Properties of an interior embedding for solving nonlinear optimization problems.

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Abstract

The paper presents a sufficient condition for the success of path-following algorithms with jumps when applied to one-parametric optimization problems. An interior embedding that always fulfils the mentioned sufficient condition is given.

Finally, the assumption of regularity in the sense of Jongen, Jonker and Twilt is analysed for the presented embedding, and its genericity is proved, provided that it is formulated on the original data of the optimization problem used for the construction of the introduced interior embedding.
Keywords: Parametric optimization, singularities, jumps, regularity

1 Introduction

Optimization problems with the following general form are considered:

$$(P) \quad \min \ \{f(x) \mid x \in M\},$$

where

$$M = \{x \in I\!R^n \mid h_i(x) = 0, i \in I, g_j(x) \geq 0, j \in J\},$$

and the index sets I and J are finite. We need different heights of differentiablility for the data functions f, g_j, $j \in J$, but it will always be sufficient to consider the functions to be three times continuously differentiable.

We will consider also optimization problems depending on a real parameter with the following form:

$$P(t) \quad \min \ \{f(x,t) \mid x \in M(t)\},$$

where

$$M(t) = \{x \in I\!R^n \mid h_i(x,t) = 0, i \in I, g_j(x,t) \geq 0, j \in J\}.$$

Here it also sufficies to suppose that $f, g_j \in C^3(I\!R^{n+1}, I\!R)$, $j \in J$.

The one-parametric optimization problems are a natural context to study the concept of embedding. For us an embedding will be a one-parametric optimization problem that connects an original problem, to be solved, with a simple one (with computable solution). For example a one parametric problem such that for the value $t = 0$ the obtained problem is trivial and and for $t = 1$ the obtained problem is the original problem to be solved. Other properties are usually supposed over the one-parametric problems, when they are interpreted as embeddings. These properties are mainly related with the use of pathfollowing techniques for solving it. The idea of pathfollowing is not new. It is mainly based on the numerical resolution of equality systems depending on one parameter (see for example [1]).

It should be mentioned that an embedding of an optimization problem can be also interpreted in other ways. For example, only as a parametric system describing some critical set of the fixed optimization problem in a fixed point of the parameter. This parametric system is not alwys associated to an optimization problem for each value of the parameter. An interesting approach in this sense, with a parametrization of the Kojima system, can be found in [15]. In this case the parametrized system is also not differentiable.

In this paper we use embeddings that are interpreted as one-parametric optimization problems. Examples of embedings representing different methods of the nonlinear programming are studied in the papers [2, 3, 4, 6].

There are two theoretical conditions ensuring that the set of generalized critical points of a one-parametric problem has a structure that is feasible for the use of pathfollowing methods. These conditions are called the regularity condition of a one-parametric problem in the sense of Jonger, Jonker and Twilt (see [12]) or in the sense of Kojima and Hirabayashi (see [14]). We use the regularity in the sense on Jongen, Jonker and Twilt (shortly JJT-regularity).

The success of a pathfollowing method (arriving $t = 1$) when being applied to a one-parametric problem is not ensured even in the case of regularity. Another type of conditions are needed to ensure that. Usually the Mangasarian Fromowitz constraint qualification is assumed to be fulfilled at each point of the parameter-dependent feasible set $M(t)$, and for each value of the parameter t. This condition eliminates the one-parametric problems, which contain singularities where it is imposible to jump to another connected component of the set of generalized critical points. The MFCQ assumption excluds also cases where a pathfollowing procedures can be succesfull.(see [6]).

We introduce here another sufficient condition, but for the success of pathfollowing methods with jumps. This condition (Condition B) allows the existence of singularities without possibilities of jump, but control in which position they appear. This is a simpler form to avoid the appearance of points during the numerical continuation, where the pathfollowing algorithm cannot continue. We can always continue a curve of critical points or jump to another connected component of this set, if we continue the curves with increasing parameter t. It must be mentioned that the consequence of the introduced Condition B for the position of the singularities without jumps can be concluded from the results presented in the textbook [5] (see chapters 6 and 7, written by J-J. Rückmann).

We present an example of an embedding which always fulfils the introduced Condition B. This embedding is constructed from a fixed problem of the form (P). The generalized critical points of this embedding lie in the interior of this fixed problem. That is the reason why we call it an interior embedding. It should be noted that the curves followed in this embedding are in general different from the central path or other curves followed in the interior point methods. In the interior point algorithms embeddings that are not related to a one-parametric differentiable optimization problem are considered.

For theoretical as well as for practical points of view the assumption of JJT-regularity of an embedding or a one-parametric problem is important.

For a general one-parametric problem this regularity is a generic condition. The JJT-regularity is supposed also in other papers for other embeddings. All these embeddings (including the embedding presented in this paper) are constructed in a fixed form taking data from an optimization problem. The set of all one-parametric problems obtained using a fixed of these constructions is always subset of the space $C^3(I\!\!R^{n+1}, I\!\!R)$, where the class $\mathcal{F}$ is defined and open and dense. Naturally, there arises the question, how reasonable is to suppose that the obtained problem is JJT-regular. With respect to this problem one usually presents a way to perturb each problem obtained with the specific embedding in order to get a regular problem (see [2, 3, 4, 6]). The density of the set of regular problems ensures for each one-parametric problem the existence of a regular problem as close as wanted. The perturbation theorem presented in the paper [16] provides an explicit way to perturb an arbitrary one-parametric problem in order to get a regular one. The perturbation theorems presented for other embeddings utilize, based on the ideas of [16], the special properties of each embedding and try to construct a simpler way to perturb.

We want to analyse the problem of the regularity of the constructed embedding in another form. We want to rewrite the regularity assumption in terms of the data of the original problem, from which the one-parametric problem is constructed. For our interior embedding the regularity of the constructed one-parametric problem results a generic assumption.

The paper is organized as follows. In the Section 2 some basic results are presented for optimization problems and for one-parametric problems. Moreover, relations between the impossibility of jump and properties of the singularities of the class of Jonger, Jonker and Twilt are presented. Section 3 provides the principal result about the Condition B. In the Section 4, we introduce the interior embedding and prove the genericity of the regularity assumption. Finally, some concluding remarks are given in the Section 5.

2 Preliminary notions

Let us consider a constrained optimization problem of the following type:

$$(P) \quad \min \ \{f(x) \mid x \in M\} \, ,$$

where

$$M = \{x \in \mathbb{R}^n \mid h_i(x) = 0, i \in I, g_j(x) \geq 0, j \in J\}.$$

For the moment it is sufficient to suppose that the appearing functions f, h_i, $i \in I = \{1, \ldots, m\}$ and $g_j, j \in J = \{1, \ldots, p\}$ belong to the class $C^2(\mathbb{R}^n, \mathbb{R})$.

Define the active index set $J_0(x)$ as

$$J_0(x) = \{j \in J \mid g_j(x) = 0\}.$$

In our investigation two constraints qualifications will play an important role. Let us recall them.

Definition 1
Let $\bar{x} \in M$

LICQ *The linear independence constraint qualification is said to hold at the point x if the vectors $\{Dh_i(\bar{x}), i \in I, Dg_j(\bar{x}), j \in J_0(\bar{x})\}$ are linearly independent.*

MFCQ *The Mangasarian Fromowitz constraint qualification is said to hold at $\bar{x}$ if the following is satisfied:*

MF1 *The vectors $\{Dh_i(\bar{x}), i \in I\}$ are linearly independent.*

MF2 *There exists a vector $\xi \in \mathbb{R}^n$ such that:*

$$\begin{aligned} Dh_i(\bar{x})\xi &= 0, \; i \in I, \\ Dg_j(\bar{x})\xi &> 0, \; j \in J_0(\bar{x}). \end{aligned}$$

Here and throughout the paper D_y^k denotes the partial derivatives of order k with respect to the variables y.

