ON ASYMPTOTICS IN CASE OF LINEAR INDEX-2-DAES

MICHAEL HANKE, EBROUL IZQUIERDO MACANA AND ROSWITHA MÄRZ

Abstract. In this paper we investigate the asymptotic behaviour of solutions of linear index-2 differential algebraic equations and their influence on the numerical integration. The DAE and its solution are decomposed into their essential components. In a similar way we proceed with Runge-Kutta methods and BDFs. In the consequence classical stability results are carried over to that case supposed certain subspaces do not rotate. On the other hand the significance of those subspaces becomes obvious.

Key words: differential-algebraic equation, stability, asymptotics, Runge-Kutta method, backward differentiation formula

1991 Mathematics Subject Classification: 65 L 06, 34 D 20

Introduction. In this paper we investigate the asymptotic properties of solutions of linear index 2 differential algebraic equations (DAEs) and their influence on the numerical integration by integration methods like BDF and IRK methods. Although numerical integration methods for index 2 DAEs have been topical for years, only little attention has been paid to the asymptotical behaviour of the continuous solutions as well as to their numerical approximation. As far as we know, first discussions on this subject for index 1 DAEs are contained in Griepentrog & März (1986). First approaches to index 2 DAEs can be found in März & Tischendorf (1993), and to very special linear Hessenberg systems in Weiner (1993).

After the basic applicability of BDF and IRK methods to general linear index 2 DAEs $A(x)x' + B(x)x = q(x)$ has been shown by proving the weak instability (for $h \to 0$), corresponding error estimates and convergence statements at least for the case of constant null spaces of $A(t)$ (cf. März (1990) for BDF, Izquierdo (1993) for IRK methods), it seems to be high time to investigate the asymptotic behaviour, too.

Already in the discussion for index 1 DAEs in Griepentrog & März (1986) the special subspace $\ker A(t)$ was of great importance. The well known example of Petzold (1982) shows that in the index 2 case, which is of interest for us, this subspace has to be assumed to be independent of t, because only problems of this kind are accessible by classical BDF and IRK methods. Remark that this does not imply any restriction from the point of view of applications. In this paper we show the relevance of further subspaces. Roughly speaking, if they are variable in the integration interval, known numerical integration methods lose their asymptotic properties. Should they remain constant, the common asymptotic stability statements for regular explicit ODEs can be applied to the case of index 2 DAEs. Those phenomena have also been observed by Ascher & Petzold (1993).

In Section 1 we investigate the DAEs themselves. By means of suitable projectors it is possible to decompose the DAE or its solutions, respectively, into essential components. Similarly to the introduction of the notion of contractive DAEs for index 1 DAEs in Griepentrog & März (1986) we provide a generalization to index 2 DAEs. Statements on the asymptotic stability follow in a natural way, known results for explicit ODEs can be generalized immediately.

In the Sections 2 and 3 the BDF and IRK methods, the latter ones together with their projected counterparts, are investigated by the same projectors in an analogous way. In case certain subspaces are constant, known asymptotic properties can be reproduced for the index 2 DAEs, too. If constancy is dropped, these properties get lost.

Section 4 presents a simple linear example in Hessenberg form, which illustrates the behaviour described above. The BDF as well as an algebraically stable IRK method generated unstable discrete solutions. In the appendix we have collected some facts from linear algebra needed in the paper.

The authors would like to mention that they consider these investigations devoted to the linear DAEs as a prerequisite for the corresponding investigations in the nonlinear case. There is no doubt that the relevance of the variable subspaces, which becomes obvious here, remains valid for the case of nonlinear DAEs - even if they are autonomous.

1. Linear continuous coefficient index-2 equations. Consider the linear equation

$$A(t)x'(t) + B(t)x(t) = q(t), \quad t \in J := [t_o, \infty) \tag{1.1}$$

with continuous coefficients.

Assume the nullspace of $A(t) \in L(\mathbb{R}^m)$ to be independent of t and denote

$$N := \ker A(t) \subset \mathbb{R}^m,$$

further

$$S(t) := \{z \in \mathbb{R}^m : B(t)z \in \operatorname{im} A(t)\} \subset \mathbb{R}^m.$$

Let $Q \in L(\mathbb{R}^m)$ be a projector onto N, $P := I - Q$.

The nullspace N of the leading coefficient matrix $A(t)$ determines what kind of functions we should accept for solutions of (1.1). Namely, $A(t)Q \equiv 0$ implies $A(t)x'(t) \equiv A(t)Px'(t) \equiv A(t)(Px)'(t)$. Thus, (1.1) may be rewritten as

$$A(t)(Px)'(t) + B(t)x(t) = q(t), \quad t \in J, \tag{1.2}$$

which shows the function space

$$C_N^1(J, \mathbb{R}^m) := \{y \in C(J, \mathbb{R}^m) : Py \in C^1(J, \mathbb{R}^m)\}$$

to be the appropriate one for (1.1). If $x \in C_N^1(J, \mathbb{R}^m)$ solves (1.1), then, for all $t \in J$, $x(t)$ belongs to

$$M_o(t) := \{z \in \mathbb{R}^m : B(t)z - q(t) \in \operatorname{im} A(t)\}.$$

Obviously, $S(t)$ is the subspace of $I\!R^m$ where the homogeneous equation solution proceed. Recall the condition

$$S(t) \oplus N = I\!R^m, \quad t \in J, \tag{1.3}$$

to characterize the class of index-1 DAEs (see the appendix for related facts from linear algebra). (1.3) implies the matrix

$$G_1(t) := A(t) + B(t)Q \tag{1.4}$$

to be nonsingular for all $t \in J$. Exactly one solution passes through each $x_o \in M_o(t_o)$ at time $t_o \in J$.

Higher index DAEs are characterized by nontrivial intersections $S(t) \cap N$ and singular matrices $G_1(t)$, respectively. $M_o(t)$ is no more filled completely by DAE solutions.

