Topological Properties of the Approximate Subdifferential

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Abstract

The approximate subdifferential introduced by Mordukhovich has attracted much attention in recent works on nonsmooth optimization. Potential advantages over other concepts of subdifferentiability might be related to its non-convexity. This motivates to study some topological properties more in detail. As the main result, it is shown that in a Hilbert space setting each weakly compact set may be obtained as the Kuratowski-Painlevé limit of the approximate subdifferentials of some family of Lipschitzian functions. As a consequence, apart from finiteness, there is no restriction on the number of connected components of the subdifferential. In the finite dimensional case, each topological type of a compact set may be realized by an approximate subdifferential of some Lipschitzian function. These are clear differences for instance to Clarke's subdifferential. The results stated above require the definition of Lipschitzian functions on a space which is enlarged by one extra dimension. Otherwise they would not hold true any longer since one can show, that for a real function the number of connected components of the approximate subdifferential is limited by two.

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1 Introduction

The approximate subdifferential was first introduced in finite dimensions by Mordukhovich [15] via normal cones to epigraphs. An equivalent definition based on Dini subdifferentials was found by Ioffe [6]. The same authors gave sevral approaches for generalizing this concept to the infinite-dimensional case [14], [7], [8].

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Being nonconvex in general, the approximate subdifferential enjoys a minimality property among a family of 'reasonable' subdifferentials (compare [6], Th. 9). On the other hand, it offers a rich calculus. We merely point out the chain rule by Jourani and Thibault [10] which works in infinite dimensions without the need of convexification. As a consequence, many promising applications in nonsmooth optimization have been reported by several authors. For instance, Glover, Craven, Flåm [2] and Glover, Craven [3] considered first order optimality conditions in Banach spaces. Other papers are concerned with metric regularity of feasible sets [9], [11], [12], [16]. It is interesting to note that, in the finite-dimensional situation, Mordukhovich [16] could give a complete characterization of metric regularity of multivalued mappings while in the Banach space setting Jourani and Thibault [11], [12] developed an approach via so-called compactly-lipschitzian mappings thereby exploiting the chain rule mentioned above.
Potential advantages of the approximate subdifferential might be related to the fact that it is not restricted to be a convex set. This motivates the question which topological properties can be expected in general. It will turn out that, even for lipschitzian functions, there is a rich variety of topological types that can occur.

2 Basic Definitions and Properties

In the sequel we shall employ the approach via Dini subdifferentials as a basis of definition. Denote by X, X^* a Banach space with its dual and let $f : X \to I\!R \cup \{-\infty, \infty\}$ be an extended-valued function.

Definition 2.1 (Dini subdifferential) *For $x \in X$ put*

$$\partial^- f(x) = \begin{cases} \{x^* \in X^* \mid x^*(h) \leq d^- f(x;h) \ \forall h \in X\} & \text{if } \mid f(x) \mid < \infty \\ \emptyset & \text{else} \end{cases}$$

where

$$d^- f(x;h) = \liminf_{\substack{u \to h \\ t \downarrow 0}} \frac{f(x+tu) - f(x)}{t}$$

refers to the lower Dini directional derivative of f at x in direction h.

For the following denote by $\mathcal{F}$ the collection of finite-dimensional subspaces of X and for any subset $S \subseteq X$ put

$$f_S(x) := \begin{cases} f(x) & \text{if } x \in S \\ \infty & \text{else} \end{cases}$$

Definition 2.2 (approximate subdifferential) *For* $z \in X$ *define*

$$\partial_a f(z) = \begin{cases} \bigcap\limits_{L \in \mathcal{F}} \limsup\limits_{\substack{x \to z \\ f(x) \to f(z)}} \partial^- f_{x+L}(x) & \text{if } \mid f(z) \mid < \infty \\ \emptyset & \text{else} \end{cases}$$

In the last definition 'lim sup' has to be understood in the Kuratowski-Painlevé sense with respect to the strong topology in X and the weak*-topology in X^*. More explicitly, for some multifunction $K : X \rightrightarrows X^*$ one has

$x^* \in \limsup\limits_{\substack{x \to z \\ f(x) \to f(z)}}$ if and only if there exist converging nets

$$x_\alpha \to z, x_\alpha^* \rightharpoonup_{w^*} x^* \text{ such that } f(x_\alpha) \to f(z) \text{ and } x_\alpha^* \in K(x_\alpha)$$

