Linear spaces for index 2 differential-algebraic equations*

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Abstract

In this paper we consider solution spaces of linear index-2-tractable differential algebraic equations. Relations between the solutions of the adjoint equations and the corresponding solution spaces are derived and, thus, a simple method for computing consistent initial values is provided.

Key words. Differential-algebraic equations, linear solution spaces, consistent initial values

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1 Introduction

We consider the homogeneous index-2-tractable differential-algebraic equation

$$A(t)y'(t) + B(t)y(t) = 0 \tag{1}$$

with the coefficients $A \in C^1(\mathcal{I}, L(\mathbb{R}^m))$, $B \in C(\mathcal{I}, L(\mathbb{R}^m))$, and $\mathcal{I} \subseteq \mathbb{R}$ a given interval.

The adjoint equation of (1) reads

$$A^T(t)\phi'(t) - (B^T(t) - A^T(t)')\phi(t) = 0. \tag{2}$$

The solution spaces of the initial equation (1) and the adjoint equation (2), respectively, can be described by means of canonical projectors, which will be described in the section 3. It is necessary to have a practicable representation of subspaces of these solution spaces for the problems (1) and (2) in order to determine the solution spaces belonging to the corresponding initial resp. boundary value problems. By

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means of this representation it becomes possible to apply transfer methods (see [1], [3], [10]), i.e. to solve boundary value problems by modified adjoint initial value problems. The solution spaces can be represented as solutions of linear systems of equations, hence, we obtain a simple method for calculating consistent initial values (section 6).

First, in section 2, we will prove that equation (2) is also index-2 tractable and then we define the matrix chains for the adjoint problem (2).

In section 4 we investigate solution representations by means of the solution of the adjoint equation. The theorem given there provides a suitable representation of solution spaces of (1) and, thus, justifies the application of transfer methods due to Abramov (see [1], [3], [10]). An analogous result for index 1 DAEs was already proved in [2].

In section 5 we apply the results to inhomogeneous systems.

Finally, in section 6, we derive relations among the representation of the solution spaces, the constraint manifolds, a suitable index reduction and the computation of consistent initial values.

At the end of the paper, in the Appendix, we gathered some statements of the theory of linear algebra.

2 The index of the adjoint equation

Let us first consider the relations between the initial equation and the adjoint problem with respect to the index.

Let the nullspace $N(t) := \ker(A(t))$ be smooth, $P(t) = A^{\dagger}(t)A(t)$ be the orthoprojector along $N(t)$ onto $N(t)^{\perp}$. The tractability with index 2 is characterized by the fact that the modified matrix pencil $(A, B - AP')$ has index 2 uniformly on $\mathcal{I}$ [5].

In order to investigate the index, we have to find the matrix pencil that is relevant for the index. For the initial equation it reads $(A, B - AP')$, which corresponds to the equation

$$A(Py)' + (B - AP')y = 0. \tag{3}$$

Let $P_\star(t) = A^{T\dagger}(t)A^T(t) = A(t)A^{\dagger}(t)$ be the projector along $\ker A^T(t)$. On account of $P_\star\phi' = (P_\star\phi)' - P_\star'\phi$, a representation of the adjoint equation (2) that is analogous to the equation (3) now reads

$$A^T(P_\star\phi)' - (B^T - A^{T\prime}P_\star)\phi = 0.$$

Hence, $(A^T, B^T - A^{T\prime}P_\star)$ is the matrix pencil to be investigated.

Theorem 1 *Let the DAE (1) be tractable with index 2. Then the DAE* $A^T\phi' - \left(B^T - A^{T\prime}\right)\phi = 0$ *has index 2, too.*

Proof. The index-2 condition reads

$$\operatorname{ind}(A, B - AP') = 2 = \operatorname{ind}\left(A^T, B^T - P'A^T\right) = \operatorname{ind}\left(A^T, -B^T + P'A^T\right),$$

where we use the special orthoprojectors $P = A^\dagger A$, $P_\star = AA^\dagger$. Because of $A^T = PA^T P_\star$, we have

$$A^{T\prime} P_\star = P' A^T P_\star + P A^{T\prime} P_\star = P' A^T + P A^{T\prime} P_\star.$$

Now we consider the matrix pencil determining the index of equation (2)

$$\left(A^T, -B^T + A^{T\prime} P_\star\right) = \left(A^T, -B^T + P'A^T + PA^{T\prime} P_\star\right).$$