Let us now recall some well-known notions. First, the notion of generalized critical points.

Definition 2 *(see [12] or [7])*
A point $\bar{x} \in M$ is called a generalized critical point (shortly g. c. point) of the problem (P) if the vectors $\{Df(\bar{x}), Dh_i(\bar{x}),\; i \in I, Dg_j(\bar{x}),\; j \in J_0(\bar{x})\}$ are linearly dependent.

Thus, if $\bar{x} \in M$ is a g.c.point of (P), then there exist u_0, λ_i, $i \in I$ and u_j, $j \in J_0(\bar{x})$ such that

$$\begin{aligned} u_0 Df(\bar{x}) - \sum_{i \in I} \lambda_i Dh_i(\bar{x}) - \sum_{j \in J_0(\bar{x})} u_j Dg_j(\bar{x}) &= 0, \\ |u_0| + \sum_{i \in I} |\lambda_i| + \sum_{j \in J_0(\bar{x})} |u_j| &> 0. \end{aligned} \tag{1}$$

If the constraint qualification LICQ is satisfied at a g.c. point $\bar{x} \in M$, then the number u_0 must be different from zero in the relations (1), and we obtain, in particular, the existence of uniquely determined λ_i, $i \in I$ and u_j, $j \in J_0(\bar{x})$ such that

$$Df(\bar{x}) - \sum_{i \in I} \lambda_i Dh_i(\bar{x}) - \sum_{j \in J_0(\bar{x})} u_j Dg_j(\bar{x}) = 0 \tag{2}$$

A g.c. point $\bar{x} \in M$ is called a stationary point of the problem (P) if there exist λ_i, $i \in I$ and $u_j \geq 0$, $j \in J_0(\bar{x})$ such that the relation (2) is satisfied.

It is well-known that the validity of the constraint qualifications LICQ or MFCQ at a local minimizer $\bar{x}$ of the problem (P) implies that $\bar{x}$ is a stationary point.

Definition 3 *(see [12] or [7])*
Let $\bar{x} \in M$ be a g.c. point of (P). It is called nondegenerated if the following conditions are fulfilled:

(ND1) *LICQ holds at $\bar{x}$. Then, there exist uniquely determined numbers $\bar{\lambda}_i$, $i \in I$ and $\bar{u}_j$, $j \in J_0(\bar{x})$ such that relation (2) holds.*

(ND2) $\bar{u}_j \neq 0$, $j \in J_0(\bar{x})$.

(ND3) $D^2 L(\bar{x})|_{T_{\bar{x}}M}$ *is non-singular.*

In the preceding definition $D^2 L(\bar{x})$ is the Hessian for the Lagrangian function defined as

$$L(x) = f(x) - \sum_{i \in I} \bar{\lambda}_i h_i(x) - \sum_{j \in J_0(\bar{x})} \bar{u}_j g_j(x)$$

and $T_{\bar{x}}M$ represents the tangential space to the set M at the point $\bar{x}$. This space the following description:

$$T_{\bar{x}}M = \{\xi \in I\!R^n \mid Dh_i(\bar{x})\xi = 0,\ i \in I,\ Dg_j(\bar{x})\xi = 0,\ j \in J_0(\bar{x})\}$$

The notation $D^2L(\bar{x})|_{T_{\bar{x}}M}$ means some matrix of the form $V^T D^2 L(\bar{x}) V$, where V is a matrix whose columns form a basis for $T_{\bar{x}}M$.

2.1 One-parametric optimization problem

In this section we deal with one-parametric optimization problems $(P(t))$ of the form:

$$(P(t)) \quad \min \ \{f(x,t) \mid x \in M(t)\},$$

where

$$M(t) = \{x \in I\!R^n \mid g_j(x,t) \geq 0,\ j \in J\}$$

Define the sets

$$\begin{aligned} \Sigma_{gc} &= \{(x,t) \in I\!R^n \times I\!R \mid x \text{ is a g.c. point of the problem } P(t)\} \\ \Sigma_{stat} &= \{(x,t) \in I\!R^n \times I\!R \mid x \text{ is a stationary point of the problem } P(t)\} \end{aligned}$$

Let us introduce the notation $z = (x,t)$. When $\bar{z} \in \Sigma_{gc}$, we say that $\bar{z}$ is a generalized critical point of the one-parametric problem $(P(t))$. Analogously we call a point $\bar{z} \in \Sigma_{stat}$ a stationary point of $(P(t))$. The activity set $J_0(\bar{z})$ is intended as the activity set of the problem $P(\bar{t})$ at the point $\bar{x}$. Finally, if the constraint qualification LICQ (or MFCQ) is satisfied at the point $\bar{x}$ belonging to the set $M(\bar{t})$ we say that LICQ (or MFCQ) holds at the point $\bar{z} = (\bar{x}, \bar{t})$.

In [12], 5 types of generalized critical points were defined. We give a short idea of the definition of these 5 types for the introduced problem $(P(t))$.

<u>Type 1</u>

$\bar{z} \in \Sigma_{gc}$ is of Type 1 if $\bar{x}$ is a nondegenerated critical point of the problem $P(\bar{t})$.

The points of the Types 2-5 represent three degenerations of Type 1.

Type 2 The violation of condition (ND2).

Type 3 The violation of condition (ND3).

Type 4 The violation of condition (ND1), but $|J_0(\bar{z})| \le n$.

Type 5 The violation of condition (ND1), with $|J_0(\bar{z})| = n+1$.

The structure of Σ_{gc} in a neighbourhood of generalized critical points of Type 1-5 is completely described in [12] (see also [7]). So, the local structure of the set Σ_{gc} is known for one-parametric problems as $(P(t))$ described by the functions $f, g_1, \ldots, g_p$, when they belong to the class $\mathcal{F}$ defined, in the same paper, as

$$\mathcal{F} = \left\{ (f, g_1, \ldots, g_p) \in C^3(\mathbb{R}^{n+1}, \mathbb{R}^{p+1}) \;\middle|\; \begin{array}{c} \text{each point of } \Sigma_{gc} \text{ belongs} \\ \text{to one of the Types } 1,2,3,4,5 \end{array} \right\}$$

In [12] it is also shown that $\mathcal{F}$ is a C^3_s-open and dense subset of the space $C^3(\mathbb{R}^{n+1}, \mathbb{R}^{p+1})$ endowed with the strong (or Whitney-) C^3_s topology. (c.f. e.g. [9] or [13] for the definition of this topology)

If the one-parametric problem $(P(t))$ belongs to $\mathcal{F}$, then, the corresponding set Σ_{gc} has a suitable structure for the use of pathfollowing methods. Pathfollowing methods are the main tools for solving one-parametric optimization problems. However, they are not successful in every cases. Sometimes it may be necessary to jump from one connected component of Σ_{gc} or Σ_{stat} to another one (see [7] and [8]).

In [8] feasible descent directions (-jumps) at bifurcation and turning points of the set Σ_{gc} were constructed under the assumption that $(P(t))$ belongs to $\mathcal{F}$. It is possible to construct jumps (if necessary) in all cases of Types 2 and 3. For some cases of Type 4 and 5 we have no jump.

We want to use the results of this investigation about the existence of jumps, so we must define the generalized critical points of Typ 4 and 5 in detail.

