Restricted Tangent Bundle on Space Curves

G. Hein,[*] H. Kurke
Humboldt-Universität zu Berlin,
Fachbereich Mathematik, Unter den Linden 6, 10099 Berlin

1 Introduction

The purpose of this paper is to investigate the restriction of the tangent bundle of $\mathbb{P}^n$ to a curve $X \subset \mathbb{P}^n$. The corresponding question for rational curves was investigated by L. Ramella [7] and F. Ghione, A. Iarrobino and G. Sacchiero [2] in the case of rational curves. Let us also mention that D. Laksov [6] proved that the restricted tangent bundle of a projectively normal curve does not split unless the curve is rational. We will show the following theorem (See 3.1):

Theorem: *In the variety of smooth connected space curves of genus $g \geq 1$ and degree $d \geq g+3$ there exists a nonempty dense open subset where the restricted tangent bundle is semistable and, moreover, simple if $g \geq 2$*

If the degree is high with respect to the genus ($d > 3g$), we get a postulation formula for the strata with a given Harder-Narasimhan polygon, following results of R. Hernández [5].
In case of plane curves the situation is simpler due to

Theorem: *If X is a smooth plane curve of degree d, the restricted tangent bundle is stable for $d \geq 3$, of splitting type $(3,3)$ for a conic, and of splitting type $(2,1)$ for a line.*

Proof: (following Bogomolov, comm. by D. Huybrechts) We denote by E the tangent bundle of $\mathbb{P}^2$ twisted by $\mathcal{O}_{\mathbb{P}^2}(-1)$. We first suppose that $d > 2$. We use the facts:

1. E is stable, $c_1(E) = 1$ and $c_2(E) = 1$.
2. If $E|_X$ is unstable, then we have a destabilizing quotient $E|_X \to L$. We define $F = ker(E \to E|_X \to L)$ and obtain a bundle F of rank 2 with $\Delta(F) = c_1(F)^2 - 4c_2(F) \geq d^2 - 3 > 0$.
3. By Bogomolov's inequality the bundle F is not semistable and if $M \subset F$ is a subbundle of maximal degree and rank 1, then $deg(M) \geq 1$, which contradicts the semistability of E.

Property 1 follows from the Euler sequence. For property 2 we use

$$\begin{aligned} c_1(F) &= c_1(E) - [X] \\ c_2(F) &= c_2(E) + deg(L) - c_1(E)\cdot[X] \qquad \text{hence:} \\ \Delta(F) &= \Delta(E) + [X]\cdot[X] + 2c_1(E)\cdot[X] - 4deg(L) \\ &= -3 + d^2 + 2(d - 2deg(L)) \\ &\geq d^2 - 3 \end{aligned}$$

Property 3 follows because $F/M \cong I \otimes M'$, where I is the sheaf of ideals of a 0-dimensional subscheme.

[*]Supported by Graduiertenkolleg "Geometrie und Nichtlineare Analysis" at the Humboldt University Berlin

We set $l = length(\mathcal{O}_{I\!P^2}/I)$ and obtain:

$$\begin{array}{rcll} c_1(F) & = & c_1(M) + c_1(M') & \\ c_2(F) & = & c_1(M)c_1(M') - l & \text{consequently:} \\ \Delta(F) & = & [c_1(M) - c_1(M')]^2 + 4l & \\ & = & [2c_1(M) - c_1(F)]^2 + 4l\ , & \end{array}$$

hence $[2deg(M) - 1 + d]^2 \geq \Delta(F) \geq d^2 - 3$. M is destabilizing, so we must have $deg(M) \geq 1$. The same proof shows that in case of $d = 2$ there can not exist a surjection $E|_X \rightarrow L$ to a linebundle of degree less than one. For $d = 1$ the statement is obvious. $\square$

The main idea we exploit in this papers is to consider degenerations of smooth curves into special reducible curves with ordinary double points and to extend the notion of the Harder-Narasimhan polygon to such curves. This idea was used by L. Ramella [7] in the case of rational curves.

2 The generalized Harder-Narasimhan polygon and the Shatz stratification

Let E be a vector bundle of rank r on a reduced curve X, with irreducible components X_i $(i = 1, \ldots, k)$. We say a sheaf $F \subset E$ is of constant rank n if $rk(F|_{X_i}) = n$ for all $i = 1, \ldots, k$. In this case we write $rk(F) = n$. We define the function

$$\begin{array}{rrcl} f_E: & \{0, \ldots, r\} & \rightarrow & \mathbb{Z} \\ & n & \mapsto & sup\{deg(F) \mid F \subset E \text{ and } rk(F) = n\} \end{array}$$

Then we define the generalized *Harder-Narasimhan polygon* $(HNP(E))$ of E as the convex hull of this function.

