90] we have proved that $s \leq n$ for any autonomous DAEs. Consequently, the degree $s$ of any non-autonomous DAE of type (2) satisfies $s \leq n$, too. □

By means of the family of constraint manifolds we can state sufficient conditions for the regularity of a DAE.

**Theorem 9.** *Let $N$ be the corresponding set, let $(M_k)$ be the family of constraint manifolds, and let $s$ be the degree of a given DAE (2). Then this DAE is regular if the condition*

*C1:* $(N \cap \{(t,x)\} \times S_{(t,x)}M_s)$ *contains exactly one element for each* $(t,x) \in M_s$

*is satisfied.*

*Under this condition, the configuration space $M$ of the DAE is given by $M = M_s$, and the corresponding vector field $v : M \to \mathbb{R}^n$ is defined for all $(t,x) \in M$ by*

$$(t, x, v(t,x)) \in (N \cap SM) \ .$$

*Furthermore, if additionally the condition*

*C2:* $(N \cap SM)$ *is a $C^k$-sub-manifold of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ and, for all* $(t,x,p) \in (N \cap SM)$, $\dim(M) = rank[pr_{1,2}(T_{(t,x,p)}(N \cap SM))]$

*holds, then the corresponding vector field is of the class $C^k$.* □

*Proof.* According to Observation 7, condition C1 yields that a differentiable mapping $x : I \to \mathbb{R}^n$ is a solution of (2) if and only if for all $t \in I$

$$(t, x(t), x'(t)) = (t, x(t), v(t, x(t)))$$

where $v : M \to \mathbb{R}^n$ is the vector field as defined in Theorem 9. Thus condition C1 implies the regularity of the DAE. Furthermore, condition C2 and the implicit function theorem yields that the mapping is differentiable and of the class $C^k$. □

## 5 DAEs of Global Index $i$

In several papers [BCP89], [Gea88], [Gri90], [LP86], [Mar90], [RR90], the properties of DAEs are characterized by an integer called the index of the problem. In this section we compare the notion of index with our concept of regularity and degree. First state the following definitions.

Let a DAE of type (2) be given. For any non-negative integer $m$, we consider, as long as the involved mappings are defined and differentiable, the nonlinear system of equations

$$\begin{aligned} F_0(t,x,p) &= 0 \\ F_1(t,x,p) &= 0 \\ &\vdots \\ F_m(t,x,p) &= 0 \end{aligned} \tag{4}$$

with

$$\begin{aligned} F_0(t,x,p) &:= F(t,x,p) \\ F_{k+1}(t,x,p) &:= R_k(t,x,p)[D_tF_k(t,x,p) + D_xF_k(t,x,p)p] \end{aligned}$$

where

$$R_k(t,x,p) \text{ is a projection along } \operatorname{im}[D-pF_k(t,x,p)] \ .$$

Let us write the system of equations (4) in the form

$$G_m(t,x,p) = 0 \tag{5}$$

with $t \in J$, $x \in U$, $p \in \mathbb{R}^n$, $G_m : J \times U \times \mathbb{R}^n \to (\mathbb{R}^n)^{m+1}$. We call the mapping $G_m$ the *derivative array* of order $m$.

As long as the derivative array $G_m$ of order $m$ is defined, we associate with it the two sets

$$L_m = \{(t,x,p) \in J \times U \times \mathbb{R}^n : \ G_m(t,x,p) = 0\}$$

and

$$P_m = \operatorname{pr}_{1,2}(L_m) \ .$$

Using a terminology which is appropriate for our purposes, we state the index definition, as given in [BCP89], [Gea88], in the following way:

**Definition 10.** Let a DAE of type (2) be given. The (global) *index* of the DAE, in the sense of [BCP89], [Gea88], is the smallest non-negative integer $i$, such that

C1 the triple $(P_i, p, J)$ is a $C^1$-sub-bundle of $J \times U$
C2 for any $(t,x) \in P_i$ there exists a unique $p \in \mathbb{R}^n$ with $(t,x,p) \in L_i$. □