Definition: The DAE (1.1) is said to be index-2-tractable if both conditions

$$\dim(S(t) \cap N) = \text{const} > 0, \tag{1.5}$$
$$S_1(t) \oplus N_1(t) = I\!R^m, \quad t \in J \tag{1.6}$$

are valid, where

$$\begin{aligned} N_1(t) &:= \ker G_1(t), \\ S_1(t) &:= \{z \in I\!R^m : B(t)Pz \in \operatorname{im} G_1(t)\}. \end{aligned}$$

Remarks:

1) It holds that $\dim N_1(t) = \dim(N \cap S(t))$.
2) Due to Lemma A.1, (1.5), (1.6) imply the matrix

   $$G_2(t) := G_1(t) + B(t)P(t)Q_1(t) \tag{1.7}$$

   to become nonsingular, but $G_1(t)$ to be singular now. Thereby, $Q_1(t)$ denotes the projector onto $N_1(t)$ along $S_1(t)$, $P_1(t) := I - Q_1(t)$. Q_1 is assumed to be C^1.
3) Applying Lemma A.1 once more we obtain the identities

   $$Q_1 = Q_1 G_2^{-1} BP, \quad Q_1 Q = 0. \tag{1.8}$$

4) Each DAE (1.1) having Kronecker index 2 is index-2 tractable (März (1989)).

The conditions (1.5), (1.6) imply the decompositions

$$\begin{aligned} I\!R^m &= N(t) \oplus P(t)S_1(t) \oplus P(t)N_1(t) \\ &= N(t) \oplus \operatorname{im} P(t)P_1(t) \oplus \operatorname{im} P(t)Q_1(t), \\ S_1(t) &= N(t) \oplus \operatorname{im} P(t)P_1(t), \end{aligned}$$

which are relevant for the index-2 case. Taking this into account we decompose the DAE solution $x \in C^1_N(J, \mathbb{R}^m)$ into

$$x = Qx + PP_1x + PQ_1x =: w + u + Pv. \tag{1.9}$$

Multiplying (1.2) by $PP_1G_2^{-1}$, $QP_1G_2^{-1}$ and $Q_1G_2^{-1}$, respectively, and carrying out a few technical computations we decouple the index-2 DAE into the system

$$u' - (PP_1)'u + PP_1G_2^{-1}Bu = PP_1G_2^{-1}q - (PP_1)'v, \tag{1.10}$$

$$-(Qv)' + (QQ_1)'(u + Pv) + w + QP_1G_2^{-1}Bu = QP_1G_2^{-1}q, \tag{1.11}$$

$$v = Q_1G_2^{-1}q. \tag{1.12}$$

Equation (1.10) represents the inherent regular ODE of the DAE system. Supposed

$$u(t_o) \in \operatorname{im} P(t_o)P_1(t_o) \tag{1.13}$$

is satisfied at some $t_o \in J$, (1.10) implies

$$u = PP_1u.$$

Further, (1.12) and (1.11) lead to

$$v = Q_1v, \quad w = Qw,$$

respectively. Thus, solving (1.10) – (1.13) and letting $x := u + Pv + w \in C^1_N$, we obtain the solutions of our DAE (1.1).

Inspired by the above decoupling we state initial conditions for (1.1) as

$$P(t_o)P_1(t_o)(x(t_o) - x^o) = 0, \quad x^o \in \mathbb{R}^m \text{ given.} \tag{1.14}$$

This yields

$$u(t_o) = P(t_o)P_1(t_o)x(t_o) = P(t_o)P_1(t_o)x^o,$$

but we should not expect $x(t_o) = x^o$ to become true.

Now, the following two assertions are easily proved by means of our decoupling.

Theorem 1.1. *Let* (1.1) *be index-2 tractable with continuously differentiable* Q_1*. Then, for each source*

$$q \in \{p \in C(J, \mathbb{R}^m) : Q_1G_2^{-1}p \in C^1(J, \mathbb{R}^m)\},$$

the initial value problems (1.1), (1.14) *are uniquely solvable with solutions* $x \in C^1_N(J, \mathbb{R}^m)$.

Remarks:

1) We ask for C^1_N solutions. Any higher regularity of solutions, say C^1, needs additional smoothness of the coefficients, projectors and sources involved. Again, the decoupled system provides some help to state correct conditions. In particular, for C^1 solutions at least $Q_1G_2^{-1}q \in C^2$, $QP_1G_2^{-1}q \in C^1$ have to be valid additionally.

2) The inherent regular ODE (1.10) is affected by the complete coefficient matrix $PP_1G_2^{-1}B - (PP_1)'$, but not only by the first term $PP_1G_2^{-1}B$. If $(PP_1)(t)$ varies quickly, the second term $(PP_1)'$ may be the dominant one. This should also be taken into account when considering the asymptotic behaviour.
3) In general, the linear DAE (1.1) appears to be much easier if the relevant subspaces N, N_1, S_1 remain constant, i.e. the two projectors Q, Q_1 are constant. In that case (1.10) – (1.12) simplifies to

$$u' + PP_1G_2^{-1}Bu = PP_1G_2^{-1}q, \tag{1.15}$$
$$-(Qv)' + w + QP_1G_2^{-1}Bu = QP_1G_2^{-1}q, \tag{1.16}$$
$$v = Q_1G_2^{-1}q, \tag{1.17}$$

which looks like the constant coefficient case.

Next we turn shortly to the homogeneous equation. For $q = 0$ the system (1.10) – (1.12) yields $v = 0$ and

$$\begin{aligned} x = u + w &= (I - (QQ_1)' - QP_1G_2^{-1}B)u \\ &= (I - (QQ_1)' - QP_1G_2^{-1}B)PP_1u. \end{aligned}$$

The matrix $\Pi(t) := (I - (QQ_1)'(t) - (QP_1G_2^{-1}B)(t))PP_1(t)$ is also a projector, and $\ker \Pi(t) = \ker PP_1(t)$.

THEOREM 1.2. *Let (1.1) be index-2 tractable, Q_1 be of class C^1. The linear subspace $M(t) \subset S(t)$, $M(t) := \operatorname{im} \Pi(t)$ contains the values of the homogeneous equation solution. Exactly one solution passes through each $x_* \in M(t_*)$ at time t_*, and it holds that $u(t_*) = PP_1(t_*)x_*$, $x(t_*) = x_*$.*

Remarks:

1) If the product PP_1 is time invariant, we have $(QQ_1)'PP_1 = (QQ_1PP_1)' = 0$, hence

$$\Pi = (I - QP_1G_2^{-1}B)PP_1.$$

Note that $(QP_1G_2^{-1}B)(t)$ is also a projector onto $\ker A(t)$.
2) It should be mentioned that the manifold $M(t)$ of consistent initial values at t remains constant supposed both projectors $Q_1 = Q_1G_2^{-1}BP$ and $QP_1G_2^{-1}B$ are constant ones. If $Q = QP_1G_2^{-1}B$, then we simply have

$$M(t) \equiv \operatorname{im}(PP_1)(t_*), \quad t \in J.$$

3) The value $x^o \in \mathbb{R}^m$ involved in the initial condition (1.14) is not expected to be a consistent initial value. What we have is

$$P(t_o)P_1(t_o)x(t_o) = P(t_o)P_1(t_o)x^o,$$

but not $x_o := x(t_o) = x^o$.