Remark The degree of F is defined by $deg(F) = \chi(F) + n\chi(\mathcal{O}_X)$. It is obvious that $f(n) < \infty$ and $HNP(E)$ coincides with the Harder-Narasimhan polygon in the case of a smooth curve X.

Theorem 2.1 *Let $X \subset I\!P^n \times S$ be a flat family of reduced curves over S, and $\mathcal{E}$ an X vector bundle of rank r. Then the map*

$$\begin{array}{rccc} HNP: & S & \longrightarrow & Polygons \\ & s & \mapsto & HNP(\mathcal{E}_s) \end{array}$$

defines a finite and locally closed stratification on S, the so-called Shatz stratification.

The theorem follows from 2.2 and 2.3, since by 2.2 there are only finitely many polygons in the image of HNP above a given P and by 2.3 the set $\{s \mid HNP(E_s) \geq P\}$ is therefore a closed set.

Lemma 2.2 *There exists an integer M, such that $f_{E_s}(n) < M$ for all $s \epsilon S$ and any $n \epsilon \{1, \ldots, r\}$*

Proof: We assume S to be connected, then $\chi(\mathcal{O}_{X_s})$ is constant. The function $h^0 : S \rightarrow \mathbb{Z}$ assigning every $s \epsilon S$ the dimension of $H^0(E_s)$ is upper semicontinuous. S is assumed to be a noetherian scheme, so there exists an upper bound M_1 of h^0. Now, for any $s \epsilon S$ and $F \subset E_s$ of rank n we have:

$$\begin{aligned} deg(F) &= \chi(F) + n\chi(\mathcal{O}_{X_s}) \\ &\leq h^0(F) + n\chi(\mathcal{O}_{X_s}) \\ &\leq h^0(E_s) + n\chi(\mathcal{O}_{X_s}) \\ &\leq M_1 + r|\chi(\mathcal{O}_{X_s})|. \end{aligned}$$

We set $M = M_1 + r|\chi(\mathcal{O}_{X_s})| + 1$. $\square$

Lemma 2.3 *Under the assumptions made above for any ν $0 \leq \nu \leq r$ the function $f : S \rightarrow \mathbb{Z}$ $s \mapsto f_{E_s}(\nu)$ is upper semicontinuous.*

Proof: We suppose S to be irreducible. We have to show that the subset $S_k = \{s \epsilon S \mid f_{\mathcal{E}_s}(\nu) \geq k\}$ is closed. Let Q be the Quot scheme over S parameterizing quotients of E with Hilbert polynomial $\chi(n) = \chi(E) - k + \nu(\chi(\mathcal{O}_X))$ The image of the natural morphism $\psi : Q \rightarrow S$ is closed. This would be enough in case of a family X of integral schemes.
Assume now that $s \epsilon S$ is in $im(\psi)$ but not in S_k. The problem occurring is that we might have different ranks over the irreducible components and we have to show that the quotient is not flatly smoothable to one of constant rank over all components. We will do this by the choice of a divisor which meets the quotient sheaf at every irreducible component in at least one point where this quotient is locally free. Let $X_s = X_1 \cup X_2 \cup \ldots \cup X_m$ be an irreducible decomposition of X_s and $E = \mathcal{E}_s$ the vector bundle on X_s.
We remark that any sheaf F on X_s which is a quotient of E has less than $N := \chi(F) + h^1(E) + 1$ torsion points (i.e. $\#supp(tors(F)) < N$). Otherwise $F' = F/tors(F)$ would be a quotient of E with $\chi(F') < -h^1(E)$, which is impossible.
Now we choose a hypersurface $H \subset \mathbb{P}^n$ which intersects X_s transversally and meets all irreducible components X_i $i = 1 \ldots m$ at least N-times. We may assume (after a restriction to a smaller open subset, if necessary) that this property holds for all points of S. We now get a semi continuous function $R : Q \rightarrow \mathbb{Z}$, assigning the minimum of the embedding dimensions of F_t at points of $X_t \cap H$ to every quotient F_t of E_t. Thus the subset $\{s' \epsilon Q \mid all\ R(s') \geq r - \nu\}$ of Q is closed, hence its image in S is closed.

However, s can not be in the image because a quotient F of E_s with the Hilbert polynomial χ must have a rank less than $r - \nu$ at one component X_s. Therefore its embedding dimension in at least one point of $X_s \cap H$ is less than $r - \nu$. $\square$

Proceeding by the same method we obtain:

Theorem 2.4 *Under the same assumptions as in 2.1 we have: The set of points $s \epsilon S$ where E_s is stable (respectively semistable) is open.*

3 The semistability of the restricted tangent bundle

We define $Hilb(d, g)$ to be the Hilbert scheme of closed subschemes $X \subset \mathbb{P}^3$ with Hilbert polynomial $\chi(\mathcal{O}_X(n)) = dn + 1 - g$. By $Hilb_0(d, g)$ we define those quotients which are smooth irreducible curves. In [1] it is proved that $Hilb_0(d, g)$ is irreducible if $d > g + 2$. Over the Hilbert scheme we have the universal curve $C(d, g)$, a closed subscheme of $Hilb(d, g) \times \mathbb{P}^3$. We consider the projection $\pi_2 : C(d, g) \rightarrow \mathbb{P}^3$ and the tangent sheaf $\Theta(-1)$ of the projective space twisted with $\mathcal{O}_{\mathbb{P}^3}(-1)$. This defines a bundle $E = \pi_2^* \Theta(-1)$. For any point $s \epsilon H(d, g)$ the sheaf E_s is the restriction of $\Theta(-1)$ to the curve parameterized by s.

