On Simple Impasse Points and Their Numerical Computation

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Abstract

Impasse Points are a well-known phenomenon in the theory of nonlinear circuits, which are often modelled by semiexplicit differential-algebraic equations (DAE's). They are a special kind of singularity of the DAE since the index-1-condition for the DEA is locally violated there. We consider a class of quasilinear DAE's more generally. We classify simple singularities as those which can be overcome by a transformation of the independent variable leading to an augmented DAE with geometrical or differential index 1. On this basis, we derive sufficient conditions for a simple singularity to be an impasse point and show a way to compute such points using standard software for index-1-DAE's. We analyse properties of solutions with a simple impasse point and study numerical properties of the used augmented systems.

1 Introduction

In Chua/Deng [2] impasse points are described as an important phenomenon found in many nonlinear circuits and systems. The best-know example is the degenerated Van-der-Pol-oscillator formulated as a semiexplicit DAE:

$$\left.\begin{array}{rcl} x' &=& f(x,y) := -y \\ 0 &=& g(x,y) := -x + y^3 - y \end{array}\right\}, \tag{1.1}$$

where $x(t), y(t) \in I\!R$. Any solution must lie on the constrained manifold

$$M := \{(x,y) \in I\!R \times I\!R : -x + y^3 - y = 0\},$$

which is illustrated in Figure 1.

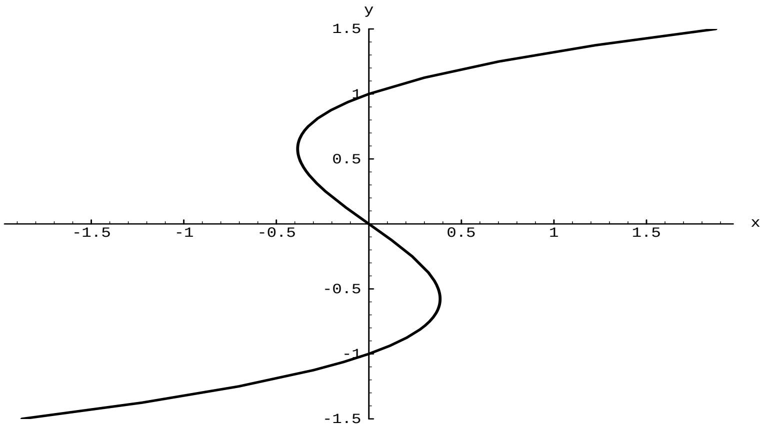

Figure 1

By the differential part of the DAE it is clear that a solution must follow the curve along the direction indicated by the arrowheads in Figure 1. Any solution must reach point A (resp. B) at some finite time $t = t_0$ if the initial point is located on the upper half plane (resp. lower half plane) of $\mathbb{R}^2$. In A (resp. B) it is impossible to continue the solution beyond $t = t_0$ and the solution does not exist for $t > t_0$. The points A and B are examples of (forward) impasse points.
The analytical background for this phenomenon is that the index-1-condition for (1.1), i.e. the nonsingularity of g'_y, is violated in A and B. Using standard software this leads to such difficulties in the neighbourhood of impasse points that it is impossible to obtain good approximations for these points. Even worse, Chua/Deng [2] describe a random small-amplitude oscillation of the numerically computed solution in the vicinity of such points. In higher dimensional problems with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$ under certain degeneracy-conditions the set of singularities forms an $n-1$-dimensional submanifold of the n-dimensional constrained manifold.
For an analytical characterization of impasse points (x_0, y_0), Chua/Deng [2] use the associated static problem

$$h(y, \tau) := g(x_0 + \tau f(x_0, y_0), y) = 0.$$

They call (x_0, y_0) an impasse point of (1.1) if this static problem has a turning point in $(y_0, 0)$.
Though the declared one-to-one-correspondence of the solutions of (1.1) and of this static equation is not true in general, our conditions for (x_0, y_0) to be a simple impasse point of (1.1) are exactly those, saying that $(y_0, 0)$ should be a simple turning point of this associated static problem.
The same set of conditions is derived in Rabier/Rheinboldt [10]. Generally this paper deals with quasilinear DAE's of the form

$$\mathfrak{A}(w)w' = G(w), \quad w \in I\!R^k, \tag{1.2}$$

where $\mathfrak{A}(w)$ has constant rank $r < k$ whenever $G(w) \in \operatorname{im} \mathfrak{A}(w)$. Here an impasse point of the DAE (1.2) is defined by conditions equivalent to those of a standard singular point in the sense of Rabier [8] of the singular reduced ordinary differential equation (ODE) resulting by using local coordinates of the hidden constrained manifold.
Recently Reißig [12, 13] has analyzed various mathematical definitions of impasse points related to different smoothness-suppositions for solutions of (1.1) very carefully. Theorem 4.4 in [13] also proves that exactly these simple-impasse-point-conditions guarantee that (x_0, y_0) is actually an impasse point of (1.1) in the sense that there is a solution approaching the impasse point, that cannot be continued through this point.
However, having characterized impasse points, the next important question is, how it will be possible to compute them and the solution branches in their vicinity numerically.
One possibility to do this is represented in Rabier/Rheinboldt [11]. Their method is based on the handling of singular points of singular ODE's. They found that the numerical difficulties can be overcome if the solution of the ODE is not computed in dependence on the original independent variable but in dependence on an artificial new one, which should lead to an approximation of the arclength-parametrization of the solution curve. The method results in an augmented regular ODE. Though for standard singular points a simple augmentation is sufficient, the emphasis of this paper is also on some "higher" singularities, for which higher dimensional augmentations are shown to be successful.

For the handling of singular points of quasilinear DAE's of the form

$$\left.\begin{array}{rcl} A(w)w' & = & f(w) \\ 0 & = & g(w) \end{array}\right\}, \tag{1.3}$$

where $w \in I\!R^{n+m}$, $f(w) \in I\!R^n$, $g(w) \in I\!R^m$, rank $A(w) = n$, they develop an efficient algorithm for a numerical way of local reduction of (1.3) to the reduced ODE in local coordinates of the constrained manifold $g^{-1}(0)$.
In the present paper we use the idea of a tranformation of the independent variable directly for DAE's of the form

$$\left.\begin{array}{rcl} A(t,w)w' & = & f(t,w) \\ 0 & = & g(t,w) \end{array}\right\}, \tag{1.4}$$

where $w \in I\!R^{n+m}$, $f(t,w) \in I\!R^n$, $g(t,w) \in I\!R^m$, rank $A(t,w) = n$. Analogously to the theory of parameter-dependent nonlinear equations, where simple singularities are exactly those, which can be overcome by a transformation of the parameter, we define simple singularities of (1.4) as singularities that can be overcome by a transformation of the independent variable.
Based on this, we are able to derive sufficient conditions for simple singularities to be an impasse point and solution properties of the DAE in the vicinity of such impasse points in a really simple and straightforward manner. The suggested transformation to an augmented DAE allows the computation of the solutions in the neighbourhood of such impasse point as well as the impasse point itself with the aid of standard software for (quasilinear) index-1-DAE's.

In § 2 we start with the basic notations and definitions.
In § 3 we describe the transformation of the independent variable and the resulting augmented DAE's. We define simple singularities and simple impasse points and discuss solution properties of (1.4) in the vicinity of such points. We show how our results for DAE's correspond to those of Rabier [8] for singular ODE's.
In § 4 we apply a result of Arnold/Strehmel/Weiner [1] and show that though the perturbation index of quasilinear systems with geometrical index 1 may be 2, the influence of the derivative of a perturbation on the solutions is essentially reduced by a factor in the size of the perturbation itself.