Let $\bar{z} = (\bar{x}, \bar{t})$ be a g.c. point of Typ 4 or 5. We have mentioned that LICQ fails to hold at $\bar{z}$ (violation of (ND1)), then there exist $\bar{u}_j, j \in J_0(\bar{z})$ such that

$$\begin{aligned} \sum_{j \in J_0(\bar{z})} \bar{u}_j Dg_j(\bar{z}) &= 0 \\ \sum_{j \in J_0(\bar{z})} |\bar{u}_j| &> 0 \end{aligned} \tag{3}$$

From the well-known alternative theorems of Farkas type (e.g. [10]) it follows that MFCQ fails to hold at $\bar{z}$ if and only if there exists a solution of the system (3) with $\bar{u}_j \geq 0$, $j \in J_0(\bar{z})$.

Let us suppose for simplicity of notation that, at the g.c. point $\bar{z} \in \Sigma_{gc}$, the activity set $J_0(\bar{z})$ has the form $\{1, \ldots, \bar{p}\}$, with $\bar{p} \leq p$.

Type 4

$\bar{z}$ is of Type 4 if the following conditions are fulfilled.

(4.1) $0 < |J_0(\bar{z})| = \bar{p} \leq n$ and it holds

$$rank \begin{pmatrix} D_x g_1(\bar{z}) \\ \vdots \\ D_x g_{\bar{p}}(\bar{z}) \end{pmatrix} = |J_0(\bar{z})| - 1$$

(4.2) Let $(\bar{u}_1, \ldots, \bar{u}_{\bar{p}})$ be an arbitrary but fixed solution of the system (3). It holds that $\bar{u}_j \neq 0$ for each $j \in J_0(\bar{z})$.

(4.3) The point $(\bar{x}, \bar{u}_1, \ldots, \bar{u}_{\bar{p}-1}, \bar{t}, 0) \in I\!R^{n+\bar{p}+1}$ is a nondegenerated g.c. point of the following problem:

$$(\hat{P}) \quad \min \left\{ \hat{F}(x, u_1, \ldots, u_{\bar{p}-1}, t, u_0) \mid \Upsilon(x, u_1, \ldots, u_{\bar{p}-1}, t, u_0) = 0 \right\},$$

where $\hat{F}(x, u_1, \ldots, u_{\bar{p}-1}, t, u_0) = t$,

$$\Upsilon(x, u_1, \ldots, u_{\bar{p}-1}, t, u_0) = \begin{bmatrix} D_x \mathcal{L}(x, u_1, \ldots, u_{\bar{p}-1}, t, u_0) \\ g_1(x,t) \\ \vdots \\ g_{\bar{p}}(x,t) \end{bmatrix}$$

and $\mathcal{L}(x, u_1, \ldots, u_{\bar{p}-1}, t, u_0) = u_0 f(z) - \sum_{j=1}^{\bar{p}-1} u_j g_j(z) - u_{\bar{p}} g_{\bar{p}}(z)$.

Type 5

$\bar{z}$ is of Type 5 if the following conditions hold:

(5.1) $|J_0(\bar{z})| = \bar{p} = n + 1$ and

$$rank \begin{pmatrix} D g_1(\bar{z}) \\ \vdots \\ D g_{\bar{p}}(\bar{z}) \end{pmatrix} = n + 1$$

(Note that the derivatives are taken also with respect to t).

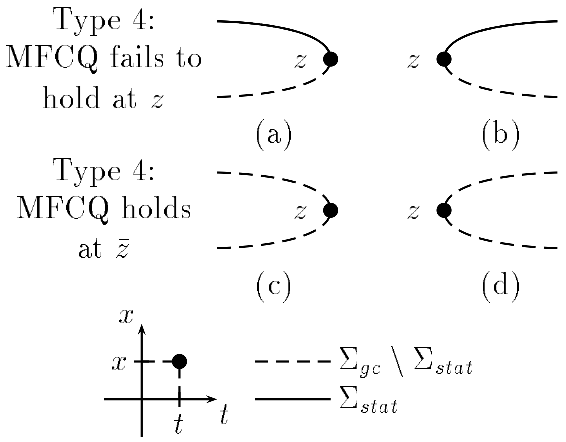


Figure 1: Point of Typ 4

(5.2) If $(\bar{u}_1, \ldots, \bar{u}_{\bar{p}})$ is a solution of the system (3), then $\bar{u}_j \neq 0$, for each $j \in J_0(\bar{z})$.

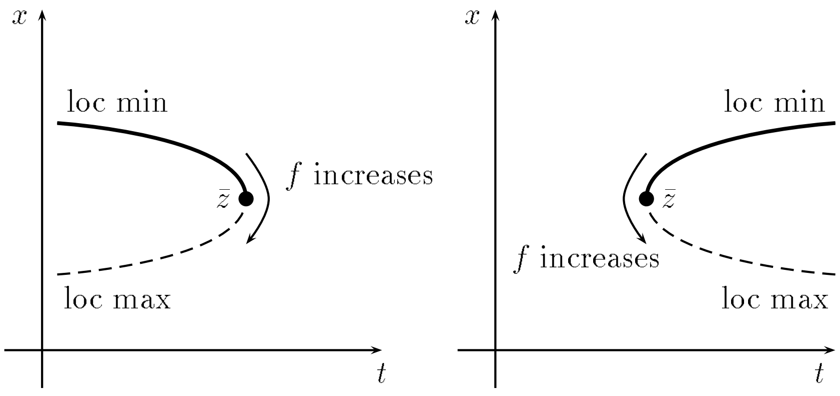
(5.3) For each vector $(\tilde{u}_1, \ldots, \tilde{u}_{\bar{p}}) \in I\!R^{\bar{p}}$ with

$$D_x f(\bar{z}) - \sum_{j=1}^{\bar{p}} \tilde{u}_j D_x g_j(\bar{z}) = 0$$

it holds that $|\{j \in J_0(\bar{z}) \mid \tilde{u}_j = 0\}| \leq 1$, where $|.|$ stands for the cardinal of this set.

According to the investigation in [12] the structure of Σ_{gc} and Σ_{stat} in a neighbourhood of a point of Type 4 is as shown in Figure 1 (for one-parametric optimization problems without equality constraints.)

It is known that the objective function $f(x,t)$ restricted to the set Σ_{gc} in a neighbourhood of a g.c. point $\bar{z}$ of Type 4 is strictly monotone at the point $\bar{z}$. If $\bar{z}$ is also an endpoint of a branch consisting of local minimizers, then Σ_{gc} consist locally of a branch of local minimizers and a branch of local maximizers. From the results of [8] we obtain that there are no possibilities of jumping at $\bar{z} \in \Sigma_{gc}$ of typ 4 if the following condition is fulfilled



(a) (b)

Figure 2: Points of Type 4 without jump.

Condition A

$\exists V_{\bar{z}} \subset I\!R^{n+1}$ such that $\forall (x_1, t_1) \in \Sigma_{stat} \cup V_{\bar{z}}$ and $\forall (x_2, t_2) \in (\Sigma_{gc} \setminus \Sigma_{stat}) \cup V_{\bar{z}}$ hold $f(x_1, t_1) < f(x_2, t_2)$.

The local structure of Σ_{gc} at the points of Type 4 without possibilities of jumping is shown in Figure 2.

The local structure of Σ_{gc} and Σ_{stat} in the neighbourhood of a point of Type 5 is shown in Figure 3.