This matrix pencil has index 2 (a simple conclusion from Theorem 24). □

Now we can construct the related matrix chains and spaces for the DAEs (1) and (2) (the arguments are omitted):

$$
\begin{array}{rlrl}
A & & A_\star &= A^T \\
B & & B_\star &= -B^T + A^{T\prime} \\
P &= A^\dagger A & P_\star &= A^{T\dagger} A^T \\
Q &= I - P & Q_\star &= I - P_\star \\
N(t) &= \ker A(t) & N_\star(t) &= \ker A_\star(t) \\
S(t) &= \{\xi \in \mathbb{R}^m : B(t)\xi \in \operatorname{im} A(t)\} & S_\star(t) &= \{\xi \in \mathbb{R}^m : B_\star(t)\xi \in \operatorname{im} A_\star(t)\} \\
B_0 &= B - AP' & B_{\star 0} &= B_\star - A_\star P_\star' = -B^T + A^{T\prime} P_\star \\
A_1 &= A + B_0 Q & A_{\star 1} &= A_\star + B_{\star 0} Q_\star = A^T - B^T Q_\star \\
N_1(t) &= \ker A_1(t) & N_{\star 1}(t) &= \ker A_{\star 1}(t) \\
S_1(t) &= \{\xi \in \mathbb{R}^m : B_0(t)P(t)\xi \in \operatorname{im} A_1(t)\} & & \\
& & S_{\star 1}(t) &= \{\xi \in \mathbb{R}^m : B_{\star 0}(t)P_\star(t)\xi \in \operatorname{im} A_{\star 1}(t)\} \\
Q_1(t) & \text{the projector onto } N_1(t) \text{ along } S_1(t) & & \\
& & Q_{\star 1}(t) & \text{the projector onto } N_{\star 1}(t) \text{ along } S_{\star 1}(t) \\
P_1 &= I - Q_1 & P_{\star 1} &= I - Q_{\star 1} \\
G_2 &= A_1 + B_0 P Q_1 & G_{\star 2} &= A_{\star 1} + B_{\star 0} P_\star Q_{\star 1}
\end{array}
$$

For Q_1 and $Q_{\star 1}$ it holds

$$Q_1 = Q_1 G_2^{-1} B_0 P \quad \text{and} \quad Q_{\star 1} = Q_{\star 1} G_{\star 2}^{-1} B_{\star 0} P_\star \qquad \text{(see e.g. [8])}.$$

The condition that A and A_1 are singular and of constant rank on $\mathcal{I}$, and G_2 is nonsingular on $\mathcal{I}$ is equivalent to the tractability with index 2 [7, Theorem 2.6]. This holds analogously for the adjoint system "$\star$".

The differentiability of A and the rank consistency imply the differentiability of the orthoprojectors P, Q, $P_\star$ and $Q_\star$.

3 The projectors Π_{can} and $\Pi_{\star can}$

Let Q_1 and $Q_{\star 1}$ be differentiable. The linear space $\operatorname{im}\Pi_{can}(t)$, where

$$\Pi_{can}(t) := (I - (QQ_1)'(t) - (QP_1G_2^{-1}B_0)(t))(PP_1)(t)$$

consists of the values of the solutions of the homogeneous equation (1) [6, Theorem 1.2], [9].

Because of $PP_1\Pi_{can} = PP_1$ and $\Pi_{can}PP_1 = \Pi_{can}$ it holds that

$$\ker \Pi_{can} = \ker PP_1. \tag{4}$$

Now we investigate the adjoint equation (2) rewritten as

$$A_\star(P_\star\phi)' + B_{\star 0}\phi = 0.$$

This equation has index 2. We split this equation by means of standard methods in order to derive a similar representation for a projector $\Pi_{\star can}$. The above equation is equivalent to

$$\underbrace{(A_\star + B_{\star 0}Q_\star)}_{A_{\star 1}}(P_\star(P_\star\phi)' + Q_\star\phi) + B_{\star 0}P_\star\phi = 0$$

and, finally, to

$$\underbrace{(A_{\star 1} + B_{\star 0}P_\star Q_{\star 1})}_{G_{\star 2}}(P_{\star 1}(P_\star(P_\star\phi)' + Q_\star\phi) + Q_{\star 1}\phi) + B_{\star 0}P_\star P_{\star 1}\phi = 0.$$

Multiplying by $P_\star P_{\star 1}G_{\star 2}^{-1}$, $Q_\star P_{\star 1}G_{\star 2}^{-1}$ and $Q_{\star 1}G_{\star 2}^{-1}$ yields the following equivalent representation of equation (2)

$$\begin{aligned}
(P_\star P_{\star 1}\phi)' - (P_\star P_{\star 1})'\, P_\star P_{\star 1}\phi + P_\star P_{\star 1}G_{\star 2}^{-1}B_{\star 0}P_\star P_{\star 1}\phi &= 0,\\
Q_\star\phi + (Q_\star Q_{\star 1})'\, P_\star P_{\star 1}\phi + Q_\star P_{\star 1}G_{\star 2}^{-1}B_{\star 0}P_\star P_{\star 1}\phi &= 0,\\
Q_{\star 1}\phi &= 0.
\end{aligned}$$

Hence, $\phi = P_\star P_{\star 1}\phi + Q_\star\phi = \left(I - (Q_\star Q_{\star 1})' - Q_\star P_{\star 1}G_{\star 2}^{-1}B_{\star 0}\right) P_\star P_{\star 1}\phi$.