2 Basic definitions and suppositions

We deal with semiexplicit autonomous DAE's of the form

$$\left.\begin{array}{rcl}\dot{x}(t) &=& f(x(t),y(t)) \\ 0 &=& g(x(t),y(t))\end{array}\right\}, \tag{2.1}$$

where $f : X \times Y \to I\!\!R^n$, $g : X \times Y \to I\!\!R^m$, $X \subseteq I\!\!R^n$, $Y \subseteq I\!\!R^m$ open nonempty sets, g continuously differentiable, f and g' Lipschitz-continuous and rank $g'(x,y) = m$ or, more generally, with quasilinear DAE's of the form

$$\left.\begin{array}{rcl}A(t,w)w' &=& f(t,w) \\ 0 &=& g(t,w)\end{array}\right\}, \qquad (2.2)\ \begin{array}{c}(a)\\(b)\end{array}$$

where $f : J \times W \to I\!\!R^n$, $g : J \times W \to I\!\!R^m$, $A : J \times W \to L(I\!\!R^{n+m}, I\!\!R^n)$ with rank $A(t,w) = n$, $J \subseteq I\!\!R$ some open interval, $W \subseteq I\!\!R^{n+m}$ open and nonempty, g continuously differentiable, A , f and g' Lipschitz-continuous with respect to w and rank $g'_w(t,w) = m$.
We say that $w \in C^1(J_0, I\!\!R^{n+m})$ with (2.2) for all $t \in J_0$, where $J_0 \subseteq J$ some interval, is a solution of (2.2), and, that $(x,y) \in C^1(J_0, I\!\!R^n) \times C^1(J_0, I\!\!R^m)$ with (2.1) for all $t \in J_0$, where $J_0 \subseteq I\!\!R$ some interval, is a solution of (2.1). Note that we have chosen a concept of solutions where all components of the solutions have to be smooth. This fits the suppositions for continuity and smoothness for f, g and A.
According to the concept of a geometrical (see Rabier/Rheinboldt [9]) or differentiation (see Gear [3]) index, but avoiding the use of higher derivatives, we say that the quasilinear DAE (2.2) has index 1 locally at some point $(t,w) \in g^{-1}(0)$ iff the matrix

$$\begin{bmatrix} A(t,w) \\ g'_w(t,w) \end{bmatrix} \in L(I\!\!R^{n+m})$$

is nonsingular. If this condition is true for some point $(t_0,w_0) \in g^{-1}(0)$, then the system

$$\begin{pmatrix} A(t,w) \\ g'_w(t,w) \end{pmatrix} w' = \begin{pmatrix} f(t,w) \\ -g'_t(t,w) \end{pmatrix}, \tag{2.3}$$

which we obtain when differentiating the constrained equations (2.2b), defines an implicit ODE in some neighbourhood of (t_0,w_0). Here the theory

of ODE's guarantees that (2.3) has a unique solution with $w(t_0) = w_0$, which then represents the unique solution of (2.2) with $w(t_0) = w_0$.
We can regard (2.1) as a special case of (2.2). The index-1-condition at some point $(x, y) \in g^{-1}(0)$ is fulfilled iff the matrix

$$g'_y(x, y) \in L(\mathbb{R}^m)$$

is nonsingular. Exploiting the special structure and using the concept of transferable DAE's from Griepentrog/März [4] or of the tractibility index (see März [6]) we end at the same condition. It guarantees that the constrained equation

$$0 = g(x, y)$$

is uniquely solvable for $y = y(x)$ locally in some neihbourhoods of x_0 in X and y_0 in Y. Hence, the DAE (2.1) is locally transferable to state form

$$\dot{x} = f(x, y(x)).$$

Let S_1 be the subset of $g^{-1}(0)$, where the index-1-condition is violated, i.e.

$$\left.\begin{array}{ll} S_1 := \{(x, y) \in g^{-1}(0) : \det g'_y(x, y) = 0\} & \text{for problem (2.1), resp.} \\ S_1 := \{(t, w) \in g^{-1}(0) : \det \binom{A(t,w)}{g'_w(t,w)} = 0\} & \text{for problem (2.2)} \end{array}\right\}. \tag{2.4}$$

Here we do not deal with higher-index-DAE's and suppose that S_1 is a lower-dimensional submanifold of $g^{-1}(0)$, i.e. (2.1) or (2.2) have index 1 on $g^{-1}(0)$ except for the lower-dimensional submanifold of singularities S_1.

3 Simple singularities

Rabier/Rheinboldt [10, 11] handle impasse points in the following way: They reduce the DAE locally to an ODE in local coordinates and treat the resulting singularity there by an augmentation of the ODE. In the simplest case of standard singular points in the sense of Rabier [8] this augmentation could be interpreted as a suitable transformation of the independent variable.
We suggest to use this augmentation technique for the DAE itself:

We determine the solution of the DAE not necessarily in dependence on the independent variable t, but on an artifical variable s. For $t = t(s)$ this leads to

$$
\begin{aligned}
\frac{\mathrm{d}x(t(s))}{\mathrm{d}s} &= \frac{\mathrm{d}t(s)}{\mathrm{d}s} \cdot f(x(t(s)), y(t(s))),\\
0 &= g(x(t(s)), y(t(s)))
\end{aligned}
$$

or

$$
\begin{aligned}
A(t(s), w(t(s)))\frac{\mathrm{d}w(t(s))}{\mathrm{d}s} &= \frac{\mathrm{d}t(s)}{\mathrm{d}s} f(t(s), w(t(s))),\\
0 &= g(t(s), w(t(s)))
\end{aligned}
$$

for (2.1) and (2.2), respectively.
We abbreviate $x(s) := x(t(s))$, $x'(s) := \frac{\mathrm{d}}{\mathrm{d}s}x(s)$, etc. and add linear conditions to normalize the vector of derivatives

$$
r^T \begin{pmatrix} t'(s) \\ x'(s) \\ y'(s) \end{pmatrix} = 1 \quad \text{or} \quad r^T \begin{pmatrix} t'(s) \\ w'(s) \end{pmatrix} = 1,
$$

where $r \in I\!R^{1+n+m}$ is suitably chosen.
Note, that for $r = (1, 0, 0)^T \in I\!R^{1+n+m}$ we get the original DAE in an autonomous form. The result is the augmented system

$$
\begin{aligned}
\begin{bmatrix} & r^T & \\ -f(x(s), y(s)) & I & 0 \end{bmatrix} \begin{pmatrix} t'(s) \\ x'(s) \\ y'(s) \end{pmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
0 &= g(x(s), y(s)),
\end{aligned} \tag{3.1}
$$

or

$$
\begin{aligned}
\begin{bmatrix} & r^T \\ -f(t(s), w(s)) & A(t(s), w(s)) \end{bmatrix} \begin{pmatrix} t'(s) \\ w'(s) \end{pmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
0 &= g(t(s), w(s))
\end{aligned} \tag{3.2}
$$

for (2.1) or (2.2), respectively.
As long as $t'(s)$ does not vanish the solutions of the augmented and the original systems are the same in phase space. For

$$
z(s) := \begin{pmatrix} t(s) \\ x(s) \\ y(s) \end{pmatrix} \in I\!R^{1+n+m} \quad \text{or} \quad z(s) := \begin{pmatrix} t(s) \\ w(s) \end{pmatrix} \in I\!R^{1+n+m},
$$

respectively, both systems represent quasilinear DAE's of the form

$$\begin{aligned} \mathcal{A}(z(s))z'(s) &= e_1 \\ 0 &= g(z(s)) \end{aligned}$$

and we can ask under which conditions we can find a suitable vector $r \in I\!R^{1+n+m}$ such that the resulting augmented quasilinear DAE has index 1 at some point $z_0 = (t_0, x_0, y_0)$ or $z_0 = (t_0, w_0)$, even if we had a singular point (x_0, y_0) or $(t_0, w_0) \in S_1$ for the original DAE.
This leads us to the following

Definition 3.1 Let $(x_0, y_0) \in S_1$ resp. $(t_0, w_0) \in S_1$ for (2.1) resp. (2.2). Then (x_0, y_0) resp. (t_0, w_0) is called a simple singularity of (2.1) resp. (2.2) iff there exists a suitable vector $r \in I\!R^{1+n+m}$ such that (3.1) resp. (3.2) have index 1.

