On a Differential-Algebraic Inclusion Model for LRS-Networks

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1 Introduction

In this paper we investigate an initial value problem for a system of algebraic and differential inclusions, which can be obtained from the loop equations of an electrical network consisting of resistors (R), inductors (L), diods (switches (S)) and voltage sources if one use the ideal current-voltage-characteristic

$$U_D(i) := \begin{cases} 0 & \text{if } i > 0 \\ [0, +\infty) & \text{if } i = 0 \end{cases} \tag{1.1}$$

for the diods. A simple example of such a network is the rectifier bridge circuit shown in Fig. 1. For the three loop currents i_1, i_2, i_3 we have the equations

Fig. 1. Rectifier brigde circuit

$$
\begin{aligned}
L_1 di_1/dt' &= U\ \sin\omega t' - R_D(2i_1 - i_2 + 2i_3) + U_D(i_1 + i_3) - \\
&\quad - U_D(i_2 - i_1 - i_3) \\
L_2 di_2/dt' &= -E - R_D(-i_1 + 2i_2 - 2i_3) + U_D(i_2 - i_1 - i_3) + \\
&\quad + U_D(i_2 - i_3) \\
4R_D i_3 &= -2R_D(i_1 - i_2) + U_D(i_1 + i_3) - U_D(i_2 - i_1 - i_3) - \\
&\quad - U_D(i_2 - i_3) + U_D(i_3)\ .
\end{aligned}
\tag{1.2}
$$

Loops containing inductors lead to differential equations and the others to algebraic equations. If we use the characteristic (1.1) we obtain a system of differential-algebraic inclusions in which the set-valued term is of the form

$$\gamma(i) = \sum_{j=1}^{4} \alpha_j \cdot b_j \tag{1.3}$$

with

$$i = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix},\ b_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ b_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix},\ b_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},\ b_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\alpha_j := U_D(\langle b_j, i\rangle), \quad j = 1, \ldots, 4\ .$$

It is obvious that $-\gamma(i)$ is the subdifferential of the indicator function ψ_C of the closed convex cone

$$C := \{i \in \mathbb{R}^3 |\ \langle b_j, i\rangle \geq 0,\ j = 1, \ldots, 4\}\ .$$

Electrical networks of the mentioned type arise in power electronics (see for instance Chua/Lin [CL75]). In the following framework it is also possible to treat electrical networks which contain capacitors (C) and constant current sources.

There are several papers which are concerned with the numerical treatment of pure ordinary differential inclusions. The first convergence results were obtained for a special class of differential inclusions by Taubert [Tau76]. Now we have for differential inclusions convergence results of different types, qualitative convergence results (see for instance Niepage/Wendt [NW87]), first order convergence results (Elliott [Ell85]) and higher order convergence results (Stewart [Ste89], Kastner-Maresch [Kas90]). Less seems to be known about numerical approaches to differential-algebraic inclusions. In this paper we extend the results of Elliott [Ell85] to a special class of differential-algebraic inclusions of a monotone type and obtain, in this way, stronger results as in Niepage [Nie88].

The outline of this paper is as follows. In Sect. 2 we formulate the continuous and discrete problems treated. In Sect. 3 we prove the existence and uniqueness of a solution for both problems and the convergence of the approximate solutions to the exact solution by means of the well-known method of Rothe (see Kačur [Kac85]). In Sect. 4 we investigate the stability of the discrete problems with respect to perturbations in the inclusions and perturbations of the initial value and give a result on the order of convergence. Finally, in Sect. 5 we present an algorithm for the solution of the discrete problems, and in Sect. 6 we give some numerical results, which we obtain by this algorithm for the example mentioned above.

2 Formulation of the Continuous and Discrete Problems

The example presented in the introduction motivates the investigation of problems of the form:

Given two single-valued maps

$$f^1 : [0,T] \times (\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \longmapsto \mathbb{R}^{n_1}$$
$$f^2 : [0,T] \times \mathbb{R}^{n_1} \longmapsto \mathbb{R}^{n_2}, \quad 0 < T < \infty, \quad n_1, n_2 \in \mathbb{N} \quad ,$$

two set-valued maps

$$\gamma^k : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \longmapsto 2^{R^{n_k}}, \quad k = 1,2 \quad \text{with}$$
$$\mathrm{dom}(\gamma^1) \cap \mathrm{dom}(\gamma^2) \neq \emptyset \quad ,$$

and two linear bounded operators $A \in B(\mathbb{R}^{n_1})$, $B \in B(\mathbb{R}^{n_2})$.

Problem **(P).** Look for a function

$$z := (x,y) \in H^1([0,T], \mathbb{R}^{n_1}) \times L^2([0,T], \mathbb{R}^{n_2})$$

satisfying the system of ordinary differential and algebraic inclusions

$$\begin{aligned} Ax'(t) &\in f^1(t, z(t)) + \gamma^1(z(t)) \\ By(t) &\in f^2(t, x(t)) + \gamma^2(z(t)) \quad \text{for a. e.} \quad t \in [0,T] \end{aligned} \tag{2.1}$$

and the initial condition

$$\begin{aligned} z(0) = z_0 &:= (x_0, y_0) \in \mathrm{dom}(\gamma^1) \cap \mathrm{dom}(\gamma^2) \\ By_0 &\in f^2(0, x_0) + \gamma^2(z_0), \quad x_0 \text{ given} \quad . \end{aligned} \tag{2.2}$$

We consider the problem (P) under the following basic assumptions:

(H1) The maps f^k, are Lipschitzean with Lipschitz constants L^k, $k = 1, 2$:

$$\|f^1(t', z') - f^1(t, z)\| \leq L^1(|t' - t| + \|z' - z\|)$$
$$\|f^2(t', x') - f^2(t, x)\| \leq L^2(|t' - t| + \|x' - x\|)$$
$$\forall t', t \in [0,T], \; \forall x', x \in \mathbb{R}^{n_1}, \; \forall z', z \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \quad .$$

In $\mathbb{R}^{n_k}$, $k = 1, 2$ we choose the Euclidean norm and, in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, the Euclidean norm of $\mathbb{R}^{n_1+n_2}$ as norm. For convenience we denote all these norms by $\|\cdot\|$ and the related scalar products by $\langle \cdot, \cdot \rangle$.

(H2) **(i)** The map $(-\gamma)$, $\gamma := (\gamma^1, \gamma^2)$ with $\mathrm{dom}(\gamma) := \mathrm{dom}(\gamma^1) \cap \mathrm{dom}(\gamma^2)$ is maximal monotone in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, i. e.

$$\langle \hat{g} - g, \hat{z} - z \rangle := \langle \hat{g}^1 - g^1, \hat{x} - x \rangle + \langle \hat{g}^2 - g^2, \hat{y} - y \rangle \leq 0$$
$$\forall \hat{z}, z \in \mathrm{dom}(\gamma), \forall \hat{g} \in \gamma(\hat{z}), \forall g \in \gamma(z) \quad ,$$

and

$$\langle h - g, u - z \rangle \leq 0 \quad \forall (z, g) \in \mathrm{graph}(\gamma)$$

for a pair $u, h \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ implies $u \in \mathrm{dom}(\gamma)$ and $h \in \gamma(u)$.

(ii) For each fixed $x_0 \in \mathbb{R}^{n_1}$ with

$$D_{x_0} := \{y \in \mathbb{R}^{n_2} \,|\, (x_0, y) \in \mathrm{dom}(\gamma)\} \neq \emptyset$$

the map $(-\gamma^2_{x_0})$, $\gamma^2_{x_0}(y) := \gamma^2(x_0, y)$, is maximal monotone in $\mathbb{R}^{n_2}$ with the effective domain of definition D_{x_0}.

Notice that, under the assumption (H2)(i) the map $(-\gamma^2_{x_0})$ is monotone on D_{x_0}.

(H3) $A = A^T > \alpha I,\ B > \beta I$ with $\alpha, \beta > 0$.

For proving the existence of a solution of (P) we will use the technique, which is known in the broad field of evolution problems under the name Rothe's method (see [Kac85]). Elliott [Ell85] has used this method for proving the existence of solutions of ordinary differential inclusions of a monotone type. This method is based on a discrete version of (2.1) with respect to the time variable t. Here we choose as discrete version of (2.1) the class of approximations for (2.1) which we want to apply for determining approximate solutions for (P). We define the approximate problems by means of a sequence of grids from $[0, T]$

$$[0, T]_N := \{t_i \in [0, T] \,|\, 0 = t_0 < t_1 \cdots < t_N = T\}, \quad N \in \mathbb{N}$$

for which the maximal step size $\tau := \max\{\tau_i := t_i - t_{i-1} \,|\, i = 1, \ldots, N\}$ converges to zero with $N \to \infty$ (denoted by $(N \in \mathbb{N})$ in the following) and by two single-valued maps

$$\begin{aligned} \phi^1 &: [0, T] \times (\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \times (\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \times [0, T] \longmapsto \mathbb{R}^{n_1} \\ \phi^2 &: [0, T] \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \times [0, T] \longmapsto \mathbb{R}^{n_2} \end{aligned}$$

as follows:

Problem $(\mathbf{P_N})$. Find an $(N+1)$-tupel of pairs

$$z_i := (x_i, y_i) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \quad i = 0, \ldots, N \ ,$$

satisfying the system of inclusions

$$\left.\begin{aligned} A\delta x_i &\in \phi^1(t_{i-1}, z_{i-1}, z_i, \tau_i) + \gamma^1(z_i) \\ By_i &\in \phi^2(t_{i-1}, x_{i-1}, x_i, \tau_i) + \gamma^2(z_i) \end{aligned}\right. i = 1, \ldots, N \tag{2.3}$$

as well as the initial Condition (2.2).

Here, δx_i denotes the difference quotient $\delta x_i := (x_i - x_{i-1})/\tau_i$. The maps $\phi^k(t, \zeta, z, \tau)$, $k = 1, 2$, are needed only for those arguments for which $t + \tau \leq T$.