Most authors restrict their interest to so-called Hessenberg form equations, i.e. to systems

$$\left.\begin{aligned} x_1' + B_{11}x_1 + B_{12}x_2 &= q_1 \\ B_{21}x_1 &= q_2 \end{aligned}\right\}. \tag{1.18}$$

In the context here this corresponds to

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad G_1 = \begin{bmatrix} I & B_{12} \\ 0 & 0 \end{bmatrix},$$

$S(t) = S_1(t) = \{(z_1^T, z_2^T)^T \in \mathbb{R}^m : B_{21}(t)z_1 = 0\}$.

Obviously, $z \in S_1(t) \cap N_1(t)$ implies $z = 0$ if and only if $B_{21}(t)B_{12}(t)$ is nonsingular, but this represents the well-known Hessenberg form index-2 condition.

Under this condition the block

$$B_{12}(t)(B_{21}(t)B_{12}(t))^{-1}B_{21}(t) =: H(t)$$

is also a projector. It projects onto $\operatorname{im} B_{12}(t)$ along $\ker B_{21}(t)$.

Denote $(B_{21}(t)B_{12}(t))^{-1}B_{21}(t) =: F(t)$ and compute

$$Q_1 = \begin{bmatrix} H & 0 \\ -F & 0 \end{bmatrix}, \quad PP_1 = \begin{bmatrix} I-H & 0 \\ 0 & 0 \end{bmatrix}, \quad PQ_1 = \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.19}$$

The manifold $M(t)$ simplifies to

$$M(t) = \operatorname{im} \begin{pmatrix} I - H(t) & 0 \\ -F(t)B_{11}(t)(I - H(t)) + F'(t)(I - H(t)) & 0 \end{pmatrix}.$$

Furthermore, the nontrivial part (i.e. dropping the zero rows) of the inherent regular ODE (1.10) reads now, with $u_1 := (I - H)x_1$,

$$u_1' + H'u_1 + (I - H)B_{11}u_1 = (I - H)q_1 - (H' + (I - H)B_{11})B_{12}(B_{21}B_{12})^{-1}q_2.$$

Emphasize once more that quickly varying subspaces may imply the dominance of the term H' within this regular ODE, which corresponds to the term $(PP_1)'u$ in (1.10).

THEOREM 1.3. *Let* (1.1) *be index-2 tractable on* $J = [t_o, \infty)$, Q_1 *be of class* C^1, $PP_1' = 0$. *Let* $PP_1G_2^{-1}B$ *be constant.*

(i) *Then the pencil* $\lambda A(t) + B(t)$ *has the eigenvalues* $\lambda_1, \ldots, \lambda_r$, $r := \operatorname{rank} PP_1(t)$, *uniformly for* $t \in J$.

(ii) $\operatorname{Re} \lambda_i < 0, \quad i = 1, \ldots, r,$

implies each homogenous equation solution to tend to zero as $t \to \infty$, *supposed the projector* $\Pi(t)$ *remains uniformly bounded.*

Proof: Due to our assumptions, the inherent regular ODE has the constant coefficient $PP_1G_2^{-1}B$. On the other hand, the nontrivial eigenvalues of $-PP_1G_2^{-1}B$ (that is, eigenvalues which do not correspond to $\ker PP_1$) are exactly the pencil eigenvalues (cf. März 1991).

Because of the solution representation

$$x(t) = \Pi(t)U(t)PP_1(t_o)x^o,$$

where $U(.)$ denotes the fundamental matrix of $u' + PP_1G_2^{-1}Bu = 0$, $U(t_o) = I$, the assertion follows immedately. □

Note that in case of the Hessenberg equation (1.18) we have simply

$$PP_1 = \begin{pmatrix} I-H & 0 \\ 0 & 0 \end{pmatrix}, \quad PP_1G_2^{-1}B = \begin{pmatrix} (I-H)B_{11}(I-H) & 0 \\ 0 & 0 \end{pmatrix}.$$

Roughly speaking, the assumptions for PP_1 and $PP_1G_2^{-1}B$ to be constant mean that there is a constant coefficient inherent regular ODE and the possible time dependence is caused by (time dependent) couplings only.

What about the contractivity of a linear index-2 DAE?
If $x \in C_N^1(J, \mathbb{R}^m)$ solves the homogeneous DAE, then (1.10) – (1.12) imply

$$\begin{aligned}
&Q_1(t)x(t) \equiv 0, \\
&y(t) := (Px)'(t) = (PP_1x)'(t) + (PQ_1x)'(t) = (PP_1x)'(t), \\
&Qy(t) = 0, \quad Q_1(t)y(t) = Q_1(t)(PP_1x)'(t) = -Q_1'(t)PP_1(t)x(t).
\end{aligned}$$

Inspired by the contractivity notion for the index-1 case in Griepentrog and März (1986) we state the following one.

Definition: The index-2 tractable DAE (1.1) is called contractive if for any $y, x \in \mathbb{R}^m$ satisfying

$$A(t)y + B(t)x = 0, \quad Qy = 0, \quad Q_1(t)y = -Q_1'(t)PP_1(t)x \tag{1.20}$$

the inequality

$$(Sy)^T Px \le -c|Px|_S^2 \tag{1.21}$$

is true with $c > 0$ and a symmetric positive definite matrix S, $|z|_S := (Sz)^T z$, $z \in \mathbb{R}^m$.

Remark: As in Griepentrog and März (1986), with this notion we aim for an inequality

$$|Px(t)|_S \le e^{-c(t-t_o)}|Px(t_o)|_S$$

for all solutions of the homogeneous DAE, what shows the component $Px = PP_1x$ to decrease in that norm.