Obviously, $\Pi_{\star can} := \left(I - (Q_\star Q_{\star 1})' - Q_\star P_{\star 1}G_{\star 2}^{-1}B_{\star 0}\right) P_\star P_{\star 1}$ is a projector and defines the space of the solutions of the adjoint homogeneous index 2 equation.

4 Justification theorem for transfer methods

In this section we introduce and prove a representation theorem for the solution subspaces of linear DAEs of index 2.

Theorem 2 *$\mathcal{M}$ is a k-dimensional linear subspace of solutions of the index 2 equation*

$$Ay' + By = 0,$$

if and only if there exists a $m \times \kappa$-dimensional matrix-valued function on $\mathcal{I}$ ϕ, $\kappa = \operatorname{rank} PP_1 - k$, and it holds that

(*i*) $\operatorname{rank} \phi(t) \equiv \kappa$,

(*ii*) *for all $t \in \mathcal{I}$ we have*
$$\mathcal{M}(t) = \left\{ \xi \in \mathbb{R}^m \,\middle|\, \phi^T(t)A(t)\xi = 0,\ (I - \Pi_{can}(t))\xi = 0 \right\},$$

(*iii*) *and ϕ is a solution of the equation (2).*

Remark 3 *This theorem is an immediate extension of the result [2] to the index 2 with similar smoothness assumptions.*

Proof. First, let $\mathcal{M}$ be a k-dimensional space of solutions of the equation (1) and let $y \in \mathcal{M}$ be chosen arbitrarily. Then, for any fixed $t \in \mathcal{I}$, $\mathcal{M}(t)$ is a k-dimensional subspace of $\mathbb{R}^m$ and it holds that $\mathcal{M}(t) \subseteq \operatorname{im} \Pi_{can}(t)$. Let us fix a t. Hence, there is a κ-dimensional basis $\{q_1, \dots, q_\kappa\}$ of the orthogonal complement of $\mathcal{M}(t)$ relative to $\operatorname{im} \Pi_{can}(t)$. Let $q := (q_1, \dots, q_\kappa)$. Then, for this fixed t, $y(t) = \Pi_{can}(t)y(t) \in \mathcal{M}(t)$ fulfils the conditions

$$\begin{cases} (I - \Pi_{can}(t))y(t) & = 0 \\ q^T \Pi_{can} y(t) & = 0. \end{cases}$$

On account of the representation of $\Pi_{can} = \Pi_{can} P = \Pi_{can} A^\dagger A$ the latter equation can be formulated as follows

$$\phi_0^T A(t)y(t) = 0, \tag{5}$$

where $\phi_0^T = q^T \Pi_{can}(t) A^\dagger(t)$. Clearly, equation (1) implies $Q_\star B y \equiv 0$. Because of $0 = Q_{\star 1}^T A_{\star 1}^T = Q_{\star 1}^T (A - Q_\star B)$ it holds

$$Q_{\star 1}^T A y = Q_{\star 1}^T Q_\star B y = 0.$$

Now equation (5) can be written in the following way

$$\phi_0^T A(t)y(t) = \phi_0^T P_{\star 1}^T(t) A(t) y(t) = \\ \phi_0^T P_{\star 1}^T(t) P_\star(t) A(t) y(t) = \phi_0^T \Pi_{\star can}^T(t) A(t) y(t) = 0.$$

Let $\phi(t_0) := \Pi_{\star can}\phi_0$. Each column vector of $\phi(t_0)$ is a consistent initial value for the adjoint problem (2). Consequently, the columns of ϕ remain linearly independent on the whole interval $\mathcal{I}$ and for the solutions $y \in C_A^1 := \{y \in C|(Py) \in C^1\}$ of (1) and ϕ of (2) we have

$$\begin{aligned}
\left(\phi^T(t)A(t)y(t)\right)' &= (\phi(t)^T A(t))'P(t)y(t) + \phi(t)^T A(t)(P(t)y(t))' \\
&= \phi^T(t)B(t)P(t)y(t) - \phi^T(t)(B(t) - A(t)P'(t))y(t) \\
&= -\phi^T(t)B(t)Q(t)y(t) + \phi^T(t)A(t)P'(t)y(t) \\
&= -(\phi^T(t)A(t))'Q(t)y(t) + \phi^T(t)A(t)P'(t)y(t) \\
&= \phi^T(t)A(t)Q'(t)y(t) + \phi^T(t)A(t)P'(t)y(t) = 0.
\end{aligned}$$

Thus, $\phi^T Ay \equiv 0$ holds true.

For the other direction of the proof we assume a linear space $\mathcal{M}$ and a function ϕ with the properties (i)–(iii) to be given.