Certainly, one always has the inclusion $\partial^- f(x) \subseteq \partial_a f(x)$. For Lipschitzian functions the main relation between the approximate and Clarke's subdifferential is $\partial_C f(x) = cocl \partial_a f(x)$, where $cocl$ refers to the convex, weak*- closure, and furthermore $\partial_a f(z)$ is weak*- compact.
When X is finite-dimensional then definition 2.2 reduces to the classical definition (see [6]):

$$\partial_a f(z) = \begin{cases} \limsup\limits_{\substack{x \to z \\ f(x) \to f(z)}} \partial^- f(x) & \text{if } \mid f(z) \mid < \infty \\ \emptyset & \text{else} \end{cases} \tag{1}$$

This relation is, of course, much easier to handle than (2.2). In particular, the expansion of 'lim sup' may be done in terms of sequences rather than nets. Unfortunately, as pointed out in [7], a formal transfer of (1) to infinite-dimensional spaces is not reasonable, since even for Lipschitzian functions the Dini subdifferential may become locally empty (see [5] for an example). This necessitates restrictions to finite-dimensional subspaces in general. For certain classes of spaces, at least, one has the following simplification due to Ioffe (compare corollary 5.1.1 in [7]):

Lemma 2.1 *Let X be a Banach space which is separable or admits an equivalent gateaux-differentiable norm. Then for each lower semicontinuous function $f : X \to I\!R$ it holds*

$$\partial_a f(z) = \limsup_{\substack{x \to z \\ f(x) \to f(z) \\ \varepsilon \downarrow 0}} \partial_\varepsilon^- f(x)$$

where the ε-Dini subdifferential is defined as

$$\partial_\varepsilon^- = \left\{ \{x^* \in X^* \mid x^*(h) \leq d^- f(x;h) + \varepsilon \|h\| \; \forall h \in X\} \text{ if } \mid f(x) \mid < \infty \right.$$

At the end of this section we cite a result from [4] (lemma 3.1), which gives sufficient conditions for specific topological types of the approximate subdifferential in terms of the local behavior of the Dini subdifferential.

Lemma 2.2 *Let X be a finite-dimensional space and f a real-valued function defined on X. Consider the following conditions:*

$(\mathbf{A1})$ $\quad \partial^- f(z) = \limsup\limits_{\substack{x \to z \\ f(x) \to f(z)}} \partial^- f(x)$

$(\mathbf{A2})$ $\quad \liminf\limits_{\substack{x \to z \\ f(x) \to f(z) \\ \partial^- f(x) \neq \emptyset}} \partial^- f(x) \neq \emptyset$

$(\mathbf{A3})$ $\quad \partial^- f(z)$ *is compact and there exists $\varepsilon > 0$ such that the relation $\partial^- f(x) \cap \partial^- f(z) \neq \emptyset$ holds for all x with $\|x - z\|, \mid f(x) - f(z) \mid < \varepsilon, \partial^- f(x) \neq \emptyset$*

Then it holds

1. $(\mathbf{A1}) \Longrightarrow \partial_a f(z)$ *is convex*
2. $(\mathbf{A2}) \Longrightarrow \partial_a f(z)$ *is star-shaped*
3. $(\mathbf{A3}) \Longrightarrow \partial_a f(z)$ *is connected*
4. *If f is locally Lipschitzian, then*
   $(\mathbf{A3}) \Longrightarrow (\mathbf{A1})$ *and*
   $(\mathbf{A2}) \Longrightarrow (\mathbf{A1})$ *or $\partial_a f(z)$ is a singleton*

By this lemma, conditions $(\mathbf{A1}), (\mathbf{A2}), (\mathbf{A3})$ imply successively weaker topological types. There are examples of lower semicontinuous functions in two variables such that (**A2**) is fulfilled but convexity violated or (**A3**) is fulfilled but star-shapedness of the approximate subdifferential fails to hold (see [4]). In this sense the indicated conditions are specific. On the other hand, the last statement of the lemma shows, that in the Lipschitzian case (**A2**) and (**A3**) even imply convexity. In this situation it seems hard, to find reasonable conditions similar to lemma 2.2 which are specific for a certain topological type (apart from convexity and compactness).

3 Results

In this section topological properties of the approximate subdifferential of Lipschitzian functions will be studied. Before proving the main result, we shall need the following lemma which allows - in a special setting - to facilitate the computation of $\partial_a f(z)$ according to definition 2.2.