We assume

(H4) The maps ϕ^k are Lipschitzean with Lipschitz constants λ^k, $k = 1, 2$:

$$\begin{aligned} \|\phi^1(t', \zeta', z', \tau') - \phi^1(t, \zeta, z, \tau)\| &\leq \lambda^1(|t' - t| + |\tau' - \tau| + \\ &\quad + \|\zeta' - \zeta\| + \|z' - z\|) \\ \|\phi^2(t', \xi', x', \tau') - \phi^2(t, \xi, x, \tau)\| &\leq \lambda^2(|t' - t| + |\tau' - \tau| + \\ &\quad + \|\xi' - \xi\| + \|x' - x\|) \end{aligned}$$

$\forall t', t, \tau'\tau \in [0,T]$ with $t+\tau \leq T, t'+\tau' \leq T$, and $\forall \xi'\xi, x', x \in \mathbb{R}^{n_1}, \forall \zeta', \zeta, z', z \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and fulfil the consistency conditions

$$\begin{aligned} \phi^1(t,z,z,0) &= f^1(t,z) \\ \phi^2(t,x,x,0) &= f^2(t,x) \end{aligned} \tag{2.4}$$

$\forall t \in [0,T], \forall x \in \mathbb{R}^{n_1},\ \forall z \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

Simple examples for the choice of ϕ^k such that (H4) is satisfied are

$$\begin{aligned} \phi^1(t,\zeta,z,\tau) &:= f^1(t,\zeta) \\ \phi^2(t,\xi,x,\tau) &:= f^2(t,\xi) \end{aligned} \tag{2.5}$$

or

$$\begin{aligned} \phi^1(t,\zeta,z,\tau) &:= f^1(t+\tau,z) \\ \phi^2(t,\xi,x,\tau) &:= f^2(t+\tau,x)\ . \end{aligned} \tag{2.6}$$

If we form the Rothe-functions

$$\begin{aligned} x_N(t) &:= x_{i-1} + (t-t_{i-1})\delta x_i \quad \text{for} \quad t_{i-1} \leq t \leq t_i \\ \bar{\zeta}_N(t) &:= z_{i-1} \qquad \text{for} \quad t_{i-1} \leq t < t_i \\ \bar{z}_N(t) &:= z_i \qquad \text{for} \quad t_{i-1} < t \leq t_i \\ \bar{\zeta}_N(t_N) &:= z_{N-1},\ \bar{z}_N(0) = z_1 \end{aligned}$$

and, analogously, $\bar{\xi}_N(t)$, $\bar{x}_N(t)$, $\bar{y}_N(t)$, as well as, for $k = 1,2$,

$$\begin{aligned} \bar{\phi}^k_N(t,\cdot,\cdot,\cdot) &:= \phi^k(t_{i-1},\cdot,\cdot,\cdot) \quad \text{for} \quad t_{i-1} \leq t < t_i \\ \bar{\phi}^k_N(t_N,\cdot,\cdot,\cdot) &:= \phi^k(t_{N-1},\cdot,\cdot,\cdot), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \bar{\tau}_N(t) &:= \tau_i \qquad \text{for} \quad t_{i-1} \leq t < t_i \\ \bar{\tau}_N(t_N) &:= \tau_N\ \ , \end{aligned}$$

the system (2.3) can be written in the following continuous form

$$\begin{aligned} Ax'_N(t) &\in \bar{\phi}^1_N(t,\bar{\zeta}_N(t),\bar{z}_N(t),\bar{\tau}_N(t)) + \gamma^1(\bar{z}_N(t)) \\ B\bar{y}_N(t) &\in \bar{\phi}^2_N(t,\bar{\xi}_N(t),\bar{x}_N(t),\bar{\tau}_N(t)) + \gamma^2(\bar{z}_N(t)) \\ &\text{for a. e.} \quad t \in [0,T] \end{aligned}$$

For the existence proof in the next section we will refer to the following lemma.

Lemma 1. *If the piecewise constant Rothe-functions $\bar{\zeta}_N$, $\bar{z}_N$ converges to a function $z \in C([0,T],\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$,*

$$\|\bar{\zeta}_N - z\|_C \to 0 \quad (N \in \mathbb{N}), \quad \|\bar{z}_N - z\|_C \to 0 \quad (N \in \mathbb{N})\ ,$$

where $\|z\|_C := \sup\{\|z(t)\|\ |t \in [0,T]\}$, then under the assumption (H4) it holds

$$\|\bar{\phi}^1_N(\cdot,\bar{\zeta}_N,\bar{z}_N,\bar{\tau}_N) - f^1(\cdot,z)\|_C \to 0 \qquad (N \in \mathbb{N})\ .$$

Proof. Because of $\|\bar{\tau}_N\|_C \to 0$ ($N \in \mathbb{N}$) the assertion follows directly from the estimation

$$\begin{aligned}
&\|\bar{\phi}_N^1(t, \bar{\zeta}_N(t), \bar{z}_N(t), \bar{\tau}_N(t)) - f^1(t, z(t))\| \\
\le\ &\|\bar{\phi}_N^1(t, \bar{\zeta}_N(t), \bar{z}_N(t), \bar{\tau}_N(t)) - \bar{\phi}_N^1(t, z(t), z(t), 0)\| + \\
&+\|\bar{\phi}_N^1(t.z(t), z(t), 0) - f^1(t, z(t))\| \\
\le\ &\lambda^1(2|\bar{\tau}_N(t)| + \|\bar{\zeta}_N(t) - z(t)\| + \|\bar{z}_N(t) - z(t)\|) \\
\le\ &\lambda^1(2\|\bar{\tau}_N\|_C + \|\bar{\zeta}_N - z\|_C + \|\bar{z}_N - z\|_C)
\end{aligned}$$

which holds $\forall t \in [0, T]$. □

Of course, a corresponding assertion holds true for the maps $\bar{\phi}_N^2$, too.

3 Existence and Uniqueness of Solutions of (P) and (P_N), Convergence of Approximate Solutions

According to the general proceeding in case of Rothe's method we first prove the existence and uniqueness as well as certain a priori estimates for the solutions of the problems (P_N). Then, utilizing compactness arguments, we conclude the existence of convergent subsequences of solutions of (P_N) for the limits of which it will be shown to be solutions of (P). For this latter conclusion we exploit the fact that a set-valued maximal monotone map M in a Hilbert space X has the property that the convergence $x_N \to x$ ($N \in \mathbb{N}$), $x_N \in \mathrm{dom}(M)$ and the weak convergence $y_N \rightharpoonup y$ ($N \in \mathbb{N}$), $y_N \in M(x_N)$ imply $x \in \mathrm{dom}(M)$, $y \in M(x)$ (see [AC84]). Then, the convergence of the whole sequence of approximate solutions follows, due to the usual argumentation, from the uniqueness of a solution of (P).

Theorem 2. *Under the assumptions (H2)-(H4) the problems (P_N) for sufficiently small τ have exactly one solution for each $x_0 \in \mathbb{R}^{n_1}$ with $D_{x_0} \neq \emptyset$ (see (H2)(ii)).*

Proof. Any monotone single-valued map M_1 from a whole Hilbert space X to X, which is hemicontinuous, is maximal monotone (see [AC84]). Hence, also the sum $M_1 + M_2$ of such a map M_1 and a set-valued maximal monotone map $M_2 : X \to 2^X$ is maximal monotone (see [Bre73]). If M_1 is even strongly monotone, then $M_1 + M_2$ is surjective, which follows immediately from a surjectivity condition for maximal monotone set-valued maps given in [Roc70] (cf. also [Bre73]). Obviously, the solution of $h \in M_1(x) + M_2(x)$ is unique then. Therefore due to (H2)(ii) there exists exactly one solution y_0 of the inclusion

$$By_0 \in f^2(0, x_0) + \gamma^2(z_0)$$

where $M_1 = B, M_2 = -\gamma_{x_0}^2$.

The maps

$$M_{1,i} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \longmapsto \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \quad i = 1, \ldots, N,\ N \in \mathbb{N}$$

with

$$M_{1,i}(z) := \left(\frac{1}{\tau_i}Ax - \phi^1(t_{i-1}, z_{i-1}, z, \tau_i), By - \phi^2(t_{i-1}, x_{i-1}, x, \tau_i)\right)$$
$$\forall z = (x, y) \in \mathrm{I\!R}^{n_1} \times \mathrm{I\!R}^{n_2}$$

are continuous in z because of (H4) and for each $\varepsilon > 0$ we have due to (H3), (H4)

$$\begin{aligned}
&\langle M_{1,i}(z') - M_{1,i}(z), z' - z\rangle = \\
&\left\langle \frac{1}{\tau_i}A(x'-x) - (\phi^1(t_{i-1}, z_{i-1}, z', \tau_i) - \phi^1(t_{i-1}, z_{i-1}, z, \tau_i), x'-x\right\rangle + \\
&+ \langle B(y'-y) - (\phi^2(t_{i-1}, x_{i-1}, x', \tau_i) - \phi^2(t_{i-1}, x_{i-1}, x, \tau_i), y'-y\rangle \\
&\geq \alpha/\tau_i\|x'-x\|^2 - \lambda^1\|z'-z\|\|x'-x\| + \beta\|y'-y\|^2 - \lambda^2\|x'-x\|\|y'-y\| \\
&\geq (\alpha/\tau - \varepsilon\lambda^1 - (\lambda^1+\lambda^2)/(4\varepsilon))\|x'-x\|^2 + (\beta - \varepsilon(\lambda^1+\lambda^2))\|y'-y\|^2
\end{aligned}$$

Thus, the maps $M_{i,1}$, $i = 1, \ldots, N$, $N \in \mathrm{I\!N}$ are strongly monotone for sufficiently small τ and, on account of (H2)(i), all systems of inclusions (2.3) are uniquely solvable. $\square$

We deduce the a priori estimates for the solutions of (P_N) with the aid of the following lemma.