Of course, such an integer does not exist in any case. But if the index can be defined for a given DAE (2), then (ii) defines a unique mapping $g : P_i \to \mathbb{R}^n$. Furthermore, by definition of the mapping $g$, a differentiable mapping $x : I \to \mathbb{R}^n$ is a solution of (2) if and only if $x$ is a solution of the ODE

$$x' = g(t,x) \qquad ((t,x) \in P_m)$$

defined on the manifold $P_i$. Thus we obtain the following

**Proposition 11.** *Let (2) be a DAE of index $i$. Then this DAE is regular. Furthermore, the configuration space $M$ is given by $M = P_i$ and the corresponding vector field $v : M \to \mathbb{R}^n$ is given by*

$$(t,x,v(t,x)) \in L_i$$

*for all $(t,x) \in M$.* □

We state now rank-conditions on the derivative arrays that insure the regularity of a DAE. (Similar conditions are given in [RR90] and by Griepentrog in the present book.)

**Theorem 12.** *Let (2) be a DAE where the mapping $F$ is of the class $C^r$, $r > 0$. Let us assume that there is a non-negative integer $k$ such that the derivative arrays $G_m$ of order $m$, $m \leq k$, exist and satisfy the conditions*

*C1: for $m < k$, $[D_xG_mD_pG_m](t,x,p)$ and $D_pG_m(t,x,p)$ have constant rank for all $(t,x,p) \in J \times U \times \mathbb{R}^n$*
*C2: the row echelon form of $D_pG_k(t,x,p)$ is*

$$\begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

*independent of $(t,x,p) \in J \times U \times \mathbb{R}^n$.*

*Then this DAE is regular and of index $i$ where $i$ is the smallest non-negative integer such that C1 and C2 hold.* □

*Proof.* For $m < k$, the condition C1 implies that for all $(t_0,x_0,p_0) \in L_m$ there is a neighbourhood $V_m$ of $(t_0,x_0,p_0)$ in $J \times U \times \mathbb{R}^n$ such that with $W_m = p_{1,2}(V_m)$,

$$L_m \cap V_m = \{(t,x,p) \in V_m : 0 = h_m(t,x,p), \quad 0 = g_m(t,x)\}$$

and

$$P_m \cap W_m = \{(t,x) \in W_m : \quad 0 = g_m(t,x)\}$$

where $h_m : V \to \mathbb{R}^{p_m}$, $g_m : W \to \mathbb{R}^{q_m}$ are $C^{r-m}$-mappings with $\mathrm{rank}[D_xg_m(t,x)] = q_m$ and $\mathrm{rank}[D_ph_m(t,x,p)] = p_m$ for all $(t,x,p) \in V_m$. Thus the sets $P_m$ are differentiable sub-bundles. The condition C2 implies that there is a mapping $h : W_k \to \mathbb{R}^n$ of class $C^{r-k}$ such that

$$L_k \cap V_k = \{(t,x,p) \in V_k : p = h_k(t,x),\ 0 = g_{k-1}(t,x)\}$$

where $V_k = W_{k-1} \times \mathbb{R}^n$. Thus the DAE (2) is regular and of index $i$. □

*Remark.* The index of a DAE allows also for categorizing the effects of perturbations on DAEs. In particular, we consider perturbations of the type

$$F(t, x, x') - z(t) = 0$$

where $z : J \to \mathbb{R}^n$ is a given $C^r$-mapping. In general, we can assume that the perturbed DAE has the same index $i$ for arbitrary perturbations $z$. For any special perturbation $z$ we denote the corresponding vector field by $v_z : M_z \to \mathbb{R}^n$. The definition of the index implies now that the vector field $v_z$ depends on $z(t)$, $Dz(t)$, ..., $D^i z(t)$ for all $(t, x) \in M_z$. This yields, however, that for higher index DAEs higher order derivatives of the perturbation influence the solutions. Consequently, the solutions of higher index problems are more sensitive with respect to perturbations than those of lower index DAEs. Results in a similar spirit have been already been derived in, e. g., [HLR89] and [Mar90]. $\square$.

