ON THE STABILITY OF THE ABRAMOV–TRANSFER FOR DAES OF INDEX 1

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Abstract

Abramov's transfer method [1] for ordinary differential equations (ODE) transfers linear boundary value problems into initial value problems and systems of linear algebraic equations that have to be solved.
K. Balla and R. März applied Abramov's transfer method to homogenized index 1 differential algebraic equations (DAE) [3].
In this paper a direct version of the transfer method for inhomogeneous differential algebraic equations is formulated and stability is proved.

Keywords: differential-algebraic equations, boundary value problems, transfer of boundary conditions, stability

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Abbreviated title: Abramov's Transfer for index-1-DAEs

1. Preliminaries. Consider the linear index 1 boundary value problem

$$A(t)y' + B(t)y = f(t), \qquad t_0 \le t \le t_1, \tag{1a}$$
$$C_0 \cdot y(t_0) = g_0, \tag{1b}$$
$$C_1 \cdot y(t_1) = g_1. \tag{1c}$$

$A \in C^1([t_0,t_1];\ \mathbb{R}^{m\times m})$, $B \in C([t_0,t_1];\ \mathbb{R}^{m\times m})$, $f \in C([t_0,t_1];\ \mathbb{R}^{m})$.
Let $Q(t)$ be a projector onto $\ker A(t)$, $P(t) = I - Q(t)$.
A necessary and sufficient condition for index 1 is that rank $A(t) = \text{const}$ and the matrix $G(t) \stackrel{\text{def}}{=} A(t) + B(t)Q(t)$ is regular for all $t \in [t_0,t_1]$. This condition does not depend on the special choice of Q.
Since $A \in C^1$, Q can be chosen in such a way that $Q \in C^1$.
Assume that the matrices C_i $(i = 0,1)$ have full rank and fulfill $C_iP(t_i) = C_i$.
Now consider the adjoint problem

$$A^T\phi' - \left(B^T - A^{T'}\right)\phi = 0, \tag{2a}$$
$$\left(\phi^T A y\right)' - \phi^T f = 0. \tag{2b}$$

The adjoint problem has index 1 iff the original problem has it.
Introduce the function h as

$$h = \phi^T A y. \tag{3a}$$

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Then the adjoint problem reads

$$(3b)\qquad A^T\phi' - \left(B^T - A^{T'}\right)\phi = 0,$$

$$(3c)\qquad h' - \phi^T f = 0.$$

Let $P_{\star s}(t) := I - Q_{\star s}(t)$, where $Q_{\star s}(t)$ is the projector onto $\ker A^T(t)$ along

$$S_\star(t) \overset{\text{def}}{=} \left\{\xi \in R^n : \quad \left(B(t)^T - A^{T'}(t)\right)\xi \in Im\ A^T(t)\right\}.$$

There are two natural ways to choose the initial conditions of the adjoint equation.

$$(4a)\qquad \phi(t_i) = P_{\star s}(t_i)\cdot A^{T+}(t_i)\cdot C_i^T \text{ and}$$

$$(4b)\qquad h(t_i) = g_i,$$

or

$$(4c)\qquad \phi(t_i) = P_{\star s}(t_i)\cdot A^{T+}(t_i)\cdot C_i^T L_i \text{ and}$$

$$(4d)\qquad h(t_i) = L_i^T g_i$$

The subscript i stands either for 0 or for 1, L_i regular, $L_i L_i^T = \left(C_i C_i^T\right)^{-1}$.
Whenever the second choice of the initial conditions looks a little bit artifical, we will see later that a transformation of the adjoint system has nice properties.
The fact that the solution $y(t)$ of the original problem (1a) belongs to

$$S_f(t) \overset{\text{def}}{=} \{\xi \in R^n : \quad B(t)\xi - f(t) \in \text{Im } A(t)\}$$

can be formulated as $R(t)B(t)y(t) = R(t)f(t)$. $R(t)$ is a differentiable projector function with $\ker[R(t)] = \text{im}[A(t)]$.
A possible choice is $R = QG^{-1}$, which leads to $Q_s(t)y(t) = Q(t)G^{-1}(t)f(t)$. $Q_s(t)$ is the unique projection onto $\ker A(t)$ along $S_0(t)$ ($f \equiv 0$).
The solution space of the initial value problem

$$\begin{aligned} A(t)y' + B(t)y &= f(t), \qquad t_0 \le t \le t_1, \\ C_i \cdot y(t_i) &= g_i \end{aligned}$$

with the initial conditions mentioned above is determined by the equations

$$\begin{aligned} \phi^T(t)A(t)y(t) &= h(t), \\ R(t)B(t)y(t) &= R(t)f(t), \qquad t \in [t_0, t_1]. \end{aligned}$$