Notation: Let ${g'_y}^0 := g'_y(x_0, y_0)$, $A^0 := A(t_0, w_0)$, ${g'_w}^0 := g'_w(t_0, w_0)$,
$f^0 := f(x_0, y_0)$ resp. $f^0 := f(t_0, w_0)$ etc.

Remark: The augmentation technique above and Definition 3.1 are not limited to DAE's of the form (2.2) where the constrained equations are given explicitly. It is easily possible to follow the same way for general quasilinear DAE's of the form

$$\bar{A}(t, w)w' = \bar{f}(t, w)$$

with rank $\bar{A}(t, w) = \text{const}$.
However, problems with impasse points are often degenerate singularly perturbed systems possesing the structure of (2.2) or even (2.1). Restricting to such problems we hope to make this paper more transparent.

Using the index-1-criteria of § 2 we obtain that $(x_0, y_0) \in S_1$ is a simple singularity of (2.1) iff

$$\text{rank} \begin{bmatrix} -f^0 & I & 0 \\ 0 & {g'_x}^0 & {g'_y}^0 \end{bmatrix} = n + m,$$

i.e.

$$\dim \ker {g'_y}^0 = 1 \quad \text{and} \quad {g'_x}^0 f^0 \notin \text{im}\, {g'_y}^0. \tag{3.3}$$

$(t_0, w_0) \in S_1$ is a simple singularity of (2.2) iff

$$\operatorname{rank} \begin{bmatrix} -f^0 & A^0 \\ g_t'^0 & g_w'^0 \end{bmatrix} = n + m,$$

i.e.

$$\dim\ker \begin{pmatrix} A^0 \\ g_w'^0 \end{pmatrix} = 1 \quad \text{and} \quad \begin{pmatrix} -f^0 \\ g_t'^0 \end{pmatrix} \notin \operatorname{im} \begin{pmatrix} A^0 \\ g_w'^0 \end{pmatrix}. \tag{3.4}$$

A good way to choose the normalizing vector $r \in \mathbb{R}^{1+n+m}$ in (3.1) or (3.2) is to take an element of the kernel of the just considered $(n+m) \times (1+n+m)$-matrices. However, each $r \in \mathbb{R}^{1+n+m}$ that is not orthogonal to that kernel is possible.

Theorem 3.1 *Let* $(x_0, y_0) \in S_1$ *resp.* $(t_0, w_0) \in S_1$ *be a simple singularity of* (2.1) *resp.* (2.2). *Let* $z \in C^1(J_s, \mathbb{R}^{1+n+m})$ *be a solution of the augmented DAE* (3.1) *resp.* (3.2) *with* $z(s_0) = z_0$, $s_0 \in J_s$, $J_s \subset \mathbb{R}$ *an open interval,* $z_0 = (t_0, x_0, y_0)$ *resp.* $z_0 = (t_0, w_0)$.
Then it holds:

(i) $t'(s_0) = 0$.

(ii) $\ker g_y'^0 = \operatorname{span}\{y'(s_0)\}$ *for* (2.1) *and* $\ker \begin{pmatrix} A^0 \\ g_w'^0 \end{pmatrix} = \operatorname{span}\{w'(s_0)\}$ *for* (2.2), *respectively.*

(iii) *Let* $(x(s), y(s)) \notin S_1$ *resp.* $(t(s), w(s)) \notin S_1$ *for all* $s \in J_s$ *with* $s \neq s_0$. *Then it holds that* $t'(s) \neq 0$ *for* $s \neq s_0$, *and* $y'(t)$ *resp.* $w'(t)$ *becomes unbounded for* $t \to t_0$.

(iv) *Let* f *and, for* (2.2) *also* A *be continuously differentiable and* g *be twice continuously differentiable.* (3.5)

Then also the solution of the augmented DAE is twice continuously differentiable and

$$t''(s_0) \neq 0 \text{ iff } g_{yy}''^0[y'(s_0), y'(s_0)] \;\notin\; \operatorname{im} g_y'^0 \quad \text{for (2.1)}, \tag{3.6}$$

$$\text{resp. } \begin{pmatrix} A_w'^0 \\ g_{ww}''^0 \end{pmatrix} [w'(s_0), w'(s_0)] \;\notin\; \operatorname{im} \begin{pmatrix} A^0 \\ g_w'^0 \end{pmatrix} \quad \text{for (2.2)}. \tag{3.7}$$

Proof: We perform the proof for the more general DAE (2.2).
Let $r = \binom{r_t}{r_w}$ be the vector chosen for normalization in (3.2). Since $z = (t, w)$ is a solution of (3.2), we have after differentiation of the constrained equation

$$\left.\begin{array}{rcl} r_t t'(s) + r_w^T w'(s) & = & 1 \\ \begin{pmatrix} -f(z(s)) \\ g_t'(z(s)) \end{pmatrix} t'(s) + \begin{pmatrix} A(z(s)) \\ g_w'(z(s)) \end{pmatrix} w'(s) & = & 0 \quad \text{for } s \in J_s \end{array}\right\} \quad (3.8) \begin{array}{c} \text{(a)} \\ \text{(b)} \end{array}$$

For $s = s_0$ it follows

$$\begin{pmatrix} -f^0 \\ g_t'^0 \end{pmatrix} t'(s_0) + \begin{pmatrix} A^0 \\ g_w'^{\,0} \end{pmatrix} w'(s_0) = 0.$$

Since z_0 is a simple singularity of (2.2), we conclude from (3.4)

$$\begin{pmatrix} -f^0 \\ g_t'^0 \end{pmatrix} \notin \operatorname{im} \begin{pmatrix} A^0 \\ g_w'^{\,0} \end{pmatrix}$$

and, hence, $t'(s_0) = 0$ and $w'(s_0) \in \ker \binom{A^0}{g_w'^{\,0}}$ have to be true, which proves (i).
Further we have $r_w^T w'(s_0) = 1$ such that $w'(s_0) \neq 0$ and $\ker \binom{A^0}{g_w'^{\,0}} = \operatorname{span}\{w'(s_0)\}$,
which proves (ii).
To prove (iii) we assume $t'(s) = 0$. Then (3.8b) implies

$$w'(s) \in \ker \begin{pmatrix} A(t(s), w(s)) \\ g_w'(t(s), w(s)) \end{pmatrix} \quad \text{and} \quad w'(s) \neq 0.$$