Then, $\phi = \Pi_{\star can}\phi$, and $A^T\phi$ is a solution of the differential equation

$$(A^T\phi)' - B^T \Pi_{\star can} A^{T\dagger} A^T \phi = 0.$$

Thus, rank $\phi^T A \equiv \kappa$. Consequently, $\mathcal{M}(t)$ is of dimension k for all $t \in \mathcal{I}$. For an arbitrarily but fixed time t we determine a basis $\{y^0(t), \dots, y^k(t)\}$ of $\mathcal{M}(t)$. Each vector of this basis is a consistent initial value for the DAE (1). The linear set spanned by the solutions to these initial values remains k-dimensional on the whole solution interval. Hence, $\mathcal{M}(t) = span\{y^0(t), \dots, y^k(t)\}$ for any t, i.e. $\mathcal{M} = span\{y^0, \dots, y^k\}$. $\square$

Corollary 4 *Provided that* $(I - \Pi_{can})y = 0$ *is valid, the conditions* $PP_1 y = 0$, $P_{\star 1}^T P_\star APP_1 y = 0$ *and* $\Pi_{\star can}^T APP_1 y = 0$ *are equivalent.*

Remark 5 *If* $Q_\star B \in C^1$ *and if* $Q_{\star 1}(t)$ *is any arbitrary differentiable projector function onto* $N_1(t)$*, then, for a more practicable realization, the space* $\mathcal{M}(t)$ *can easily be described according to Theorem 13 and Remark 14 as a solution of the system of equations*

$$\begin{aligned}
\phi^T(t)A(t)y(t) &= 0, \\
Q_\star(t)B(t)y(t) &= 0, \\
Q_{\star 1}^T(t)\left[P_\star(t)B(t) - (Q_\star(t)B(t))'\right]y(t) &= 0.
\end{aligned}$$

5 Application to inhomogeneous systems

As the inhomogeneous systems are more general than the homogeneous ones, we consider the inhomogeneous index 2 DAE

$$Ay' + By = f. \tag{6}$$

This DAE is homogenized by a simple trick [3], and then we can apply Theorem 2 to this special case.

$$\hat{A}\hat{y}' + \hat{B}\hat{y} = 0, \qquad \hat{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} B & -f \\ 0 & 0 \end{pmatrix}. \tag{7}$$

The vector $\hat{y}$ is by one dimension larger than y, that means $(m+1)$. The DAEs (6) and (7) are equivalent in the sense that $(y^T, 1)^T$ is a solution of (7) in case y is a solution of (6) and reversely.

In order to apply Theorem 2, the projectors and matrices have to be computed. Obviously,

$$\hat{P} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}, \hat{A}_1 = \begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix}, \hat{Q}_1 = \begin{pmatrix} Q_1 & -Q_1 G_2^{-1} f \\ 0 & 0 \end{pmatrix}, \hat{G}_2 = \begin{pmatrix} G_2 & -B_0 P Q_1 G_2^{-1} f \\ 0 & 1 \end{pmatrix}$$

holds, which implies

$$\hat{\Pi}_{can} = \begin{pmatrix} \Pi_{can} & QP_1G_2^{-1}f + (QQ_1G_2^{-1}f)' + (I - (QQ_1)')PQ_1G_2^{-1}f \\ 0 & 1 \end{pmatrix}.$$

Now, let $\widehat{\mathcal{M}}$ be a $(k+1)$-dimensional solution space of the homogenized DAE (7). By Theorem 2 there exists a function $\hat{\phi}$ with the properties mentioned in the theorem, $\hat{\kappa} = \kappa$. Let $\hat{\phi}^T = (\phi^T | -h)$ be a solution of the equation that is adjoint to the homogenized DAE (7). Then $(\phi^T | -h)^T$ satisfies the differential equations

$$\begin{aligned} A^T\phi' - (B^T - A^{T'})\phi &= 0, \\ h' - \phi^T f &= 0. \end{aligned}$$

Now $\hat{y}$ is a solution of (7) if

$$(I - \Pi_{can})\, y = QP_1G_2^{-1}f + (QQ_1G_2^{-1}f)' + (I - (QQ_1)')\, PQ_1G_2^{-1}f.$$

These conditions are now retransformed to the non-homogenized system (6),

$$\begin{aligned} (I - \Pi_{can})y &= QP_1G_2^{-1}f y_{m+1}, \\ \phi^T A y &= h y_{m+1}. \end{aligned}$$

Taking into account that $y_{m+1} \equiv 1$ has to be valid, i.e. that the space $\widehat{\mathcal{M}}_{y_{m+1}\equiv 1}$ is only k-dimensional, we obtain

Theorem 6 *$\mathcal{M}$ is a k-dimensional affine set of solutions of the equation (6) if and only if there exist a $m \times \kappa$-dimensional matrix-valued function ϕ, $\kappa = \operatorname{rank} PP_1 - k$, and a vector-valued function h on $\mathcal{I}$ such that*