Lemma 3.1 *Let* X *be a Hilbert space,* $C \subseteq X$ *a weakly compact subset and* $f : X \rightarrow I\!R$ *defined by* $f(z) = \min\{\langle z, x\rangle \mid x \in C\}$. *Denote* $E(z) = \{x \in C \mid \langle z, x\rangle = f(z)\}$. *Then it holds*

1. $d^- f(z;h) = \min\{\langle x, h\rangle \mid x \in E(z)\}$ (2)
2. $\sharp E(z) = 1 \Rightarrow E(z) = \partial^- f(z); \quad \sharp E(z) \geq 2 \Rightarrow \partial^- f(z) = \emptyset$ (3)
3. $\limsup\limits_{z \to 0, \varepsilon \downarrow 0} \partial^-_\varepsilon f(z) = \limsup\limits_{z \to 0} \partial^- f(z)$ (4)

Proof: Since f is a concave function, its directional derivative exists and coincides with $d^- f(z;h)$. On the other hand, the value of the directional derivative computes from (2), see e.g. [1], proposition 4.4.
Concerning the first relation of (3) one concludes from $E(z) = \{x\}$ and from (2) that $d^- f(z;h) = \langle x, h\rangle \; \forall h \in X$, hence $\partial^- f(z) = \{x\}$. For the second relation note, that nonemptyness of $\partial^- f(z)$ implies (by definition) $d^- f(z;h) \geq -d^- f(z;-h)$. Now, choosing $x^1, x^2 \in E(z), x^1 \neq x^2$ and assuming $\partial^- f(z) \neq \emptyset$ we obtain from (2) for arbitrary $h \in X$

$$\begin{aligned} \min\{\langle x^1, h\rangle, \langle x^2, h\rangle\} &\geq \min_{x \in E(z)} \langle x, h\rangle = d^- f(z;h) \geq -d^- f(z;-h) \\ &= -\min_{x \in E(z)} \langle x, -h\rangle = \max_{x \in E(z)} \langle x, h\rangle \\ &\geq \max\{\langle x^1, h\rangle, \langle x^2, h\rangle\} \end{aligned}$$

This inequality holding for all $h \in X$ one gets the contradiction $x^1 = x^2$.
Finally, for proving (4), first observe that the inclusion $' \supseteq'$ is trivially fulfilled because of $\partial^-_\varepsilon \supseteq \partial^-$. For the reverse inclusion let

$$x \in \limsup_{z \to 0, \varepsilon \downarrow 0} \partial^-_\varepsilon f(z)$$

This means existence of nets $z_\lambda \to 0, x_\lambda \to x, \varepsilon_\lambda \to 0, x_\lambda \in \partial^-_{\varepsilon_\lambda} f(z_\lambda), \varepsilon_\lambda > 0$. Let U be a (strong) open neighborhood of 0 and V be a weak open neighborhood of x. Then V contains a base neighborhood of the type

$$W = \{y \in X \mid |\langle y - x, y^i \rangle| < \varepsilon, y^i \in X \ i = 1, \ldots, n\}$$

and for $\lambda \geq \lambda_0$ one has $|\langle x_\lambda - x, y^i \rangle| < \varepsilon/2$ $(i = 1, \ldots, n)$. This implies $y \in W \subseteq V$ whenever $\|y - x_\lambda\| < \varepsilon/(2 \max \|y^i\|)$. Summarizing, it follows the existence of an index λ^* such that $z_{\lambda^*} \in U$ and $x_{\lambda^*} + B_{\varepsilon_{\lambda^*}} \subseteq V$. The last relation yields $E(z_{\lambda^*}) \subseteq V$. In fact, since $x_{\lambda^*} \in \partial^-_{\varepsilon_{\lambda^*}} f(z_{\lambda^*})$, one has for any $v \in E(z_{\lambda^*})$ and $h \in X$ that (compare definition of $\partial^-_\varepsilon f$ and (2)):

$$\langle x_{\lambda^*}, h \rangle \leq \min\{\langle v', h \rangle \mid v' \in E(z_{\lambda^*})\} + \varepsilon \|h\| \leq \langle v, h \rangle + \varepsilon \|h\|$$