Lemma 3. *For each $N \in \mathrm{I\!N}$, $N \geq 2$, let $(a_i^N)_{i=1,\ldots,N}$, $(b_i^N)_{i=1,\ldots,N}$, $(\tau_i^N)_{i=1,\ldots,N}$ be finite sequences of nonnegative real numbers satisfying the following inequalities:*

(i)

$$\begin{aligned}&a_i^N \leq (1 + c_1\tau_{i-1}^N)a_{i-1}^N + c_2\tau_i^N a_i^N + b_i^N \\ &i = 2, \ldots, N,\ N \in \mathrm{I\!N},\ N \geq 2\ ,\end{aligned}$$

with real constants c_1, c_2 independent of i and N,

(ii)

$$c_2\tau < 1$$

with $\tau := \sup\{\tau_i^N \mid i = 1, \ldots, N,\ N \in \mathrm{I\!N},\ N \geq 2\}$ and

(iii)

$$\sum_{j=1}^{N} \tau_j^N \leq T, \quad N \in \mathrm{I\!N},\ N \geq 2$$

with a real constant T, independent of N.

Then there are real constants R_1, R_2 independent of i and N such that the following estimates hold:

$$a_i^N \leq R_1 a_1^N + R_2 \sum_{j=2}^{i} b_j^N, \quad i = 2, \ldots, N,\ N \in \mathrm{I\!N},\ N \geq 2\ . \tag{3.1}$$

Proof. Summing up the inequalities (i) yields

$$a_i^N \le A_i^N + (c_1+c_2)\sum_{j=2}^{i-1} \tau_j^N a_j^N + c_2\tau_i^N a_i^N$$

with

$$A_i^N = (1+c_1\tau_1^N)a_1^N + \sum_{j=2}^{i} b_j^N, \quad i=2,\dots,N,\ N\in\mathbb{N},\ N\ge 2\ .$$

The nonnegative numbers A_i^N are monotone increasing for each $N \in \mathbb{N}$, $N \ge 2$, and, hence, the well-known discrete Gronwall lemma, which obviously is true for nonequidistant sequences of grids, too, implies the claimed estimates (3.1) with

$$\begin{aligned} R_1 &= (1+c_1\tau_1^N)/(1-c_2\tau)\cdot\exp((c_1+c_2)/(1-c_2\tau)\cdot T) \\ R_2 &= 1/(1-c_2\tau)\cdot\exp((c_1+c_2)/(1-c_2\tau)\cdot T)\ . \end{aligned}$$

□

Lemma 4. *Under the assumptions (H2)-(H4) and the condition for the stepsizes* **(S1)**

$$q_1 \le \tau_i/\tau_{i-1}, \quad i=2,\dots,N,\ N\in\mathbb{N}$$

with a constant $q_1 > 0$ there exists, for sufficiently small τ, a constant $c > 0$ independent of i and N such that we have for the solutions of (P_N)

$$\begin{gathered} \|\delta x_i\| \le c, \quad \|x_i\| \le c, \quad \|y_i\| \le c, \\ \|g^1(z_i)\| \le c, \quad \|g^2(z_i)\| \le c, \quad i=1,\dots,N,\ \ N\in\mathbb{N} \end{gathered} \tag{3.2}$$

where $g^1(z_i)$, $g^2(z_i)$ denote the uniquely determined selctors of $\gamma^1(z_i)$, $\gamma^2(z_i)$ for the solutions of (2.3), respectively.

Proof. In this proof, $k_1, k_2\dots$ denote positive real generic constants independent of i and N.

(i) First we prove the uniform boundedness of the difference quotients δx_i, $i = 1,\dots,N$, $N \in \mathbb{N}$. To this end, we show that the assumption (i) of Lemma 3 is fulfilled for

$$a_i^N := \|\delta x_i\|_A^2 + 1/\tau_i \cdot \|\Delta y_i\|_B^2$$

where

$$\begin{aligned} \|\delta x_i\|_A^2 &= \langle A\delta x_i, \delta x_i\rangle \\ \|\Delta y_i\|_B^2 &= \langle B(y_i-y_{i-1}), y_i-y_{i-1}\rangle\ . \end{aligned}$$

Subtracting the corresponding $(i-1)$-th inclusion from the i-th one in (2.3) and forming the scalar products of the first difference with δx_i and the second difference with $\Delta y_i/\tau_i$ and adding them, we obtain with $d = (1+2/q_1)^2$ and an arbitrary $\varepsilon > 0$:

$$
\begin{aligned}
&\|\delta x_i\|_A^2 + 1/\tau_i\|\Delta y_i\|_B^2 \\
&\le \|\delta x_i\|_A\|\delta x_{i-1}\|_A \\
&+\lambda^1(\tau_{i-1} + |\tau_i - \tau_{i-1}| + \|\Delta z_{i-1}\| + \|\Delta z_i\|)\|\delta x_i\| \\
&+\lambda^2/\tau_i(\tau_{i-1} + |\tau_i - \tau_{i-1}| + \|\Delta x_{i-1}\| + \|\Delta x_i\|)\|\Delta y_i\| \\
\le\ & 1/2\|\delta x_i\|_A^2 + 1/2\|\delta x_{i-1}\|_A^2 + \lambda^1 \cdot d/2 \cdot \tau_i + \lambda^1/(2\alpha)\cdot\tau_i\|\delta x_i\|_A^2 \\
&+\lambda^1\varepsilon(1/(q_1\alpha)\cdot\tau_{i-1}\|\delta x_{i-1}\|_A^2 + 1/\alpha\cdot\tau_i\|\delta x_i\|_A^2 \\
&+1/(q_1\beta)\|\Delta y_{i-1}\|_B^2/\tau_{i-1} + 1/\beta\|\Delta y_i\|_B^2/\tau_i) \\
&+\lambda^1/(2\varepsilon\alpha)\cdot\tau_i\|\delta x_i\|_A^2 + \lambda^2 d/(4\varepsilon)\cdot\tau_i + 2\lambda^2\varepsilon/\beta\cdot\|\Delta y_i\|_B^2/\tau_i \\
&+\lambda^2/(4\varepsilon\alpha)\cdot(1/q_1\tau_{i-1}\|\delta x_{i-1}\|_A^2 + \tau_i\|\delta x_i\|_A^2)\ .
\end{aligned}
$$

If we choose $\varepsilon > 0$ such that

$$\lambda^1\varepsilon/(q_1\beta) \le 1/2 \quad \text{and} \quad \varepsilon(\lambda^1 + 2\lambda^2)/\beta \le 1/2$$

we get for $i = 2, \dots, N$, $N \in \mathbb{N}$

$$
\begin{aligned}
\|\delta x_i\|_A^2 + 1/\tau_i\|\Delta y_i\|_B^2 \le\ & (1+c_1\tau_{i-1})(\|\delta x_{i-1}\|_A^2 + 1/\tau_{i-1}\cdot\|\Delta y_{i-1}\|_B^2) \\
&+c_2\tau_i\|\delta x_i\|_A^2 + b_i
\end{aligned}
$$

with

$$
\begin{aligned}
c_1 &= (2\lambda^1\varepsilon + \lambda^2/(2\varepsilon))/(\alpha q_1) \\
c_2 &= (\lambda^1(1+2\varepsilon+1/\epsilon) + \lambda^2/(2\varepsilon))/\alpha \\
b_i &= (\lambda^1 + \lambda^2/(2\varepsilon))\cdot d\cdot\tau_i\ .
\end{aligned}
$$

Hence, the estimation (3.1) is valid for this a_i^N for sufficiently small τ ($c_2\tau < 1$), and for the uniform boundedness of δx_i it suffices to show the boundedness of a_1^N. With an arbitrary fixed selector $g^1(z_0) \in \gamma^1(z_0)$

$$
\begin{aligned}
A\delta x_1 \in\ & \phi^1(0, z_0, z_1, \tau_1) - \phi^1(0, z_0, z_0, 0) \\
&+\gamma^1(z_1) - g^1(z_0) + g^1(z_0) + f^1(0, z_0)
\end{aligned}
$$

implies the inequality

$$
\begin{aligned}
\|\delta x_1\|_A^2 \le\ & \lambda^1(\|\Delta z_1\| + \tau_1 + \|g^1(z_0)\| + \|f^1(0,z_0)\|)\|\delta x_1\| \\
&+\quad \langle g^1(z_1) - g^1(z_0), \delta x_1\rangle\ .
\end{aligned}
$$

Thus, because of

$$
\begin{aligned}
1/\tau_1\|\Delta y_1\|_B^2 \le\ & \lambda^2(\|\Delta x_1\| + \tau_1)\|\Delta y_1\|/\tau_1 + \\
&\langle g^2(z_1) - g^2(z_0), \Delta y_1\rangle 1/\tau_1
\end{aligned}
$$

for each $\varepsilon > 0$ we obtain

$$\|\delta x_1\|_A^2 + 1/\tau_1 \|\Delta y_1\|_B^2 \le k_1 \varepsilon (\|\delta x_1\|_A^2 + 1/\tau_1 \|\Delta y_1\|_B^2) \\ + k_2 \tau_1/\varepsilon (\|\delta x_1\|_A^2 + 1/\tau_1 \|\Delta y_1\|_B^2) + k_3/\varepsilon \ .$$

Consequently, also $a_1^N = \|\delta x_1\|_A^2 + 1/\tau_1 \|\Delta y_1\|_B^2$ is bounded for sufficiently small τ.