What we want to do is solving the system (3b), (3c) firstly with one pair of initial conditions for $i = 0$ from t_0 into the direction t_1, and secondly for $i = 1$ from t_1 into the direction t_0. We get two matrix-valued functions ϕ_l, ϕ_r and two vector-valued functions h_l, h_r.
We call an analytical procedure a "transfer of the boundary conditions" if the procedure leads to a system of initial value problems and the following system of algebraic equations

$$(5a)\qquad \phi_l^T(t)A(t)y(t) = h_l(t),$$

$$(5b)\qquad \phi_r^T(t)A(t)y(t) = h_r(t),$$

$$(5c)\qquad R(t)B(t)y(t) = R(t)f(t), \qquad t \in [t_0, t_1].$$

In the case of explicit regular ordinary differential equations ($A \equiv P \equiv P_\star \equiv I$), equation (5c) disappears.
The solution space of the system (1a)–(1c) is given by all solutions of (5a)–(5c).
What we have got are two matrix-valued differential equations with initial conditions that have to be solved. Then we have to determine the solutions of systems of algebraic equations on a grid to obtain the solutions of the problem (1a)–(1c).

2. Transformation of the adjoint equation. In this section a transformation due to Abramov [1] of the system (2a), (2b) is introduced.
Take a variable ψ and consider the following transformation

$$\psi = \phi \cdot T, \tag{6}$$

where $T \in C^1([t_0, t_1];\ I\!R^{m\times m})$. We will say something about the special choices of T later.
Now it holds

$$\begin{aligned}\left(A^T\psi\right)' &= \left(A^T\phi\right)' \cdot T + A^T\phi \cdot T' \\ &= B^T\phi \cdot T + A^T\phi \cdot T' \\ &= B^T\psi + A^T\psi \cdot \underbrace{T^{-1}T'}_{w}\end{aligned}$$

$$\Rightarrow \left(A^T\psi\right)' - B^T\psi - A^T\psi \cdot w = 0. \tag{7}$$

$$\begin{aligned} 0 &= \left(\phi^T Ay\right)' - \phi^T f \\ &= \left(T^{T^{-1}}\psi^T Ay\right)' - T^{T^{-1}}\psi^T f \\ &= \left(T^{T^{-1}}\right)'\psi^T Ay + T^{T^{-1}}\left(\psi^T Ay\right)' - T^{T^{-1}}\psi^T f \\ \Leftrightarrow 0 &= T^T\left(T^{T^{-1}}\right)'\psi^T Ay + \left(\psi^T Ay\right)' - \psi^T f \\ &= -\left(T^T\right)' T^{T^{-1}}\psi^T Ay + \left(\psi^T Ay\right)' - \psi^T f \\ &= -w^T\psi^T Ay + \left(\psi^T Ay\right)' - \psi^T f.\end{aligned}$$

Set

$$h := \psi^T Ay, \tag{8a}$$

then

$$\Rightarrow h' = w^T h + \psi^T f. \tag{8b}$$

Remark: The h in (8a) is different from the h in (3a), but it has to fulfill the same function in the resulting algebraic equations. This is the reason for the same notation h.
In the following we consider two choices of the transformation T.