Theorem 3.1 *If $g \geq 1$ and $d > g + 2$, then for a general $s \epsilon H_0(d, g)$ the vector bundle E_s is semistable.*

The proof of the theorem divides into three steps:

Step 1: We show that the statement is true for $g = 1$ and $d \geq 4$. (See 4.5)

Step 2: Under the assumption that in $Hilb_0(d, g)$ there exists a point s parameterizing a smooth curve Y with:
- E_s is semistable
- $H^1(\mathcal{O}_Y(1)) = 0$
- E_s is not isomorphic to a direct sum of two vector bundles,
we show that there exists a point t in $Hilb(d + 1, g + 1)$ parameterizing a curve X satisfying:
- E_t is semistable

- $H^1(\mathcal{O}_X(1)) = 0$
- $dim(End(E_t)) = 1$.

Step 3: The curve X obtained in the previous step corresponds to a smooth point in $Hilb(d+1, g+1)$ and is in the closure of $Hilb_0(d+1, g+1)$, because of $H^1(\mathcal{O}_X(1)) = 0$ (see [4] 1.2). However semistability is an open condition (2.4) and, hence holds for an open subset of $Hilb_0(d+1, g+1)$ too. The same applies to $dim(End(E_t)) = 1$ and $H^1(\mathcal{O}_X(1)) = 0$ which implies that on a nonempty open subset of $Hilb_0(d+1, g+1)$ the necessary conditions of step 2 are fulfilled.

The rest of this section is devoted to the proof of the second step.

Let X be a connected curve with two ordinary double points and two irreducible components Y and Z of genus g_Y and 0 (i.e. $Z \cong \mathbb{P}^1$), which intersect in two points P and Q. Then we have an exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Z \to k(P) \oplus k(Q) \to 0$$

Hence $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) - 1$ or $g_X = g_Y + 1$.

Lemma 3.2 *Let E be a vector bundle of rank n on X such that $E_Y = E \otimes \mathcal{O}_Y$ is semistable of degree d and $E_Z = E \otimes \mathcal{O}_Z$ is globally generated and of degree 1. Let F be a sheaf of constant rank r with maximal degree among subsheaves of constant rank r. If F is destabilizing then F is a subbundle of E and F_Z is of degree 1.*

Proof: If F_Y and F_Z are the subbundles of E_Y and E_Z generated by the images of F in E_Y resp. E_Z, then $F \subset \tilde{F} = E \cap (F_Y \times F_Z) \subset E_Y \times E_Z$ and $\tilde{F}/F$ has finite support. Since F has maximal degree it follows that $F = \tilde{F}$ and we obtain an exact sequence

$$0 \to F \to F_Y \times F_Z \to F \otimes \mathcal{O}_D \to 0$$

(with $D = P + Q$, as a subscheme of X) If F is destabilizing then $r(deg(E_Y) + 1) < n(deg(F_Y) + deg(F_Z) - [l(F \otimes \mathcal{O}_D) - r \cdot l(\mathcal{O}_d)])$
$r(deg(E_Y)+1) < r \cdot deg(E_Y) + n \cdot deg(F_Z) - n[l(F \otimes \mathcal{O}_D) - r \cdot l(\mathcal{O}_d)]$ hence $r + n[l(F \otimes \mathcal{O}_D) - r \cdot l(\mathcal{O}_d)] < n deg(F_Z)$.

Since $deg(F_Z) \leq 1$ this implies $deg(F_Z) = 1$ and $l(F \otimes \mathcal{O}_D) = r \cdot l(\mathcal{O}_d)$, i.e. F is a subbundle. □

Let V be a vector space of dimension 4. From now on we consider a fixed smooth curve Y of genus $g_Y \geq 1$ and a quotient $V \otimes \mathcal{O}_Y \to E_Y$ for which we suppose:

(i) E_Y is a semistable vector bundle of rank 3 and degree d.

(ii) E_Y is not decomposable.

(iii) The morphism $Y \to \mathbb{P}(V^\vee)$ defined by the surjection $V \otimes \mathcal{O}_Y \to E_Y$ is an embedding. Therefore we will identify Y with its image in $\mathbb{P}(V^\vee)$.