THEOREM 1.4. *Let* (1.1) *be index-2 tractable on* $J = [t_o, \infty)$*,* Q_1 *belong to* C^1*,* $\Pi(t)$ *be uniformly bounded on* J *and* (1.1) *be contractive. Then, for each homogeneous equation solution it holds that*

$$|x(t)|_S \le \gamma e^{-c(t-t_o)}|x(t_o)|_S, \quad t \ge t_o, \tag{1.22}$$

where γ *is a bound of* $|\Pi(t)|_S \cdot |PP_1(t_o)|$.

Proof: We have $x(t) = \Pi(t)U(t)PP_1(t_o)x(t_o)$, further $x(t) = \Pi(t)PP_1(t)x(t) = \Pi(t)Px(t)$.
□

Not surprisingly we obtain

COROLLARY 1.5. *Let (1.1) be index-2 tractable on $J = [t_o, \infty)$ with $Q_1 \in C^1$ and $\Pi(t)$ uniformly bounded. Then, if the condition*

$$(Su)^T\{(PP_1)'(t) - (PP_1G_2^{-1}B)(t)\}u \leq -c|u|_S^2, \tag{1.23}$$

is satisfied for all $u \in \operatorname{im} PP_1(t)$, $t \in J$, the estimation (1.22) is valid.

Proof: It may be checked easily that (1.20), (1.23) lead to (1.21), i.e. (1.23) implies contractivity. □

Note that (1.23) suffices to hold for $u \in \operatorname{im} PP_1(t)$. There is no need for demanding that inequality for all $u \in \mathbb{R}^m$.

For Hessenberg form DAEs (1.18), relation (1.21) applies to the first components only, i.e. it means that

$$(S_{11}y_1)^T x_1 \leq -c(S_{11}x_1)^T x_1$$

should be satisfied for (cf. (1.20))

$$\begin{aligned} y_1 + B_{11}(t)x_1 + B_{12}(t)x_2 &= 0, \\ B_{21}(t)x_1 \qquad\qquad &= 0, \\ y_2 = 0, \quad H(t)y_1 &= -H'(t)(I - H(t))x_1. \end{aligned}$$

Now, (1.23) simplifies to

$$(S_{11}u_1)^T\{-H'(t) - (I - H(t)B_{11}(t)(I - H(t))\}u_1 \leq -c|u_1|_{S_{11}}^2,$$
$$\text{for all } u_1 \in \operatorname{im}(I - H(t)), \quad t \in J.$$

Once more we learn the constant subspace case $H'(t) \equiv 0$ to become much easier to handle.

As a direct consequence of well-known theorems on stability one may obtain their counterparts for linear index 2 DAE's, e.g. the well-known Poincaré-Lyapunov Theorem (e.g. Verhulst (1990)).

2. BDF applied to linear index-2 DAEs. The k-step BDF applied to (1.1) reads

$$A(t_\ell)\frac{1}{h}\sum_{j=0}^{k}\alpha_j x_{\ell-j} + B(t_\ell)x_\ell = q(t_\ell), \quad \ell \geq k. \tag{2.1}$$

At each step, equation (2.1) provides an approximation x_ℓ of the exact solution value $x(t_\ell)$, $t_\ell := t_o + \ell h$. Recall that the nullspace of $A(t)$ is assumed to be constant.

Supposed (1.1) is index-2 tractable we may decouple (2.1) and (1.1) simultaneously (cf. (1.10) - (1.12)). This gives now

$$\begin{aligned} &\frac{1}{h}\sum_{j=0}^{k}\alpha_j u_{\ell-j} + \frac{1}{h}\sum_{j=1}^{k}\alpha_j(PP_{1,\ell} - PP_{1,\ell-j})u_{\ell-j} + \\ &+ PP_{1,\ell}G_{2,\ell}^{-1}B_\ell u_\ell = PP_{1,\ell}G_{2,\ell}^{-1}q_\ell - \frac{1}{h}\sum_{j=1}^{k}\alpha_j(PP_{1,\ell} - PP_{1,\ell-j})v_{\ell-j}, \end{aligned} \tag{2.2}$$

$$-\frac{1}{h}\sum_{j=0}^{k}\alpha_j Qv_{\ell-j}-\frac{1}{h}\sum_{j=1}^{k}\alpha_j(QQ_{1,\ell}-QQ_{1,\ell-j})(u_{\ell-j}+Pv_{\ell-j})+$$

$$+w_\ell+QP_{1,\ell}G_{2,\ell}^{-1}B_\ell u_\ell=QP_{1,\ell}G_{2,\ell}^{-1}q_\ell, \tag{2.3}$$

$$v_\ell=Q_{1,\ell}G_{2,\ell}^{-1}q_\ell, \tag{2.4}$$

where we have again used the above decomposition, i.e.

$$x_\ell=PP_{1,\ell}x_\ell+PQ_{1,\ell}x_\ell+Qx_\ell=:u_\ell+Pv_\ell+w_\ell. \tag{2.5}$$

In particular, supposed the inhomogeneity q vanishes identically, the Q_1-components $v(t_\ell)$ and v_ℓ are both zero, and it results

$$\frac{1}{h}\sum_{j=0}^{k}\alpha_j u_{\ell-j}+\frac{1}{h}\sum_{j=1}^{k}\alpha_j P(P_{1,\ell}-P_{1,\ell-j})u_{\ell-j}+PP_{1,\ell}G_{2,\ell}^{-1}B_\ell u_\ell=0 \tag{2.6}$$

to approximate

$$u'-PP_1'u+PP_1G_2^{-1}Bu=0 \tag{2.7}$$

and, further,

$$w_\ell=-QP_{1,\ell}G_{2,\ell}^{-1}B_\ell u_\ell+\frac{1}{h}\sum_{j=1}^{k}\alpha_j Q(Q_{1,\ell}-Q_{1,\ell-j})u_{\ell-j}$$

for

$$w=-QP_1G_2^{-1}Bu+QQ_1'u.$$

Now it becomes clear (not surprisingly in this context) that the BDF applied to (1.1) generates exactly the same BDF applied to the inherent regular ODE (2.7) if and only if the projector $PP_1(t)$ does not vary with t. For a constant projector PP_1 all known results concerning the asymptotic stability (e.g. A-stability, B-stability), for regular linear ODEs remain valid for index-2 DAEs, too.