2.1. First Transformation. The transformation T is chosen such that

$$\psi^T A\left(A^T\psi\right)' = 0 \tag{9}$$

holds.
This leads to $\psi^T AA^T\psi = \Phi = const = C_0^T C_0 \quad (C_1^T C_1)$ or, in case the second choice of initial conditions is used, $\psi^T AA^T\psi = I$ in the solution.
Fortunately there is no need to compute the transformation T explicitly, but it is possible to get an expression for w:

$$w = -\left(\psi^T AA^T\psi\right)^{-1}\psi^T AB^T\psi.$$

Put w into the equations (7) and (8b), so you get the following equations

$$\left(A^T\psi\right)' - \left(I - A^T\psi\left(\psi^T AA^T\psi\right)^{-1}\psi^T A\right)B^T\psi = 0, \tag{10a}$$

$$h' + \psi^T BA^T\psi\left(\psi^T AA^T\psi\right)^{-1}h - \psi^T f = 0. \tag{10b}$$

The differential equation for h is completely decoupled from the equation for ψ. This is of interest for theoretical investigations like the existence of solutions, but for the numerical solution of the resulting algebraic equations the same grid is needed.
The existence of a solution of equation (10a) on the whole interval $[t_0, t_1]$ is proved in [3], theorem 2.1. The existence of a solution of equation (10b) follows from the fact that the equation is linear in h.

2.2. Homogenization. Choose T such that

$$\psi^T A \left(A^T \psi\right)' + hh^{T'} = 0. \tag{11}$$

With $W := \psi^T A A^T \psi + hh^T = const$ (in the solution) this leads to

$$w = -W^{-1}\left(\psi^T A B^T \psi - h f^T \psi\right).$$

and hence to

$$\begin{aligned} \left(A^T \psi\right)' - \left(I - A^T \psi W^{-1} \psi^T A\right) B^T \psi + A^T \psi W^{-1} h f^T \psi &= 0, && \text{(12a)} \\ h' + \psi^T B A^T \psi W^{-1} h - (1 - h^T W^{-1} h) \psi^T f &= 0. && \text{(12b)} \end{aligned}$$

The latter transformation is equivalent to the approach by Balla/März [3] via homogenization of the DAE (1a)–(1c). The equations are coupled via the matrix W.
In this case $L_i L_i^T = \left(C_i C_i^T + g_i g_i^T\right)^{-1}$ is valid for the initial conditions and leads to $W = I$.
The existence of a solution of the equations (12a), (12b) on the whole interval $[t_0, t_1]$ is proved directly in [3] for an equivalent formulation.

The result is an algorithm for the transfer of index 1 DAEs:

1. Solve the transformed adjont problem (10a), (10a) or (12a), (12b) with the relevant initial conditions from the left to the right and vice versa.
2. Solve the linear algebraic system (5a)–(5c) with ψ instead of ϕ.

This algorithm is feasible on the whole interval $[t_0, t_1]$.

2.3. Hessenberg systems. For linear index 1 Hessenberg systems

$$\begin{aligned} u' + B_{11} u + B_{12} v &= f_1, && \text{(13a)} \\ B_{21} u + B_{22} v &= f_2 && \text{(13b)} \end{aligned}$$

it is easy to verify that the special structure can be exploited.
Let ϕ be partitioned as $(\phi_1^T, \phi_1^T)^T$. Then the transfer equations for the first transformation are

$$\begin{aligned} \phi_1' - \left(I - \phi_1 (\phi_1^T \phi_1)^{-1} \phi_1^T\right)\left(B_{11}^T \phi_1 + B_{21}^T \phi_2\right) &= 0, \\ B_{12}^T \phi_1 + B_{22}^T \phi_2 &= 0, \\ h' + (\phi_1^T B_{11} + \phi_2^T B_{21}) \phi_1 (\phi_1^T \phi_1)^{-1} h - (\phi_1^T f_1 + \phi_2^T f_2) &= 0. \end{aligned}$$

In this case the algebraic equations are

$$\begin{aligned} \phi_{1,l}^T u &= h_l, \\ \phi_{1,r}^T u &= h_r, \\ B_{21} u + B_{22} v &= f_2. \end{aligned}$$

An analogous consideration, but with the homogenized equation, leads to similar results.