From this it follows $\|x_{\lambda^*} - v\| \leq \varepsilon$, hence $E(z_{\lambda^*}) \subseteq x_{\lambda^*} + B_{\varepsilon_{\lambda^*}}$.
Since $C \setminus V$ is weakly compact, continuity of f implies existence of some (strong) open neighborhood U' of z_λ such that $E(y) \subseteq V \ \forall y \in U'$. As a consequence of a theorem by Preiss ([17], Th. 2.4) the lipschitz continuous function f is gateaux-differentiable on a dense subset of X. Now, $U \cap U'$ is a nonempty open set which consequently must contain some point y such that f possesses some gateaux derivative y^* at y. Then, by (3)

$$\partial^- f(y) = \{y^*\} = E(y) \subseteq V$$

Summarizing, to each pair U, V of (strong, weak) open neighborhoods of $0, x$ we can assign $y_{[U,V]} \in U, y^*_{[U,V]} \in V$ such that $y^*_{[U,V]} \in \partial^- f(y_{[U,V]})$. In this way one obtains converging nets $y_\mu \to 0, y^*_\mu \to x, y^*_\mu \in \partial^- f(y_\mu)$ $(\mu = [U, V])$, hence

$$x \in \limsup_{z \to 0} \partial^- f(z)$$

as desired. $\square$.

Combining (4) in lemma 3.1 with lemma 2.1 one gets

Corollary 3.1 *Under the assumptions of lemma 3.1 the approximate subdifferential of f (defined in the lemma) computes as*

$$\partial_a f(0) = \limsup_{z \to 0} \partial^- f(z)$$

As it was stated in section 2 the approximate subdifferential of a locally lipschitzian function defined on some Banach space is weak*- compact. In turn, the following theorem shows, that in a Hilbert space setting each weakly compact set may be approximated by subdifferentials of lipschitzian functions.

Theorem 3.1 *For any weakly compact subset K of some Hilbert space $(\mathcal{H}, \langle\rangle)$ there exists a family $f_u : \mathbb{R} \times \mathcal{H} \to \mathbb{R} \, (u \in \mathbb{R})$ of lipschitzian functions with some common modulus L, such that $\lim_{u \to 0} \partial_a f_u(0,0) = \{0\} \times K$ holds in the sense of Kuratowski-Painlevé convergence.*

Proof: Let c be a constant with $\|x\|^2 < c \; \forall x \in K$ (which exists by weak compactness of K) and fix a parameter u with $0 < u \le 1/c$. In $\mathbb{R} \times \mathcal{H}$ (which is a Hilbert space by canonical extension of the inner product) consider the ellipsoidal surface

$$E_u = \{(t,x) \in \mathbb{R} \times \mathcal{H} \mid u\|x\|^2 + (t-1)^2 = 1\}$$

as well as the subset

$$K_u = \{(t,x) \in \mathbb{R} \times \mathcal{H} \mid x \in K, t = 1 - \sqrt{1 - u\|x\|^2}\}$$

Note, that K_u is correctly defined and $K_u \subseteq E_u$. Furthermore, the function $f_u : \mathbb{R} \times X \to \mathbb{R}$ with

$$f_u(\alpha, z) = \min\{\langle(\alpha, z), (t,x)\rangle \mid (t,x) \in \bar{co} K_u\}$$

is correctly defined as well (by weak compactness of $\bar{co} K_u$). Since

$$K_u \subseteq [0,1] \times B(0; \sqrt{c}) \subseteq B((0,0); \sqrt{c+1})$$

one has $\bar{co} K_u \subseteq B((0,0); \sqrt{c+1})$, hence $\sqrt{c+1}$ is a modulus of lipschitz continuity for f_u (not depending on u).

Next we show validity of the following relations which are essential for proving the assertion of the theorem:

$$pr(\partial_a f_u(0,0) \mid \{0\} \times \mathcal{H}) = \{0\} \times K \tag{5}$$

$$pr(\partial_a f_u(0,0) \mid I\!R \times \{0\}) \subseteq [0, 1 - \sqrt{1-uc}] \times \{0\} \tag{6}$$