Given two different points P and Q of Y we denote by $Z(P,Q)$ the line in $\mathbb{P}(V^\vee)$ through P and Q and by $X(P,Q)$ the union of Y and $Z(P,Q)$. Again we define $E_{X(P,Q)}$ to be the restriction of the tangent bundle of $\mathbb{P}(V^\vee)$ twisted by $\mathcal{O}_{\mathbb{P}(V^\vee)}(-1)$ to $X(P,Q)$. Restricting the Euler sequence to X gives:

$$\begin{array}{rl} \text{a surjection} & V \otimes \mathcal{O}_{X(P,Q)} \to E_{X(P,Q)} \\ \text{with} & E_Y = E_{X(P,Q)} \otimes \mathcal{O}_Y \\ \text{and} & E_{Z(P,Q)} = E_{X(P,Q)} \otimes \mathcal{O}_{Z(P,Q)} \cong \mathcal{O}_Z(1) \oplus \mathcal{O}_Z^{\oplus 2}. \end{array}$$

We will show that for two general points P and Q of Y the bundle $E_{X(P,Q)}$ is semistable. Of course we have to choose P and Q such that $Z(P,Q)$ meets Y in exactly these two points and, moreover, quasi transversally, i.e. not tangentially. But this is always possible because Y is not a strange curve. We will call the corresponding line $Z(P,Q)$ the *bisecant* to Y, determined by P and Q.

We have $E_{Z(P,Q)} \cong \mathcal{O}_Z(1) \oplus \mathcal{O}_Z^{\oplus 2}$ and therefore a canonic subbundle of rank and degree 1 in $E_{Z(P,Q)}$, namely the tangent bundle of $Z(P,Q)$ twisted with $\mathcal{O}(-1)$. This defines a one-dimensional subspace of $E \otimes k(P)$, which we denote by $T(P,Q)$. We will frequently use the following obvious lemma:

Lemma 3.3 *Let $P_0, P_1, P_2, \ldots, P_m$ be different points of Y. Then we have one dimensional subspaces $T(P_0, P_i)$ of $E \otimes k(P_0)$ and*

$$dim[\sum_{i=1}^{m} T(P_0, P_i)] = dim W \quad ,$$

where $W \subset \mathbb{P}(V^\vee)$ is the linear subspace spanned by the points P_i.

Lemma 3.4 *Let Y and E_Y be as before and suppose that, for two given points P and Q, the bundle $E_{X(P,Q)}$ is not semistable. Then there exists a subbundle $F_Y \subset E_Y$ such that:*

$$\begin{array}{lrcl} (i) & \mu(F_Y) + \frac{1}{rk(F_Y)} & > & \mu(E_Y) + \frac{1}{rk(E_Y)} \\ (ii) & T(P,Q) & \subset & F_Y \otimes k(P) \\ (iii) & T(Q,P) & \subset & F_Y \otimes k(Q) \ . \end{array}$$

Proof: Let $F \subset E_{X(P,Q)}$ be a sheaf with constant rank and $\mu(F) > \mu(E_{X(P,Q)})$. Because of 3.2, F is a subbundle.
We set $F_Y = F \otimes \mathcal{O}_Y$ *and* $F_Z = F \otimes \mathcal{O}_{Z(P,Q)}$ We have: $\mu(F) = \mu(F_Y) + \mu(F_Z) \geq \mu(E_Y) + \mu(E_Z) = \mu(E)$ hence $deg(F_Z) = 1$ and therefore: $\mu(F_Y) + \frac{1}{rk(F)} \geq \mu(E_Y) + \frac{1}{n}$, $T(P,Q) \subset F \otimes k(P) = F_Y \otimes k(P)$ and $T(Q,P) \subset F \otimes k(Q) = F_Y \otimes k(Q)$. □

Analogously we obtain:

Lemma 3.5 *Let Y and E_Y be as above and suppose that, for two given points P and Q, the bundle $E_{X(P,Q)}$ contains a destabilizing subbundle of constant rank 2. Then there exists a subbundle $F_Y \subset E_Y$ of rank 2 and a plane $H \subset \mathbb{P}(V^\vee)$ such that:*

$$\begin{array}{lrcl} (i) & \mu(F_Y) + \frac{1}{2} & > & \mu(E_Y) + \frac{1}{rk(E_Y)} \\ (ii) & P \epsilon H & and & Q \epsilon H \\ (iii) & \Theta_H(-1) \otimes k(P) & = & F_Y \otimes k(P) \\ (iv) & \Theta_H(-1) \otimes k(Q) & = & F_Y \otimes k(Q) \ . \quad \Box \end{array}$$

We denote by $\Theta_Y(-1)$ the tangent bundle of Y twisted with $\mathcal{O}_{\mathbb{P}(V^\vee)}(-1)$. $\Theta_Y(-1)$ is a sublinebundle of E_Y of degree $2 - 2g_Y - d$.