3. Stability considerations. Without loss of generality only the transfer equations with initial conditions on the left are regarded in the following.
Look at the system (10a), (10b) with small perturbations on the right-hand side

$$
\begin{aligned}
&\text{(14a)} \qquad \left(A^T\tilde{\psi}\right)' - \left(I - A^T\tilde{\psi}\left(\tilde{\psi}^T A A^T \tilde{\psi}\right)^{-1}\tilde{\psi}^T A\right) B^T\tilde{\psi} &=& \ \varepsilon,\\
&\text{(14b)} \qquad \tilde{h}' + \tilde{\psi}^T B A^T \tilde{\psi}\left(\tilde{\psi}^T A A^T \tilde{\psi}\right)^{-1}\tilde{h} - \tilde{\psi}^T f &=& \ \delta,
\end{aligned}
$$

where $\tilde{\psi}, \tilde{h}$ are the exact solutions of the perturbed equations.
Assume that ε, δ are sufficiently smooth and introduce the following transformation

$$
\tilde{\psi} = \Psi K, \quad \tilde{h} = K^T\rho. \tag{15}
$$

K has to be chosen such that

$$
\Psi^T A \left(A^T\Psi\right)' = 0.
$$

With this property it is possible to compute K and to transform (14a), (14b) into

$$
\begin{aligned}
&\text{(16a)} \qquad \left(A^T\Psi\right)' - \left(I - A^T\Psi\left(\Psi^T A A^T \Psi\right)^{-1}\Psi^T A\right)\tilde{B}^T\Psi &=& \ 0,\\
&\text{(16b)} \qquad \rho' + \Psi^T\tilde{B}A^T\Psi\left(\Psi^T A A^T\Psi\right)^{-1}\rho - \Psi^T\tilde{f} &=& \ 0.
\end{aligned}
$$

$\tilde{B}, \tilde{f}$ are defined as

$$
\begin{aligned}
\tilde{B} &= AA^T\tilde{\psi}\left(\tilde{\psi}^T A A^T \tilde{\psi}\right)^{-1}\varepsilon^T + B,\\
\tilde{f} &= AA^T\tilde{\psi}\left(\tilde{\psi}^T A A^T \tilde{\psi}\right)^{-1}\delta + f.
\end{aligned}
$$

The equations (16a), (16b) are the transfer equations of the DAE

$$
A\tilde{y}' + \tilde{B}\tilde{y} = \tilde{f}. \tag{17}
$$

The DAE (17) has index 1, too: The matrix pencil (A, B) is regular and has index 1. Then, for each matrix $W \in L(\mathbb{R}^m)$ the matrix pencil $(A, B + AW)$ is regular and has index 1 ([5], appendix A, theorem 13). The projections of this DAE coincide with the projections of the DAE (1a):
The matrix A does not change. This implies that P is the same projection for both DAEs. For the projections R and Q_s one can argue that, for the spaces S_0, $\tilde{S}_0$,

$$
\begin{aligned}
\tilde{S}_0(t) &= \left\{\xi \in R^n : \tilde{B}(t)\xi \in Im\ A(t)\right\}\\
&= \left\{\xi \in R^n : B(t)\xi + A(t)W(t)\varepsilon^T\xi \in Im\ A(t)\right\} \ = \ S_0(t)
\end{aligned}
$$

is true with a matrix $W = A^T\tilde{\psi}\left(\tilde{\psi}^T A A^T\tilde{\psi}\right)^{-1}$. The arguments are skipped for brevity.

In the case of homogenization the perturbed transfer equations are not related to a perturbed homogenized problem. There are additional couplings of the equations (12a), (12b) with the factors ε and δ.

In the following parts we are interested in bounds for the perturbation-term
$\left(\tilde{\psi}^T A A^T\tilde{\psi}\right)^{-1}\tilde{\psi}^T A$.

3.1. Bounds for the perturbation term. Unless it leads to confusions, the arguments of the matrix functions are suppressed again.
Denote $\left\|A^T(t_0)\tilde{\psi}(t_0)\right\|_2 = w\ (\in I\!R),\ v := \left\|\left(\tilde{\psi}^T AA^T\tilde{\psi}\right)^{-1}\right\|_2,\ v_0 = v(t_0)$,
$\varepsilon_0 := \sup_{t_0\leq t\leq t_1}\|\varepsilon(t)\|_2$ and $\varepsilon_0|t_1 - t_0| \leq \sqrt{w^2 + \frac{1}{v_0}} - w$. If we take the initial conditions (4c), then $v_0 = 1$. For the definition and properties of the Dini derivatives $D^\star$ see Appendix A.
Using