On the other hand, as mentioned in Section 1, varying subspaces may cause the term PP_1' to appear dominantly in the inherent regular ODE itself. For instance, the backward Euler method provides then

$$\frac{1}{h}(u_\ell-u_{\ell-1})-\frac{1}{h}P(P_{1,\ell}-P_{1,\ell-1})u_{\ell-1}+PP_{1,\ell}G_{2,\ell}^{-1}B_\ell u_\ell=0,$$

which shows that $u(t_\ell)-u_\ell\to 0$ ($\ell\to\infty$) may or may not happen. Ascher and Petzold (1993) observed this phenomenon in case of index-2 Hessenberg systems (1.18). However, this is not surprising since we cannot expect any discretization method to follow the subspaces precisely without profound information on the inner structure.

Naturally, similar arguments apply to Runge-Kutta methods, too.

3. Implicit Runge-Kutta methods and their projected counterparts applied to linear index-2 DAEs. According to the originally conceived method for the numerical solution of ordinary differential equations, an implicit Runge-Kutta (IRK) method can be realized for the DAE (1.1) in the following way (Petzold (1986)). Given an approximation x_ℓ of the solution of (1.1) at $t_{\ell-1}$, a new approximation x_ℓ at $t_\ell = t_{\ell-1} + h$ is obtained via

$$x_\ell = x_{\ell-1} + h \sum_{i=1}^{s} b_i X'_{\ell i}, \tag{3.1}$$

where $X'_{\ell i}$ is defined by

$$\begin{aligned} A(t_{\ell i})X'_i + B(t_{\ell i})X_{\ell i} &= q(t_{\ell i}), \quad i = 1, \ldots, s, \\ t_{\ell i} &= t_{\ell-1} + c_i h \end{aligned} \tag{3.2}$$

and the internal stages are given by

$$X_{\ell i} = x_{\ell-1} + h \sum_{j=1}^{s} a_{ij} X'_{\ell j}, \quad i = 1, \ldots, s. \tag{3.3}$$

The coefficients a_{ij}, b_i, c_i determine the IRK method, and s is the number of stages. We combine the coefficients in the matrix $\ = (a_{ij})_{i,j=1}^{s}$ and the vectors $b = (b_1, \ldots, b_s)^T$, $c = (c_1, \ldots, c_s)^T$. A condition for $X'_{\ell 1}, \ldots, X'_{\ell s}$ to be uniquely defined by (3.2) – (3.3) is the nonsingularity of , which we will assume in the following. Denoting $\hat{\ } = {}^{-1} = (\hat{a}_{ij})_{i,j=1}^{s}$ and $\varrho = 1 - \sum_{i=1}^{s} \sum_{j=1}^{s} b_i \hat{a}_{ij}$, (3.1) – (3.3) is equivalent to

$$x_\ell = \varrho x_{\ell-1} + \sum_{i=1}^{s} \sum_{j=1}^{s} b_i \hat{a}_{ij} X_{\ell j}, \tag{3.4}$$

$$A(t_{\ell i}) \sum_{j=1}^{s} \hat{a}_{ij}(X_{\ell j} - x_{\ell-1}) + hB(t_{\ell i})X_{\ell i} = hq(t_{\ell i}), \ i = 1, \ldots, s. \tag{3.5}$$

The special class of IRK methods (IRK(DAE)) having coefficients

$$b_i = a_{si}, \quad i = 1, \ldots, s, \quad c_s = 1 \tag{3.6}$$

is shown to stand out from all IRK methods in view of their applicability to DAEs (Griepentrog and März (1986)). Because of $\varrho = 0$ in that case, the new value $x_\ell = X_{\ell_s}$ always belongs to the manifold $M_o(t_\ell)$, i.e.

$$B(t_\ell)x_\ell - q(t_\ell) \in \operatorname{im} A(t_\ell). \tag{3.7}$$

For Hessenberg equations (1.18), relation (3.7) simplifies to

$$B_{21}(t_\ell)x_{1,\ell} = q_2(t_\ell). \tag{3.8}$$

In general, if (3.6) is not given, we have $\varrho \neq 0$, and (3.7) resp. (3.8) are no longer true. Since this behaviour is a source of instability (for $h \to 0$), Ascher and Petzold (1991) propose another version for the application of IRK methods to index-2 Hessenberg DAEs (1.18), the so-called Projected IRK (PIRK). Actually, now the first component of (3.4) is replaced by

$$\hat{x}_{1,\ell} = \varrho \hat{x}_{1,\ell-1} + \sum_{i=1}^{s}\sum_{j=1}^{s} b_i \hat{a}_{ij} \hat{X}_{1,\ell j} + B_{12}(t_\ell)\lambda_\ell \tag{3.9}$$

and λ_ℓ is determined by

$$B_{21}(t_\ell)\hat{x}_{1,\ell} = q_2(t_\ell). \tag{3.10}$$

If we multiply (3.9) by $B_{21}(t_\ell)$, λ_ℓ can be eliminated:

$$(I - H(t_\ell))\hat{x}_{1,\ell} = \varrho(I - H(t_\ell))\hat{x}_{1,\ell-1} + \sum_{i=1}^{s}\sum_{j=1}^{s} b_i \hat{a}_{ij}(I - H(t_\ell))\hat{X}_{1,\ell j}. \tag{3.11}$$

On the other hand, (3.10) is equivalent to

$$H(t_\ell)\hat{x}_{1,\ell} = B_{12}(t_\ell)(B_{21}(t_\ell)B_{12}(t_\ell))^{-1}q_2(t_\ell). \tag{3.12}$$

It should be mentioned that for IRK(DAEs) the projected version is exactly the same as the original one, since (3.6) implies $\lambda_\ell = 0$ in (3.9), (3.10).

Comparing (3.11) – (3.12) with (1.19), an immediate generalization of PIRK methods to fully implicit linear index-2 systems (1.1) is given by[1]

$$PP_1(t_\ell)\hat{x}_\ell = \varrho PP_1(t_\ell)\hat{x}_{\ell-1} + \sum_{i=1}^{s}\sum_{j=1}^{s} b_i \hat{a}_{ij} PP_1(t_\ell)\hat{X}_{\ell j}, \tag{3.13}$$

$$Q_1(t_\ell)\hat{x}_\ell = Q_1(t_\ell)G_2^{-1}(t_\ell)q(t_\ell). \tag{3.14}$$

By (3.5), the internal stages $\hat{X}_{\ell j}$ do not depend on $Q\hat{x}_{\ell-1}$ such that there is no need to compute them here.

Suppose again that (1.1) is index-2 tractable. Then we can decouple (3.5) in the same fashion as (1.10) – (1.12).