Lemma 3.6 *For any 1-dimensional subspace $W \subset V$, corresponding to a point $P \epsilon \mathbb{P}(V^\vee)$, we denote by $L^P \subset E_Y$ the subbundle of E_Y generated by the image of $W \otimes \mathcal{O}_Y \to V \otimes \mathcal{O}_Y \to E_Y$. The sublinebundle $L^P \subset E_Y$ satisfies:*

$$\begin{array}{lrcl} (i) & L^P \otimes k(Q) & = & T(Q,P) \text{ for all points } Q \epsilon Y \text{ with } Q \neq P \ , \\ (ii) & L^P \otimes k(P) & = & \Theta_Y(-1) \otimes k(P) \quad \text{if } P \epsilon Y \ , \\ (iii) & deg(L^P) & = & 0 \quad \text{if } P \not\epsilon Y \ , \\ & & = & 1 \quad \text{if } P \epsilon Y \ . \end{array}$$

Proof: obvious

We will now prove that for $rk(E_Y) = 3$ and two general points P and Q of Y the bundle $E_{X(P,Q)}$ has no destabilizing subbundle of rank one or two. Assuming the contrary we will derive the existence of certain subsheaves of E_Y by using 3.3 and 3.4 which leads to contradictions. The proof splits into three cases, depending on $deg(E_Y)$ modulo 3.

Lemma 3.7 *Let E_Y be a semistable bundle on Y of rank 3 and degree $d = 3k$. If E_Y is indecomposable, then $E_{X(P,Q)}$ has no destabilizing sheaf of constant rank 1 for two general points P and Q of Y.*

Proof: We take four points P, Q_1, Q_2, Q_3 of Y which span $I\!P(V^\vee)$ and define bisecants $Z(P, Q_I)$ to Y. If $E_{X(P,Q_i)}$ were not semistable for $i = 1, 2, 3$, we would have 3 linebundles $L_i \subset E_Y$ with $deg(L_i) = k$ and $L_i \otimes k(P) = T(P, Q_i)$ by 3.4. We define $E' = L_1 + L_2 + L_3$. By 3.3 E' is a sheaf of E_Y of rank 3 and degree at least $3k$. However this would imply that $E_Y = L_1 \oplus L_2 \oplus L_3$ $\square$

Lemma 3.8 *Let E_Y be a semistable and indecomposable vector bundle on Y of rank 3 and degree $d = 3k (k > 1)$. Then $E_{X(P,Q)}$ has no destabilizing sheaf of constant rank 2 for two general points P and Q of the curve Y.*

Proof: As before we assume that for all pairs P, Q of points of Y there exits a sheaf $F_Y \subset E_Y$ of rank 2 and degree $2k$, see 3.5. We distinguish three cases.

Case 1: E_Y has no sublinebundle of degree k.
We choose points P, Q_1, Q_2 of Y defining bisecants $Z(P, Q_i)$ and rank 2 subbundles F_1, F_2 of rank 2 and degree $2k$ according to 3.5, such that $T(P, Q_1) \subset F_1 \otimes k(P)$ $T(P, Q_2) \not\subset F_1 \otimes k(P)$ and $T(P, Q_2) \subset F_2 \otimes k(P)$.
We define $F = F_1 + F_2$ and G to be the kernel of the surjection $F_1 \oplus F_2 \to F$. Then G must have rank 1 and we have an injection from G to F_2, hence an injection from G to E_Y, therefore $deg(G) < k$. But this gives $deg(F) > 3k$, which is impossible.

Case 2: E_Y has two (or more) sublinebundles L_1 and L_2 of degree k. We take a point P of Y such that $L_1 \otimes k(P) \neq L_2 \otimes k(P)$ in $E_Y \otimes k(P)$. We define $W = L_1 \otimes k(P) + L_2 \otimes k(P)$. Now we choose a point Q in Y such that $T(P, Q) \not\subset W$ and both points define a bisecant. Again we suppose, there were an $F \subset E_Y$ of rank 2 and $deg F = 2k$ and $T(P, Q) \subset F \otimes k(P)$. This implies that at most one of the linebundles L_i can be contained in F. We suppose $L_1 \not\subset F$ and find $E_Y = F \oplus L_1$ as before.

Case 3: E_Y has exactly one sublinebundle L of degree k.
If there were a bundle $F \subset E_Y$ of rank 2 and degree $2k$ not containing L, then we would have $E_Y = L \oplus F$. So we can assume that all subbundles of E_Y with degree $2k$ and rank 2 contain L.
For any $P \epsilon Y$ we denote the line through P with direction $L \otimes k(P)$ in P by $Z(L, P)$. We choose two points $P, Q \epsilon Y$ such that $Z(L, P)$ and $Z(L, Q)$ differ from the line through P and Q. However, because of 3.5 these three lines are in a plane $H \subset I\!P(V^\vee)$. We denote the intersection point $Z(L, P) \cap Z(L, Q)$ by Q_0.