(*i*) $\operatorname{rank} \phi(t) \equiv \kappa$, *(ii) for all $t \in \mathcal{I}$ we have*

$$\mathcal{M}(t)=\Big\{\xi \in \mathbb{R}^m \,\Big|\, \phi^T(t)A(t)\xi = h(t),$$
$$(I-\Pi_{can}(t))\xi = \Big(QP_1G_2^{-1}f + (QQ_1G_2^{-1}f)'$$
$$+\left(I-(QQ_1)'\right)PQ_1G_2^{-1}f\Big)(t)\Big\},$$

(iii) and ϕ, h fulfil the system of equations

$$A^T\phi' - (B^T - A^{T\prime})\phi = 0,$$
$$h' - \phi^T f = 0.$$

This has already been proved.

Remark 7 *If $Q_\star B$, $Q_\star f \in C^1$ and if $Q_{\star 1}(t)$ is any differentiable projector function onto $N_1(t)$, then, for a practicable realization, the set $\mathcal{M}(t)$ can easily be described by Theorem 13 and Remark 14 of the next section as the solution of the system of equations*

$$\phi^T(t)A(t)y(t) = h(t),$$
$$Q_\star(t)B(t)y(t) = Q_\star(t)f(t),$$
$$Q_{\star 1}^T(t)\left[P_\star(t)B(t) - (Q_\star(t)B(t))'\right]y(t) = Q_{\star 1}^T(t)\left[P_\star(t)f(t) - (Q_\star(t)f(t))'\right].$$

6 Representation of $\operatorname{im}\Pi_{can}$ by a reduction step

The fact that $A_{\star 1}^T$ is the leading coefficient matrix after a reduction of the equation (1) permits to prove the following reduction theorem

Theorem 8 *Let $Q_\star B$ and $Q_{\star 1}^T[P_\star B - (Q_\star B)']$ be differentiable. Then the DAE $Ay' + By = 0$ is equivalent to the implicit regular ordinary differential equation*

$$(A - Q_\star B - Q_{\star 1}^T[P_\star B - (Q_\star B)'])y'$$
$$+\left(P_{\star 1}^T(P_\star B - (Q_\star B)') - \left(Q_{\star 1}^T[P_\star B - (Q_\star B)']\right)'\right)y = 0 \quad (8)$$

with the additional conditions

$$Q_{\star 1}^T(\tilde t)\left[P_\star(\tilde t)B(\tilde t) - (Q_\star(\tilde t)B(\tilde t))'\right]y(\tilde t) = 0, \quad \tilde t \in \mathcal{I}, \qquad (9a)$$
$$Q_\star(\hat t)B(\hat t)y(\hat t) = 0, \quad \hat t \in \mathcal{I}. \qquad (9b)$$

Proof. By Theorem 1 the DAE (2)

$$A^T\phi' - \left(B^T - A^{T\prime}\right)\phi = 0$$

has index 2, too.

Multiplying the DAE $Ay' + By = 0$ by $Q_\star$ from the left yields

$$Q_\star By = Q_\star B_0 y = 0,$$

hence,

$$\begin{aligned} \underbrace{(A - Q_\star B)}_{A_{\star 1}^T} y' + (P_\star B - (Q_\star B)')\, y &= 0, \\ Q_\star(\hat{t})B(\hat{t})y(\hat{t}) &= 0, \end{aligned} \tag{10}$$

for each $\hat{t} \in \mathcal{I}$. The other direction of the equivalence to equation (1) becomes clear when multiplying the DAE (10) by $Q_\star$ from the left

$$(Q_\star By)' = Q_\star' Q_\star By.$$

Due to the condition (9b) at time $\hat{t}$ it holds that $Q_\star By \equiv 0$ and we obtain the given DAE [5, A. B, Corollary 7].

The leading matrix in equation (10) is $A_{\star 1}^T$. Multiplication by $Q_{\star 1}^T$ provides equation (9a) and, finally, the equivalent system

$$\begin{aligned} (A - Q_\star B - Q_{\star 1}^T\, [P_\star B - (Q_\star B)'])y' & \\ + \left(P_{\star 1}^T\, (P_\star B - (Q_\star B)') - \left(Q_{\star 1}^T\, [P_\star B - (Q_\star B)']\right)'\right) y &= 0, \\ Q_{\star 1}^T(\tilde{t}) \left[P_\star B(\tilde{t}) - (Q_\star(\tilde{t})B(\tilde{t}))'\right] y(\tilde{t}) &= 0, \\ Q_\star(\hat{t})B(\hat{t})y(\hat{t}) &= 0, \end{aligned}$$

$\tilde{t} \in \mathcal{I}$. Equivalence can be proved analogously to the above.