$$\left(\tilde{\psi}^T AA^T\tilde{\psi}\right)' = \tilde{\psi}^T A\left(A^T\tilde{\psi}\right)' + \left(\tilde{\psi}^T A\right)' A^T\tilde{\psi} \overset{(14a)}{=} \tilde{\psi}^T A\varepsilon + \varepsilon^T A^T\tilde{\psi}$$

and

$$\left|D^\star\left\|\tilde{\psi}^T AA^T\tilde{\psi}\right\|_2\right| \leq \left\|\left(\tilde{\psi}^T AA^T\tilde{\psi}\right)'\right\|_2 \leq 2\|\varepsilon\|_2\left\|A^T\tilde{\psi}\right\|_2,$$

then

$$\begin{aligned} 2\left\|A^T\tilde{\psi}\right\|_2\left|D^\star\left\|A^T\tilde{\psi}\right\|_2\right| &\leq 2\|\varepsilon\|_2\left\|A^T\tilde{\psi}\right\|_2 \\ \Rightarrow \qquad \left|D^\star\left\|A^T\tilde{\psi}\right\|_2\right| &\leq \varepsilon_0. \end{aligned}$$

Because of $w > 0$ it holds that $\left\|A^T(t)\tilde{\psi}(t)\right\|_2 \leq w + \varepsilon_0(t - t_0)$. Hence, with

$$\left[\left(\tilde{\psi}^T AA^T\tilde{\psi}\right)^{-1}\right]' = -\left(\tilde{\psi}^T AA^T\tilde{\psi}\right)^{-1}\left(\tilde{\psi}^T AA^T\tilde{\psi}\right)'\left(\tilde{\psi}^T AA^T\tilde{\psi}\right)^{-1}$$

it follows

$$\begin{aligned} \|D^\star v\|_2 &\leq \left\|\left[\left(\tilde{\psi}^T AA^T\tilde{\psi}\right)^{-1}\right]'\right\|_2 \leq 2v^2\varepsilon_0\left(w + \varepsilon_0(t - t_0)\right) \\ \Rightarrow \left\|D^\star\left(\frac{1}{v}\right)\right\|_2 &\leq 2\varepsilon_0\left(w + \varepsilon_0(t - t_0)\right). \end{aligned}$$

Let $v(t_0) = v_0 > 0$,

$$\begin{aligned} \frac{1}{v_0} - \frac{1}{v(t)} &\leq 2\varepsilon_0 w(t - t_0) + 2\varepsilon_0^2(t - t_0)^2, \text{ this means} \\ v(t) &\leq \frac{1}{\frac{1}{v_0} - 2\varepsilon_0 w(t - t_0) - 2\varepsilon_0^2(t - t_0)^2}. \end{aligned}$$

Now the following inequality can be concluded

$$\left\|\left(\tilde{\psi}^T(t)A(t)A^T(t)\tilde{\psi}(t)\right)^{-1}\tilde{\psi}^T(t)A(t)\right\|_2 \leq \frac{w + \varepsilon_0(t - t_0)}{\frac{1}{v_0} - 2\varepsilon_0 w(t - t_0) - 2\varepsilon_0^2(t - t_0)^2}. \tag{18}$$

This inequality is sharp, there is no better estimation at $t = t_0$.

3.2. Stability theorem. Consider the state space form of the index 1 DAE (1a)

$$u' = (P'P_s - PG^{-1}B)u + P(I + P')G^{-1}f, \tag{19a}$$

$$y = P_s u + QG^{-1}f. \tag{19b}$$

The state space form of the perturbed problem reads

$$\tilde{u}' = (P'P_s - P\tilde{G}^{-1}\tilde{B})\tilde{u} + P(I+P')\tilde{G}^{-1}\tilde{f}, \tag{20a}$$