(ii) On account of

$$x_i = x_0 + \sum_{j=1}^{i} \tau_j \delta x_j$$

we have

$$\|x_i\| \le \|x_0\| + Tc, \quad i = 1, \cdots, N, \ N \in \mathbb{N} \ .$$

(iii) From

$$g^1(z_i) = A\delta x_i - (\phi^1(t_{i-1}, z_{i-1}, z_i, \tau_i) - \phi^1(0, z_0, z_0, 0)) + f^1(0, z_0)$$

we conclude

$$\|g^1(z_i)\| \le \|A\|c + \lambda^1(t_i + \|z_{i-1} - z_0\| + \|z_i - z_0\|) + \|f^1(0, z_0)\| \\ \le k_4 + k_5 \|y_i - y_0\|_B \quad \text{for} \quad i = 1, \ldots, N, \ N \in \mathbb{N} \ .$$

Moreover, with an arbitrary $\varepsilon > 0$, it holds

$$\|y_i - y_0\|_B^2 \le \lambda^1(t_i + \|x_{i-1} - x_0\| + \|x_i - x_0\|)\|y_i - y_0\| \\ + (\|g^1(z_i)\| + \|g^1(z_0)\|)\|x_i - x_0\| \\ \le k_6 \varepsilon \|y_i - y_0\|_B^2 + k_7/\varepsilon + k_8 \ .$$

Consequently, also the y_i and the $g^1(z_i)$ are uniformly bounded and, hence, the $g^2(z_i)$, too.

□

For the corresponding Rothe-functions Lemma 4 implies

$$x_N \in W^{1,\infty}([0,T], \mathbb{R}^{n_1}) \text{ with } x_N'(t) = \delta x_i \text{ for a. e. } t \in [0,T]$$
$$\bar{\zeta}_N, \bar{z}_N \in L^\infty([0,T], \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$$
$$\bar{\xi}_N, \bar{x}_N \in L^\infty([0,T], \mathbb{R}^{n_1}), \quad \bar{y}_N \in L^\infty([0,T], \mathbb{R}^{n_2})$$

and these functions are uniformly bounded in the norms of the designed Banach spaces. In particular it holds therefore

$$x_N \in H^1([0,T], \mathbb{R}^{n_1}), \quad \bar{y}_N \in L^2([0,T], \mathbb{R}^{n_2})$$

and the uniform boundedness of these functions in the designed Hilbert spaces. Hence, there are elements

$$x \in H^1([0,T], \mathbb{R}^{n_1}), \quad y \in L^2([0,T], \mathbb{R}^{n_2})$$

and weakly convergent subsequences $(x_N)_{N\in \mathbb{N}'\subset \mathbb{N}}$, $(\bar{y}_N)_{N\in \mathbb{N}'\subset \mathbb{N}}$ with

$$x_N \rightharpoonup x \quad (N\in \mathbb{N}') \quad \text{in} \quad H^1([0,T],\mathbb{R}^{n_1})$$

and

$$\bar{y}_N \rightharpoonup y \quad (N\in \mathbb{N}') \quad \text{in} \quad L^2([0,T],\mathbb{R}^{n_2}) \ .$$

Consequently, $x_N \to x$ $(N \in \mathbb{N}')$ in $C([0,T],\mathbb{R}^{n_1})$ and $x_N \rightharpoonup x$ $(N \in \mathbb{N}')$ in $L^2([0,T],\mathbb{R}^{n_1})$. Since

$$\|\bar{x}_N(t) - x_N(t)\| \le \tau \|\delta x_i\| \quad \text{for} \quad t_{i-1} \le t \le t_i$$

and

$$\|\bar{\xi}_N(t) - x_N(t)\| \le \tau \|\delta x_i\| \quad \text{for} \quad t_{i-1} \le t \le t_i$$

we have

$$\|\bar{x}_N - x\|_C \to 0 \quad (N\in \mathbb{N}'), \quad \|\bar{\xi}_N - x\|_C \to 0 \quad (N\in \mathbb{N}') \ .$$

Next we show, that the limit x is Lipschitz continuous and the limit y is Hölder continuous with the exponent 1/2 and that

$$\|\bar{y}_N - y\|_C \to 0 \qquad (N\in \mathbb{N}') \ .$$

Since the x_N are uniformly Lipschitz continuous

$$\|x_N(t') - x_N(t)\| \le C|t'-t|, \quad \forall t',t\in[0,T] \ ,$$

it follows

$$\|x(t') - x(t)\| \le C|t'-t|, \quad \forall t',t\in[0,T]$$

with the same Lipschitz constant C. We have

$$\begin{aligned}\|\bar{y}_N(t) - \bar{y}_{N'}(t)\|_B^2 \le \lambda^2/\beta^{1/2}(&\|\bar{\xi}_N(t) - \bar{\xi}_{N'}(t)\| + 2\max(\tau,\tau') + \\ &\|\bar{x}_N(t) - \bar{x}_{N'}(t)\|)\|\bar{y}_N(t) - \bar{y}_{N'}(t)\|_B + \\ &(\|g^1(\bar{z}_N(t))\| + \|g^1(\bar{z}_{N'}(t))\|)\|\bar{x}_N(t) - \bar{x}_{N'}(t)\|\end{aligned}$$

and the $\|g^1(\bar{z}_N(t))\|$ are uniformly bounded in t and $N\in\mathbb{N}'$. Hence $\{\bar{y}_N(t)\}_{N\in\mathbb{N}'}$ is a Cauchy sequence in $\mathbb{R}^{n_2}$ for each t and for the limit function $\tilde{y}:\ [0,T]\to \mathbb{R}^{n_2}$ it holds

$$\|\bar{y}_N - \tilde{y}\|_C \to 0 \qquad (N\in \mathbb{N}') \ .$$

The analogous estimation for $\|\bar{y}_N(t') - \bar{y}_N(t)\|_B^2$ yields the Hölder continuity of $\tilde{y}$ with the exponent 1/2. Since $\|\bar{y}_N - \tilde{y}\|_C \to 0$ $(N\in\mathbb{N}')$ implies $\bar{y}_N \to \tilde{y}$ $(N\in\mathbb{N}')$ in $L^2([0,T],\mathbb{R}^{n_2})$ we have $\tilde{y} = y$. Now Lemma 1 provides

$$\begin{aligned}\|\bar{\phi}^1_N(\cdot,\bar{\zeta}_N,\bar{z}_N,\bar{\tau}_N) - f^1(\cdot,z)\|_C \to 0 \quad (N\in\mathbb{N}')\\ \|\bar{\phi}^2_N(\cdot,\bar{\xi}_N,\bar{x}_N,\bar{\tau}_N) - f^2(\cdot,x)\|_C \to 0 \quad (N\in\mathbb{N}')\end{aligned}$$

and thus it holds

$$g(\bar{z}_N) \rightharpoonup \tilde{g}(z) \quad (N\in\mathbb{N}') \quad \text{in} \quad L^2([0,T],\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$$

with

$$\tilde{g}(z(t)) = (Ax'(t) - f^1(t, z(t)), By(t) - f^2(t, x(t))) \ .$$

The set-valued map γ is maximal monotone in $L^2([0,T], \mathrm{I\!R}^{n_1} \times \mathrm{I\!R}^{n_2})$, too, as shown in [AC84]. Because of $\bar{z}_N \to z$ $(N \in \mathrm{I\!N}')$ in $L^2([0,T], \mathrm{I\!R}^{n_1} \times \mathrm{I\!R}^{n_2})$, we have according to the property of maximal monotone maps, which was mentioned at the beginning of this section, $\tilde{g}(z) \in \gamma(z)$, i. e. $z = (x, y)$ is a solution of (P).

Finally, we want to show the unique solvability of (P), from which it follows the uniform convergence of the whole sequence of the Rothe-functions formed by the solutions of (P_N).

Theorem 5. *Under the assumptions (H1)-(H3) the problem (P) has at most one solution for each* $x_0 \in \mathrm{I\!R}^{n_1}$ *with* $D_{x_0} \neq \emptyset$.

Proof. Let $z_1(\cdot)$, $z_2(\cdot)$ be two solutions of (P). Subtracting the corresponding inclusions (2.1) valid for z_1, z_2, forming the scalar products of the differences with $\Delta x(t) := x_1(t) - x_2(t)$ and $\Delta y(t) := y_1(t) - y_2(t)$, respectively, and adding them, we obtain for arbitrary $\varepsilon > 0$:

$$\begin{aligned}
&1/2 \cdot \frac{d}{dt} \|\Delta x(t)\|_A^2 + \|\Delta y(t)\|_B^2 \\
&\leq L^1 \|\Delta z(t)\| \, \|\Delta x(t)\| + L^2 \|\Delta x(t)\| \cdot \|\Delta y(t)\| \\
&\leq L^1 \varepsilon (1/\alpha \|\Delta x(t)\|_A^2 + 1/\beta \cdot \|\Delta y(t)\|_B^2) \\
&\quad + L^2 \varepsilon/\beta \cdot \|\Delta y(t)\|_B^2 + (L^1 + L^2)/(4\varepsilon\alpha) \|\Delta x(t)\|_A^2 \\
&= R \|\Delta x(t)\|_A^2 + (L^1 + L^2)\varepsilon/\beta \|\Delta y(t)\|_B^2
\end{aligned}$$

with $R = (L^1 \varepsilon + (L^1 + L^2)/(4\varepsilon))/\alpha$.
Choosing ε such that $(L^1 + L^2)\varepsilon/\beta < 1$, provides by integrating the estimates

$$\|\Delta x(t)\|_A^2 \leq \|\Delta x(0)\|_A^2 + 2R \int_0^t \|\Delta x(t')\|_A^2 \, dt'$$

and by means of Gronwall's Lemma (see [Kac85])

$$\|\Delta x(t)\|_A \leq \|\Delta x(0)\|_A e^{Rt} \qquad \forall t \in [0, T] \ .$$

This shows that the x-component of the solution is uniquely determined by the initial value x_0. For the difference $\Delta y(t)$ we have the estimates

$$\begin{aligned}
\|\Delta y(t)\|_B^2 &\leq L^2 \|\Delta x(t)\| \cdot \|\Delta y(t)\| + \langle\!\langle g^2(z_1(t)) - g^2(z_2(t)), \Delta y(t) \rangle\!\rangle \\
&\leq 0 \qquad\qquad \forall t \in [0, T]
\end{aligned}$$

in case of $\|\Delta x(t)\| = 0$ and therefore the y-component of the solution is uniquely determined by the initial value x_0, too. □

Since (H4) can be fulfilled under the assumption (H1), and the condition (S1) can be fulfilled by choosing equidistant grids for instance, we obtain

Theorem 6.