Here, $pr(\cdot \mid M)$ denotes projection onto a closed subspace M of $I\!R \times \mathcal{H}$. To show first the inclusion $\{0\} \times K \subseteq pr(\partial_a f_u(0,0) \mid \{0\} \times \mathcal{H})$ it is sufficient to verify the relation $K_u \subseteq \partial_a f_u(0,0)$ because of $pr(K_u \mid \{0\} \times \mathcal{H}) = \{0\} \times K$. So let $(t,x) \in K_u$. For $n \in I\!N$ put $\alpha_n = (1-t)/n$ and $z_n = -(u/n)x$. Then, the Kuhn-Tucker conditions (recall that for $u > 0$ E_u is a regular surface) yield that the linear function $\langle(\alpha_n, z_n), \cdot\rangle$ restricted to E_u attains its unique minimum at the point (t,x). Since $(t,x) \in K_u \subseteq E_u$ and by a convexity argument one has

$$\begin{aligned} \{(t,x)\} &= argmin\{\langle(\alpha_n, z_n), (t', x')\rangle \mid (t', x') \in K_u\} \\ &= argmin\{\langle(\alpha_n, z_n), (t', x')\rangle \mid (t', x') \in \bar{co}K_u\} \end{aligned}$$

Therefore, with the notation introduced in lemma 3.1, $E(\alpha_n, z_n) = \{(t,x)\}$ and (3) imply $\partial^- f_u(\alpha_n, z_n) = \{(t,x)\}$. Together with $(\alpha_n, z_n) \to (0,0)$ and using the trivial sequence $(t_n, x_n) \equiv (t,x)$ corollary 3.1 provides

$$(t,x) \in \limsup_{(\alpha,z)\to(0,0)} \partial^- f_u(\alpha, z) = \partial_a f_u(0,0)$$

Before proving the reverse inclusion of (5) and (6) we show first the relation

$$(t^*, x^*) \in \partial^- f_u(\alpha, z) \Longrightarrow x^* \in K, \quad t^* \in [0, 1 - \sqrt{1-|u|c}] \tag{7}$$

In fact, from (3) in lemma 3.1 one has

$$\{(t^*, x^*)\} = argmin\{\langle(\alpha, z), (t', x')\rangle \mid (t', x') \in \bar{co}K_u\} \tag{8}$$

and by convexity of $\bar{co}K_u$ there exists a sequence $\{(t_n, x_n)\} \subseteq K_u$ such that $\langle(\alpha, z), (t_n, x_n)\rangle \to_n \langle(\alpha, z), (t^*, x^*)\rangle$. Weak compactness of $\bar{co}K_u$ then yields existence of a weakly convergent subsequence $(t_{n_l}, x_{n_l}) \rightharpoonup_l (\tau, y) \in \bar{co}K_u$. Consequently, $\langle(\alpha, z), (\tau, y)\rangle = \langle(\alpha, z), (t^*, x^*)\rangle$, hence $(\tau, y) = (t^*, x^*)$ because of (8). It results $x_{n_l} \rightharpoonup_l x^*$ with $x_{n_l} \in K$ (since

$(t_{n_l}, x_{n_l}) \in K_u$ and by definition of K_u). Weak compactness of K yields $x^* \in K$, as desired. On the other hand, from the definition of K_u it follows that $0 \leq t_{n_l} \leq 1 - \sqrt{1-uc}$, which proves the second implication in (7).

Now, let

$$(t, x) \in \partial_a f_u(0,0) = \limsup_{(\alpha,z)\to(0,0)} \partial^- f_u(\alpha, z)$$

(compare corollary 3.1), i.e. there exist nets $(\alpha_\lambda, z_\lambda) \to (0,0), (t_\lambda, x_\lambda) \rightharpoonup (t,x)$ with $(t_\lambda, x_\lambda) \in \partial^- f_u(\alpha_\lambda, z_\lambda)$. (7) yields $x_\lambda \in K$ and $t_\lambda \in [0, 1 - \sqrt{1-uc}]$, whence $x \in K$ by weak compactness of K and $t \in [0, 1 - \sqrt{1-uc}]$. Thus, (5) and (6) are completely proved.

In order to verify the assertion of the theorem it suffices to show the relation

$$\limsup_{u\to 0} \partial_a f_u(0,0) \subseteq \{0\} \times K \subseteq \liminf_{u\to 0} \partial_a f_u(0,0) \tag{9}$$

which directly implies equality between all three sets. Considering first some element $(t, x) \in \limsup_{u\to 0} \partial_a f_u(0,0)$, one has converging nets $u_\lambda \to 0$, $(t_\lambda, x_\lambda) \rightharpoonup (t,x)$, $(t_\lambda, x_\lambda) \in \partial_a f_{u_\lambda}(0,0)$. Then (5) implies $x_\lambda \in K$ which means $x \in K$ because of weak compactness of K. Furthermore, the nets u_λ, t_λ being real valued, one can extract converging sequences $u_n \to 0, t_n \to t$ as well as $(t_n, x_n) \in \partial_a f_{u_n}(0,0)$. The latter provides $t_n \in [0, 1 - \sqrt{1-u_n c}]$ with reference to (6). Therefore $t = 0$ as was to be proved.