In the remainder of this section we compare the notions of index and degree and show that the constraint manifolds $M_k$ are identical with the sets $P_{k-1}$. Thus the derivative arrays allow for a computation of the constraint manifolds.

**Observation 13.** Let (2) be a DAE of degree $s$, let $N$ be the corresponding set, and let $(M_k)$ be the family of constraint manifolds. Furthermore we assume that the derivative arrays exist up to and including order $s$. Obviously, by definition of the derivative array $G_0$, we have

$$L_0 = N$$

and thus

$$P_0 = M_1 \quad .$$

Now the set $N \cap SM_1$ can be characterized by

$$N \cap SM_1 = \{(t, x, p) : \exists \underline{p} \in \mathbb{R}^n \text{ such that } (t, x, p, \underline{x}, \underline{p}) \in SN, p = \underline{x}\} \quad .$$

Because the set $SN$ is given by the solutions of

$$\begin{aligned} 0 &= F(t, x, p) \\ 0 &= D_t F(t, x, p) + D_x F(t, x, p)\underline{x} + D_p F(t, x, p)\underline{p} \quad , \end{aligned}$$

we obtain that $N \cap SM_1$ is given by the solutions of the following system of equations

$$\begin{aligned} 0 &= F(t, x, p) \\ 0 &= R(t, x, p)[D_t F(t, x, p) + D_x F(t, x, p)p] \end{aligned} \qquad (6)$$

where $R(t, x, p)$ is a projection along $\mathrm{im}[D_p F(t, x, p)]$. However, (6) is equivalent with the derivative array of order 1. Thus we have

$$L_1 = N \cap SM_1$$

and

$$P_1 = M_2 \quad .$$

$\square$

This process can be continued and we obtain the following

**Proposition 14.** *Let (2) be a DAE of degree s. Furthermore, let the derivative arrays exist up to and including order s. Then we have*

$$L_k = N \cap SM_k$$

*and*

$$P_k = M_{k+1}$$

*for all* $k \in \{0, \ldots, s\}$. □

## 6 Index Transformations

In several papers [BCP89], [Gea88], [LP86] a straightforward index reduction through differentiation the algebraic constraints is suggested. Differentiating a DAE of index $i$ $i$-times yields an ODE which is often called the underlying ODE of the problem. Now Theorem 3 implies that the configuration space $M$ of the given DAE is an invariant manifold of the underlying ODE and that the restriction of the underlying ODE to the configuration space is equivalent to the corresponding vector field. Thus the reduction by differentiation turns constraints into integral invariants. It is well-known (see e. g. [EFLR90]) that these integral invariants are not fulfilled under discretization. Therefore, most of the current proposals for solving higher index DAEs numerically combine discretization of the underlying ODE with some sort of stabilizing use of the constraints [EFLR90], [Bau72].

We follow here a somewhat different approach which does not require the underlying ODE and which is based on index reductions that maintain the invariants during reduction; i. e., we look for reductions that satisfy the following

**Definition 15.** Let (2) be a DAE of index $i$, $i > 1$. Furthermore, let $S : \mathbb{R}^n \to \mathbb{R}^n$ and $T : J \times U \times \mathbb{R}^n \to J \times U \times \mathbb{R}^n$ be two arbitrary differentiable mappings. Then we call the DAE

$$F^{\#}(t, x, x') = 0 \tag{7}$$

with $F^{\#} = S \circ F \circ T$ a *transformed DAE*, if the corresponding vector fields of the original and the transformed DAE are identical. If the transformed DAE (7) is a DAE of index $j$, $j < i$, then we call the transition from the DAE (2) to the DAE (7) an *index transformation of degree* $m$ with $m = i - j$. □

In this section we show that any DAE of index $i$, $i > j$, can be transformed to a DAE of index one. Within the framework of regular DAEs, the DAEs of index 1 are especially well understood. For such DAEs consistent initial values can be computed easily. Moreover, methods for the numerical integration of index one problems are well investigated [BCP89], [GM86], [HLR89].