We decompose

$$\begin{aligned} X_{\ell i} &= PP_1(t_{\ell i})X_{\ell i} + PQ_1(t_{\ell i})X_i + QX_{\ell i} =: U_{\ell i} + PV_{\ell i} + W_{\ell i}. \\ x_\ell &= PP_1(t_\ell)x_\ell + PQ_1(t_\ell)x_\ell + Qx_\ell =: u_\ell + Pv_\ell + w_\ell. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{1}{h}\sum_{j=1}^{s} \hat{a}_{ij}(U_{\ell j} - u_{\ell-1}) + P\Big\{\frac{1}{h}\sum_{j=1}^{s} \hat{a}_{ij}(P_1(t_{\ell i}) - P_1(t_{\ell j}))U_{\ell j}\Big\} \\ &\qquad - P\{\frac{1}{h}\sum_{j=1}^{s} \hat{a}_{ij}(P_1(t_{\ell i}) - P_1(t_{\ell-1}))x_{\ell-1}\Big\} + PP_1G_2^{-1}B(t_{\ell i})U_{\ell i} \\ &\qquad = PP_1G_2^{-1}q(t_{\ell i}) - P\Big\{\frac{1}{h}\sum_{j=1}^{s}(P_1(t_{\ell i}) - P_1(t_{\ell j}))V_{\ell j}\Big\}, \end{aligned} \tag{3.15}$$

[1] Recall that the nullspace of $A(t)$ is assumed to be constant.

$$\frac{1}{h}\sum_{j=1}^{s}\hat{a}_{ij}(QV_{\ell j}-Qv_{\ell-1})-Q\Big\{\frac{1}{h}\sum_{j=1}^{s}(Q_1(t_{\ell i})-Q_1(t_{\ell j}))(U_{\ell j}+PV_{\ell j})\Big\}$$
$$+Q\Big\{\frac{1}{h}\sum_{j=1}^{s}(Q_1(t_{\ell i})-Q_1(t_{\ell-1}))(u_{\ell-1}+Pv_{\ell-1})\Big\}+W_{\ell i}+$$
$$+QP_1G_2^{-1}B(t_{\ell i})U_{\ell i}=QP_1G_2^{-1}q(t_{\ell i}), \tag{3.16}$$

$$V_{\ell i}=Q_1G_2^{-1}q(t_{\ell i}). \tag{3.17}$$

The recursion (3.4) can be decomposed simply by multiplying by the projections.

$$\begin{aligned} u_\ell &= \varrho u_{\ell-1}+\varrho P(P_1(t_\ell)-P_1(t_{\ell-1}))(u_{\ell-1}+Pv_{\ell-1})+\sum_{i=1}^{s}\sum_{j=1}^{s}b_i\hat{a}_{ij}U_{\ell j} \\ &\quad +P\sum_{i=1}^{s}\sum_{j=1}^{s}b_i\hat{a}_{ij}(P_1(t_\ell)-P_1(t_{\ell j}))(U_{\ell j}+PV_{\ell j}) \end{aligned} \tag{3.18}$$

$$\begin{aligned} Pv_\ell &= \varrho Pv_{\ell-1}+\varrho P(Q_1(t_\ell)-Q_1(t_{\ell-1}))(u_{\ell-1}+Pv_{\ell-1})+\sum_{i=1}^{s}\sum_{j=1}^{s}b_i\hat{a}_{ij}PV_{\ell j} \\ &\quad +P\sum_{i=1}^{s}\sum_{j=1}^{s}b_i\hat{a}_{ij}(Q_1(t_\ell)-Q_1(t_{\ell j}))(U_{\ell j}+PV_{\ell j}) \end{aligned} \tag{3.19}$$

$$w_\ell = \varrho Qw_{\ell-1}+\sum_{i=1}^{s}\sum_{j=1}^{s}b_i\hat{a}_{ij}W_{\ell j} \tag{3.20}$$

Assume that the inhomogeneity q vanishes identically. Then v does, too. Moreover, the $V_{\ell i}$'s are identical to zero. According to (3.19) this is no longer true for $Q_1Pv_\ell = v_\ell$. This means in general that the resulting x_ℓ has a nontrivial component $Q_1(t_\ell)x_\ell$ in contrary to the exact solution with $Q_1(t_\ell)x(t_\ell)=0$.

Now, (3.15) reduces to

$$\begin{aligned} \frac{1}{h}\sum_{j=1}^{s}\hat{a}_{ij}(U_{\ell j}-u_{\ell-1}) &+ P\Big\{\frac{1}{h}\sum_{j=1}^{s}\hat{a}_{ij}(P_1(t_{\ell i})-P_1(t_{\ell j}))U_{\ell j}\Big\} \\ &- P\Big\{\frac{1}{h}\sum_{j=1}^{s}(P_1(t_{\ell i})-P_1(t_{\ell-1}))x_{\ell-1}\Big\} \\ &+ PP_1G_2^{-1}B(t_{\ell i})U_{\ell i}=0, \end{aligned} \tag{3.21}$$

which should approximate

$$u'-PP_1'u+PP_1G_2^{-1}Bu=0. \tag{3.22}$$

Similarly, (3.16) yields

$$\begin{aligned} W_{\ell i} &= -QP_1G_2^{-1}B(t_{\ell i})U_{\ell i}+Q\Big\{\frac{1}{h}\sum_{j=1}^{s}(Q_1(t_{\ell i})-Q_1(t_{\ell j}))U_{\ell j}\Big\} \\ &\quad -Q\Big\{\frac{1}{h}\sum_{j=1}^{s}(Q_1(t_{\ell i})-Q_1(t_{\ell-1}))u_{\ell-1}\Big\} \end{aligned} \tag{3.23}$$

for

$$w = -QP_1G_2^{-1}Bu + QQ_1'u.$$

In complete analogy to the case of BDF method, the IRK method applied to (1.1) generates exactly the same IRK applied to the inherent regular ODE (3.19) if and only if PP_1 does not vary with t.