Concerning the second inclusion let $x \in K$. We have to show that $(0, x) \in \liminf_{u\to 0} \partial_a f_u(0,0)$. To this aim consider an arbitrary net $u_\lambda \to 0$. Put $x_\lambda \equiv x$ and choose - in virtue of (5) and (6) - some t_λ with $(t_\lambda, x) \in \partial_a f_{u_\lambda}(0,0)$ and $0 \leq t_\lambda \leq 1 - \sqrt{1-u_\lambda c}$. Obviously, $(t_\lambda, x_\lambda) \rightharpoonup (0, x)$, as desired. □

It is clear that a similar approximation result as in theorem 3.1 does not hold, for instance, for Clarke's subdifferential being restricted to convexity. This will become even clearer from the following topological derivations of the theorem.

Corollary 3.2 *Let X be a Hilbert space with $\dim X \geq 2$. Then, given any $n \in \mathbb{N}$ there is a lipschitz continuous function $f : X \to \mathbb{R}$ such that the number of connected components of $\partial_a f(0)$ is at least n.*

Proof: First we remark that, without loss of generality, we may restrict considerations to the case $X = I\!\!R \times \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space of dimension greater than zero. In fact, choose an element $x \in X$, $(x \neq 0)$, denote by $\mathcal{H}$ the orthogonal complement of x and consider the mapping $\phi : X = span(x) \oplus \mathcal{H} \to I\!\!R \times \mathcal{H}$ defined by $\phi(y) = \phi(y_x + y_H) = (y_x\|x\|, y_H)$. Clearly, ϕ is an isometric isomorphism between Hilbert spaces (leaving invariant the inner product). Now consider a lipschitz continuous function $f : I\!\!R \times \mathcal{H} \to I\!\!R$ as well as the (lipschitz continuous) composition $\tilde{f} := f \circ \phi : X \to I\!\!R$. From the definitions one easily deduces the relation $\partial^- \tilde{f}(y) = \phi^{-1}[\partial^- f(\phi(y))]$ which implies the corresponding equation for the approximate subdifferentials, in particular $\partial_a \tilde{f}(0) = \phi^{-1}[\partial_a f(0)]$. With ϕ being a homeomorphism (w.r.t. the weak topologies) the connectivity structures of the approximate subdifferentials of f and $\tilde{f}$, respectively, coincide.
So, let $X = I\!\!R \times \mathcal{H}$. In $\mathcal{H}$ define K to be the union of n pairwise disjoint, closed balls (recall that $dim\, \mathcal{H} \geq 1$). Then K is a (weakly) compact subset and there exist (weakly) open sets U_i which are pairwise disjoint and satisfy $K \subseteq \cup_{i=1}^n U_i$ and $K \cap U_i \neq \emptyset$ $(i = 1, \ldots, n)$. Consequently, $\{0\} \times K \subseteq \cup_{i=1}^n (I\!\!R \times U_i)$ with pairwise disjoint, (weakly) open sets $I\!\!R \times U_i$. By (5) in theorem 3.1 there is a lipschitzian function $f_u : I\!\!R \times \mathcal{H} \to I\!\!R$ (for some fixed $u \in I\!\!R$), such that $\partial_a f_u(0,0) \subseteq \cup_{i=1}^n (I\!\!R \times U_i)$ and $\partial_a f_u(0,0) \cap (I\!\!R \times U_i) \neq \emptyset$ $(i = 1, \ldots, n)$. Therefore, $\partial_a f_u(0,0)$ contains at least n connected components. $\square$

The corollary shows that, apart from finiteness, there is no restriction on the number of connected components of the approximate subdifferential. Recall that, by convexity, all classical subdifferentials have only one connected component. In finite dimensions a much stronger result may be derived from theorem 3.1, which was first proved directly in [4] (theorem 3.2).