To motivate our index transformation technique, we make the following observation.

**Observation 16.** Let (2) be a DAE of index $i$, $i > 1$, let $N$ be the corresponding set, and let $(M_k)$ be the family of constraint manifolds. Furthermore, let $k$ be an arbitrary integer with $m \in \{1, \ldots, i-1\}$. We consider now a mapping $Q : J \times U \times \mathbb{R}^n \to J \times U \times \mathbb{R}^n$ that satisfies the condition

C: $Q(t, x, \cdot)$ is a projection onto $\{(t, x)\} \times S_{(t,x)} M_m$.

Obviously, if a differentiable mapping $x : I \to \mathbb{R}^n$ is a solution of (2) then we have

$$(t, x(t), x'(t)) = Q(t, x(t), x'(t))$$

for all $t \in I$; i. e., $Q$ acts on the solutions of (2) like the identity. This motivates the investigation of a new DAE

$$F^{\#}(t, x, x') = 0 \tag{8}$$

with $F^{\#} \circ Q$.

Let $N^{\#}$ be the corresponding set of the DAE (8) and let $(M_k^{\#})$ be the family of constraint manifolds of this DAE. Then, by definition of the mapping $F^{\#}$, we have

$$N^{\#} = Q^{-1}(N \cap SM_m) \ .$$

However this implies

$$M_1^{\#} = M_{m+1}$$

and

$$N^{\#} \cap SM_1^{\#} = N \cap SM_{m+1} \ .$$

□

As a consequence of Observation 16 we obtain:

**Theorem 17.** *Let (2) be a DAE of index $i$, $i > 1$, and let $k$ be an arbitrary integer with $m \in \{1, \ldots, i-1\}$. Then the DAE (8), as defined in Observation 16, is a DAE of index $j$ with $j = i - m$. Furthermore, the DAE (8) is a transformed DAE of (2) and thus the transition from (2) to (8) is an index transformation of degree $m$.* □

We like to demonstrate this result by means of the following problem:

Let (2) be a DAE of index $i$, $i > 1$, and let us assume that the constraint manifold $M_m$, $m < i$, is given by

$$M_m = \{(t, x) \in J \times U : g(t, x) = 0\}$$

with $g : J \times U \to \mathbb{R}^m$ and $\operatorname{rank}[D_x g(t, x)] = m$ for all $(t, x) \in J \times U$. We like to remark that, according to the proof of Theorem 12, such a mapping $g$ can be found at least locally if the rank-condition C1 of Theorem 12 is satisfied.

The restricted tangent space $S_{(t,x)} M_m$ at $(t, x)$ can be identified with the solutions $p \in \mathbb{R}^n$ of

$$D_t g(t, x) + D_x g(t, x) p = 0 \ .$$

To obtain a transformed DAE, we need a mapping $Q$ which satisfies the condition C of Observation 16. Under the above assumptions, such a mapping can be obtained by

$$Q(t,x,p) = (t,x,[P(t,x)p - D_x g(t,x)^+ D_t g(t,x)])$$

where $P(t,x)$ is a projection onto $\ker[D_x g(t,x)]$ and $D_x g(t,x)^+$ denotes the Moore-Penrose inverse of $D_x g(t,x)$. $P(t,x)$ and $D_x g(t,x)^+$ are, for example, given by

$$D_x g(t,x)^+ = D_x g(t,x)^T [D_x g(t,x) D_x g(t,x)^T]^{-1}$$

and

$$P(t,x) = I - D_x g(t,x)^+ D_x g(t,x) \ .$$

By means of the above defined mapping $Q$, the transformed DAE is given by

$$F(t,x[P(t,x)x' - D_x g(t,x)^+ D_t g(t,x)]) = 0 \ .$$

According to Theorem 17, this DAE is a DAE of index $j$ with $j = i - m$.