It remains to show that $\hat{G}_{\star 2} = A^T - B^T Q_\star - \left(B^T P_\star - (B^T Q_\star)'\right) Q_{\star 1}$ is nonsingular.

$$\begin{aligned} \hat{G}_{\star 2} &= A^T - B^T Q_\star - \left(B^T P_\star - (B^T Q_\star)'\right) Q_{\star 1} \\ &= G_{\star 2} + (B^T Q_\star)' Q_{\star 1} - A^{T\prime} P_\star Q_{\star 1} \\ &= G_{\star 2} - A_{\star 1}' Q_{\star 1} + A^T P_\star' Q_{\star 1} \\ &= G_{\star 2} + A_{\star 1} Q_{\star 1}' Q_{\star 1} + A^T P_\star' Q_{\star 1} \\ &= G_{\star 2}\, (I + P_{\star 1}(Q_{\star 1}' + P_\star P_\star')Q_{\star 1})\,. \end{aligned}$$

The two matrices of the product above are nonsingular, hence $\hat{G}_{\star 2}$ is so, too. The differential equation (8) is an implicitly given regular differential equation. □

Corollary 9 *Under analogous smoothness assumptions the DAE* $A^T\phi' - \left(B^T - A^{T\prime}\right)\phi = 0$ *is equivalent to the implicitly given ODE*

$$\begin{aligned} &\left(A^T + QB_0^T + Q_1^T \left[P\left(B^T - A^{T\prime}\right) - (QB_0^T)'\right]\right)\phi' \\ &- \left(P_1^T \left(P\left(B^T - A^{T\prime}\right) - (QB_0^T)'\right) - \left(Q_1^T\left[P\left(B^T - A^{T\prime}\right) - (QB_0^T)'\right]\right)'\right)\phi = 0 \end{aligned} \tag{11}$$

with the conditions

$$Q_1^T(\tilde{t})\left[P(\tilde{t})\left(B^T(\tilde{t})-A^{T\prime}(\tilde{t})\right)-\left(Q(\tilde{t})B_0^T(\tilde{t})\right)'\right]\phi(\tilde{t})=0,\quad \tilde{t}\in\mathcal{I},$$
$$Q(\hat{t})\left(B^T(\hat{t})-A^{T\prime}(\hat{t})\right)\phi(\hat{t})=0,\quad \hat{t}\in\mathcal{I}.$$

Proof. It holds that $(B-A')Q=B_0Q$. The other steps of the proof are the same as in Theorem 8. □

Remark 10 *The equations (9a) and (9b) can also be regarded as additional equations instead of the initial conditions. They determine the manifold where the solutions of (8) belong to. For determining the manifold the smoothness assumption can be weakened, i.e. $Q_\star B\in C^1$ is sufficient.*

Lemma 11 *The conditions*

$$PP_1y_0=PP_1y^0 \tag{13a}$$
$$Q_{\star1}^T\left[P_\star B-(Q_\star B)'\right]y_0=0 \tag{13b}$$
$$Q_\star By_0=0 \tag{13c}$$

make up a complete set of initial values for the differential equation (8). This includes the fact that these equations uniquely determine y_0 and do not contradict each other. Furthermore, they provide a consistent initial value for equation (1), i.e. $y_0=\Pi_{can}y^0$. (All functions and matrices are assumed at a certain time t.)

Remark 12 *An analogous statement applies to the adjoint problem (2) and the conditions*

$$P_\star P_{\star1}\phi_0=P_\star P_{\star1}\phi^0,$$
$$Q_1^T\left[P\left(B^T-A^T\right)-\left(QB_0^T\right)'\right]\phi_0=0,$$
$$Q\left(B^T-A^{T\prime}\right)\phi_0=0.$$

Proof. From $Q_\star By_0=0$ it follows that $By_0\in\operatorname{im}P_\star=\operatorname{im}AA^\dagger$, and, consequently, $Q_1y_0=0$. We consider the second equation (13b) in the formulation of the lemma.

$$\begin{array}{rrcll}
& Q_{\star1}^T\left[P_\star B-(Q_\star B)'\right]y_0 & = & 0 & \\
\Leftrightarrow & \left[P_\star B-(Q_\star B)'\right]y_0 & \in & \operatorname{im}P_{\star1}^T=\operatorname{im}A_{\star1}^TG_{\star2}^{-1T}=\operatorname{im}A_{\star1}^T & \\
\overset{(13c)}{\Leftrightarrow} & P_\star By_0-(Q_\star B)'y_0 & = & Az-Q_\star Bz, & \text{for a } z\in\mathbb{R}^m \\
\Leftrightarrow & \left\{\begin{array}{r} P_\star By_0 \\ Q_\star(Q_\star B)'y_0 \end{array}\right. & \begin{array}{c} = \\ = \end{array} & \begin{array}{l} Az \\ Q_\star Bz \end{array} & \\
\overset{(13c)}{\Leftrightarrow} & \left\{\begin{array}{r} By_0 \\ (Q_\star B)'y_0 \end{array}\right. & \begin{array}{c} = \\ = \end{array} & \begin{array}{l} Az \\ Q_\star Bz \end{array} &
\end{array}$$