$$\tilde{y} = P_s\tilde{u} + Q\tilde{G}^{-1}\tilde{f}. \tag{20b}$$

Recall that

$$\begin{aligned}\tilde{G} = A + \tilde{B}Q &= A + BQ + AA^T\tilde{\psi}\left(\tilde{\psi}^T AA^T\tilde{\psi}\right)^{-1}\varepsilon^T Q \\ &= G + AA^T\tilde{\psi}\left(\tilde{\psi}^T AA^T\tilde{\psi}\right)^{-1}\varepsilon^T Q,\end{aligned}$$

hence

$$\|\tilde{G}(t) - G(t)\|_2 \le \|A(t)\|_2 \cdot \|\varepsilon(t)\|_2 \cdot \|Q(t)\|_2 \cdot \frac{w + \varepsilon_0(t-t_0)}{\frac{1}{v_0} - 2\varepsilon_0 w(t-t_0) - 2\varepsilon_0^2(t-t_0)^2}.$$

Equation (20a) is a regular perturbation of (19a), and (20b) is a regularly perturbed assignment of (19b). Now all theorems and properties from the theory of regular perturbations for ODEs can be applied to the underlying ODEs (19a), (20a), and this leads to the following definition and statement:

Definition 1. *A regular perturbation of an index 1 DAE is a perturbation that leads to a regular perturbation of the underlying ODE (19a) and a regular perturbation of the assignment (19b).*

Theorem 3.1. *The boundary condition transfer (10a), (10b) is stable in the sense that small right-hand side perturbations of the transfer equations represent a regular perturbation of the original index 1 DAE of the same size.*

4. Numerical examples. For the integration of the transfer equations the code DASSL [9] was used. This is the reason why the matrix A in the examples has to be constant so that we do not have to use an explicit derivative of the matrix A.

The tolerances were chosen as $atol = rtol = 1d{-}6$. The intervals were divided into equidistant subintervals and DASSL produced output at the gridpoints. The resulting system of linear equations was solved with a precision of $1d-8$ at every gridpoint.

4.1. An ODE example. To demonstrate that the transfer of the boundary conditions works well even in the ODE case, an ODE example ([2], page 121, example 3.13, page 169, example 4.13) was testet. The example is constructed in such a way that, in an ODE with an obvious dichtomy, the components of the variables rotate.

The example is

$$y' + \begin{pmatrix} \lambda\cos(2\omega t) & -\omega - \lambda\sin(2\omega t) \\ \omega - \lambda\sin(2\omega t) & -\lambda\cos(2\omega t) \end{pmatrix} y = 0, \qquad 0 < t < \Pi.$$

The eigenvalues of the matrix above are $\pm\sqrt{\lambda^2 - \omega^2}$, while the kinematic eigenvalues (they come from the system with the obvious dichotomy) are $\pm\lambda$. If the amount of ω increases, the eigenvalues drift away from the kinematic eigenvalues, for $\omega > \lambda$ they become imaginary and do not yield any information about the dichotomy, which does not change with ω.

A solution is given by

$$y(t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

for $\alpha, \beta \in \mathbb{R}$.

The authors of [2] describe that for $\lambda = 1$ and large ω the Riccati method performs poorly, for $\omega = 1$ and λ large the Riccati method performs well. Exact numerical results are not given in the book.

With the matrices

$$C_0 = (\cos(\omega t), -\sin(\omega t)), \quad C_1 = (\sin(\omega t), \cos(\omega t))$$

for the boundary conditions and $\lambda \geq 0$ the problem is well posed, the components of the fundamental matrix

$$Y(t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix}$$

are integrated into the stable directions.
The solution was computed in the interval $[0.001; \Pi - 0.001]$ with exact boundary conditions. The other values were chosen as $\alpha = 1, \beta = 1$.
If $\hat{y}$ denotes the computed solution and y denotes the solution, then the regarded errors are

$$\text{error} = \max_i \frac{\|\hat{y}(t_i) - y(t_i)\|_2}{\|y(t_i)\|_2}.$$

These relative errors of the transfer method are given in the following table:

ω / λ	1	10	20	100	200	1000
1	8.1D-06	1.5D-05	2.5D-05	7.7D-04	1.3D-03	1.4D-02
2	8.3D-06	2.3D-05	1.3D-04	7.5D-04	6.2D-03	1.7D-02
5	3.2D-04	1.1D-05	1.3D-04	7.3D-04	5.1D-03	1.6D-02
10	3.4D-04	3.8D-05	3.2D-05	7.1D-04	2.7D-03	1.5D-02
100	8.2D-04	8.9D-04	8.9D-04	2.6D-04	4.5D-04	1.7D-03
200	1.4D-03	1.4D-03	1.5D-03	1.6D-03	4.2D-03	1.2D-03

The transfer method performs well, even in the stiff case of $\lambda = 200$.