(i) *Under the assumptions (H1)-(H3) the problem (P) has exactly one solution* $z(t) = (x(t), y(t))$, $t \in [0, T]$, *for each* $x_0 \in \mathbb{R}^{n_1}$ *with* $D_{x_0} \neq \emptyset$, *where the* x*-component of the solution is Lipschitz continuous and the* y*-component of the solution is Hölder continuous with the exponent* $1/2$.

(ii) *If (H4) and the stepsize condition (S1) hold additionally, then the Rothe-functions* x_N, $\bar{\xi}_N$, $\bar{x}_N$ *respectively* $\bar{y}_N$, *which were formed by the solutions of* (P_N) *for sufficiently small* τ, *converge uniformly on* $[0, T]$ *to the* x*- and* y*-component of the solution of (P), respectively.*

4 Stability, order of convergence

In this section we first investigate the sensitivity of the solutions of (P_N) with respect to perturbations δ_i^1, $i = 1, \ldots, N$, δ_i^2, $i = 0, 1, ..., N$, in the inclusions (2.3). Next, for an important special case, which the introductory example also belongs to, we present improved stability inequalities, which consider perturbations δ_0^1 of the initial value x_0, too. For this special case we formulate as usually by means of the improved stability inequalities, a statement about the order of convergence for a practical important class of solutions. Besides the problems (P_N) we consider perturbed versions of the form

Problem $(\hat{\mathbf{P}}_{\mathbf{N}})$. Find an $(n+1)$-tuple of pairs $\hat{z}_i := (\hat{x}_i, \hat{y}_i) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $i = 0, 1, \ldots, N$, which satisfies the system of inclusions

$$\begin{aligned} A\delta\hat{x}_i &\in \phi^1(t_{i-1}, \hat{z}_{i-1}, \hat{z}_i, \tau_i) + \delta_i^1 + \gamma^1(\hat{z}_i) \\ B\hat{y}_i &\in \phi^2(t_{i-1}, \hat{x}_{i-1}, \hat{x}_i, \tau_i) + \delta_i^2 + \gamma^2(\hat{z}_i) \\ & i = 1, \ldots, N \ , \end{aligned} \tag{4.1}$$

and the initial condition

$$\hat{z}_0 := (x_0, \hat{y}_0) \in \mathrm{dom}(\gamma) \tag{4.2}$$

with $B\hat{y}_0 \in f^2(0, x_0) + \delta_0^2 + \gamma^2(\hat{z}_0)$, x_0 be given.

Thereby consider $\delta_i^1 \in \mathbb{R}^{n_1}$, $\delta_i^2 \in \mathbb{R}^{n_2}$ to be given. The deviations of the solutions of (4.1) from those of (2.3) we denote by

$$e_i^x := \hat{x}_i - x_i, \quad e_i^y := \hat{y}_i - y_i, \quad e_i^z := \hat{z}_i - z_i, \quad i = 0, 1, \ldots, N \ .$$

Then we have

$$e_0^x = 0 \qquad \text{and} \qquad \|e_0^y\|_B \le \|\delta_0^2\| / \beta^{1/2} \ .$$

Multiplying the inclusions in (4.1) and (2.3) by τ_i, subtracting then each inclusion of (2.3) from the corresponding one of (4.1), forming the scalar products of this differences with e_i^x and e_i^y, respectively, and adding them finally yields for each $\varepsilon > 0$ the following estimates

$$\begin{aligned}
\|e_i^x\|_A^2 + \tau_i \|e_i^y\|_B^2 \le\ & \|e_i^x\|_A \|e_{i-1}^x\|_A + \tau_i \lambda^1 (\|e_{i-1}^z\| + \|e_i^z\|)\|e_i^x\| \\
& + \tau_i \lambda^2 (\|e_{i-1}^x\| + \|e_i^x\|)\|e_i^y\| + \tau_i \|\delta_i^1\|\, \|e_i^x\| + \tau_i \|\delta_i^2\|\, \|e_i^y\| \\
\le\ & 1/2 \cdot \|e_i^x\|_A^2 + 1/2 \|e_{i-1}^x\|_A^2 + \tau_i \lambda^1 \varepsilon (\|e_{i-1}^x\|^2 + \|e_i^x\|^2 \\
& + \|e_{i-1}^y\|^2 + \|e_i^y\|^2) + \tau_i \lambda^1/(2\varepsilon) \|e_i^x\|^2 \\
& + 2\tau_i \lambda^2 \varepsilon \|e_i^y\|^2 + \tau_i \lambda^2/(4\varepsilon)(\|e_{i-1}^x\|^2 + \|e_i^x\|^2) \\
& + \tau_i \|\delta_i^1\|\, \|e_i^x\| + \tau_i/(4\varepsilon) \cdot \|\delta_i^2\|^2 + \varepsilon \tau_i \|e_i^y\|^2 \\
& \qquad \text{for} \quad i = 1, \ldots, N \ .
\end{aligned}$$

Assuming the restriction for the stepsizes

(S2) $\tau_i/\tau_{i-1} \le q_2, \quad i = 2, \ldots, N, \quad N \in \mathbb{N},$

with a constant $q_2 > 0$ and choosing $\varepsilon > 0$ such that

$$\varepsilon(\lambda^1 + 2\lambda^2 + 1)/\beta \le 1/2 \quad \text{and} \quad \lambda^1 q_2 \varepsilon/\beta \le 1/2$$

we obtain

$$\begin{aligned}
\|e_i^x\|_A^2 + \tau_i \|e_i^y\|_B^2 \le\ & (1 + c_1 \tau_{i-1})(\|e_{i-1}^x\|_A^2 + \tau_{i-1}\|e_{i-1}^y\|_B^2) \\
& + c_2 \tau_i \|e_i^x\|_A^2 + \tau_i (\|\delta_i^1\|\, \|e_i^x\|_A/\alpha^{1/2} + \|\delta_i^2\|^2/(4\varepsilon))
\end{aligned}$$

for $i = 2, \ldots, N$ and

$$\begin{aligned}
\|e_1^x\|_A^2 + \tau_1 \|e_1^y\|_B^2 \le\ & c_3 \tau_1 \|\delta_1^1\|\, \|e_1^x\|_A + c_4 \|e_0^x\|^2 \\
& + c_5 \tau_1 (\|\delta_1^2\|^2 + \|e_0^y\|^2)
\end{aligned} \tag{4.3}$$

with real constants $c_k > 0$, $k = 1, \ldots, 5$ independent of i and N, and τ_1 sufficiently small. For sufficiently small τ Lemma 3 can be applied to $a_i^N := \|e_i^x\|_A^2 + \tau_i \|e_i^y\|_B^2$ and, hence, (3.2) holds true and we obtain the estimates

$$\|e_i^x\|_A^2 + \tau_i \|e_i^y\|_B^2 \le c\Big(\sum_{j=1}^{i} \tau_j \|\delta_j^1\| \cdot E_i^x + \tau_1 \|\delta_0^2\|^2 + \sum_{j=1}^{i} \tau_j \|\delta_j^2\|^2\Big)$$
$$\text{for} \quad i = 1, \ldots, N$$

with a constant $c > 0$ independent of i and N, and $E_i^x := \max\{\|e_j^x\|_A \,|\, j = 1, \ldots, i\}$, $i = 1, \ldots, N$.

From this estimates we easily conclude the following statement:

Theorem 7. *Under the assumptions (H2)-(H4) and the restriction (S2) for the stepsizes, the following stability inequalities hold for sufficiently small* τ

(i)

$$\|e_0^y\| \le \|\delta_0^2\|/\beta$$

(ii)

$$\|e_i^x\| \leq K_1(\sum_{j=1}^{i} \tau_j \|\delta_j^1\| + (\tau_1\|\delta_0^2\|^2 + \sum_{j=1}^{i} \tau_j \|\delta_j^2\|^2)^{1/2}) \tag{4.4}$$

(iii)

$$\|e_i^y\| \leq \tau_i^{-1/2} K_1(\sum_{j=1}^{i} \tau_j \|\delta_j^1\| + (\tau_1\|\delta_0^2\|^2 + \sum_{j=1}^{i} \tau_j \|\delta_j^2\|^2)^{1/2})$$

for $i = 1, \ldots, N$ with a constant $K_1 > 0$ independent of i and N.

This stability inequalities we shall improve under a specification of the standard assumptions (H2), (H3).

Let

$$C = C(\varrho) := \{z \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \,|\, \langle b_j, z\rangle \geq \varrho_j\,,\ j \in J := \{1, \ldots, s\}\}$$

with $b_j \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $\varrho_j \in \mathbb{R}$, $j \in J$, $\varrho = (\varrho_1, \ldots, \varrho_s)^T$ be a nonempty convex polyhedron and let

$$\psi_C(z) := \begin{cases} 0 & \text{for } x \in C \\ +\infty & \text{for } x \notin C \end{cases}$$

be the indicator function of C. It is well-known, that the subdifferential

$$\partial\psi_C(z) := \{h \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \,|\, \langle h, y - z\rangle \leq 0\ \forall y \in C\}$$

of ψ_C is maximal monoton (see [AC84]). With the aid of the rules for computing subdifferentials (see [Aub84]) one easily shows

$$\partial\psi_C(z) = \{h = -\sum_{j\in J} \alpha_j b_j \,|\, \alpha_j \geq 0,\ (\langle b_j, z\rangle - \varrho_j) \cdot \alpha_j = 0,\ j \in J\}$$
$$\text{for } z \in C\ \ .$$

This respresentation makes obvious that the map $\gamma = -\partial\psi_C$ satisfies (H2)(ii), too. We then specify (H2) and (H3) as follows

(H2')

$$\begin{aligned} &\gamma = -\partial\psi_C \quad \text{with} \\ &C = C(\varrho) := \{z \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \,|\, \langle b_j, z\rangle \geq \varrho_j, j \in J\} \\ &b_j \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \varrho_j \in \mathbb{R} \quad \text{for} \quad j \in J \\ &\varrho = (\varrho_1, \ldots, \varrho_s)^T\ \ . \end{aligned}$$

(H3')

$$A = A^T \geq \alpha I, \quad B = B^T \geq \beta I, \quad \alpha, \beta > 0\ \ .$$

For the improvement of the stability inequalities we first show

Lemma 8. *Let $C = C(\varrho)$ and $C' = C(\varrho')$ be two nonempty convex polyhedra in the $\mathbb{R}^m$ generated by the same vectors $b_j \in \mathbb{R}^m$, $j \in J$. Then for the solutions of*

$$z \in h - \partial\psi_C(z) \quad \textit{respectively} \quad z' \in h - \partial\psi_{C'}(z') \tag{4.5}$$

the estimates

$$\|z - z'\| \le K\|\varrho - \varrho'\|$$

holds true with a constant $K > 0$ independent of h.