By the relation $G_2^{-1}BQ = Q + P_1PP'Q$, which can be easily derived, we now conclude the relation $Qy_0 + P_1PP'Qy_0 + G_2^{-1}BPy_0 = P_1Pz$ from $By_0 = Az$. Multiplying this equation by QP_1 from the left yields, after some technical transformations,

$$Qy_0 - (QQ_1)'Qy_0 + QP_1G_2^{-1}BPy_0 = -QQ_1z.$$

Let us have a look at the right-hand side of this equation

$$\begin{aligned} -QQ_1z = -QQ_1G_2^{-1}B_0z = -QQ_1G_2^{-1}Q_\star Bz &= -QQ_1G_2^{-1}(Q_\star B)'y_0 \\ &= -(QQ_1G_2^{-1}Q_\star B)'y_0 = -(QQ_1)'y_0. \end{aligned}$$

Inserting into the equation provides $Qy_0 = -QP_1G_2^{-1}BPy_0 - (QQ_1)'Py_0$, hence altogether

$$y_0 = \Pi_{can}y_0 = \Pi_{can}PP_1y_0 = \Pi_{can}y^0.$$

Analogously it is shown that also the equations for the adjoint problem have full rank. Now it remains to show that the equations (13a)-(13c) do not contradict each other. It is sufficient to show that $\operatorname{rank} PP_1 + \operatorname{rank} Q_{\star1} + \operatorname{rank} Q_\star = m$. This sum has at least the value m. Supposed it is larger, then, because of $\operatorname{rank} P_\star P_{\star1} + \operatorname{rank} Q_{\star1} + \operatorname{rank} Q_\star = m$, it holds that $\operatorname{rank} PP_1 > \operatorname{rank} P_\star P_{\star1}$. Since now $\operatorname{rank} Q = m - \operatorname{rank} A = \operatorname{rank} Q_\star$, it follows that $\operatorname{rank} Q_1 < \operatorname{rank} Q_{\star1}$. Obviously, $\operatorname{rank} P_\star P_{\star1} + \operatorname{rank} Q_1 + \operatorname{rank} Q = \operatorname{rank} P_\star P_{\star1} + \operatorname{rank} Q_1 + \operatorname{rank} Q_\star \geq m$, however. This contradicts $\operatorname{rank} P_\star P_{\star1} + \operatorname{rank} Q_{\star1} + \operatorname{rank} Q_\star = m$ and $\operatorname{rank} Q_1 < \operatorname{rank} Q_{\star1}$. □

Let

$$\hat{S}_1(t) := \left\{\xi \in \mathbb{R}^m | \xi \in \ker Q_\star(t)B(t) \cap \ker\left(\hat{Q}_{\star1}^T(t)(P_\star(t)B(t) - (Q_\star(t)B(t))'\right)\right\}.$$

Theorem 13 $\hat{S}_1 = \operatorname{im}\Pi_{can}$.

Proof. $\hat{S}_1 \subseteq \operatorname{im}\Pi_{can}$ follows immediately from the proof of Lemma 11.

On the other hand, Lemma 11 and the relation (4) imply $\hat{S}_1 + \ker\Pi_{can} = \mathbb{R}^m$, hence $\dim\hat{S}_1 \geq m - \dim\ker\Pi_{can}$. We have $\operatorname{im}\Pi_{can} \oplus \ker\Pi_{can} = R^m$, $\dim\operatorname{im}\Pi_{can} + \dim\ker\Pi_{can} = m$, and thus $\dim\operatorname{im}\Pi_{can} = m - \dim\ker\Pi_{can}$. The dimensions of $\hat{S}_1$ and $\operatorname{im}\Pi_{can}$ coincide. □

Consequently, the space of all solutions of the homogeneous index 2 DAE (1) can be represented as the image space of the projector Π_{can} resp. the kernel of the projector $(I - \Pi_{can})$, or as the solution set of a system of equations and as the kernel of a matrix, respectively.

Remark 14 *Let $\hat{Q}_{\star1}(t)$ be any other differentiable projector function $Q_{\star1}(t)$ onto $N_{\star1}(t) = \ker A_{\star1}(t)$. Because of $\hat{Q}_{\star1}(t)Q_{\star1}(t) = Q_{\star1}(t)$ and $Q_{\star1}(t)\hat{Q}_{\star1}(t) = \hat{Q}_{\star1}(t)$ the statements of Lemma 11 and Theorem 13 hold with $\hat{Q}_{\star1}(t)$, too.*

Remark 15 *The above process corresponds to the index reduction considered in [4, Theorem 13].*

A Linear Algebra

Definition 16 *The matrix A has index k if k is the smallest natural number for which* $\operatorname{im} C^k = \operatorname{im} C^{k+1}$ *holds true.*

Lemma 17 *For $C \in L(\mathbb{R}^m)$ it holds that* $\operatorname{im} C^k = \operatorname{im} C^{k+1} \Leftrightarrow \ker C^k = \ker C^{k+1}$.