4.2. An index 1 example. Consider the following example:
Let the matrices E and F be defined as follows

$$E = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \\ 1 & 32 & 243 & 1024 & 3125 \end{pmatrix}, \quad F = \begin{pmatrix} -1 & -2 & 1 & 2 & 3 \\ 1 & 4 & 1 & 4 & 9 \\ -1 & -8 & 1 & 8 & 27 \\ 1 & 16 & 1 & 16 & 81 \\ -1 & -32 & 1 & 32 & 243 \end{pmatrix}.$$

The matrices A and B are defined as

$$A = E \cdot \operatorname{diag}(1, 1, 1, 0, 0) \cdot F^{-1}, \; B = E \cdot \operatorname{diag}(0, 0, t+1, t+2, (t+1)^2) \cdot F^{-1}.$$

The solution y is given by

$$y(t) = \begin{pmatrix} \sin(t) + \cos(5t) \\ t^2 + 3 \\ e^{-t} \\ \cos(t) \cdot e6t \\ \frac{1}{t+1} \end{pmatrix}.$$

The right-hand side q is defined by

$$Ay' + By = q.$$

Due to the Kronecker canonical normal form this DAE has index 1.

The boundary conditions were chosen as

$$\begin{pmatrix} \frac{7}{5} & \frac{1}{12} & -\frac{11}{24} & -\frac{1}{12} & \frac{7}{120} \\ \frac{7}{10} & \frac{7}{6} & -\frac{19}{24} & -\frac{1}{6} & \frac{11}{120} \end{pmatrix} \cdot y(0) = \begin{pmatrix} \frac{7}{6} \\ \frac{10}{3} \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} -\frac{1}{10} & \frac{4}{3} & \frac{19}{24} & -\frac{4}{3} & \frac{37}{120} \end{pmatrix} \cdot y(1) &= \\ & \frac{439}{80} + \frac{19}{24} \cdot e^{-1} - \frac{4}{3} e \cos(1) - \frac{1}{10}(\cos(5) + \sin(1)) \ . \end{aligned}$$

4.2.1. Results with the first choice of the initial conditions. Firstly, the results of the computations with the initial conditions (4a), (4b) are regarded.
The maximal error taken over all five components in the interval $[0, 1]$ was $2.035D-05$.
The product $\psi^T AA^T \psi$ was monitored, too. At $t = 0(1)$ this matrix was computed and used for comparing with the matrix at the actual time. The Euclidian norm of the difference was divided by the Euclidian norm of the comparison matrix. The norm of the maximal errors may be much larger with this choice of initial conditions.
The maximal (relative) error of the constant matrix during the computation from the left to the right was $1.017d-07$, in the other direction $1.828d-6$.

4.2.2. Results with the second choice of the initial conditions. Now the results with the initial conditions (4c), (4d) are presented.
Here the error over all components was at most $7.676d-5$.
The product $\psi^T AA^T \psi$ was monitored, too. Here the comparison matrix was really an approximation of the identity matrix. The absolute errors of the constant matrix had nearly the same values as at the other initial conditions.

Some further examples were tested, too. The results were as good as those of the examples above.
REMARK 1. *The restriction $C_i P(t_i) = C_i$ is due to the choice of the initial condition*

$$\phi(t_i) = P_{\star s}(t_i) \cdot A^{T+}(t_i) \cdot C_i^T \ (\cdot L_i).$$

For the considered form of the transfer equations this restriction is essential, there exists a simple example that shows that:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} y' + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad y_1(0) - y_2(0) = 0.$$

The solution is $y_0 \equiv y_1 \equiv 1$.
The transfer equations simplify to

$$\begin{aligned} \psi_1' &= 0 \\ \psi_2 &= 0 \\ h' - \psi_2 &= 0. \end{aligned}$$