For constant PP_1, the homogeneous equation solution

$$x(t) = \Pi(t)u(t) = (I - (QP_1G_2^{-1}B)(t))u(t), \tag{3.24}$$

$$u' + PP_1G_2^{-1}Bu = 0, \tag{3.25}$$

$$u(t_o) \in \operatorname{im} PP_1, \tag{3.26}$$

is approximated at t_ℓ by

$$x_\ell = u_\ell + Pv_\ell + w_\ell, \tag{3.27}$$

$$u_\ell = \varrho u_{\ell-1} + \sum_{i=1}^{s}\sum_{j=1}^{s} b_i\hat{a}_{ij}U_{\ell j}, \tag{3.28}$$

$$\frac{1}{h}\sum_{j=1}^{s}\hat{\alpha}_{ij}(U_{\ell j} - u_{\ell-1}) + (PP_1G_2^{-1}B)(t_{\ell i})U_{\ell i} = 0,\ i = 1,\dots,s, \tag{3.29}$$

$$Pv_\ell = \varrho Pv_{\ell-1}, \tag{3.30}$$

$$w_\ell = \varrho Qw_{\ell-1} - \sum_{i=1}^{s}\sum_{j=1}^{s} b_i\hat{a}_{ij}(QP_1G_2^{-1}B)(t_{\ell j})U_{\ell j}. \tag{3.31}$$

Starting with a consistent initial value x_o (with $v_o = 0$) the components v_ℓ vanish also.

For IRK(DAEs), (3.31) provides simply

$$\begin{aligned} w_\ell &= -(QP_1G_2^{-1}B)(t_\ell)U_{\ell s}, \quad u_\ell = U_{\ell s}, \\ x_\ell &= (I - (QP_1G_2^{-1}B)(t_\ell))u_\ell, \end{aligned}$$

that is, the approximation x_ℓ belongs even to the solution manifold $M(t_\ell)$.

Turn shortly to PIRK methods (3.13), (3.14). For homogeneous equations, (3.14) yields $\hat{v}_\ell = Q_1(t_\ell)\hat{x}_\ell = 0$. The decoupled systems (3.21), (3.23) remain valid also for the "^" values. Again, the PIRK method applied to (1.1) generates exactly the underlying IRK method applied to the inherent regular ODE (3.19) if and only if PP_1 does not vary with t. In this case, (3.30) is replaced by

$$\hat{v}_\ell = 0. \tag{3.32}$$

Since $Q\hat{x}_\ell$ does not depend on $Q\hat{x}_{\ell-1}$, Ascher & Petzold did not use a recursion for this component for Hessenberg systems (1.19). Nevertheless, if one is interested in approximations for Qx_ℓ, a recursion like (3.31) appears again. In that case, the only difference between PIRK and IRK methods is the determination of $Q_1\hat{x}_\ell$ ((3.32) versus (3.29)). Note again that PIRK and IRK are identical for IRK (DAE).

4. A numerical counterexample. In the previous sections we learnt that BDF and Runge-Kutta methods preserve their stability behaviour if PP_1 is constant. The following example shows that these properties get lost if PP_1 varies with time. Consider the DAE

$$A(t)x'(t) + B(t)x(t) = 0, \quad t \geq 0 \tag{4.1}$$

with

$$A(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} \lambda & -\beta & -1 \\ \beta\eta t(1-\eta t) - \eta & \lambda & -\eta t \\ 1 - \eta t & 1 & 0 \end{pmatrix},$$

where $\beta, \lambda, \eta \in I\!R$ are constant. Note that (4.1) is a Hessenberg index 2 system. One easily computes (using $A(t) \equiv P$)

$$A_1(t) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\eta t \\ 0 & 0 & 0 \end{pmatrix},$$

such that

$$N_1(t) = \{x \in I\!R^3 \mid x_1 - x_3 = 0,\ x_2 - \eta t x_3 = 0\}.$$

Compute the projections $P_1(t) = \begin{pmatrix} \eta t & -1 & 0 \\ \eta t(\eta t - 1) & 1 - \eta t & 0 \\ \eta t - 1 & -1 & 1 \end{pmatrix}$,

$$PP_1(t) = \begin{pmatrix} \eta t & -1 & 0 \\ \eta t(\eta t - 1) & 1 - \eta t & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Taking into account that $q = 0$ in (4.1), (1.10) reads

$$u' + \mathcal{M}(t)u = 0, \tag{4.2}$$

where $u(t) = (PP_1x)(t)$ and

$$\mathcal{M}(t) = PP_1G_2^{-1}B - (PP_1)' = \begin{pmatrix} \lambda\eta t - \eta + \eta^2 t & -\lambda - \eta & 0 \\ (\lambda+\eta)\eta t(\eta t - 1) + \eta(1 - 2\eta t) & (\lambda+\eta)(1-\eta t) + \eta & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The subspace $M(t)$ of Theorem 1.2 is given by

$$M(t) = \operatorname{im}\Pi(t) = \{x \in I\!R^3 \mid (\eta t - 1)x_1 - x_2 = 0,\ \beta x_2 + x_3 = 0\}.$$

Since $\dim\operatorname{im}(PP_1(t)) \equiv 1$, (4.2) reduces to the scalar ODE

$$u_1' + \lambda u_1 = 0 \tag{4.3}$$

together with the assertions

$$u_2(t) = (\eta t - 1)u_1(t), \quad u_3(t) = 0. \tag{4.4}$$

Consequently, the asymptotic stability of (4.2) is governed by the sign of λ (independently of $\eta \in I\!R$). The parameter η measures the change of $N_1(t)$. β serves only for mixing the P component with the nullspace components. The complete solution of (4.1) can be easily computed using (1.10) – (1.12) now. If $x^o \in I\!R^3$ is a consistent initial value at $t = 0$ (i.e., $x_1^o + x_2^o = 0$, $x_3^o + \beta x_2^o = 0$), the solution of (4.1) is

$$x(t) = \begin{pmatrix} x_1^o e^{-\lambda t} \\ x_1^o(\eta t - 1)e^{-\lambda t} \\ x_1^o \beta(1 - \eta t)e^{-\lambda t} \end{pmatrix}.$$

(4.1) was solved using the 5-step BDF method and an algebraically stable 2-stage Runge-Kutta method introduced by Crouzeix (cf. Hairer/Nørsett & Wanner (1993), p. 207) with $\rho \approx -0.73$. The figures show the norm of the numerical solution at the end of the interval $[0, T]$ for different values of η und λ. Note that, for $\eta = 0$, (4.1) represents a constant coefficient system. The results indicate that the asymptotic behaviour of the numerical solution depends not only on the asymptotic stability of the inherent differential equation (4.3) (controlled by λ), but also on the geometry of the problem (controlled by η). Note that we were not able to construct a similar example for the Radau IIA-type methods.