Hence, (2.2) would not have index 1 locally at $(t(s), w(s))$, and $(t(s), w(s))$ would be a singular point of (2.2).
Now we prove (iv). We can write (3.8) in matrix-form

$$\mathfrak{A}(z(s)) z'(s) = e_1 \quad \text{for } s \in J_s,$$

where we have $\mathfrak{A} \in C^1(J_0 \times W_0, L(I\!R^{1+n+m}))$ and $\mathfrak{A}(z_0)$ nonsingular. In some neighbourhood of z_0, $\mathfrak{A}(z)$ is invertible and $[\mathfrak{A}(z)]^{-1}$ depends continuously differentiably on z.
Hence, in some neighbourhood $Z_0 \subseteq J_0 \times W_0$ of z_0, (3.8) is equivalent to the explicit ODE

$$z'(s) = [\mathfrak{A}(z(s))]^{-1} e_1 =: b(z)$$

with $b \in C^1(Z_0, I\!R^{1+n+m})$, having a unique solution $z \in C^2(J_s, I\!R^{1+n+m})$ with $z(s_0) = z_0$.
Now, we can differentiate (3.8b) with respect to s. Evaluating the resulting differential equation at $s = s_0$ and using $t'(s_0) = 0$ we obtain

$$\begin{pmatrix} -f^0 \\ g_t'^0 \end{pmatrix} t''(s_0) + \begin{pmatrix} A^0 \\ g_w'^{\,0} \end{pmatrix} w''(s_0) + \begin{pmatrix} A_w'^{\,0} \\ g_{ww}''^{\,0} \end{pmatrix} [w'(s_0), w'(s_0)] = 0,$$

which implies the assertion

$$t''(s_0) = 0 \quad \text{iff} \quad \begin{pmatrix} A_w'^{\,0} \\ g_{ww}''^{\,0} \end{pmatrix} [w'(s_0), w'(s_0)] \in \operatorname{im} \begin{pmatrix} A^0 \\ g_w'^{\,0} \end{pmatrix}.$$

□

If the conditions (3.5) and (3.6) resp. (3.7) in Theorem 3.1(iv) are fulfilled, we have the interesting case that (x_0, y_0) and (t_0, w_0) are impasse points of (2.1) and (2.2), respectively, since the solutions of (2.1) or (2.2) approaching (x_0, y_0) or w_0 for $t \to t_0$ cannot be continued for $t > t_0$ in the case that $t_0 = t(s_0)$ is a local maximum for $t(s)$, or cannot be continued for $t < t_0$ in the case that $t_0 = t(s_0)$ is a local minimum for $t(s)$.

We summarize all the conditions in the following

Definition 3.2 Let the further smoothness-properties of (3.5) hold. Then we call $(x_0, y_0) \in g^{-1}(0)$ or $(t_0, w_0) \in g^{-1}(0)$ simple impasse point of (2.1) or (2.2), resp., iff they are simple singularities of (2.1) or (2.2), (i.e. (3.3) or (3.4), resp., hold) and (3.6) or (3.7), resp., are fulfilled.

Remark: Now, simple impasse points (x_0, y_0) of (2.1) are characterized by the following set of conditions:

$$\begin{aligned} &\dim \ker g_y'^{\,0} = 1 \quad \text{and} \quad g_x'^{\,0} f^0 \notin \operatorname{im} g_y'^{\,0} \\ &\text{and} \quad g_{yy}''^{\,0}[v_0, v_0] \notin \operatorname{im} g_y'^{\,0} \quad \text{for} \quad \ker g_y'^{\,0} = \operatorname{span}\{v_0\}. \end{aligned}$$

These are exactly the conditions that characterize a simple turning point of the nonlinear parameter-dependent equation

$$h(y, \tau) := g(x_0 + \tau f(x_0, y_0), y) = 0$$

at $(y, \tau) = (y_0, 0)$.

Chua/Deng [2] used this equation for a definition of impasse points.

The equation $h(y,\tau) = 0$ describes the solutions of the DAE

$$\begin{aligned} x' &= f(x_0, y_0) \\ 0 &= g(x,y) \end{aligned}$$

and we see that, locally around (x_0, y_0), the investigation of this systems gives qualitatively correct answers in the case of simple turning points and, of course, in the case that $(x_0, y_0) \in g^{-1}(0) \setminus S_1$.
For more general singularities it is not sufficient to consider this simple approximation of the original DAE to obtain characterizations of the singularity.
To illustrate this, let us take an example with a funnel, which we found in Takens [14]:

Example :

$$n = 2,\ m = 1, \qquad \begin{aligned} x_1' &= x_2 + 3y \\ x_2' &= 1 \\ 0 &= y^2 + x_1 \end{aligned}$$

Here we have

$$S_1 := \{(x_1, x_2, y) : x_1 = -y^2, 2y = 0\} = \{(x_1, x_2, y) : x_1 = 0, y = 0\}.$$

The points $(x_1, x_2, y) = (0, x_2, 0)$ with $x_2 \neq 0$ are simple impasse points. The origin $(0,0,0)$ is a funnel, i.e. for every $\varepsilon > 0$ and for every neighbourhood U of $(0,0,0)$ there exists an open and nonempty subset $W \subseteq U$ such that every solution of the DAE with initial value from W reaches $(0,0,0)$ in a time less than ε.
In $(0,0,0)$ we have

$$h(y,\tau) := y^2 = 0,$$

having the solution set $\{(y,\tau) = (0,\tau),\ \tau \in \mathbb{R}\}$, which consists of singular points only.

Now we want to analyse the situation in the neighbourhood of a simple impasse point in more detail.

- The augmented system (3.1) resp. (3.2) has a unique solution z with $z(s_0) = z_0$, which is twice continuously differentiable with respect to s and fulfils $t'(s_0) = 0$ and $t''(s_0) \neq 0$ such that $t(s)$ has a local extremum at $t(s_0) = t_0$.

- Because of $t''(s_0) \neq 0$ it holds that $t'(s) \neq 0$ for $s \neq s_0$ in some neighbourhood of s_0, say $(s_0 - \varepsilon, s_0 + \varepsilon)$ for some $\varepsilon > 0$.

- Without loss of generality let t_0 be a local maximum, i.e. $t(s) < t_0$ for $s \neq s_0$, $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$. Since $t'(s) \neq 0$ for $s \neq s_0$, the solutions of the augmented system for $s \in (s_0 - \varepsilon, s_0)$ and $s \in (s_0, s_0 + \varepsilon)$ represent the only solutions of the original DAE with $t \in (t(s_0 - \varepsilon), t_0)$ and $t \in (t(s_0 + \varepsilon), t_0)$. They approach (x_0, y_0) resp. w_0 for $t \to t_0$ and they have no continuation for $t > t_0$.

- From Theorem 3.1(iii) we know that the derivative of the solutions with respect to t , i.e. $y'(t)$ resp. $w'(t)$ becomes unbounded for $t \to t_0$.

- During a numerical computation of the solutions of (3.1) resp. (3.2) the simple impasse point does not play an outstanding role. As long as $t'(s) > 0$, we compute solutions corresponding to the solutions of the original DAE. After passing $t'(s_0) = 0$, the trajectory of the augmented system represents a second branch of solutions of (2.1) and (2.2), respectively, terminating in the impasse point, in negative time-direction.

- If we want to compute the impasse point, all we have to do is to check the sign of $t'(s)$ and, after a detected change in the sign, to solve the scalar equation $t'(s) = 0$ by some Newton-like method.

We note that we have continous and smooth dependence of an reached simple impasse point on parameters in the problem, especially on the initial values if they are stated in a correct way. This is based on corresponding properties for the used quasilinear DAE's (3.1) and (3.2) which we summarize in the appendix and follows at the end with the Implicit Function Theorem for the scalar equation $t'(s) = 0$ if t and therefore also t' depend on the mentioned parameters.

At the end of this chapter we want to show that simple impasse points are exactly those which correspond to standard singular points in the sense of [8] of the implicit ODE's arising from the DAE either by differentiation of the constrained equation or using local coordinates of the constrained manifold $g^{-1}(0)$.