Proof. We prove the assertion for an ϱ' which differs from ϱ only in the k-th component, $\varrho'_j = \varrho_j$ for $j \neq k$. The general case can be concluded immediately by applying the triangle inequality. The uniquely determined solutions of the inclusions (4.5) can be represented as

$$z = h + \sum_{j \in J} \alpha_j b_j, \quad z' = h + \sum_{j \in J} \alpha'_j b_j$$

with nonnegative coefficients $\alpha_j, \alpha'_j \ge 0$, $j \in J$. These coefficients $\alpha, \alpha' \in \mathbb{R}^s$ are solutions of the linear complementary problems: Find $\alpha \in \mathbb{R}^s$ with

$$\alpha \ge 0, \quad M\alpha + q \ge 0, \quad \langle \alpha, M\alpha + q\rangle = 0 \tag{4.6}$$

with the matrix $M = (\langle b_k, b_j\rangle)_{k,j\in J}$ and the vectors $q = (\langle b_k, h\rangle - \varrho_k)_{k\in J}$, $q = (\langle b_k, h\rangle - \varrho'_k)_{k\in J}$, respectively.
The matrix M is the Gram-matrix of the vectors $b_1, \ldots, b_s$ and therefore copositive plus, i. e. for all $\alpha \ge 0$ it holds $\langle \alpha, M\alpha\rangle \ge 0$ and $(M+M^T)\alpha = 0$ if $\langle \alpha, M\alpha\rangle = 0$ and $\alpha \ge 0$. Hence, for each $q \in \mathbb{R}^s$, the Lemke-algorithm (see [Lem68], [CD68]) provides a complementary basic feasible solution of $w = q + M\alpha$, where in particular the b_j, which belong to the components $\alpha_j > 0$, are linearly independent. We denote the index set of the linearly independent b_j by J^+. Thus, there are solutions of (4.6) whose positive components α^+ satisfying the system

$$M^+\alpha^+ + q^+ = 0, \quad S\alpha^+ + q^0 \ge 0, \quad \alpha^+ \ge 0 \tag{4.7}$$

and, conversely, each solution of (4.7) that is completed by zero-components is a solution of (4.6). We have put

$$\begin{aligned} M^+ &:= (\langle b_k, b_j\rangle)_{k,j\in J^+}, & S &:= (\langle b_k, b_j\rangle)_{k\in J\setminus J^+, j\in J^+} \\ \alpha^+ &:= (\alpha_j)_{j\in J^+}, & q^+ &:= (\langle b_j, h\rangle - \varrho_j)_{j\in J^+} \\ q^0 &:= (\langle b_k, h\rangle - \varrho_k)_{k\in J\setminus J^+} \ . \end{aligned}$$

Furthermore, let

$$q(s) := q + s(\varrho'_k - \varrho_k)e_k, \quad s \in [0,1] \tag{4.8}$$

with the k-th unit vector e_k, $e_{k,j} = \delta_{kj}$, $j \in J$. In general, it will not be possible to choose in (4.7) the same index set $J^+(s) = J^+$ (and, thus, the same matrices $M^+(s)$, $S(s)$) for all $s \in [0,1]$. However, if there are solutions of (4.7) for two different $s_1, s_2 \in [0,1]$, $s_1 < s_2$, with the same index set J^+, also

$$\alpha(s) := ((s_2 - s)\alpha(s_1) + (s - s_1)\alpha(s_2))/(s_2 - s_1)$$

are solutions of (4.6) for $q(s)$ from (4.8) with the index set J^+, for all $s \in [s_1, s_2]$. Namely, we have

$$\alpha^0(s) := (\alpha_k(s))_{k \in J \setminus J^+} = 0, \quad \alpha^+(s) \geq 0$$

and

$$\begin{gathered} M^+((s_2 - s)\alpha^+(s_1) + (s - s_1)\alpha^+(s_2))/(s_2 - s_1) = \\ -(s_2 - s)/(s_2 - s_1)q^+(s_1) - (s - s_1)/(s_2 - s_1)q^+(s_2) = -q^+(s) \\ S((s_2 - s)\alpha^+(s_1) + (s - s_1)\alpha^+(s_2))/(s_2 - s_1) \leq \\ -(s_2 - s)/(s_2 - s_1) \cdot q^0(s_1) - (s - s_1)/(s_2 - s_1) \cdot q^0(s_2) = -q^0(s) \ . \end{gathered}$$

Furthermore, if $\alpha^+(s_n)$, $n \in \mathbb{N}$ are solutions of (4.7) with the same index set $J^+(s_n) = J^+$ for a convergent sequence $s_n \to s_0$, $s_n \in [0,1]$, then $\alpha^+(s_0) = \lim \alpha^+(s_n)$ is a solution of (4.7) with the index set J^+. Hence, a fixed index set J^+ can be maintained in the interval

$$[\inf(s \in [0,1] \,|\, J^+(s) = J^+), \sup(s \in [0,1] \,|\, J^+(s) = J^+)] \ .$$

This interval will not degenerate if there are at least two $s_1 \neq s_2 \in [0,1]$ with $J^+(s_1) = J^+(s_2) = J^+$. The number of these non-degenerated intervals is finite and with their union the interval $[0,1]$ is exhausted. Consequently, there are finitely many points $s_r \in (0,1)$, $r = 1, \ldots, R$ such that there exists a solution of (4.6) for (4.8) which has, in the subintervals $[s_{r-1}, s_r]$, $r = 1, \ldots, R+1$, $s_0 = 0$, $s_{R+1} = 1$, the representation

$$\alpha^0(s) = 0, \quad \alpha^+(s) = -(M^+)^{-1} q^+(s)$$

with the same regular matrix M^+.

Let z_r, $r = 1, \ldots, R$ be the solutions of the inclusion $z \in h - \partial\psi_{C(\varrho)}(z)$ for $\varrho = (\varrho_1, \ldots, \varrho_{k-1}, \varrho_k + s_r(\varrho'_k - \varrho_k), \varrho_{k+1}, \ldots, \varrho_s)$ and $z_0 = z$, $z_{R+1} = z'$. Then it follows that

$$\begin{aligned} \|z' - z\| &\leq \sum_{r=1}^{R+1} \|z_r - z_{r-1}\| = \sum_{r=1}^{R+1} \| \sum_{j \in J^+} (\alpha_j^+(s_r) - \alpha_j^+(s_{r-1})) b_j \| \\ &\leq K_k |\varrho'_k - \varrho_k| = K_k \|\varrho' - \varrho\| \end{aligned}$$

with a constant $K_k > 0$. □

Lemma 9. *Under the assumptions (H1), (H2'), (H3'), (H4) the estimates*

$$\|e_i^y\| \leq K_1(\|e_{i-1}^x\| + \|e_i^x\| + \|\delta_i^2\|), \quad i = 1, \ldots, N$$

and

$$\|e_0^y\| \leq K_1(\|e_0^x\| + \|\delta_0^2\|)$$

are valid with a constant $K_1 > 0$ independent of i and N. $e_0^x = \hat{x}_0 - x_0$ is a perturbation of the initial value x_0.

Proof. Under the assumptions (H2'), for given x_i, $\hat{x}_i$, $i = 0, 1, \ldots, N$ the subsystems of the algebraic inclusions in (2.3) and (4.1), respectively, are of the form

$$\begin{array}{l} By_i \in \phi^2(t_{i-1}, x_{i-1}, x_i, \tau_i) - \partial\psi_C(y_i) \\ B\hat{y}_i \in \phi^2(t_{i-1}, \hat{x}_{i-1}, \hat{x}_i, \tau_i) + \delta_i^2 - \partial\psi_{\hat{C}}(\hat{y}_i) \end{array} \tag{4.9}$$

with

$$\begin{array}{l} C := \{y \in \mathbb{R}^{n_2} \,|\, \langle b_j^2, y\rangle \geq \varrho_j - \langle b_j^1, x_i\rangle, j \in J\} \\ \hat{C} := \{y \in \mathbb{R}^{n_2} \,|\, \langle b_j^2, y\rangle \geq \varrho_j - \langle b_j^1, \hat{x}_i\rangle, j \in J\} \;. \end{array}$$