Corollary 3.3 *Given any compact subset $K \subseteq I\!\!R^n$ there exists a lipschitz continuous function $f : I\!\!R^{n+1} \to I\!\!R$ such that $\partial_a f(0)$ is homeomorphic with K.*

Proof: In the finite dimensional case the set K_u defined in theorem 3.1 is compact and so in the definition of f_u one may replace $\bar{co}K_u$ by K_u.

Doing so and repeating the same line of argumentation as in the theorem one arrives at the sharper (compared to (5) and (6)) relation $\partial_a f_u(0) = K_u$. Now it is immediately seen that the function $x \mapsto (x, 1-\sqrt{1-u\|x\|^2})$ is a bijection from K to K_u which is continuous in both directions (with projection $(x,t) \mapsto x$ as inverse function). $\square$
vspace0.3cmAs a consequence, not only the number of connected components of the approximate subdifferential is free - as in the infinite-dimensional setting of corollary 3.2 - but the topological type itself. For instance, the homeomorphic copy of a Cantor set may be realised. Furthermore, in [4] it was proved that each compact set of $I\!R^n$ may be exactly realised (not only as a homeomorphic copy) by the so-called partial approximate subdifferential of some locally lipschitzian function. This last concept was introduced by Jourani and Thibault [9] in the context of functions depending on a parameter.
Revisiting the previous results one could ask, whether it is really necessarry to introduce one extra dimension for defining the lipschitzian function in theorem 3.1 and corollary 3.3. Theorem 3.2 below indicates, that for real continuous functions $f : I\!R^1 \to I\!R^1$ the approximate subdifferential either is an interval or the disjoint union of two intervals, which obviously is a topological restriction. Thus, in general, the domain of f cannot coincide with the space which the compact sets in theorem 3.1 and lemma 3.3 are considered to be subsets of. Furthermore, this result answers the question why corollary 3.2 is only valid when $dim\, X \geq 2$.
Before proving the mentioned univariate statement of theorem 3.2 we need two additional lemmas. To this aim, for functions $f : I\!R \to I\!R$ we introduce the following notation ($z \in I\!R$):

$$d^l(z) = \liminf_{\substack{t \downarrow 0 \\ u \to -1}} t^{-1}(f(z+tu) - f(z)); \;\; d^r(z) = \liminf_{\substack{t \downarrow 0 \\ u \to 1}} t^{-1}(f(z+tu) - f(z)),$$

where improper values $\pm\infty$ are allowed. Obvioiusly, it holds

$$\partial^- f(z) = [-d^l(z), d^r(z)] \tag{10}$$

with the interval to be interpreted appropriately for improper values.

Lemma 3.2 *Let $f : I\!R \to I\!R$ be continuous and $t_1, t_2 \in I\!R$, $t_1 < t_2$. Then*

$$\begin{aligned} d^r(t_1) < -d^l(t_2) \ &\Rightarrow\ \forall c \in (d^r(t_1), -d^l(t_2))\ \exists \bar{t} \in (t_1, t_2) : \\ &\qquad c \in [-d^l(\bar{t}), d^r(\bar{t})] \qquad (11) \\ d^r(t_1) > -d^l(t_2) \ &\Rightarrow\ \forall c \in (-d^l(t_2), d^r(t_1))\ \exists \bar{t} \in (t_1, t_2) : \\ &\qquad c \in [d^r(\bar{t}), -d^l(\bar{t})] \qquad (12) \end{aligned}$$

Proof: Concerning (11) let $c \in (d^r(t_1), -d^l(t_2))$. Then, the function $f(x) - cx$, restricted to $x \in [t_1, t_2]$ attains its minimum in a point of the open interval (t_1, t_2). To see this, assume that t_1 is a minimizer. It follows

$$f(t + t_1) - f(t_1) \geq ct \quad \forall t \in [0, t_2 - t_1]$$

This however means

$$\liminf_{\substack{t \downarrow 0 \\ u \to 1}} t^{-1}(f(t_1 + ut) - f(t_1)) \geq c$$

leading to the contradiction $c \leq d^r(t_1)$. In the same way (t_2 may be excluded as a minimizer. As a consequence, there exists $\bar{t} \in (t_1, t_2)$, such that $f(x) - cx \geq f(\bar{t}) - c\bar{t}\ \forall x \in [t_1, t_2]$. Choose $\delta > 0$ with $(\bar{t} - \delta, \bar{t} + \delta) \subseteq (t_1, t_2)$ to get

$$f(t + \bar{t}) - f(\bar{t}) \geq ct \quad \forall t \in (-\delta, \delta)$$

which immediately implies (11). For (12) consider the maximum (rather than minimum) of $f(x) - cx$ and repeat the same arguments. $\square$