If $\bar{z} \in \Sigma_{gc}$ is of Type 5 and the MFCQ is fulfilled at $\bar{z}$, then there exists a continuation for Σ_{gc} as well as for Σ_{stat}. If the MFCQ is not fulfilled at a g.c. point $\bar{z}$ of Type 5, then there exists neither a continuation nor possibilities to jump to another connected component. From the investigations in [12] is easy to conclude that if the MFCQ fails to hold at $\bar{z}$ of Type 5, then $\bar{z}$ is an endpoint of a branch consisting of local minimizers. We conclude from the following proposition:

Proposition 1
Assume $(P(t)) \in \mathcal{F}$. If at the g.c. point $\bar{z}$ there are no possibilities to jump, then the following conditions are fulfilled:

a) the MFCQ fails to hold at $\bar{z}$.

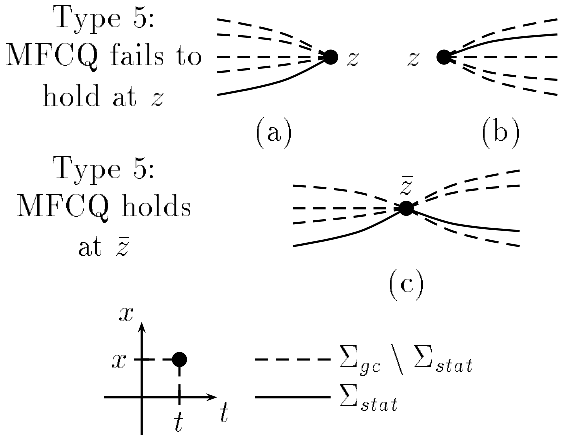


Figure 3: Points of Type 5.

b) $\bar{z}$ is of Type 4 or 5.

c) $\bar{z}$ is an endpoint of a branch consisting of local minimizers.

d) If $\bar{z}$ is of Type 4, then the Condition A is fulfilled.

In our paper we must distinguish between the cases a) and b) in the Figures 2 and 3. We introduce the following notion.

Definition 4
Assume $(P(t)) \in \mathcal{F}$ and $\bar{z} \in \Sigma_{gc}$. We call $\bar{z}$ a turning point in negative position (positive position) if there exists a neighbourhood V of $\bar{z}$ such that

$$\forall (x,t) \in \Sigma_{gc} \cup V, \text{ holds } t \leq \bar{t} \ (t \geq \bar{t}).$$

In the Figures 2 a) and 3 a) the point $\bar{z}$ is a turning point in negative position. In the Figures 2 b) and 3 b) turning points in positive position are shown.

Since the constraint qualification LICQ implies the MFCQ, we can conclude that for a problem $(P(t)) \in \mathcal{F}$ the MFCQ fails to hold only at points of Type 4 or 5. We have seen that MFCQ fails to hold at $\bar{z} = (\bar{x}, \bar{t})$ if and only if there exists a $\bar{u} \in I\!R^{|J_0(\bar{z})|}$ such that:

$$
\begin{aligned}
\sum_{j\in J_0(\bar z)} \bar u_j Dg_j(\bar z) &= 0 \\
\sum_{j\in J_0(\bar z)} \bar u_j &> 0 \\
\bar u_j &\geq 0,\ j\in J_0(\bar z)
\end{aligned}
\tag{4}
$$

If the MFCQ is not satisfied at $\bar z$, then $\bar z$ is of Type 4 or 5, and the conditions (4.1-2) and (5.1-2), respectively, imply that:

1) There exists a $\bar u \in I\!R^{|J_0(\bar z)|}$, with $\bar u \neq 0$, such that the solution set of the system (4) is of the form $\{\lambda \bar u \mid \lambda > 0\}$

2) $\bar u_j > 0,\ j \in J_0(\bar z)$.

Let $\bar u$ be fixed as in the preceding condition 1). Using condition (4.3) it is not difficult to verify that

$$\sum_{j\in J_0(\bar z)} \bar u_j D_t g_j(\bar z) \neq 0.$$

The same inequality follows from (5.1) for the case that $\bar z$ is of Type 5. We summarize the preceding conclusions in the following proposition.

Proposition 2
Assume that $(P(t)) \in \mathcal{F}$. If the MFCQ fails to hold at $\bar z \in \Sigma_{gc}$, then

a) $\bar z$ is a point of Type 4 or 5.

b) The system (4) has a solution set of the form $\{\lambda \bar u \mid \lambda > 0\}$, where $\bar u \in I\!R^{|J_0(\bar z)|}$ is a nonzero vector.

c) $\forall j \in J_0(\bar z)$ it holds $\bar u_j > 0$.

d) $\sum_{j\in J_0(\bar z)} \bar u_j D_t g_j(\bar z) \neq 0$.

We note that the sign of the expression $\sum_{j\in J_0(\bar z)} \bar u_j D_t g_j(\bar z)$ does not depend on the fixed $\bar u \in I\!R^{|J_0(\bar z)|}$ satisfying the system (4). Let us introduce the following condition of the problem $(P(t)) \in \mathcal{F}$.

Condition B

At each point $\bar{z} \in \Sigma_{gc}$ where the MFCQ is not satisfied, the following inequality holds:

$$\sum_{j \in J_0(\bar{z})} \bar{u}_j D_t g_j(\bar{z}) > 0,$$

where $\bar{u} \in I\!R^{|J_0(\bar{z})|}$ is a vector selected as in Proposition 2 b).

3 Main Result

Our main result in this paper is the following:

Theorem 1

Assume $(P(t)) \in \mathcal{F}$ and Condition B to be satisfied. Then, every turning point $\bar{z} \in \Sigma_{gc}$ without possibilities to jump is a turning point in positive position.

Proof:

We divide the proof into two parts. In each part we deal with one of the Types 4 or 5.

Part I:

In this part we prove the assertion of the theorem for the case that $\bar{z}$ is a point of Type 5. Let us suppose for simplicity that $J_0(\bar{z}) = \{1, \ldots, \bar{p}\}$. Let $\tilde{\jmath}$ be a fixed index of $J_0(\bar{z})$. Let us suppose that $\tilde{\jmath} = \bar{p}$. The conditions (5.1) and (5.2) imply that the vectors $\{D_x g_j(\bar{z}), j = 1, \ldots, \bar{p}-1\}$ are linearly independent. Then we obtain a uniquely determined solution $\bar{u}^{\bar{p}} = (\bar{u}_1^{\bar{p}}, \ldots, \bar{u}_{\bar{p}-1}^{\bar{p}})$ of the following system:

$$D_x f(\bar{z}) - \sum_{j=1}^{\bar{p}-1} \bar{u}_j^{\bar{p}} D_x g_j(\bar{z}) = 0.$$

From condition (5.3) we conclude that $\bar{u}_j^{\bar{p}} \neq 0$ for $j = 1, \ldots, \bar{p}-1$. Let us consider the following system of equations:

$$\begin{bmatrix} D_x f(x,t) - \sum_{j=1}^{\bar{p}-1} u_j^{\bar{p}} D_x g_j(x,t) \\ g_1(x,t) \\ \vdots \\ g_{\bar{p}-1}(x,t) \end{bmatrix} = 0. \tag{5}$$

From the implicit function we obtain an interval of the form $(\bar t-\epsilon,\bar t+\epsilon)$ and uniquely determined functions:

$$(x^{\bar p}(t),u^{\bar p}(t)):(\bar t-\epsilon,\bar t+\epsilon)\to I\!R^{n+\bar p-1}$$

such that $x^{\bar p}(\bar t)=\bar x$, $u^{\bar p}(\bar t)=\bar u^{\bar p}$, and for each $t\in(\bar t-\epsilon,\bar t+\epsilon)$ the vector $(x^{\bar p}(t),u^{\bar p}(t),t)$ is a solution of the system (5).