We now see that for a general point P' of Y not contained in H the line $Z(L, P')$ must intersect with $Z(L, P)$ and $Z(L, Q)$. This is possible only if $Q_0 \epsilon Z(L, P')$ for all $P' \epsilon Y$. But this immediately yields: $L = L^{Q_0}$ $\square$

Now we come to the case of $deg(E_Y) = 3k+1$. For numerical reasons $E_{X(P,Q)}$ can not have a destabilizing sheaf of constant rank 2 (see 3.4). So only the subsheaves of constant rank 1 have to be considered:

Lemma 3.9 *Let E_Y be a stable bundle on Y of rank 3 and degree $d = 3k + 1$. If, moreover, $d \geq 5$, then $E_{X(P,Q)}$ has no destabilizing sheaf of constant rank 1, for two general points P and Q of Y.*

Proof: We take a general hyperplane H of $I\!P(V^\vee)$, such that $Y \cap H = \{P, Q_1, \ldots, Q_{d-1}\}$ consists of d different points in general position. Moreover, we take a point Q of Y which is not contained in H. Now we suppose that for all $i = 1, \ldots, d-1$ there is a sublinebundle L_i of E_Y with $deg(L_i) = k$ and $T(P, Q_i) = L_i \otimes k(P)$ in $E \otimes k(P)$, see 3.4. We see that $L_1 + L_2 \cong L_1 \oplus L_2$, therefore $F = L_1 + L_2 + L_3$ is of rank 2 or 3.

Case 1: $rk(F) = 2$. Here we have a non-zero morphism $L_3 \to L_1 + L_2 \cong L_1 \oplus L_2$. However, from the choice of the points Q_i it follows that neither $L_3 \subset L_1$ nor $L_3 \subset L_2$ holds, hence $L_1 \cong L_2$.
Let now L be a sublinebundle of E_Y such that $deg(L) = k$ and $T(P, Q) = L \otimes k(P)$. We see that $G = L + L_1 + L_2 \cong L_1 \oplus L_1 \oplus L$. We obtain a short exact sequence $0 \to G \to E_Y \to T \to 0$, where T is a

torsion sheaf of length one. So we have $dim(Ext^1(T, E_Y)) = 3$ and $dim(Hom(G, E_Y) \geq 5$, which implies $dim(End(E)) \geq 2$. This is impossible because a stable sheaf is simple.

Case 2: $rk(F) = 3$.
This is only possible when $F \cong L_1 \oplus L_2 \oplus L_3$. Now we consider the linebundle $L_4 \subset E_Y$ of degree k with $T(P, Q_4) = L_4 \otimes k(P)$. (Here we really need the assumption $d \geq 5$.) It follows from the construction that L_4 is even a sublinebundle of F and as before we get $L_1 \cong L_2$. The rest we conclude like in the first case. □

We need the statement of the previous lemma also for the case of $g = 1$ and $d = 4$.

Lemma 3.10 *Let Y be an elliptic curve embedded in $\mathbb{P}(V^\vee)$ as a curve of degree 4, and E_Y be stable. Then, for two general point P and Q of Y, the bundle $E_{X(P,Q)}$ has no destabilizing sheaf of constant rank one.*

Proof: Let $Q(E_Y, 2, 3)$ be the Quot scheme of Quotients $E_Y \longrightarrow F$ with $deg(F) = 3$ and $rk(F) = 2$. Considering the kernel of these surjections we obtain a morphism $\phi : Q(E_Y, 2, 3) \longrightarrow Pic^1(Y)$. For a linebundle L of degree one we have:

$$\begin{array}{rcll} Hom(E_Y, L) & \cong & Ext^1(L, E_Y)^\vee & \text{(Serre duality)} \\ Hom(E_Y, L) & = & 0 & E_Y \text{ is stable, hence:} \\ dim(Hom(L, E_Y)) & = & 1\ . & \end{array}$$

The same argument shows that $Q(E_Y, 2, 3)$ is smooth of dimension 1 and so $\phi : Q(E_Y, 2, 3) \longrightarrow Pic^1(Y)$ is an isomorphism. On the other hand, we have a family of sublinebundles of E_Y parameterized by Y (3.6). So it follows that all linebundles $L \subset E_Y$ of degree 1 are the linebundles L^P for a $P \epsilon Y$. □

Now we come to the easy case where $deg(E_Y) = 3k + 2$. Here we see immediately that for all points P and Q the bundle $E_{X(P,Q)}$ is semistable. But we have to show a little bit more:

Lemma 3.11 *Let E_Y be stable. Then, for any two points P and Q of Y, we have $dim(End(E_{X(P,Q)}) = 1$*