(H3') implies the existence of a decomposition $B = L^T L$ with a regular matrix L. y_i and $\hat{y}_i$ are solutions of the inclusions (4.9) iff

$$u_i := Ly_i, \quad \hat{u}_i := L\hat{y}_i, \quad i = 1, \ldots, N$$

are solutions of the inclusions

$$\begin{array}{l} u_i \in (L^T)^{-1}\phi^2(t_{i-1}, x_{i-1}, x_i, \tau_i) - \partial\psi_D(u_i) \\ \hat{u}_i \in (L^T)^{-1}\phi^2(t_{i-1}, \hat{x}_{i-1}, \hat{x}_i, \tau_i) - (L^T)^{-1}\delta_i^2 - \partial\psi_{\hat{D}}(\hat{u}_i) \\ \qquad\qquad i = 1, \ldots, N \end{array} \tag{4.10}$$

with

$$\begin{array}{l} D := \{u \in \mathbb{R}^{n_2} \,|\, \langle (L^T)^{-1}b_j^2, u\rangle \geq \varrho_j - \langle b_j^1, x_i\rangle, j \in J\} \\ \hat{D} := \{u \in \mathbb{R}^{n_2} \,|\, \langle (L^T)^{-1}b_j^2, u\rangle \geq \varrho_j - \langle b_j^1, \hat{x}_i\rangle, j \in J\} \;. \end{array}$$

Then the first estimates follow immediately by Lemma 8 and the fact that $(I + \partial\psi_D)^{-1}$ is a single-valued non-expansive operator (see [AC84]). The latter estimate we obtain analogously from the inclusions of the initial conditions. □

Let us remark,that under the assumptions (H1), (H2'), (H3') the y-component of the solution of (2.1) is Lipschitz-continuous, as one can show by an analogous proof. From Theorem 7 and Lemma 9 we obtain

Theorem 10. *Under the assumptions (H1), (H2'), (H3'), (H4) and the restriction (S2) for the stepsizes, for sufficiently small τ the following stability inequalities hold for the deviations e_i^x, e_i^y of the solutions of the problems $(\hat{P}_N)$ with the general initial condition*

$$\hat{z}_0 := (\hat{x}_0, \hat{y}_0) \in dom(\gamma) \tag{4.2'}$$

with $B\hat{y}_0 \in f^2(0, \hat{x}_0) + \delta_0^2 + \gamma^2(\hat{z}_0)$, $\hat{x}_0$ be given, from those of problem (P_N):

$$\|e_0^y\| \leq K_2(\|e_0^x\| + \|\delta_0^2\|)$$

$$\|e_i^x\| \leq K_2\Big(\sum_{j=1}^{i} \tau_j\|\delta_j^1\| + \big(\|e_0^x\|^2 + \tau_1\|\delta_0^2\|^2 + \sum_{j=1}^{i} \tau_j\|\delta_j^2\|^2\big)^{1/2}\Big)$$

$$\|e_i^y\| \leq K_2\Big(\sum_{j=1}^{i} \tau_j\|\delta_j^1\| + \|\delta_i^2\| + \big(\|e_0^x\|^2 + \tau_1\|\delta_0^2\|^2 + \sum_{j=1}^{i} \tau_j\|\delta_j^2\|^2\big)^{1/2}\Big)$$

for $i = 1, \ldots, N$ with a constant $K_2 > 0$ independent of i and N.

For solutions $z(t)$ of (P) satisfying the inclusions (4.1), (4.2) with $\hat{z}_i := z(t_i)$, $i = 0, \ldots, N$ and perturbations δ_i^1, δ_i^2 of the form

$$\begin{array}{ll} \|\delta_i^1\| = 0(1),\ i \in S_N, & \|\delta_i^1\| = 0(\tau_i), \quad i \in \{1, \ldots, N\} \backslash S_N, \\ \|\delta_i^2\| = 0(\tau_i),\ i = 1, \ldots, N, & \|\delta_0^2\| = 0(\tau_1) \end{array} \tag{4.11}$$

where $S_N \subset \{1, \ldots, N\}$ are index sets whose numbers of elements are bounded by a constant $L \in \mathbb{N}$ independent of N, card $S_N \leq L$ (compare [Ell85]), we obtain immediately from Theorem 10

Corollary 11. *Under the assumptions (H1), (H2'), (H3'), (H4) and the restriction (S2) for the stepsizes we have for solutions $z(t)$ of (P) satisfying the system of inclusions (4.1), (4.2) with perturbations δ_i^1, δ_i^2 of the form (4.11) and the solutions z_i, $i = 0, \ldots, N$ of (P_N):*

$$\|z(t_i) - z_i\| = 0(\tau) \ .$$

5 Solution of the Problems (P_N)

We consider the solution of the problems (P_N) only under the special hypothesis (H2'), (H3'). In this case we have, for each $i = 1, \ldots, N$, to solve a system of inclusions of the following type

$$\begin{array}{l} Ax \in F_i^1(z) + \tau_i \gamma^1(z) \\ By \in F_i^2(x) + \gamma^2(z), \quad z = (x, y) \end{array} \tag{5.1}$$

with

$$\begin{array}{l} F_i^1(z) = Ax_{i-1} + \tau_i \phi^1(t_{i-1}, z_{i-1}, z, \tau_i) \\ F_i^2(x) = \phi^2(t_{i-1}, x_{i-1}, x, \tau_i) \\ \gamma^1(z) = \sum\limits_{j \in J} \alpha_j b_j^1, \quad \gamma^2(z) = \sum\limits_{j \in J} \alpha_j b_j^2 \\ \alpha \geq 0, \quad \langle b_j^1, x \rangle + \langle b_j^2, y \rangle \geq \varrho_j \\ (\langle b_j^1, x \rangle + \langle b_j^2, y \rangle - \varrho_j)\alpha_j = 0, \quad j \in J \ . \end{array}$$

The problem (5.1) is then equivalent with the following nonlinear complementary problem:

Find $\alpha \geq 0$ with

$$q(z) + M\alpha \geq 0, \quad \langle \alpha, q(z) + M\alpha \rangle = 0 \tag{5.2}$$

with

$$\begin{array}{c} M = (\tau_i \langle \tilde{b}_k^1, b_j^1 \rangle + \langle \tilde{b}_k^2, b_j^2 \rangle)_{j,k \in J} \\ q(z) = (\langle \tilde{b}_k^1, F_i^1(z) \rangle + \langle \tilde{b}_k^2, F_i^2(x) \rangle - \varrho_k)_{k \in J} \end{array}$$

and

$$\tilde{b}_k^1 := A^{-1} b_k^1, \quad \tilde{b}_k^2 := B^{-1} b_k^2, \quad k \in J \ .$$

First we consider the case F_i^1 independent of z and F_i^2 independent of x. Therefore q is independent of z and (5.2) becomes a linear complementary problem. If the matrix M is diagonal, the solution of (5.2) is

$$\alpha_j = -\min(q_j, 0)/M_{jj}, \quad j \in J \tag{5.3}$$

and with $J_i^+ := \{j \in J \mid \alpha_j > 0\}$ we have for the solution of (5.1)

$$\begin{aligned} x_i &= A^{-1}F_i^1 + \tau_i \sum_{j \in J_i^+} \alpha_j \tilde{b}_j^1 \\ y_i &= B^{-1}F_i^2 + \sum_{j \in J_i^+} \alpha_j \tilde{b}_j^2, \quad i = 1, \dots, N \ . \end{aligned} \tag{5.4}$$

Analogously we obtain

$$y_0 = B^{-1}f^2(0, x_0) + \sum_{j \in J_0^+} \alpha_j \tilde{b}_j^2 \tag{5.5}$$

with α_j from (5.3) with

$$q_j = \langle \tilde{b}_j^2, f^2(0, x_0) \rangle + \langle b_j^1, x_0 \rangle - \varrho_j, \quad j \in J \ .$$

In the general linear case we determine a complementary basic feasible solution of $w = q + M\alpha$ by the algorithm of Lemke [Lem68], which terminates after finitely many iterations, because the matrix M is copositive plus. The corresponding solutions of (5.1) we compute with (5.4). In order to reduce the number of calls of the algorithm of Lemke, we first try in each step to use the index set J^+ of the step before. We use the following notations to sketch the whole procedure in the form of an algorithm:

l-alg$(q, M \rightarrow J^+, \alpha^+)$ - computing of the index set J^+ and the positive coefficients $\alpha^+ > 0$ by the algorithm of Lemke,
solve$(x_{i-1}, \tau_i, \alpha^+, J^+ \rightarrow z_i)$ - computing of $z_i = (x_i, y_i)$ by (5.4),
chol$(A \rightarrow L_A)$ - performing of the Cholesky-decomposition $L_A^T L_A$ of the symmetric positively definite matrix A.
split$(M, J^+ \rightarrow M^+, S)$ - splitting of the matrix M

$$M = \begin{pmatrix} M^+ & S^T \\ S & M^0 \end{pmatrix}$$

with $M_{j,k}^+ := M_{j,k}$ for $j, k \in J^+$ and $S_{j,k} := M_{j,k}$ for $j \in J \backslash J^+$, $k \in J^+$,
split$(q, J^+ \rightarrow q^+, q^0)$ - splitting of the vector q

$$q = \begin{pmatrix} q^+ \\ q^0 \end{pmatrix}$$

with $q_j^+ := q_j$ for $j \in J^+$, and $q_j^0 := q_j$ for $j \in J \backslash J^+$,

start: chol($A \to L_A$); chol($B \to L_B$);
  for $k \in J$ do
    begin $\tilde{b}_k^1 := L_A^{-1}(L_A^T)^{-1} b_k^1$; $\tilde{b}_k^2 := L_B^{-1}(L_B^T)^{-1} b_k^2$;
      $q_k := \langle \tilde{b}_k^2, f^2(0, x_0) \rangle + \langle b_k^1, x_0 \rangle - \varrho_k$
    end;
  for $k, j \in J$, $l = 1, 2$ do $M_{k,j}^l := \langle \tilde{b}_k^l, b_j^l \rangle$;
  l-alg($q, M^2 \to J^+, \alpha^+$);
  solve($x_0, 0, \alpha^+, J^+ \to z_0$);
  split($\tau_1 M^1 + M^2, J^+ \to M^+, S$);
  chol($M^+ \to L^+$);
  $i := 1$;
next: for $k \in J$ do $q_k := \langle b_k^1, x_{i-1} \rangle + \tau_i \langle \tilde{b}_k^1, \phi^1 \rangle + \langle \tilde{b}_k^2, \phi^2 \rangle - \varrho_k$;
repeat: split($q, J^+ \to q^+, q^0$);
  $\alpha^+ := -(L^+)^{-1}(L^{+T})^{-1} q^+$;
  if $q^0 + S\alpha^+ \not\geq 0$ then
    begin l-alg($q, \tau_i M^1 + M^2 \to J^+, \alpha^+$);
      solve($x_{i-1}, \tau_i, \alpha^+, J^+ \to z_i$);
      split($\tau_i M^1 + M^2, J^+ \to M^+, S$);
      chol($M^+ \to L^+$);
      $i := i + 1$;
      if $i \leq N$ then goto next else goto end
    end;
  if $\alpha^+ \not\geq 0$ then
    begin $J^+ := J^+ \setminus \{ j \in J^+ \mid \alpha_j \leq 0 \}$;
      split($\tau_i M^1 + M^2, J^+ \to M^+, S$);
      chol($M^+ \to L^+$);
      goto repeat
    end;
  solve($x_{i-1}, \tau_i, \alpha^+, J^+ \to z_i$);
  $i := i + 1$,
  if $i \leq N$ then goto next;
end: □