If $g_{\bar p}(x^{\bar p}(t),t)\geq 0$, then $(x^{\bar p}(t),t)$ is a generalized critical point. It is not difficult to prove that

$$\frac{d}{dt}g_{\bar p}(x^{\bar p}(\bar t),\bar t)\neq 0.$$

The preceding inequality implies that the curve $(x^{\bar p}(t),t)$ belongs to Σ_{gc} either for $t\in(\bar t,\bar t+\epsilon)$ or for $t\in(\bar t-\epsilon,\bar t)$. Taking into account the condition (5.3) (especially that $\bar u_j^{\bar p}\neq 0$ for $j=1,\ldots,\bar p-1$) easily provides that, in a neighbourhood of $\bar z$, Σ_{gc} is reduced to the feasible parts of the curves $(x^{\tilde\jmath}(t),t)$, for $\tilde\jmath\in J_0(\bar z)$.

Now it is sufficient to show that it holds $\forall\ \tilde\jmath\in J_0(\bar z)$

$$\frac{d}{dt}g_{\tilde\jmath}(x^{\tilde\jmath}(\bar t),\bar t)>0.$$

We will show this inequality for $\tilde\jmath=\bar p$.

Let us now calculate this quantity:

$$\frac{d}{dt}g_{\bar p}(x^{\bar p}(\bar t),\bar t)=D_xg_{\bar p}(\bar x,\bar t)\ \dot x^{\bar p}(\bar t)+D_tg_{\bar p}(\bar x,\bar t). \tag{6}$$

According to the definition of $\dot x^{\bar p}(\bar t)$ the following equality is obtained for each $j=1,\ldots,\bar p-1$:

$$D_xg_j(\bar z)\dot x^{\bar p}(\bar t)=-D_tg_j(\bar z) \tag{7}$$

From the Proposition 1 we know that the MFCQ is not fulfilled at $\bar z$. Now, using Proposition 2, we obtain the existence of $\bar u_j>0,\ j\in J_0(\bar z)$, such that:

$$\sum_{j=1}^{\bar p}\bar u_jD_xg_j(\bar z)=0. \tag{8}$$

Multiplying equation (8) by $\dot x^{\bar p}(\bar t)$ and substituting the relations (7) yields

$$-\sum_{j=1}^{\bar p-1}\bar u_jD_tg_j(\bar z)+\bar u_{\bar p}D_xg_{\bar p}(\bar z)\dot x^{\bar p}(\bar t)=0.$$

Now we substitute the preceding relation in equation (6) and obtain the following equation:

$$\frac{d}{dt} g_{\bar{p}}(x^{\bar{p}}(\bar{t}), \bar{t}) = \bar{u}_{\bar{p}} \left(\sum_{j \in J_0(\bar{z})} \bar{u}_j D_t g_j(\bar{z}) \right)$$

Condition B and the inequality ($\bar{u}_{\bar{p}} > 0$) imply the desired result:

$$\frac{d}{dt} g_{\bar{p}}(x^{\bar{p}}(\bar{t}), \bar{t}) > 0.$$

Part II

Now we consider the case of $\bar{z}$ being of Type 4. Let us suppose again that $J_0(\bar{z}) = \{1, \ldots, \bar{p}\}$.

Since $\bar{z}$ is a g.c. point where there are no possibilities to jump, Proposition 1 implies that MFCQ fails to hold at $\bar{x} \in M(\bar{t})$. Now we fix a vector $\bar{u} \in I\!R^{\bar{p}}$ satisfying the conditions mentioned in Proposition 2. Condition (4.3) is fulfilled for the vector $\bar{u}$. We write this condition explicitly. The following notation, introduced in Part I of this proof, will be used: $u^{\bar{p}} = (u_1, \ldots, u_{\bar{p}-1})$. Then we have then to deal with the following optimization problem.

$$(\hat{P}) \quad \min \left\{ \hat{F}(x, u^{\bar{p}}, t, u_0) \mid (x, u^{\bar{p}}, t, u_0) \in \hat{M} \right\},$$

where

$$\hat{M} = \left\{ (x, u^{\bar{p}}, t, u_0) \in I\!R^{n+\bar{p}+1} \mid \Upsilon(x, u^{\bar{p}}, t, u_0) = 0 \right\}.$$

Since the point $(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0)$ is a nondegenerated generalized critical point of $(\hat{P})$, the following conditions are fulfilled:

1. $$rank\,(D\Upsilon(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0)) = n + \bar{p}$$

2. $\exists \lambda_j,\ j = 1, \ldots, n + \bar{p}$ such that $\lambda = (\lambda_1, \ldots, \lambda_{n+\bar{p}}) \neq 0$ and
$$D\hat{F}(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0) = \lambda^T D\Upsilon(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0).$$

3. The number
$$D^2 \hat{L}(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0)|_{T_{(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0)} \hat{M}} \tag{9}$$
is not equal to zero, where $\hat{L}$ is the Lagrangian corresponding to the problem $(\hat{P})$.

The preceding condition 1. implies that the set $\hat{M}$ is a differentiable manifold of dimension one in a neighbourhood of the point $(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0)$.

The points of Type 4 are always turning points. The position of a point of Type 4 is decided by the sign of the expression (9). If it is positive (negative), then $\bar{z}$ is a turning point in positive (negative) position. We have to show that (9) is strictly positive.

First of all, we present the Jacobian matrix of the map Υ at the point $(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0)$.

$$D\Upsilon(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0) =$$
$$\begin{pmatrix} D_x^2 \mathcal{L}(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0) & -D_x^T g_1(\bar{z}) \cdots - D_x^T g_{\bar{}-1}(\bar{z}) & D_{xt}^2 \mathcal{L}(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0) & D_x^T f(\bar{z}) \\ D_x g_1(\bar{z}) & & D_t g_1(\bar{z}) & \\ \vdots & 0 & \vdots & 0 \\ D_x g_{\bar{p}}(\bar{z}) & & D_t g_{\bar{p}}(\bar{z}) & \end{pmatrix}$$

Condition B implies the relation:

$$\sum_{j=1}^{\bar{p}} \bar{u}_j D_t g_j(\bar{z}) = -D_t \mathcal{L}(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0) > 0 \qquad (10)$$

Let the structure of the gradient of $\hat{F}$ be known

$$\begin{aligned} D_t \hat{F}(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0) &= 1 \\ D_{(x, u^{\bar{p}}, u_0)} \hat{F}(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0) &= 0_{n+\bar{p}} \end{aligned}$$

Here $0_{n+\bar{p}}$ stands for the zero of $\mathbb{R}^{n+\bar{p}}$.

Now we have to calculate the Lagrange multipliers λ. After a little calculation the following relation is verified:

$$D\hat{F}(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0) = \frac{1}{-D_t \mathcal{L}(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0)} (0_n, \bar{u}) D\Upsilon(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0),$$

which means that the uniquely determined multiplier λ has the form $(0_n, \bar{u})$, where $\bar{u} \in \mathbb{R}^{\bar{p}}$ is the fixed vector at the beginning of this part.