We first recall the definition of standard singular points of Rabier [8]. For later comparisons we introduce also simple singular points in this context. Consider the implicit ODE

$$\left.\begin{array}{rcl} \mathfrak{A}(x)x' & = & G(x) \\ x(0) & = & x_0 \end{array}\right\} \tag{3.9}$$

with $G \in C^1(X_0, I\!R^k)$, $\mathfrak{A} \in C^1(X_0, L(I\!R^k))$, $x_0 \in X_0 \subseteq I\!R^k$, X_0 open.

Definition 3.3 x_0 is a simple singular point of (3.9) iff the following conditions hold:

$$\left.\begin{array}{ll} \text{(i)} & \mathfrak{A}(x_0) \text{ is singular and } \dim\ker\mathfrak{A}(x_0) = 1 \\ & \text{let } \ker\mathfrak{A}(x_0) = \operatorname{span}\{e_0\} \\ \text{(ii)} & G(x_0) \notin \operatorname{im}\mathfrak{A}(x_0) \end{array}\right\} \tag{3.10}$$

x_0 is a standard singular point of (3.9) iff it holds additionally that

$$\text{(iii)} \quad \mathfrak{A}'(x_0)[e_0, e_0] \notin \operatorname{im}\mathfrak{A}(x_0). \tag{3.11}$$

Of course, the qualitative results obtained in [8] for problems of the form (3.9) could also be obtained by applying the technique used in Theorem 3.1. We show the equivalences mentioned above for the more general DAE (2.2). The first mentioned way to derive an implicit ODE from (2.2) is the differentiation of the constrained equation leading to the $(n+m)$-dimensional implicit ODE (2.3), which then can be transformed into the autonomous form:

$$\left.\begin{array}{l} \mathfrak{A}_d(t,w)\binom{t'}{w'} = G_d(t,w) \quad \text{with} \\ \mathfrak{A}_d(t,w) = \begin{pmatrix} 1 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & A(t,w) \\ 0 & \vdots & g_w'(t,w) \end{pmatrix}, \quad G_d(t,w) = \begin{pmatrix} 1 \\ f(t,w) \\ -g_t'(t,w) \end{pmatrix}. \end{array}\right\} \tag{3.12}$$

For the second way we need a local representation of the constrained manifold

$$g^{-1}(0) = \{(t,w) \in J_0 \times W_0 : g(t,w) = 0\}$$

by local coordinates.

To simplify the computations we first make the usual transformation of (2.2) into an autonomous DAE

$$\begin{aligned} t' &= 1 \\ A(t,w)w' &= f(t,w) \\ 0 &= g(t,w), \end{aligned}$$

which clearly has the form

$$\begin{aligned} \bar{A}(\bar{w})\bar{w}' &= \bar{f}(\bar{w}) \\ 0 &= g(\bar{w}) \end{aligned} \tag{3.13}$$

where $\bar{w} = \binom{t}{w}$, $\bar{A}(\bar{w}) = \begin{pmatrix} 1 & 0 \\ 0 & A(\bar{w}) \end{pmatrix}$, $\bar{f}(\bar{w}) = \binom{1}{f(w)}$.
Let $\bar{W}_0 = J_0 \times W_0$, $\bar{n} = n + 1$.
Now, let $\bar{w}_0 \in g^{-1}(0)$ and let $g^{-1}(0)$ in the neighbourhood of $\bar{w}_0$ be represented by local coordinates in the following form:

$$\left.\begin{array}{l} \text{Let} \quad \bar{w}(\xi_0) = w_0 \ \text{ and } \ g(\bar{w}(\xi)) \equiv 0 \ \text{ for all } \ \xi \in U_0 \subset I\!R^{\bar{n}}, \\ \text{where } \ U_0 \ \text{ some open neighbourhood of } \ \xi_0 \in I\!R^{\bar{n}}, \\ \bar{w} \in C^2(U_0, I\!R^{\bar{n}+m}) \ \text{ and } \ \operatorname{rank} w'(\xi) = \bar{n} \ \text{ for all } \ \xi \in U_0. \end{array}\right\} \tag{3.14}$$

Then (3.13) is reduced to

$$\bar{A}(\bar{w}(\xi(t)))\bar{w}'(\xi(t)) \cdot \xi'(t) = \bar{f}(\bar{w}(\xi(t))) \tag{3.15}$$

$$\left.\begin{aligned} \text{and with } \ \mathfrak{A}_\ell(\xi) &:= \bar{A}(\bar{w}(\xi)) \cdot \bar{w}'(\xi) \\ \text{and } \ G_\ell(\xi) &:= \bar{f}(\bar{w}(\xi)) \end{aligned}\right\}. \tag{3.16}$$

We see that (3.1) is an $\bar{n}$-dimensional system of the form (3.9).
Now we can state the following:

Theorem 3.2 *Let* $(t_0, w_0) = \bar{w}_0 \in g^{-1}(0)$, *let* (3.5) *and* (3.13) *be true. Then it holds*

(i) $\bar{w}_0$ *is a simple singularity of* (2.2), *i.e.* (3.4) *holds*
iff $\bar{w}_0$ *is a simple singular point of* (3.12)
iff ξ_0 *is a simple singular point of* (3.15)

(ii) $\bar{w}_0$ *is a simple impasse point of* (2.2)
iff $\bar{w}_0$ *is a standard singular point of* (3.12)
iff ξ_0 *is a standard singular point of* (3.15).

Proof. 1) We start proving the equivalence of the stated conditions for (2.2) and (3.12). We have:
$\bar{w}_0$ is a simple singular point of (3.12)
iff $\dim\ker\mathfrak{A}_d(\bar{w}_0)=1$ and $G_d(\bar{w}_0)\notin \mathrm{im}\,\mathfrak{A}_d(\bar{w}_0)$
iff $\dim\ker\binom{A(\bar{w}_0)}{g'_w(\bar{w}_0)}=1$ and $\binom{f(\bar{w}_0)}{-g'_t(\bar{w}_0)}\notin \mathrm{im}\binom{A(\bar{w}_0)}{g'_w(\bar{w}_0)}$
iff $\bar{w}_0$ is a simple singularity of (2.2).

Let now $\bar{w}_0$ be a simple singularity of (2.2) resp. a simple singular point of (3.12).
Then we have: $\bar{w}_0$ is a standard singular point of (3.12)
iff $\mathfrak{A}'_d(\bar{w}_0)[\bar{v}_0,\bar{v}_0]\notin \mathrm{im}\,\mathfrak{A}_d(\bar{w}_0)$ with $\ker\mathfrak{A}_d(\bar{w}_0)=\mathrm{span}\{\bar{v}_0\}$.

But $\bar{v}_0\in\ker\mathfrak{A}_d(\bar{w}_0)$ iff $\bar{v}_0=(0,v_0)\in I\!R\times I\!R^{n+m}$ and $v_0\in\ker\binom{A(\bar{w}_0)}{g'_w(\bar{w}_0)}$.
Thus, it follows that $\bar{w}_0$ is a standard singular point of (3.12)
iff $\binom{A'(\bar{w}_0)}{g''_{ww}(\bar{w}_0)}[v_0,v_0]\notin \mathrm{im}\binom{A(\bar{w}_0)}{g'_w(\bar{w}_0)}$
iff $\bar{w}_0$ is a simple impasse point of (2.2).