*Remark.* For semi-explicit DAEs in Hessenberg form [BCP89] the above described index transformation technique yields formulations which are equivalent in essence to the ones proposed by Gear in [Gea88]. This fact has been explored further in [Chu69]. Finally we like to remark that problems concerning index reduction methods are also discussed by Griepentrog in the present book. □

## 7 Linear Time-Varying DAEs

In this section we apply our results to linear DAEs

$$A(t)x' = B(t)x + q(t)$$

where $A, B : J \to L(\mathbb{R}^n)$ are differentiable matrix functions. These linear problems probably represents the most extensively studied DAEs in the literature, (see e. g., [BCP89], [GM86], [Mar90]). To simplify the discussion, we restrict ourselves to the case $q(t) = 0$, i. e., we consider from now on the DAE

$$A(t)x' = B(t)x \ . \tag{9}$$

According to Definition 8, the fibres of the constraint manifolds $M_1$ of this problem are given by

$$M_1(t) = \{x \in \mathbb{R}^n : B(t)x \in \mathrm{im}[A(t)]\} \ .$$

Let $R : J \to L(\mathbb{R}^n)$ be a differentiable projector function with $\ker[R(t)] = \mathrm{im}[A(t)]$ on $J$. Then the fibres of the constraint manifold $M_1$ are also characterized by

$$M_1(t) = \ker[R(t)B(t)] \ .$$

Let us assume now that the mapping $RB$ has a differentiable nullspace and let $P : J \to L(\mathbb{R}^n)$ be a differentiable projector function with $\mathrm{im}[P(t)] = \ker[R(t)B(t)]$ on $J$; i. e., let $P(t)$ be a projector onto $M_1(t)$.

In the sequel we apply the results of Sect. 4 through 6 to the DAE (9). First we note that the intersection of the corresponding set $N$ and the restricted tangent bundel $SM_1$ is given by the solutions $(t, x, p)$ of

$$A(t)p + B(t)x = 0 \tag{10.1}$$

$$p \in S_{(t,x)}M_1 \ . \tag{10.2}$$

Thus, according to Theorem 9 and Proposition 11, the DAE (9) is of index one if and only if (10) has a unique solution $p \in \mathbb{R}^n$ for all $(t, x) \in M_1$. Now (10.2) can be replaced by

$$P(t)p + D_t P(t)x = p \tag{11}$$

and, because of linearity, we reformulate the resulting system of equations to

$$\begin{aligned} 0 &= A(t)p \\ 0 &= [I - P(t)]p \ . \end{aligned}$$

However, the unique solvability of this system of equations with respect to $p$ is equivalent to the condition

$$\det[A(t) + R(t)B(t)] \neq 0 \ . \tag{12}$$

Thus (9) is of index one if and only if (12) holds. This result has been derived in other papers before (see, e. g., [Gri90], [RR90]) and is equivalent to many other index one criteria [GM86], [Mar90], [RR90].

As already suggested in [Rei90], higher index DAEs can be characterized by means of the index transformation technique introduced in Sect. 6. Because this approach is discussed in detail by Griepentrog in the present book, we like to sketch her only the basic idea.