We state now an expression for the Langrangian of $(\hat{P})$ at the point $(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0)$

$$
\begin{aligned}
\hat{L}(x,u^{\bar{p}},t,u_0) &= t + \frac{1}{D_t\mathcal{L}(\bar{x},\bar{u}^{\bar{p}},\bar{t},0)}\sum_{j=1}^{\bar{p}}\bar{u}_j g_j(z) \\
&= t - \frac{1}{D_t\mathcal{L}(\bar{x},\bar{u}^{\bar{p}},\bar{t},0)}\mathcal{L}(x,\bar{u}^{\bar{p}},t,0).
\end{aligned}
\tag{11}
$$

By $(w_x, w_{u^{\bar{p}}}, w_t, w_{u_0}) \in I\!R^{n+\bar{p}+1}$ we denote a vector that generates the tangent space to $\hat{M}$ at the point $(\bar{x},\bar{u}^{\bar{p}},\bar{t},0)$. Then it follows by definition:

$$
D\Upsilon(\bar{x},\bar{u}^{\bar{p}},\bar{t},0)\begin{pmatrix} w_x \\ w_{u^{\bar{p}}} \\ w_t \\ w_{u_0} \end{pmatrix} = 0_{n+\bar{p}}. \tag{12}
$$

A short analysis shows that $w_t = 0$ and $w_{u_0} \neq 0$. When we multiply the equation (12) by the vector $(w_x, 0_{\bar{p}})$, we obtain the following relation

$$
w_x^T\ D_x^2\mathcal{L}(\bar{x},\bar{u}^{\bar{p}},\bar{t},0)\ w_x + w_x^T\ D_x^T f(\bar{z})w_{u_0} = 0. \tag{13}
$$

We observe for (11) that the Lagrangian $\hat{L}(x,u^{\bar{p}},t,u_0)$ of the problem $(\hat{P})$ at the point $(\bar{x},\bar{u}^{\bar{p}},\bar{t},0)$ does not depends on the variables $u^{\bar{p}} \in I\!R^{\bar{p}-1}$ and u_0. From this observation and taking into account that $w_t = 0$ we derive the next simple expression for (9).

$$
(w_x, w_{u^{\bar{p}}}, w_t, w_{u_0})^T D^2\hat{L}(\bar{x},\bar{u}^{\bar{p}},\bar{t},0)\begin{pmatrix} w_x \\ w_{u^{\bar{p}}} \\ w_t \\ w_{u_0} \end{pmatrix} = -\frac{w_x^T D_x^2\mathcal{L}(\bar{x},\bar{u}^{\bar{p}},\bar{t},0)w_x}{D_t\mathcal{L}(\bar{x},\bar{u}^{\bar{p}},\bar{t},0)}.
$$

Now we substitute the relation (13) in the preceding equation and obtain:

$$
D^2\hat{L}(\bar{x},\bar{u}^{\bar{p}},\bar{t},0)|_{T_{(\bar{x},\bar{u}^{\bar{p}},\bar{t},0)}\hat{M}} = \frac{w_x^T D_x^T f(\bar{z})w_{u_0}}{D_t\mathcal{L}(\bar{x},\bar{u}^{\bar{p}},\bar{t},0)}. \tag{14}
$$

From the well-known properties of the tangent space we conclude that there exist maps $(x(\tau), u^{\bar{p}}(\tau), t(\tau), u_0(\tau))$ defined over a neighbourhood of zero $(-\epsilon,\epsilon) \subset I\!R$ and such that:

- $\Upsilon(x(\tau), u^{\bar{p}}(\tau), t(\tau), u_0(\tau)) =$, for all $\tau \in (-\epsilon,\epsilon)$

- $(x(0), u^{\bar{p}}(0), t(0), u_0(0)) = (\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0)$
- $(\dot{x}(0), \dot{u}^{\bar{p}}(0), \dot{t}(0), \dot{u}_0(0)) = (w_x, w_{u^{\bar{p}}}, w_t, w_{u_0})$

We have mentioned that $w_{u_0} \neq 0$. From (14) we note that $w_x^T D_x^T f(\bar{z})$ is either different of zero. We show that the product $w_x^T D_x^T f(\bar{z}) w_{u_0}$ is less than zero. Condition A now becomes involved. Let us suppose that $w_{u_0} = -1$. Other cases are reduced to this one. The following equation holds for all $\tau \in (-\epsilon, \epsilon)$:

$$u_0(\tau) D_x^T f(x(\tau), t(\tau)) - \sum_{j=1}^{\bar{p}-1} u_j(\tau) D_x^T g_j(x(\tau), t(\tau)) - \bar{u}_{\bar{p}} D_x^T g_{\bar{p}}(x(\tau), t(\tau)) = 0 \tag{15}$$

Since $w_{u_0} = -1$ and $\bar{u}_j > 0,\ j = 1, \ldots, \bar{p}$, then the following inequalities hold for $\tau < 0$:

$$\begin{array}{rcl} u_0(\tau) & > & 0 \\ \dfrac{u_j(\tau)}{u_0(\tau)} & > & 0,\ j =, \ldots, \bar{p} - 1 \\ \dfrac{u_{\bar{p}}}{u_0(\tau)} & > & 0 \end{array}$$

From the above relations and the equation (15) it follows that the points $(x(\tau), t(\tau))$ are stationary points of $P(t(\tau))$ for $\tau < 0$.

Since $(\bar{x}, \bar{t})$ is a point of Type 4 without possibilities to jump, then, for $\tau < 0$, $(x(\tau), t(\tau))$ are local minima (stationary points) and, for $\tau > 0$, local maxima of $(P(t(\tau))$.

Condition A implies now that

$$D_t f(x(0), t(0)) = w_x^T D_x^T f(\bar{z}) > 0$$

Taking into account the inequality (10) and the relation (14) we obtain the desired result:

$$D^2 \hat{L}(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0)|_{T_{(\bar{x}, \bar{u}^{\bar{p}}, \bar{t}, 0)} \hat{M}} > 0.$$

This concludes the proof. □.

As mentioned in the introduction the result 1 can be induced from the results about singularities and jumps presented in the 6th and 7th chapters of the textbook [5]. In these chapters, written by J-J. Rückmann, the obtained results related with the theorem 1 are obtained with use of the so called normal forms of the singularities (see for example [11]. In this section we have presented a complete formulation of the result and we have given a directed proof.

4 An interior embedding.

In this section we use Theorem 1 for the analysis of a specific parametric optimization problem. It will be a parametric problem whose critical sets lie in the interior of the feasible set of a fixed optimization problem. This parametric problem holds the Condition B introduced in the latter section.

We consider an optimization problem whose feasible set is described by inequalities only. For our later investigations we need to introduce a notation expressing the dependence on the problem of the functions defining it. Let $f, g_j \in C^3(I\!R^n, I\!R)$, $j = 1, \dots, p$, be given functions. Let us denote the maps $(f, g) = (f, g_1, \dots, g_p) \in C^3(I\!R^n, I\!R^{p+1})$. The following notation is used throughout this section:

$$P(f,g) \quad \min \{f \mid g_j(x) \geq 0, j \in J\}, \tag{16}$$

where $J = \{1, \dots, p\}$ as in the latter section.

A point $x \in I\!R^n$ with $g_j(x) > 0$ for all $j \in J$ is called an interior point of the problem $P(f,g)$ described by (16). Let x^0 be an interior point of the problem (16) and $q \in I\!R^p$ be a fixed vector such that $g_j(x^0) > q_j > 0$, for all $j \in J$.