2) Now we prove the equivalence of the stated conditions for (3.12) and (3.15). Let $g^{-1}(0)$ be represented by local coordinates $\bar{w}\in C^2(U_0,I\!R^{\bar{n}+m})$ with (3.14).
From $g(\bar{w}(\xi))\equiv 0$ for $\xi\in U_0$ we derive

$$g'_w(\bar{w}(\xi))\cdot\bar{w}'(\xi)\equiv 0 \quad \text{for} \quad \xi\in U_0,$$

and for $\xi=\xi_0$ we obtain

$$g'^{\,0}_{\bar{w}}\cdot\bar{w}'(\xi_0)=0.$$

Hence, we have $\mathrm{im}\,\bar{w}'(\xi_0)\subseteq\ker g'^{\,0}_{\bar{w}}$ and because of
$\dim\mathrm{im}\,\bar{w}'(\xi_0)=\dim\ker g'^{\,0}_{\bar{w}}=\bar{n}$ this yields $\mathrm{im}\,\bar{w}'(\xi_0)=\ker g'^{\,0}_{\bar{w}}$.
Now we see

$$\begin{aligned} e_0\in\ker\mathfrak{A}_\ell(\xi_0) &\Leftrightarrow \bar{A}(\bar{w}_0)\bar{w}'(\xi_0)e_0=0\\ &\Leftrightarrow \bar{w}'(\xi_0)e_0\in\ker\bar{A}(\bar{w}_0)\\ &\Leftrightarrow \bar{v}_0:=\bar{w}'(\xi_0)e_0\in\ker\begin{pmatrix}\bar{A}(\bar{w}_0)\\ g'_{\bar{w}}(\bar{w}_0)\end{pmatrix}=\ker\mathfrak{A}_d(\bar{w}_0).\end{aligned}$$

Since $w'(\xi_0)$ has full rank $\bar{n}$, $e_0\neq 0$ induces also $\bar{v}_0:=w'(\xi_0)e_0\neq 0$. On the other hand, for $\bar{v}_0\in\ker\mathfrak{A}_d(\bar{w}_0)$ the overdetermined equation

$$\bar{w}'(\xi_0)e_0=\bar{v}_0$$

for determining e_0 from $\bar v_0$ has a unique solution, since $\bar v_0 \in \ker g'_{\bar w}(\bar w_0) = \operatorname{im} \bar w'(\xi_0)$.
This proves the equivalence of (3.10)(i) for (3.12) and (3.15). Further we have

$$\begin{aligned}
G_\ell(\bar w_0) \in \operatorname{im} \mathfrak{a}_\ell(\bar w_0) \quad &\Leftrightarrow \quad \bar f(\bar w_0) \in \operatorname{im}(\bar A(\bar w_0) \cdot \bar w'(\xi_0)) \\
&\Leftrightarrow \quad \exists\, \xi \in I\!R^{\bar n} : \bar f(\bar w_0) = \bar A(\bar w_0) \bar w'(\xi_0)\xi \\
&\Leftrightarrow \quad \exists\, \xi \in I\!R^{\bar n} : \bar f(\bar w_0) = \bar A(\bar w_0) \bar w'(\xi_0)\xi \\
&\qquad\quad \text{and} \quad {g'_{\bar w}}^0 \bar w'(\xi_0)\xi = 0 \\
&\Leftrightarrow \quad \exists\, \bar v \in I\!R^{\bar n+m} : \bar f(\bar w_0) = \bar A(\bar w_0)\bar v, 0 = {g'_{\bar w}}^0 \bar v \\
&\Leftrightarrow \quad f(\bar w_0) = A(\bar w_0)v,\ -g_t(\bar w_0) = g'_w(\bar w_0)v \\
&\qquad\quad \text{since} \quad \bar v = (1, v) \\
&\Leftrightarrow \quad G_d(\bar w_0) \in \operatorname{im} \mathfrak{a}_d(\bar w_0).
\end{aligned}$$

Thus we have the equivalence of (3.10)(ii) for (3.12) and (3.15).

To prove the equivalence of (3.11) for (3.12) and (3.15) we state

$$\begin{aligned}
&\mathfrak{a}'_\ell(\xi_0)[e_0, e_0] \in \operatorname{im} \mathfrak{a}(\xi_0) \\
&\Leftrightarrow \exists\, \xi \in I\!R^{\bar n} : \mathfrak{a}'_\ell(\xi_0)[e_0, e_0] \mathfrak{a}(\xi_0)\xi \\
&\Leftrightarrow \exists\, \xi \in I\!R^{\bar n} : \bar A'(\bar w_0)[\bar w'(\xi_0)e_0, \bar w'(\xi_0)e_0] + \bar A(\bar w_0)\bar w''(\xi_0)[e_0, e_0] = \bar A(\bar w_0)\bar w'(\xi_0)\xi \\
&\Leftrightarrow \exists\, \xi \in I\!R^{\bar n} : \bar A'(\bar w_0)[\bar w'(\xi_0)e_0, \bar w'(\xi_0)e_0] + \bar A(\bar w_0)\bar w''(\xi_0)[e_0, e_0] = \bar A(\bar w_0)\bar w'(\xi_0)\xi \\
&\qquad g''_{\bar w\bar w}(\bar w_0)[\bar w'(\xi_0)e_0, \bar w'(\xi_0)e_0] + g'_{\bar w}(\bar w_0) w''(\xi_0)[e_0, e_0] = 0 = {g'_{\bar w}}^0 \bar w'(\xi_0)\xi \\
&\Leftrightarrow \exists\, \xi \in I\!R^{\bar n} : \binom{\bar A'^0}{{g''_{\bar w\bar w}}^0}[\bar v_0, \bar v_0] = \binom{\bar A^0}{{g'_{\bar w}}^0}(\bar w''(\xi_0)[e_0, e_0] + \bar w'(\xi_0)\xi) \\
&\Leftrightarrow \exists\, \bar v \in I\!R^{\bar n+m} : \binom{\bar A'^0}{{g''_{\bar w\bar w}}^0}[\bar v_0, \bar v_0] = \binom{\bar A^0}{{g'_{\bar w}}^0}(\bar w''(\xi_0)[e_0, e_0] + \bar v) \\
&\Leftrightarrow \binom{\bar A'^0}{{g''_{\bar w\bar w}}^0}[\bar v_0, \bar v_0] \in \operatorname{im} \binom{\bar A^0}{{g'_{\bar w}}^0} \\
&\Leftrightarrow \mathfrak{a}_d^{0\prime}[v_0, v_0] \in \operatorname{im} \mathfrak{a}_d^0 \ \text{ since } \ \bar v_0 = (0, v_0).
\end{aligned}$$

$\square$

4 Numerical properties of the augmented systems

The augmented systems (3.1) and (3.2) for the DAE's (2.1) and (2.2), respectively, are both quasilinear autonomous systems of the form

$$\begin{aligned} \mathcal{A}(z)z' &= a(z) \\ 0 &= g(z) \end{aligned} \tag{4.1}$$

with $a \in C(Z_0, I\!R^{\bar{n}})$, $g \in C^1(Z_0, I\!R^m)$, $\mathcal{A} \in C(Z_0, (I\!R^{\bar{n}+m}, I\!R^{\bar{n}}))$ with rank $\mathcal{A}(z) = \bar{n}$ for all $z \in Z_0 \subseteq I\!R^{n+m}$ open and nonempty, where the condition that (4.1) has geometrical index 1 is fulfilled, i.e.

$$\begin{pmatrix} \mathcal{A}(z) \\ g'(z) \end{pmatrix} \quad \text{is nonsingular for} \quad z \in Z_0. \tag{4.2}$$