Proof: On the one hand this follows from the fact that the only endomorphisms of the stable bundle E_Y are the multiplications and on the other hand that an endomorphism of $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$, which is the multiplication at two different points P and Q of $\mathbb{P}^1$, is itself a multiplication. □

4 A result of R. Hernández

Here we give a short review on a result of R. Hernández [5], which was the starting point for this work. For a fixed smooth curve X of genus g Hernández considered the Quot scheme $Q(m, n, d)$ of quotients from $\mathcal{O}_X^{\oplus m}$ of rank n $(n < m)$ and degree d. We denote by $Q_0(m, n, d) \subset Q(m, n, d)$ those quotients E which are vector bundles and satisfy $H^1(X, E) = 0$. Its is obvious that Q_0 is an open subset whose existence is given by the following lemma.

Lemma 4.1 *$Q_0(m, n, d)$ is nonempty if and only if $d > ng$.*

Proof: By the Riemann-Roch theorem it follows that $d > ng$ is necessary. (The only exception occurs when $g = 0$). It remains to show the that the condition is sufficient.
We proceed by induction on n. For $n = 1$ we have to show that a general line bundle L on X with $deg(L) > g$ is globally generated. For a general linebundle L of degree $d > g$ it is well known that $H^1(L) = 0$.

Let $Z = \{\alpha \epsilon Pic^{d-1} \,|\, h^0(\alpha) \geq d - g + 1\}$. We show that $codim(Z, Pic^{d-1}) \geq 2$. To do so, we regard the surjection $pr : S^{d-1}X \longrightarrow Pic^{d-1}$. Now the fibre of this morphism is at least of dimension $d - g$ over Z. At the other hand we have $codim(pr^{-1}(Z), S^{d-1}X) \geq 1$. Combining these two facts we get the stated codimension.

Now for any line bundle $L \epsilon Pic^d$ we define the following map:

$$\begin{array}{cccc} \phi_L : & X & \longrightarrow & Pic^{d-1} \\ & p & \mapsto & L(-p) \end{array}$$

We can choose L such that $im(\phi_L) \cap Z = \emptyset$, which means that L is base-point free and, therefore, generated by its global sections.

Now we assume that the assertion is true for $n-1$, so that we have a quotient $E \epsilon Q_0(m-1, n-1, d-g)$. Dualizing we get a short exact sequence:

$$0 \to E^\vee \to \mathcal{O}_X^{\oplus m-1} \to L \to 0 \ .$$

On the other hand, we take an effective divisor D of degree g, such that $h^0(\mathcal{O}_X(D)) = 1$ and a section $s \epsilon H^0(L(D))$ not vanishing at the points of D. This yields the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to & \mathcal{O}_X^{\oplus m-1} & \to & \mathcal{O}_X^{\oplus m} & \to & \mathcal{O}_X & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & L & \to & L(D) & \to & \mathcal{O}_D & \to 0 \end{array}$$

Denoting the kernel of the vertical morphism in the middle by G we obtain the kernel-cokernel sequence:

$$0 \to E^\vee \to G \to \mathcal{O}_X(-D) \to 0$$

and we obviously conclude $G^\vee \epsilon Q_0(m, n, d))$ $\square$

Let V be a vector space of dimension n and $Q(V, r, d)$ be the Quot scheme of quotients of $V \otimes \mathcal{O}_X$ of degree d and rank r. As before we denote by $Q_0(V, r, d)$ the open subset where the quotients have no first cohomology and are locally free.
Now we fix a convex polygon $P = \{(0,0); (r_1, d_1); (d_2, r_2) \ldots (r_l, d_l)\}$ with $r_l = r$ and $d_l = d$ and consider the subset $Q_0(V, P)$ in $Q_0(V, r, d)$ which parameterizes quotients E with $HNP(E) = P$. By 2.1 $Q_0(V, P)$ is by locally closed. We set $r_0 = d_0 = 0$ and $r_{-1} = r_l - n$. Under the assumption that $Q_0(V, P) \neq \emptyset$ we have (See [5]):

Theorem 4.2 *$Q_0(V, P)$ is smooth irreducible and*

$$dim(Q_0(V, P)) = \sum_{i=0}^{l-1} [(r_{i+1} - r_{i-1})(d_l - d_i) + (r_i - r_{i-1})(r_l - r_i)(1-g)]$$

Let X be a smooth curve of genus g and V a vector space of dimension m. As before we denote by $Q_0(V, n, d)$ the quotients $V \otimes \mathcal{O}_X \to E$ that are bundles and satisfy $h^1(E) = 0$. (So we obviously require $m > n$ and $d > ng$ by 4.1.)
We define now two convex polygons by: $P_{min} = \{(0,0); (n, d)\}$ and $P_{max} = \{(0,0); (1, d-1-g(n-1)); (n, d)\}$.