Now we define a parametric optimization problem which depends on the preceding data:

$$P^{int}_{(x^0,q)}(f,g,t) \quad \min \left\{ f^{int}(f,x,t) \mid g^{int}_j(g,x,t) \geq 0, j \in J \right\}, \tag{17}$$

where

$$\begin{aligned} f^{int}(f,x,t) &= tf(x) + (1-t)\|x - x^0\|^2 && (18) \\ g^{int}_j(g,x,t) &= g_j(x) - (1-t)q_j && (19) \end{aligned}$$

For each $j \in J$ the inequality:

$$D_t g^{int}_j(g,x,t) = q_j > 0$$

holds. Condition B follows immediately for this parametric problem. For the problem $P^{int}_{(x^0,q)}(f,g,0)$ the point x^0 is a nondegenerated generalized critical point, since the following three conditions hold:

$$\begin{aligned} g_j^{int}(g,x^0,0) &> 0,\ \forall j \in J && (20)\\ D_x f^{int}(f,x^0,0) &= 0 && (21)\\ D_x^2 f^{int}(f,x^0,0) &= 0 && (22)\end{aligned}$$

The point $(x^0,0)$ can be used as a starting point in a pathfollowing procedure for the problem $P^{int}_{(x^0,q)}(f,g,t)$. Let us assume that the regularity condition $P^{int}_{(x^0,q)}(f,g,t) \in \mathcal{F}$ holds. The direct use of Theorem 1 allows us to remark the following.

Remark 1
A pathfollowing procedure with jumps would not arrive at the value $t = 1$ if it was applied to the parametric problem $P^{int}_{(x^0,q)}(f,g,t)$, only in the case that a jump (with a starting feasible descent direction) were not successful. If we suppose, for instance, that the feasible set of the original problem $P(f,g)$ is compact, then a pathfollowing procedure with jumps will be successful.

We want to discuss now the basic assumption $P^{int}_{(x^0,q)}(f,g,t) \in \mathcal{F}$. Recall some well-known facts about the class $\mathcal{F}$. First of all, we mention the perturbation theorem presented in the paper [16].

Theorem 2
Let $(\bar{f},\bar{g}_1,\ldots,\bar{g}_p) \in C^3(I\!R^{n+1}, I\!R^{p+1})$ be fixed, then each measurable subset of the set

$$\left\{ (A,b,c^1,\ldots,c^p,d) \;\middle|\; \begin{array}{l} (\bar{f}(x,t) + 0.5x^TAx + b^Tx, \bar{g}_1(x,t) + c_1^Tx + d_1, \ldots \\ \ldots, \bar{g}_p(x,t) + c_p^Tx + d_p) \notin \mathcal{F} \end{array} \right\}$$

has Lebesgue measure zero.

Here the Lebesgue measure is defined on the space of the parameters

$$(A,b,c^1,\ldots,c^p,d) \in I\!R^{0.5n(n+1)} \times I\!R^n \times I\!R^{pn} \times I\!R^p$$

The other well-known fact is the local C_s^3 stability of the singularities involved in the class $\mathcal{F}$. Let us introduce first some notations. For a fixed neighbourhood $U \subset I\!R^k$ and a fixed positive number r we define the corresponding C_s^3 neighbourhood as the following subset of $C^3(I\!R^k, I\!R)$

$$V_s^3(U,r) := \left\{ h \in C^3(I\!R^k, I\!R) \;\middle|\; \begin{array}{l} |D^s_{(\alpha_1,\ldots,\alpha_s)}h(y)| < r, \forall s \in \{0,1,2,3\}, \\ \forall \alpha_i \in \{y_1,\ldots,y_k\} \text{ and } \forall y \in U \end{array} \right\}.$$

The local result we have in mind is the following:

Proposition 3 *(see e.g. [5])*
Let $(\bar{f}, \bar{g}_1, \ldots, \bar{g}_p) \in C^3(I\!R^{n+1}, I\!R)$ be fixed functions and $(\bar{x}, \bar{t})$ be a fixed point. Then there exists an open neighbourhood $U_{(\bar{x},\bar{t})}$ of $(\bar{x}, \bar{t})$ and a positive number $r_{(\bar{x},\bar{t})}$ such that for each functions $(\tilde{f}, \tilde{g}_1, \ldots, \tilde{g}_p) \in C^3(I\!R^{n+1}, I\!R)$ with

$$\begin{array}{rcl} (\bar{f} - \tilde{f})(x,t) & \in & V_s^3(U_{(\bar{x},\bar{t})}, r_{(\bar{x},\bar{t})}) \\ (\bar{g}_j - \tilde{g}_j)(x,t) & \in & V_s^3(U_{(\bar{x},\bar{t})}, r_{(\bar{x},\bar{t})}), \;\; \forall j \in J \end{array}$$

the following property holds:
If $(\bar{x}, \bar{t})$ is a generalized critical point of Type $\nu \in \{1,2,3,4,5\}$ (resp. is not a g.c. point) of the one-parametric problem formed with the data $(\bar{f}, \bar{g}_1, \ldots, \bar{g}_p)$, then the one-parametric problem with the data $(\tilde{f}, \tilde{g}_1, \ldots, \tilde{g}_p)$ has only g.c. points of Type ν or 1 (resp. has no g.c. point) in the neighbourhood $U_{(\bar{x},\bar{t})}$.

The proof of Proposition 3 is based on the following fact, which is proved by use of continuity arguments and the relations defining the 5 types of g.c. points of the class $\mathcal{F}$.

Proposition 4
Let $x_n, t_n, f^n, g_1^n, \ldots, g_p^n$ be sequences with $f^n, g_1^n, \ldots, g_p^n \in C^3(I\!R^{n+1}, I\!R)$ such that:

- $(x_n, t_n) \to (\bar{x}, \bar{t})$.
- $D^s_{(\alpha_1,\ldots,\alpha_s)} f^n(x^n, t^n) \to D^s_{(\alpha_1,\ldots,\alpha_s)} \bar{f}(\bar{x}, \bar{t})$ *for all* $s \in \{0,1,2,3\}$ *and* $\alpha_i \in \{x_1, \ldots, x_n, t\}$.
- $D^s_{(\alpha_1,\ldots,\alpha_s)} g_j^n(x^n, t^n) \to D^s_{(\alpha_1,\ldots,\alpha_s)} \bar{g}_j(\bar{x}, \bar{t})$ *for all* $j \in J$, $s \in \{0,1,2,3\}$ *and* $\alpha_i \in \{x_1, \ldots, x_n, t\}$.

Then the following conclusions hold:

1. *If $(\bar{x}, \bar{t})$ is a g.c. point of Type $\nu \in \{1, 2, 3, 4, 5\}$ of the one-parametric problem with the data $(\bar{f}, \bar{g}) \in C^3(\mathbb{R}^{n+1}, \mathbb{R}^{p+1})$, then, for sufficiently large n, the point (x_n, t_n) is of Type ν or 1 if this point is a g.c. point of the one parametric problem with the data $(f^n, g^n) \in C^3(\mathbb{R}^{n+1}, \mathbb{R}^{p+1})$.*

2. *If $(\bar{x}, \bar{t})$ is not a g.c. point of the one parametric problem with data $(\bar{f}, \bar{g}) \in C^3(\mathbb{R}^{n+1}, \mathbb{R}^{p+1})$ then for sufficiently large n the point (x_n, t_n) is not a g.c. point of the one parametric problem with data $(f^n, g^n) \in C^3(\mathbb{R}^{n+1}, \mathbb{R}^{p+1})$.*

We note that the class $\mathcal{F}$ can be proved to be open and dense by an usual procedure of differential topology, which is based on Theorem 2 and Proposition 3.

In other similar papers about specific parametrizations (e.g. [2, 3, 4, 6]) the obtained parametrizations are always assumed to belong to the class $\mathcal{F}$. This class $\mathcal{F}$ of Jongen, Jonker and Twilt is an open and dense set of the space $C^3(\mathbb{R}^{n+1}, \mathbb{R}^{p+1})$ endowed with the C^3_s topology.