In both systems, (3.1) and (3.2), the kernel of $\mathcal{A}(z)$ depends on z in general. Thus, the question which part of the derivative $z'(s)$ really influences $\mathcal{A}(z(s))z'(s)$ depends on the solution itself. It is not possible to split the solution $z(s)$ into those components that have to be differentiated and into others that do not have to. The tractability-index (März [6]) is not defined and we cannot expect that analytical and numerical properties of DAE's which are index 1-tractable are applicable here, too. In partikular, the solutions of (4.1) need not depend continously on time-dependent perturbations of the right-hand side, which already becomes obvious from the following example of Hairer/Lubich/Roche [5]:

Example : Let $r = 1$, $k = 3$, $z \in I\!R^3$, $z = (z_1, z_2, z_3)^T$,
$\mathcal{A}(z) := (1, -z_3, z_2) \in L(I\!R^3, I\!R^1)$, $a(z) := 0$, $g(z) := \binom{z_2}{z_3} \in I\!R^2$.
For the initial condition $z_1(0) = 0$ we get the unique solution $z \equiv 0$.
Let $\hat{z}$ be the solution of the perturbed equation

$$\left.\begin{aligned} \mathcal{A}(\hat{z}(t))\hat{z}'(t) &= a(\hat{z}(t)) + \delta(t), & z_1(0) = 0 \\ 0 &= g(\hat{z}(t)) + \vartheta(t), & t \in [0, T]. \end{aligned}\right\} \tag{4.3}$$

For $\delta(t) \equiv 0$, $\vartheta(t) = (\varepsilon \sin wt, \varepsilon \cos wt)^T$, $q = (\delta, \vartheta^T)^T$ we get $\hat{z}(t) = (\varepsilon^2 wt, -\varepsilon \sin wt, -\varepsilon \cos wt)$ and e.g. for $w := \frac{1}{\varepsilon^2}$ we see that $\|q\| \to 0$ for $\varepsilon \to 0$, but $\|z - \hat{z}\| \not\to 0$.

The DAE in this example has perturbation index 2, since we only obtain an estimation of the form

$$\|z - \hat{z}\| \leq C_0\|q\| + C_1\|q'\| \tag{4.4}$$

with some constants $C_0 > 0$, $C_1 > 0$, where the term $\|q'\|$ cannot be omitted.
Here $\|z\|$ denotes the Tschebyscheff-Norm of z in $C([0,T], I\!R^k)$. However, if we look at this example in more detail, we see that, for our special choice of q, we could replace the constant C_1 by $c_1 \cdot \|q\|$ such that the dangerous term $\|q'\|$ in the estimation (4.4) appears multiplied by the norm of the perturbation itself.
Applying a result of Arnold/Strehmel/Weiner [1] for index-2-Hessenberg-systems we can prove an analogous property for system (4.1) in general. We start repeating the result from [1].
Consider index-2-Hessenberg-systems

$$\left.\begin{array}{rcll} u'(t) & = & f(u(t), v(t)), & t \in [t_0, T],\ u(t_0) = u_0,\ v(t_0) = v_0, \\ 0 & = & g(u(t)), & g(u_0) = 0,\ g'(u_0) f(u_0, v_0) = 0. \end{array}\right\} \qquad (4.5)$$

Let f and f'_v be continuously differentiable, g be twice continuously differentiable, let $(u,v) : [t_0, T] \to I\!R^{n_u} \times I\!R^{n_v}$ be a solution of (4.5) and let the index-2-condition

$$g'(u) f'_v(u,v) \quad \text{nonsingular and} \quad \|[g'(u) f'_v(u,v)]^{-1}\| \leq M < \infty$$

be satisfied in a neighbourhood of the solution.
We compare the solution of (4.5) with that of the perturbed system

$$\left.\begin{array}{rcll} \hat{u}'(t) & = & f(\hat{u}(t), \hat{v}(t)) + \delta(t), & t \in [t_0, T],\ \hat{u}(t_0) = \hat{u}_0,\ \hat{v}(t_0) = \hat{v}_0 \\ \vartheta(t) & = & g(\hat{u}(t)) & \end{array}\right\} \qquad (4.6)$$

with the perturbation $\delta \in C([t_0, T], I\!R^{n_u})$, $\vartheta \in C^1([t_0, T], I\!R^{n_v})$. Although such systems have perturbation index 2 in general, the differential components u are less effected by the derivative of the perturbations.

In [1] the following is proved.

Theorem 4.1 *Let f be linear in v. Then there is a constant $C_0 > 0$ so that for all functions $\hat{u}$, $\hat{v}$ with (4.6), $\delta \in C([t_0, T], I\!R^{n_u})$, $\vartheta \in C^1([t_0, T], I\!R^{n_v})$, the estimate*

$$\|\hat{u}(t) - \hat{v}(t)\| \leq C_0(\|\hat{u}_0 - u_0\| +$$
$$+ \max_{\tau \in [t_0, t]} (\| \int_{t_0}^{\tau} P(s)\delta(s) \mathrm{d}s\| + \|\vartheta(\tau)\| + \|\vartheta(\tau)\|(\|\delta(\tau)\| + \|\vartheta'(\tau)\|)))$$

is satisfied for all $t \in [t_0, T]$ provided that $\|\delta\| \leq C_\delta$ and that the right-hand side of the estimate is sufficiently small. (The constant C_0 may depend on C_δ.)
Here $P(t) := I - (f'_v[g', f'_v]^{-1}g')(u(t), v(t))$.

In the same paper [1] also a discrete analogon for Theorem 4.1 for the implicit Euler-method is proved. The higher perturbation index of the DAE leads to weak instabilities of the numerical integration method though they appear again multiplied by the norm of the perturbation.

We will now write the system (4.1) as special index-2-Hessenberg-system. Following the tranformation used in Hairer/Lubich/Roche [5] we can do this at least locally in a neighbourhood of each point $z_0 \in Z_0$. This neighbourhood is determined only by the condition that the same $\bar{n}$ columns of $\mathcal{A}(z)$ have to be linearly independent. For simplicity let us assume that the first $\bar{n}$ columns of $\mathcal{A}(z)$ are linearly independent.
Then we split $z = (z_1, z_2)$, $z_1 \in I\!\!R^{\bar{n}}$, $z_2 \in I\!\!R^m$, and correspondingly $\mathcal{A}(z) = (\mathcal{A}_{11}(z), \mathcal{A}_{12}(z))$.
Since $\mathcal{A}_{11}(z) \in L(I\!\!R^{\bar{n}})$ is invertible, we can multiply the first block-row of (4.1) by $\mathcal{A}_{11}(z)^{-1}$ and obtain

$$z_1' = -\mathcal{A}_{11}(z)^{-1}\mathcal{A}_{12}(z)z_2' + \mathcal{A}_{11}(z)^{-1}a(z).$$

With the new variable $zp_2 := z_2'$ (4.1) is equivalent to

$$\left.\begin{array}{rcl} z_1' &=& -\mathcal{A}_{11}(z)^{-1}\mathcal{A}_{12}(z)zp_2 + \mathcal{A}_{11}(z)^{-1}a(z) \\ z_2' &=& zp_2 \\ 0 &=& g(z). \end{array}\right\}. \tag{4.7}$$

With $u := z$, $v := zp_2$ we see, that (4.7) has Hessenberg form. The index-2-condition means that the matrix

$$(-g'_{z_1}\mathcal{A}_{11}^{-1}\mathcal{A}_{12} + g'_{z_2}) \in L(I\!\!R^m)$$

is nonsingular, which is equivalent to the geometrical index-1-condition (4.2) for (4.1).
Now we are able to study the effect of perturbations

$$\begin{pmatrix} \delta \\ \vartheta \end{pmatrix} \in C([s_0, S], I\!\!R^{\bar{n}}) \times C^1([s_0, S], I\!\!R^m).$$