To carry out an index transformation of degree one (as suggested in Observation 16), we need a projector $Q(t, x) : \mathbb{R}^n \to \mathbb{R}^n$ onto the restricted tangent space $S_{(t,x)}M_1$ for all $(t, x) \in M_1$. However, because of (11), such a projector is given by

$$Q(t, x)p = P(t)p + D_t P(t)x \ .$$

According to Observation 16 we obtain the transformed DAE

$$A(t(P(t)x' + [B(t) + A(t)D_t P(t)]x = 0 \tag{13}$$

which is of index $i - 1$ if and only if (9) is of index $i$. This implies for example that (9) is of index two if and only if (13) is of index one. Now (13) is of index one if and only if

$$\det[A(t)P(t) + R_1(t)\{B(t) + A(t)D_t P(t)\}] \neq 0$$

where $R_1 : J \to L(\mathbb{R}^n)$ is a projector function with $\ker[R_1(t)] = \mathrm{im}[A(t)P(t)]$ on $J$.

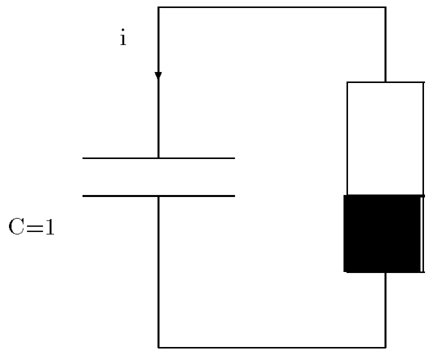


Fig. 1

## 8 A Simple RC-Circuit

Throughout this section let us consider the RC-circuit of Fig.1, where we assume the voltage-current relation $R$ of the non-linear resistor to be given by

$$R = \{(t, u_R, i_R) \in \mathbb{R} \times \mathbb{R}^2 : 0 = g(t, u_R, i_R)\}$$

with $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ of class $C^1$ and $\operatorname{rank}[D_2 g \; D_3 g](t, u_R, i_R) = 1$ for all $(t, u_R, i_R) \in \mathbb{R} \times \mathbb{R}^2$. This implies that the set $R$ is a differentiable manifold and that the time behaviour of the RC-circuit can be modelled by the DAE

$$q' = i \tag{14.1}$$

$$0 = g(t, q, -i) \ . \tag{14.2}$$

Obviously, equation (14.2) cannot be solved globally for the variable $i$ in general. Consequently, the state equations (3) do not exist in the general case. Therefore, we apply now the theory of regular DAEs to the DAE (14).

Clearly, by definition, the constraint manifold $M_1$ is given by

$$M_1 = \{(t, q, i) \in \mathbb{R} \times \mathbb{R}^2 : 0 = g(t, q, -i)\} \ .$$

A point $(t, q, i) \in M_1$ is now an element of the set $M_2$ if and only if

$$(N_{(t,q,i)} \cap S_{(t,q,i)} M_1) \neq \emptyset \tag{15}$$

where

$$N_{(t,q,i)} = \{(\underline{q}, \underline{i}) \in \mathbb{R}^2 : \underline{q} = i\}$$

and

$$S_{(t,q,i)}M_1 = \{(\underline{q}, \underline{i}) : 0 = D_t g(t,q,-i) + D_q g(t,q,-i)\underline{q} - D_i g(t,q,-i)\underline{i}\} \ .$$

It is readily shown that we have

$$(N_{(t,q,i)} \cap S_{(t,q,i)}M_1) = \emptyset \tag{16}$$

if and only if $D_i g(t,q,-i) = 0$ and $D_q g(t,q,-i)^{-1} D_t g(t,q,-i) \neq -i$. This is illustrated in Fig. 2a. In the sequel we call a point $(t,q,i) \in M_1$ satisfying (16) a singular point.

Obviously, in case (15) holds, we have either

(i) $D_i g(t,q,-i) = 0$ and $D_q g(t,q,-i)^{-1} D_t g(t,q,-i) = -i$ (Fig. 2b)

or
(ii) $D_i g(t,q,-i) \neq 0$. (Fig. 2c)

Fig. 2a

Fig. 2b

Fig. 2c

For (ii) we conclude that the set

$$N_{(t,q,i)} \cap S_{(t,q,i)} M_1$$

contains exactly one element, while for (i) this set is diffeomorphic to the real line $\mathbb{R}$. Therefore, we call a point $(t, q, i) \in M_1$ regular of (ii) holds, and degenerated if (i) holds.