With the initial conditions (4a), (4b)

$$\begin{aligned} \psi(0) &= P_{\star s}(0) \cdot A^{T+}(0) \cdot C_0^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ h(0) &= 0 \end{aligned}$$

we get $\psi_1 \equiv 1, \psi_2 \equiv h \equiv 0$ and this leads finally to $y_1 \equiv 0$ and $y_2 \equiv 1$.
When using the initial conditions for the P-components only, $y_1(0) = 1$, the transfer method provides the correct results.
REMARK 2. *Methods that preserve quadratic invariants (see [4]) can be used to solve the system of transfer equations.*

A. The Dini derivative . Let M be a differentiable matrix-valued function. $m(t) := \|M(t)\|_2$. The classical concept of derivatives cannot be applied to m due to corners originating from the norm. This is why the Dini derivatives are used as a generalization for continuous functions m [6]

$$\begin{aligned} D^+ m(t) &:= \limsup_{h>0} \frac{m(t+h)-m(t)}{h}, \\ D^- m(t) &:= \limsup_{h<0} \frac{m(t+h)-m(t)}{h}. \end{aligned}$$

Furthermore, some nice properties of the Dini derivatives are needed.

From the triangle inequality it follows

$$\begin{aligned} \|M(t+h)\|_2 - \|M(t)\|_2 &\le \|M(t+h)-M(t)\|_2, \\ \|M(t)\|_2 - \|M(t+h)\|_2 &\le \|M(t+h)-M(t)\|_2. \end{aligned} \tag{21}$$

Dividing the inequalities (21) by $h > 0$ $(-h > 0)$ yields

$$\left|D^\star \|M(t)\|_2\right| \le \|M'(t)\|_2 \,.$$

Here $D^\star$ stands for one of the Dini derivatives.

In the Euclidian norm for any matrix M it holds

$$\begin{aligned} \|M\|_2^2 &= \rho(M^T M) \;=\; \rho\left(T \cdot d \cdot T^{-1}\right), \\ \|M^T M\|_2 &= \sqrt{\rho(M^T M M^T M)} \;=\; \sqrt{\rho\left(T \cdot d^2 \cdot T^{-1}\right)}. \end{aligned}$$

$\rho(\cdot)$ denotes the largest eigenvalue of the argument. T is the matrix that transforms $M^T M$ into a diagonal-matrix d consisting of the nonnegative eigenvalues of $M^T M$. Let λ be the greatest of these eigenvalues. Then $\lambda = \|M\|_2^2$. On the other hand, $\|M^T M\|_2 = \sqrt{\lambda^2} = \lambda$.

The next property of Dini derivatives is needed, too:

$$|D^\star \|M\|_2^2| = 2 \cdot \|M\|_2 \, |D^\star \|M\|_2| \,.$$

Firstly we prove the chain rule for Dini derivatives. For any continuous function m we have

$$\begin{aligned} D^\star m^2 &= \limsup_{h\neq 0} \frac{m^2(t+h)-m^2(t)}{h} \\ &= \limsup_{h\neq 0} \left(m(t+h)\frac{m(t+h)-m(t)}{h} + m(t)\frac{m(t+h)-m(t)}{h} \right) \\ &= \limsup_{h\neq 0} \left(m(t+h)+m(t)\right) \frac{m(t+h)-m(t)}{h} \\ &= \lim_{h\neq 0} \left(m(t+h)+m(t)\right) \cdot \limsup_{h\neq 0} \frac{m(t+h)-m(t)}{h} \\ &= 2 \cdot m \cdot D^\star m. \end{aligned}$$

Now we can set $m = \|M\|_2$.

The next property we need is $D^\star\left(\frac{1}{m}\right) = -\frac{D^\star m}{m^2}$:

$$\begin{aligned} D^\star\left(\frac{1}{m}\right) &= \limsup_{h\neq 0} \left(\frac{1}{m(t+h)} - \frac{1}{m(t)}\right) \\ &= \limsup_{h\neq 0} \frac{m(t)-m(t+h)}{m(t)m(t+h)} \\ &= -\frac{D^\star m}{m^2}. \end{aligned}$$

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