Theorem 4.3 *A convex polygon P from $(0,0)$ to (n, d) arises in the image of* $HNP : Q_0(V, n, d) \to$ *Polygons* $E \mapsto HNP(E)$ *if and only if $P_{min} \leq P \leq P_{max}$.*

Proof: Let $V \otimes \mathcal{O}_X \to E$ be a point in the Quot scheme, then $P_{min} \leq P$ holds by definition of $HNP(E)$. Let $F \subset E$ be a sheaf of E. We can assume $E' = E/F$ to be a vector bundle. (Otherwise E would have a subshaef of same rank as F but with a higher degree.) E' is generated by its global sections and satisfies $h^1(E') = 0$, hence by 4.1: $deg(E') \geq rk(E')g + 1$ which implies $deg(F) \leq d - 1 - gn + rk(F)g$

and therefore $P \leq P_{max}$.

Conversely given any convex polygon $P = \{(0,0);(r_1,d_1);\ldots;(n,d)\}$ of length l with $P_{min} \leq P \leq P_{max}$. Then, again by 4.1, there exist semistable bundles $E_i \quad i = 1,\ldots,l$ with $rk(E_i) = r_i - r_{i-1}$ and $deg(E_i) = d_i - d_{i-1}$ which are globally generated and satisfy $h^1(E_i) = 0$.

Their direct sum is therefore a quotient with the given Harder-Narasimhan polygon. $\square$

Example: Let X be an elliptic curve and let V be a 4-dimensional vector space. We consider the open subset Q_0 of $Quot(V \otimes \mathcal{O}_X, 3, 9)$
$dim(Q_0) = 36$, and the possible Harder-Narasimhan polygons (beside P_{min}) are:

$$\begin{array}{ll}
P_1 = \{(0,0);(1,4);(3,9)\} & codim(HNP^{-1}(P_1), Q_0) = 3 \\
P_2 = \{(0,0);(1,5);(3,9)\} & codim(HNP^{-1}(P_2), Q_0) = 6 \\
P_3 = \{(0,0);(1,6);(3,9)\} = P_{max} & codim(HNP^{-1}(P_3), Q_0) = 9 \\
P_4 = \{(0,0);(2,7);(3,9)\} & codim(HNP^{-1}(P_4), Q_0) = 3 \\
P_5 = \{(0,0);(1,4);(2,7);(3,9)\} & codim(HNP^{-1}(P_5), Q_0) = 4\,.
\end{array}$$

Let now $X \subset \mathbb{P}^m \times S$ be a flat family of smooth curves over S. Then we have the Quot scheme $Q = Quot(V \otimes \mathcal{O}_X, n, d)$ together with a morphism $\pi : Q \to S$. Again we define Q_0 to be the open subset of Q that parameterizes bundles with vanishing first cohomology. Let P be a convex polygon given by $\{(0,0),(r_1,d_1),\ldots,(r_l,d_l)\}$ with $r_l = n$ and $d_l = d$, then we can define the subset $Q_0(V,P)$ of Q_0 as before. By 4.2 and 4.3 we know the fibres of $\pi|_{Q_0(V,P)} : Q_0(V,P) \to S$. Neither their existence nor their dimension depend on $s \epsilon S$, which gives:

Theorem 4.4 *$Q_0(V,P) \neq \emptyset$ if and only if $P_{min} \leq P \leq P_{max}$.*

$$dim(Q_0(V,P)) = \sum_{i=0}^{l-1}[(r_{i+1} - r_{i-1})(d_l - d_i) + (r_i - r_{i-1})(r_l - r_i)(1-g)] + dim S$$

If S is irreducible (resp. smooth), then $Q_0(V,P)$ is so.

Corollary 4.5 *If $d > 3g$, then the general curve in $Hilb_0(d,g)$ has a semistable restricted tangent bundle.*

References

[1] L. Ein , Hilbert schemes of smooth space curves, Ann. Sc. Ec. Norm. Sup., 4. sér., 19, (1986), p. 469-478

[2] F. Ghione, A. Iarrobino and G. Sacchiero, Restricted tangent bundles of rational curves in $\mathbb{P}^r$, preprint, (1988)

[3] A. Grothendieck, Techniques de construction et théorèmes d'existence en géométrie algébrique, IV. Les schémas de Hilbert, Seminaire Bourbaki 221 (1960/61)

[4] R. Hartshorne and A. Hirschowitz, Smoothing algebraic space curves, Lect. Notes Math 1124 (1978), 98-131

[5] R. Hernandez, On Harder-Narasimhan stratification over Quot schemes, J. reine ang. Math 371 (1986), 114-24

[6] D. Laksov, Indecomposability of restricted tangent bundles, Astérisque 87/88, (1981), p. 207-219

[7] L. Ramella, Sur le schéma de Hilbert de courbes rationales, These de doctorat, Nice, (1988)