Let $\hat{z}$ be a solution of the perturbed system

$$\left.\begin{array}{rcl}\mathcal{A}(\hat{z})\hat{z}' & = & a(\hat{z})+\delta \\ \vartheta & = & g(\hat{z})\end{array}\right\}. \tag{4.8}$$

Since (4.8) is equivalent to

$$\left.\begin{array}{rcl}\hat{z}_1' & = & -\mathcal{A}_{11}(\hat{z})^{-1}\mathcal{A}_{12}(\hat{z})\widehat{zp_2}+\mathcal{A}_{11}(\hat{z})^{-1}a(\hat{z})+\mathcal{A}_{11}(\hat{z})^{-1}\delta \\ \hat{z}_2' & = & zp_2 \\ \vartheta & = & g(\hat{z})\end{array}\right\}, \tag{4.9}$$

we derive from Theorem 4.1 the estimate

$$\begin{aligned}&\|\hat{z}(s)-z(s)\| \leq C_0\{\|\hat{z}_0-z_0\|+\\ &\quad+\max_{\sigma\in[s_0,s]}(C_1\|\delta(\sigma)\|+\|\vartheta(\sigma)\|+\|\vartheta(\sigma)\|(C_2\|\delta(\sigma)\|+\|\vartheta'(\sigma)\|))\}\end{aligned} \tag{4.10}$$

with constants $C_0, C_1, C_2 > 0$, provided $\|\delta\|$ and the right-hand side of (4.10) are sufficiently small.

As in the example above the dangerous term $\|\vartheta'\|$ appears mutliplied with $\|\vartheta\|$.
Let us summarize this result in

Theorem 4.2 *Let $z, \hat{z} \in C^1([s_0, S], I\!R^{\bar{n}+m})$ be a solution of* (4.1) *and* (4.8), *respectively, $\mathcal{A}$, a be continuously differentiable, g be twice continuously differentiable.*
Then there exist positive constants C_0, C_1, C_2 such that the estimate (4.10) *holds for $s \in [s_0, S]$, provided that $\|\delta\|$ and the right-hand side of* (4.10) *are sufficiently small.*

Because of the more special structure of system (3.1) we can find $\bar{n}$ linearly independent columns of $\mathcal{A}(z)$, which are constant and independent of z. Therefore, (3.1) is globally equivalent to an index-2-Hessenberg system.
We have used the BDF for numerical computations of solutions of (3.1) and (3.2) and found that, though the perturbation index of these systems may be 2, the code made for index-1-systems works very well without any difficulties.

5 Appendix: Some properties of quasilinear DAE's

Again, we consider quasilinear DAE's of the form

$$\left.\begin{array}{rcl} A(t,w)w' &=& f(t,w) \\ 0 &=& g(t,w) \end{array}\right\}, \tag{5.1}$$

where $w \in I\!R^{n+m}$, $f(t,w) \in I\!R^n$, $g(t,w) \in I\!R^m$, rank $A(t,w) = n$ and suppose A and f to be once and g be twice continuously differentiable. Let $J_0 \subseteq J$ be some compact interval, $w_\star(.) \in C^1(J_0, I\!R^{n+m})$ be a solution of (5.1), and let the geometrical index-1-condition be true at the solution curve.

Let $t_0 \in J_0$ and $w_0 = w_\star(t_0)$.

Let $K_0 \in L(I\!R^{n+m}, I\!R^n)$ with rank$K_0 = n$ and

$$\left[\begin{array}{c} K_0 \\ g'_w(t_0,w_0) \end{array}\right] nonsingular.$$

Obviously, one possibility to choose K_0 is $K_0 = A(t_0,w_0)$.

Now, we can look at the solutions of (5.1) and the initial conditions

$$K_0(w(t_0) - w^0) = 0 \tag{5.2}$$

for arbitrary $w^0 \in I\!R^{n+m}$. Then a consistent inital value $w_c = w_c(w^0)$ has to solve the partly nonlinear system

$$\left.\begin{array}{rcl} K_0(w_c - w^0) &=& 0 \\ g(t_0,w_c) &=& 0 \end{array}\right\}. \tag{5.3}$$

Assertion 5.1: *There exists a neighbourhood $U_0 \subseteq I\!R^{n+m}$ of w_0 such that (5.3) has a unique solution $w_c(w^0)$ in U_0 for $w^0 \in U_0$ with $w_c(w_0) = w_0$. $w_c(.)$ depends continuously differentiable on w^0 and it holds*

$$\frac{\partial w_c(w_0)}{\partial w^0} = \left[\begin{array}{c} K_0 \\ g'^{\,0}_w \end{array}\right]^{-1} \left[\begin{array}{c} K_0 \\ 0 \end{array}\right].$$

Especially, it follows that

$$\mathrm{rank}\frac{\partial w_c(w_0)}{\partial w^0} = n.$$

The proof follows immediately as an application of the Implicit Function Theorem.
Now, let the coefficients of the considered quasilinear DAE's additionally depend on a real parameter λ . We consider

$$\left.\begin{array}{rcl} A(t,w,\lambda)w' & = & f(t,w,\lambda) \\ 0 & = & g(t,w,\lambda) \end{array}\right\}, \tag{5.4}$$

and suppose that A, f, g, g'_w, g'_t are continuously differentiable. Again, let $J_0 \subseteq J$ be some compact interval, $w_\star(.) \in C^1(J_0, \mathbb{R}^{n+m})$ be a solution of (5.4), and let the geometrical index-1-condition be true at the solution curve. Let t_0, w_0, K_0 be like before and

$$\left[\begin{array}{c} K_0 \\ g'_w(t_0, w_0, \lambda_0) \end{array}\right] \mathit{nonsingular.}$$

Again, we state initial conditions like (5.2) for arbitrary $w^0 \in \mathbb{R}^{n+m}$. A consistent initial value $w_c = w_c(\lambda, w^0)$ now has to solve the parameter-dependent system

$$\left.\begin{array}{rcl} K_0(w_c - w^0) & = & 0 \\ g(t_0, w_c, \lambda) & = & 0 \end{array}\right\}. \tag{5.5}$$

Now, it holds

Theorem 5.1 *Let $w_\star(.) \in C^1(J_0, \mathbb{R}^{n+m})$ be a solution of (5.4) for $\lambda = \lambda_0$, $J_0 \subseteq J$ some compact interval. Let the geometrical index-1-condition be true at the solution curve, let $t_0 \in J_0$, $w_0 = w_\star(t_0)$, and $K_0 \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ with $\binom{K_0}{g'_w{}^0}$ nonsingular.*
Then there exist neighbourhoods U of w_0 and Λ of λ_0 such that for all $w^0 \in U$ and for all $\lambda \in \Lambda$ the DAE (5.4) with initial condition (5.2) has a unique solution $w(.;\lambda, t_0, w^0)$ with $w(t_0;\lambda, t_0, w^0) \in U$.
This solution depends continuously differentiably on λ and w^0, and it holds:

$$\frac{\partial w(t;\lambda_0, t_0, w_0)}{\partial \lambda} = w_\lambda(t),$$

where $w_\lambda(.)$ is the solution of

$$w'_\lambda(t) = F'_w(t, w_\star(t), \lambda_0) w_\lambda(t) + F'_\lambda(t, w_\star(t), \lambda_0),$$

$$w_\lambda(t_0) = \frac{\partial w_c(w^0, \lambda)}{\partial \lambda} = \left[\begin{array}{c} K_0 \\ g'_w{}^0 \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ -g'_\lambda{}^0 \end{array}\right].$$

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