The general system (5.1) one can solve iteratively by

$$\begin{aligned} Ax^{n+1} &\in F_i^1(z^n) + \tau_i \gamma^1(z^{n+1}) \\ By^{n+1} &\in F_i^2(x^n) + \gamma^2(z^{n+1}) \end{aligned} \tag{5.6}$$

with $z^1 := z_{i-1}$ (see [Ell85]). The iterations z^n are computable with the algorithm mentioned above. The iteration process (5.6) is convergent for sufficiently small τ_i, because it holds

$$\begin{aligned} \|\Delta x^{n+1}\|_A^2 + \tau_i \|\Delta y^{n+1}\|_B^2 &\leq \tau_i (\lambda^1 \|\Delta z^n\| \, \|\Delta x^{n+1}\| + \lambda^2 \|\Delta x^n\| \, \|\Delta y^{n+1}\|) \\ &\leq \tau_i^{1/2} \cdot c_1 (\|\Delta x^n\|_A^2 + \tau_i \|\Delta y^n\|_B^2) \\ &\quad + \varepsilon c_2 (\|\Delta x^{n+1}\|_A^2 + \tau_i \|\Delta y^{n+1}\|_B^2) \end{aligned}$$

with constants $c_1, c_2 > 0$ independent of n and each $\varepsilon > 0$. Hence, the sequence $(z^n)_{n \in \mathbb{N}}$ is a Cauchy-sequence whose limit z is a solution of (5.1), because F_i^l,

$l = 1, 2$ are continuous and $-\gamma$ is maximal monoton.

6 Numerical Results

In this section we report some results of test calculations with the proposed procedure for two approximations of the example in the introduction. For the tests we have first transformed the system (1.2) with $t = \omega t'$ and then chosen the parameters $\omega L_1 = \omega L_2 = R_D = U = 1$, $E = 0.5$ (case A) and $E = 0.2$ (case B), respectively, $T = 2\pi$ and $x_0 = (0,0)^T$ ($x(t) = (i_1(t), i_2(t))^T$, $y(t) = i_3(t)$). Obviously, then $y_0 = 0$. The exact solutions of problem (P) can be obtained and have in these two cases the representations:

Case A:

$$
\begin{aligned}
\forall t \in [0, t^A] &: x_1(t) = x_2(t) = y_1(t) = 0 \ ; \\
\forall t \in [t^A, t^B] &: x_2(t) = -0.5E + 0.25(\sin t - \cos t) + k \cdot e^{-(t-t^A)} \\
&\ x_1(t) = x_2(t) \\
&\ y_1(t) = 0 \ ; \\
\forall t \in [t^B, t^A + \pi] &: x_1(t) = x_2(t) = y_1(t) = 0 \ ; \\
\forall t \in [t^A + \pi, 2\pi] &: x_2(t) = -0.5E - 0.25(\sin t - \cos t) + k \cdot e^{-(t-t^A-\pi)} \\
&\ x_1(t) = -x_2(t) \\
&\ y_1(t) = x_2(t) \ ;
\end{aligned}
$$

Case B:

$$
\begin{aligned}
\forall t \in [0, t^A] &: x_1(t) = x_2(t) = y_1(t) = 0 \ ; \\
\forall t \in [t^A, t^A + \pi] &: x_2(t) = -0.5E + 0.25(\sin t - \cos t) + k \cdot e^{-(t-t^A)} \\
&\ x_1(t) = x_2(t) \\
&\ y_1(t) = 0 \ ; \\
\forall t \in [t^A + \pi, t^B] &: x_1(t) = 0.5(\sin t - \cos t] + (e^{-\pi} - 1)k \cdot e^{-(t-t^A-\pi)} \\
&\ x_2(t) = -E + (1 + e^{-\pi}) \cdot k \cdot e^{-(t-t^A-\pi)} \\
&\ y_1(t) = -0.5E - 0.25(\sin t - \cos t) + k \cdot e^{-(t-t^A-\pi)} \ ; \\
\forall t \in [t^B, 2\pi] &: x_2(t) = -0.5E - 0.25(\sin t - \cos t) + k \cdot e^{-(t-t^A-\pi)} \\
&\ x_1(t) = x_2(t) \\
&\ y_1(t) = x_2(t) \ .
\end{aligned}
$$

with $t^A = \arcsin E$, $k = 0.25(E + \sqrt{1 - E^2})$ and t^B the zero in $[\pi, 2\pi]$ of $F(t) := 0.5E - 0.25(\sin t - \cos t) - k \cdot e^{-(t-t^A)}$.

We have used the constant stepsize $\tau = 2\pi/N$ and the approximation (2.5) (denoted by M1) and (2.6) (denoted by M2), respectively. In these cases the systems (2.3) are

M1:

$$\begin{aligned} x_{1,i} &\in x_{1,i-1} + \tau(\sin t_{i-1} - 2x_{1,i-1} + x_{2,i-1} - 2y_{1,i-1}) + \tau(\alpha_1 - \alpha_2) \\ x_{2,i} &\in x_{2,i-1} + \tau(-E + x_{1,i-1} - 2x_{2,i-1} + 2y_{1,i-1}) + \tau(\alpha_2 + \alpha_3) \\ y_{1,i} &\in 0.5(x_{2,i-1} - x_{1,i-1}) + 0.25(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4) \\ &\qquad \text{for} \quad i = 1, 2, \ldots, N \end{aligned}$$

M2:

$$\begin{aligned} x_{1,i} &\in x_{1,i-1} + \tau(\sin t_i - 2x_{1,i} + x_{2,i} - 2y_{1,i}) + \tau(\alpha_1 - \alpha_2) \\ x_{2,i} &\in x_{2,i-1} + \tau(-E + x_{1,i} - 2x_{2,i} + 2y_{1,i}) + \tau(\alpha_2 + \alpha_3) \\ y_{1,i} &\in 0.5(x_{2,i} - x_{1,i}) + 0.25(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4) \\ &\qquad \text{for } i = 1, 2, \ldots, N. \end{aligned}$$

The three tables show the absolut errors $err_j = \max\{|x_j(t_i) - x_{j,i}| \mid i = 1, \ldots, N\}$, $j = 1, 2$, $err_3 = \max\{|y_1(t_i) - y_{1,i}| \mid i = 1, \ldots, N\}$ and the corresponding numerical determined orders of convergence m_j, $j = 1, 2, 3$ by which the theoretical result of Corollary 11 is well confirmed.

Table 1. Example M1

N/100	$err_1 = err_2 = err_3$		$m_1 = m_2 = m_3$	
	case A	case B	case A	case B
2^2	3.82×10^{-3}	4.30×10^{-3}	–	–
2^3	1.93×10^{-3}	2.15×10^{-3}	0.9836	0.9978
2^4	9.72×10^{-4}	1.08×10^{-3}	0.9918	0.9968
2^5	4.87×10^{-4}	5.40×10^{-4}	0,9959	0.9985
2^6	2.44×10^{-4}	2.70×10^{-4}	0.9980	0.9987
2^7	1.22×10^{-4}	1.35×10^{-4}	0.9990	0.9998
2^8	6.11×10^{-5}	6.76×10^{-5}	0.9996	0.9998

Table 2. Example M2, case A

N/100	$err_1 = err_2$	err_3	$m_1 = m_2$	m_3
2^2	1.58×10^{-3}	1.57×10^{-3}	-	-
2^3	7.91×10^{-4}	7.86×10^{-4}	0.9972	0.9971
2^4	3.96×10^{-4}	3.93×10^{-4}	0.9984	0.9985
2^5	1.98×10^{-4}	1.97×10^{-4}	0.9992	0.9993
2^6	9.90×10^{-5}	9.84×10^{-5}	0.9996	0.9996
2^7	4.95×10^{-5}	4.92×10^{-5}	0.9997	0.9998
2^8	2.48×10^{-5}	2.46×10^{-5}	0.9998	0.9998

Table 3. Example M2, case B

N/100	err_1	err_2	err_3	m_1	m_2	m_3
2^2	2.35×10^{-3}	1.70×10^{-3}	1.70×10^{-3}	-	-	-
2^3	1.18×10^{-3}	8.53×10^{-4}	8.51×10^{-4}	0.9967	0.9973	0.9973
2^4	5.89×10^{-4}	4.27×10^{-4}	4.26×10^{-4}	0.9979	0.9986	0.9985
2^5	2.95×10^{-4}	2.14×10^{-4}	2.13×10^{-4}	0.9987	0.9993	0.9993
2^6	1.47×10^{-4}	1.07×10^{-4}	1.07×10^{-4}	0.9998	0.9996	0.9996
2^7	7.37×10^{-5}	5.34×10^{-5}	5.33×10^{-5}	0.9999	0.9998	0.9998
2^8	3.68×10^{-5}	2.67×10^{-5}	2.66×10^{-5}	1.0000	0.9999	0.9999

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