Definition 18 *The matrix pencil (A, B) is called regular if*
$p(c) := \det(cA + B) \not\equiv 0$.

Definition 19 *The index of a matrix pencil (A, B) is the index of the matrix $(cA + B)^{-1}A$.*

Remark 20 *This definition also implies that*

$$\begin{aligned}
\operatorname{ind}(A, B) &:= \operatorname{ind}\left((cA + B)^{-1}A\right) \\
&= \min\left\{k \,\middle|\, \ker\left((cA + B)^{-1}A\right)^k = \ker\left((cA + B)^{-1}A\right)^{k+1}\right\} \\
&\overset{\text{L17}}{=} \min\left\{k \,\middle|\, \operatorname{im}\left((cA + B)^{-1}A\right)^k = \operatorname{im}\left((cA + B)^{-1}A\right)^{k+1}\right\}.
\end{aligned}$$

Lemma 21 *Let X, Y, $M \in L(\mathbb{R}^m)$, and let M be nonsingular. Then*
$\operatorname{im} MX = \operatorname{im} MY \Leftrightarrow \operatorname{im} X = \operatorname{im} Y \Leftrightarrow \operatorname{im} XM^{-1} = \operatorname{im} YM^{-1}$.

The uniquely determined Moore-Penrose-inverse $C^\dagger$ of a matrix $C \in L(\mathbb{R}^n, \mathbb{R}^m)$ is defined by its properties:

$$CC^\dagger C = C, \quad C^\dagger CC^\dagger = C^\dagger, \quad CC^\dagger = \left(CC^\dagger\right)^T, \quad C^\dagger C = \left(C^\dagger C\right)^T.$$

We can easily reconstruct that $\left(C^T\right)^\dagger = \left(C^\dagger\right)^T$ holds. $CC^\dagger$ is a projector onto $\operatorname{im} C$ along $(\operatorname{im} C)^\perp$, $C^\dagger C$ is a projector onto $(\ker C)^\perp$ along $\ker C$.

Lemma 22 *For $C \in L(\mathbb{R}^m)$ we have* $(\operatorname{im} C^T)^\perp = \ker C$.

Lemma 23 *For regular matrix pencils (A, B), A, $B \in L(\mathbb{R}^m)$ we have*
$\operatorname{ind}(A, B) = \operatorname{ind}(A^T, B^T)$.

Proof.

$$
\begin{aligned}
& \ker\left[\left(\left(cA^T+B^T\right)^{-1}A^T\right)^k\right] &=\;& \ker\left[\left(\left(cA^T+B^T\right)^{-1}A^T\right)^{k+1}\right] \\
\overset{L17}{\Leftrightarrow}\;& \operatorname{im}\left[\left(\left(cA^T+B^T\right)^{-1}A^T\right)^k\right] &=\;& \operatorname{im}\left[\left(\left(cA^T+B^T\right)^{-1}A^T\right)^{k+1}\right] \\
\overset{L21}{\Leftrightarrow}\;& \operatorname{im}\left[\left(A^T\left(cA^T+B^T\right)^{-1}\right)^k\right] &=\;& \operatorname{im}\left[\left(A^T\left(cA^T+B^T\right)^{-1}\right)^{k+1}\right] \\
\Leftrightarrow\;& \operatorname{im}\left[\left(A^T\left(cA^T+B^T\right)^{-1}\right)^k\right]^\perp &=\;& \operatorname{im}\left[\left(A^T\left(cA^T+B^T\right)^{-1}\right)^{k+1}\right]^\perp \\
\Leftrightarrow\;& \operatorname{im}\left[\left(\left((cA+B)^{-1}A\right)^k\right)^T\right]^\perp &=\;& \operatorname{im}\left[\left(\left(A^T(cA+B)^{-1}\right)^{k+1}\right)^T\right]^\perp \\
\overset{L22}{\Leftrightarrow}\;& \ker\left[\left((cA+B)^{-1}A\right)^k\right] &=\;& \ker\left[\left(A^T(cA+B)^{-1}\right)^{k+1}\right]
\end{aligned}
$$

Thus, $\operatorname{ind}(A^T,B^T)=k \Leftrightarrow \operatorname{ind}(A^T,B^T)=k$. $\square$

Theorem 24 *[5, A. A, Theorem 16] If* $\operatorname{ind}(A,B)=2$, *then* $(A, B+ASA)$ *is regular with index 2 for each matrix* $S\in L(\mathbb{R}^m)$.

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