According to the remark in Sect. 4, we conclude that the degree $s$ of the DAE (14) cannot exceed $s = 2$.

We discuss now the results obtained so far by means of the specific voltage-current relations given in Fig. 3a-d. To simplify the discussion, we consider here only time-invariant voltage-current relations. As a consequence of this, the constraint manifolds $M_i$ are trivial sub-bundles which we denote by

$$M_i = \mathbb{R} \times Q_i \ .$$

Clearly, by considering the manifolds $Q_i$ instead of the manifolds $M_i$, we can reduce the dimension of the problem by one. Therefore, to obtain good drawings, this fact is explored in the subsequent examples.

Fig. 3a

Fig. 3b

Fig. 3c

Fig. 3d

*Example 1.* First we consider the RC-circuit of Fig. 1 with the voltage-current relation of the nonlinear resistor as depicted in Fig. 3a. Clearly, we have

$$Q_2 = Q_1 \;\; ;$$

that is, every point $(q, i) \in Q_1$ is a regular point. Consequently, the DAE (14) is regular and of index one. □

*Example 2.* We now consider the voltage-current relation $R$ of Fig. 3b. We obtain

$$Q_2 = Q_1 \setminus \{(-1,-1),(1,1)\} \;\; ;$$

that is, $(-1,-1)$ and $(1,1)$ are singular points and all the other points belonging to $Q_1$ are regular. Thus, the DAE (14) is in this case regular and of index two. The vector field on $Q_2$ is depicted schematically in Fig. 4a. Obviously, no solution can be extended beyond the two points $(-1,-1)$ and $(1,1)$. Therefore, such points are called impasse points [Chu69]. Impasse points are an important phenomenon found in many circuits (see, e. g., [Chu69]) in connection with the so-called jump behaviour. □

*Example 3.* Let the voltage-current relation $R$ be given by Fig. 3c with $g(u_R, i_R) = i_R^2 + u_R$. Similar to Example 1 we obtain

$$Q_2 = Q_1 \ .$$

However, the point $(0,0)$ is a degenerated point and, as a consequence of this, the condition C1 of Theorem 9 does not hold because

$$N_{(0,0)} \cap T_{(0,0)} Q_2$$

is diffeomorphic to the real line $\mathbb{R}$. Clearly, the two mappings $x_i : I \to \mathbb{R}^2$ $(i = 1, 2)$ defined by

$$x_1(t) = (0,0) \quad \text{and} \quad x_2(t) = (-t^2/4, -t/2) \quad (t \in I)$$

are solutions of the DAE (14) through the point $(0,0)$. Now, because of

$$x_1'(0) \neq x_2'(0) \ ,$$

we conclude that the DAE (14) with the characteristics of Fig. 3c cannot be regular. □

*Example 4.* We now consider the voltage-current relation $R$ of Fig. 3d. The resulting circuit is equivalent to the circuit which we obtain by replacing the nonlinear resistor by a dc-voltage source $u(t) = U$. With expection of the point $(U, 0)$, all the other points belonging to $Q_1$ are singular and we get therefore

$$Q_2 = \{(U,0)\} \ .$$

The point $(U,0)$ is degenerated. But because $\dim(Q_2) = 0$, we have

$$(N_{(U,0)} \cap T_{(U,0)} Q_2) = \{(0,0)\} \ .$$

Accordingly, the Condition C1 of Theorem 9 is satisfied and the DAE (14) is regular with the index two (Fig. 4b). □

Fig. 4a

Fig. 4b

## 9 Concluding Remarks

This paper has described a number of theoretical results concerning the relation of DAEs to vector fields. Based on this, we have formulated an existence and uniqueness result and have derived an index transformation technique for higher-index DAEs. Further studies about the relation of DAEs to vector fields on special fields, such as bifurcation and stability theory, are still needed.

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