FIG. 1. *5-step BDF, $T = 40$, $\beta = 1$, $\lambda = -10, \ldots, +10$, $\eta = -10 \ldots + 10$, $h = 0.01$*

FIG. 2. *RKM of Crouzeix, $T = 10$, $\beta = 1$, $\lambda = -10 \ldots + 10$, $\eta = -80 \ldots + 80$, $h = 0.01$*

Appendix: Basic linear algebra lemma. A basic connection between the spaces appearing at the tractability index and the choice of the corresponding projectors is given by the following Lemma, which may be directly obtained from Theorem A.13. and Lemma A.14. in Griepentrog & März (1986).

Lemma A.1 *Let* $\bar{A}, \bar{B}, \bar{Q} \in L(I\!R^m)$ *be given* $\bar{Q}^2 = \bar{Q}$, $\mathrm{im}\,(\bar{Q}) = \ker(\bar{A})$, *i.e. let* $\bar{Q}$ *be a projector onto* $\ker(\bar{A})$. *Call* $\bar{S} := \{z \in I\!R^m : \bar{B}z \in \mathrm{im}\,(\bar{A})\}$. *Then the following conditions are equivalent:*

(i) *The matrix* $\bar{G} := \bar{A} + \bar{B}\bar{Q}$ *is nonsingular.*

(ii) $I\!R^m = \bar{S} \oplus \ker(\bar{A})$.

(iii) $\bar{S} \cap \ker(\bar{A}) = \{0\}$.

If $\bar{G}$ *is nonsingular, then the relation*

$$\bar{Q}_s = \bar{Q}\bar{G}^{-1}\bar{B}$$

holds for the canonical projector $\bar{Q}_s$ *(canonical means:* $\bar{Q}_s$ *projects* $I\!R^m$ *onto* $\ker(\bar{A})$ *along* $\bar{S}$*).*

Proof:

(i) $\rightarrow$ (ii) At first, the space $I\!R^m$ can be described as $\bar{S} + \ker(\bar{A})$, because

$$z = (I - \bar{Q}\bar{G}^{-1}\bar{B})z + \bar{Q}\bar{G}^{-1}\bar{B}z =: z_1 + z_2 \qquad (*)$$

holds for any $z \in I\!R^m$.

Now z_2 obviously lies in $\ker(\bar{A})$, because $\bar{Q}$ is a projector onto $\ker(\bar{A})$. For z_1 we obtain

$$\bar{B}z_1 = (I - \bar{B}\bar{Q}\bar{G}^{-1})\bar{B}z = \bar{A}\bar{G}^{-1}\bar{B}z \in \mathrm{im}\,(\bar{A}),$$

i.e. $z_1 \in \bar{S}$.

It remains to show that $\bar{S} \cap \ker(\bar{A}) = \{0\}$. For that, let $x \in \bar{S} \cap \ker(\bar{A})$. Then $x = \bar{Q}x$ holds and there exists a $z \in I\!R^m$ so that

$$\bar{A}z = \bar{B}x = \bar{B}\bar{Q}x \quad \text{and so} \quad \bar{G}^{-1}\bar{A}z = \bar{G}^{-1}\bar{B}\bar{Q}x,$$

i.e. $(I - \bar{Q})z = \bar{Q}x$, so $0 = \bar{Q}x = x$.

(ii) $\rightarrow$ (iii) This holds trivially because of the definition.

(iii) $\rightarrow$ (i) Let $x \in I\!R^m$ be chosen such that $\bar{G}x = 0$, i.e. $\bar{B}\bar{Q}x = -\bar{A}x$ and so $\bar{Q}x \in \bar{S}$. On the other hand, $\bar{Q}x$ lies in $\ker(\bar{A})$. Thus, $x \in \ker(\bar{Q})$ holds because of the assumption. That means $\bar{A}x = 0$, hence $x \in \mathrm{im}(\bar{Q})$. Therefore $x = 0$ must hold, and $\bar{G}$ is nonsingular.

Because of the uniqueness of the partition $(*)$, the latter assertion follows immediately.

$\square$

References.

U. Ascher & L.R. Petzold (1991), "*Projected implicit Runge-Kutta methods for differential-algebraic equations*", SIAM J. Numer. Anal. 28, 1097-1120.

U. Ascher & L.R. Petzold (1993), "Stability of computational methods for constrained dynamic systems", SIAM J. Sci. Comp. 14, 95-120.

K.E. Brenan, S.L. Campbell & L.R. Petzold (1989), "*Numerical solution of initial-value problems in differential-algebraic equations*", North Holland, New York-Amsterdam-London.

E. Griepentrog & R. März (1986), "*Differential-algebraic equations and their numerical treatment*", Teubner Texte zur Mathematik 88, Teubner, Leipzig.

E. Hairer, S.P. Nørsett & G. Wanner (1993), "*Solving ordinary differential equations I*", 2nd edition, Springer, Berlin.

E. Izquierdo Macana (1993), "*Numerische Approximation von Algebro-Differentialgleichungen mit Index 2 mittels impliziter Runge-Kutta-Verfahren*", Doctoral thesis, Humboldt-Univ. Berlin, Fachbereich Mathematik.

R. März (1989), "*Index-2 differential-algebraic equations*", Results in Mathematics 15, 148-171.

R. März (1990), "*Higher-index differential-algebraic equations: Analysis and numerical treatment*", Banach Center Publications 24, 199-222.

R. März (1991), "*On quasilinear index 2 differential-algebraic equations*". Preprint 269, Humboldt-Univ. Fachbereich Mathematik.

R. März (1992), "*Numerical methods for differential-algebraic equations*", Acta Numerica, 141-198.

R. März & C. Tischendorf (1993) "Solving more general index-2 differential algebraic equations", Preprint 93-6, to appear in Computers and Mathematics with Applications.

L.R. Petzold (1986), "*Order results for implicit Runge-Kutta methods applied to differential algebraic systems*", SIAM J. Numer. Anal. 23, 837-852.

F. Verhulst (1990), "Nonlinear differential equations and dynamical systems", Springer, Berlin.

R. Weiner (1993), "Stability investigations for semi-explicit index-2-systems", Lecture given at the Oberwolfach Conference "Differential-Algebraic Equations: Theory and Application in Technical Simulation", June 13-19, 1993.