Lemma 3.3 *Let $f : I\!R \to I\!R$ be continuous and $\partial^- f(z) \neq \emptyset$ for $z \in I\!R$. Then $\partial_a f(z)$ is a closed (possibly unbounded) interval.*

Proof: By assumption, there is some $e \in \partial^- f(z) \subseteq \partial_a f(z)$. It will be sufficient to show that $a \in \partial_a f(z)$, $a < e$ implies $(a, e) \subseteq \partial_a f(z)$ (the proof is running along similar lines for $a > e$). According to the definition, there exist sequences

$$t_k \to 0, \quad a_k \to a, \quad a_k \in \partial^- f(z + t_k) \qquad (13)$$

Let $c \in (a, e)$ be arbitrarily given. We have to show that $c \in \partial_a f(z)$. For $k \geq k_0$ one has $c > a_k$ and, without loss of generality, $c \notin \partial^- f(z + t_k) \cup$

$\partial^- f(z)$ (since otherwise the assertion $c \in \partial_a f(z)$ follows immediately). Combining (13) with (10) one gets

$$-d^l(z+t_k) \le a_k \le d^r(z+t_k) < c < -d^l(z) \le e \le d^r(z) \quad \forall k \ge k_0 \quad (14)$$

<u>1. case:</u>
There is a negative subsequence $t_{k_p} < 0$. Application of (11) to the relation $d^r(z+t_{k_p}) < -d^l(z)$ (see (14) yields existence of $\tau_p \in (t_{k_p}, 0)$ with $c \in \partial^- f(z+\tau_p)$ (see (10). The assertion follows from $\tau_p \to 0$.
<u>2. case:</u>
There is a positive subsequence $t_{k_p} > 0$. Fix an arbitrary $\bar{c} \in (c, e)$ and apply (12) to the relation $d^r(z) > -d^l(z+t_{k_p})$. Hence, there exists $\tau_p \in (0, t_{k_p})$ with $\bar{c} \le -d^l(z+\tau_p)$. Next choose $q \ge p$ to fulfill $t_{k_q} < \tau_p$. Application of (11) to the relation $d^r(z+t_{k_q}) < c < \bar{c} \le -d^l(z+\tau_p)$ finally provides $c \in \partial^- f(z+\mu_p)$ for some $\mu_p \in (t_{k_q}, \tau_p)$. The assertion follows from $\mu_p \to 0$.
<u>3. case:</u>
$t_k = 0 \quad \forall k \ge k_0$. Then $a_k \in \partial^- f(z)$ thus $a \in \partial^- f(z)$ by closedness and $(a, e) \subseteq \partial^- f(z) \subseteq \partial_a f(z)$ by convexity of the Dini subdifferential. □

Theorem 3.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then $\partial_a f(z)$ contains at most two connected components (for all $z \in \mathbb{R}$).*

Proof: Negating the assertion means existence of elements $a < b < c$ with $a, b, c \in \partial_a f(z)$ and which are contained in three pairwise different connected components. By definition there are sequences

$$t_k, \tau_k, \mu_k \to 0, \quad a_k \to a,\ b_k \to b,\ c_k \to c,$$
$$a_k \in \partial^- f(z+t_k),\ b_k \in \partial^- f(z+\tau_k),\ c_k \in \partial^- f(z+\mu_k)$$

Two of the sequences t_k, τ_k, μ_k must contain subsequences with equal sign (≥ 0 or ≤ 0). Without loss of generality assume $t_k, \tau_k \le 0$. If $t_{\bar{k}} = 0$ for some $\bar{k}$, then $a_{\bar{k}} \in \partial^- f(z)$ and lemma 3.3 would yield a contradiction with the assumed number of connected components of $\partial_a f(z)$. Hence one may operate with sequences $t_k \to 0\,(t_k < 0)$ and $\tau_k \to 0\,(\tau_k \le 0)$ in a way which is very similar to the proof of lemma 3.3, to obtain that $c \in (a, b)$ implies $c \in \partial_a f(z)$. Hence, $[a, b] \subseteq \partial_a f(z)$ which contradicts the assumption, that a and b come from different connected components of $\partial_a f(z)$. □

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