In the mentioned papers [2, 3, 4, 6] perturbation theorems, for the final form of the parametrization to be used there were presented. As a consequence of the density of the set $\mathcal{F}$, there always exists a small perturbation, that brings our parametric problem inside the class $\mathcal{F}$. The purpose of the mentioned perturbation theorems is to find a specific regularization that uses the special structure of the parametrizations considered. All these perturbations results are based on the ideas of Theorem 2.

Here we do not want to deal with the question: How can we perturb a fixed one parametric optimization problem (obtained by a selected parametrization) in order to get a JJT-regular problem? For the defined parametrization $P^{int}_{(x^0,q)}(f, g, t)$ we study the question: How reasonable with respect to the data $(f, g) \in C^3(\mathbb{R}^n, \mathbb{R}^{p+1})$ is the assumption that the corresponding parametric problem $P^{int}_{(x^0,q)}(f, g, t)$ belongs to the class $\mathcal{F}$? In other words, we want to study the meaning of the assumption $P^{int}_{(x^0,q)}(f, g, t) \in \mathcal{F}$ in terms of the original problem data $(f, g) \in C^3(\mathbb{R}^n, \mathbb{R}^{p+1})$. It is easily noticed that that the properties of the class $\mathcal{F}$ to be open and dense do not provide an answer to this question. The form in which our specific parametrization uses the problem data $(f, g) \in C^3(\mathbb{R}^n, \mathbb{R}^{p+1})$ for the construction of a one-parametric problem plays now an important role.

We note that there exists a mapping from the space $C^3(I\!R^n, I\!R^{p+1})$ into the space $C^3(I\!R^{n+1}, I\!R^{p+1})$ which is involved in the construction of each specific parametrization. For our example we define this mapping to depend on the fixed vectors x_0 and q. This mapping (called $T_{(x^0,q)}$) is defined in the following way:

$$T_{(x^0,q)} : C^3(I\!R^n, I\!R^{p+1}) \mapsto C^3(I\!R^{n+1}, I\!R^{p+1})$$

with

$$T_{(x^0,q)}(f, g_1, \ldots, g_p) := \begin{pmatrix} tf + (1-t)\|x - x^0\|^2 \\ g_1 - (1-t)q_1 \\ \vdots \\ g_p - (1-t)q_p \end{pmatrix}$$

We note that the one-parametric problem constructed with the data $T_{(x^0,q)}(f, g_1, \ldots, g_p)$ is nothing else but $P^{int}_{(x^0,q)}(f, g, t)$. We analyze how large the set $T^{-1}_{(x^0,q)}(\mathcal{F})$ is.

We obtained the following result.

Theorem 3
Let $(x^0, q) \in I\!R^n \times I\!R^q$ be fixed. Then the set $T^{-1}_{(x^0,q)}(\mathcal{F})$ is a generic subset of $C^3(I\!R^n, I\!R^{p+1})$, endowed with the C^3_s (or Whitney) topology.

First of all we recall that a generic set is a set that contains a countable intersection of open and dense sets.

<u>Proof:</u>
Let us first introduce some notations. Given subsets $M_1 \subset I\!R^n$ and $M_2 \subset I\!R$ we define the set $\mathcal{F}|_{M_1 \times M_2} \subset C^3(I\!R^{n+1}, I\!R^{p+1})$ as follows:

$$\mathcal{F}|_{M_1\times M_2} := \left\{ (f,g) \in C^3(I\!R^{n+1}, I\!R^{p+1}) \;\middle|\; \begin{array}{l} \text{each point of the set} \\ \Sigma_{gc} \cap M_1 \times M_2 \\ \text{is of Type 1-5.} \end{array} \right\}$$

In the right-hand part of the above formula the set Σ_{gc} is intended to be the set corresponding to the one-parametric optimization problem with the data $(f, g) \in C^3(I\!R^{n+1}, I\!R^{p+1})$. We will always write $\mathcal{F}|_{M_2}$ instead of $\mathcal{F}|_{I\!R^n \times M_2}$

The idea of the proof is the following. The class $\mathcal{F}$ can be obtained as a countable intersection

$$\mathcal{F} = \cap_{n=1}^{\infty} \mathcal{F}|_{[-n,n]}.$$

We show that the set $T^{-1}_{(x^0,q)}(\mathcal{F}|_{[-n,n]}) \subset C^3(\mathbb{R}^n, \mathbb{R}^{p+1})$ is open and dense for each $n = 1, 2, ...$ The result is obtained from the relations

$$\cap_{n=1}^{\infty} T^{-1}_{(x^0,q)}(\mathcal{F}|_{[-n,n]}) = T^{-1}_{(x^0,q)}(\cap_{n=1}^{\infty} \mathcal{F}|_{[-n,n]}) = T^{-1}_{(x^0,q)}(\mathcal{F})$$

Let $n \in \{1, 2, ...\}$ be fixed. As mentioned before, in order to prove that the set $T^{-1}_{(x^0,q)}(\mathcal{F}|_{[-n,n]})$ is open and dense, we only need to state results analogous to Theorem 2 and Proposition 3. As in the case of the class $\mathcal{F}$, the standard techniques of the differential topology can be used to state the density and openness of $T^{-1}_{(x^0,q)}(\mathcal{F}|_{[-n,n]})$.

We have to prove the following two claims then:

<u>Claim I:</u>(Perturbation theorem)
Let $(\bar{f}, \bar{g}_1, \dots, \bar{g}_p) \in C^3(\mathbb{R}^n, \mathbb{R}^{p+1})$ be fixed, then each measurable subset of the set

$$\left\{ (A, b, c^1, \dots, c^p, d) \;\middle|\; \begin{array}{c} T_{(x^0,q)}(\bar{f} + 0.5x^T Ax + b^T x, \bar{g}_1 + c_1^T x + d_1, .. \\ \dots, \bar{g}_p + c_p^T x + d_p) \notin \mathcal{F}|_{[-n,n]} \end{array} \right\}$$

has Lebesgue measure zero.

Here the Lebesgue measure is refered on the space of parameters

$$(A, b, c^1, \dots, c^p, d) \in \mathbb{R}^{0.5n(n+1)} \times \mathbb{R}^n \times \mathbb{R}^{pn} \times \mathbb{R}^p.$$

<u>Claim II:</u>
Let $(\bar{f}, \bar{g}_1, \dots, \bar{g}_p) \in C^3(\mathbb{R}^n, \mathbb{R})$ be fixed functions and $\bar{x}$ be a fixed point such that $T_{(x^0,q)}(\bar{f}, \bar{g}_1, \dots, \bar{g}_p) \in \mathcal{F}|_{\{\bar{x}\}\times[-n,n]}$. Then there exists an open neighbourhood $U_{\bar{x}}$ of $\bar{x}$ and a positive number $r_{\bar{x}}$ such that, for each mapping $(\tilde{f}, \tilde{g}_1, \dots, \tilde{g}_p) \in C^3(\mathbb{R}^n, \mathbb{R})$ with

$$\begin{array}{rcl} (\bar{f} - \tilde{f})(x) & \in & V_s^3(U_{\bar{x}}, r_{\bar{x}}) \\ (\bar{g}_j - \tilde{g}_j)(x) & \in & V_s^3(U_{\bar{x}}, r_{\bar{x}}), \; \forall j \in J, \end{array}$$

it holds that $T_{(x^0,q)} \in \mathcal{F}(\tilde{f}, \tilde{g}_1, \dots, \tilde{g}_p)|_{U_{\bar{x}}\times[-n,n]}$.

<u>Proof of Claim I:</u>
Let us first introduce the notation

$$\mathcal{A} = (A, b, c^1, \dots, c^p, d) \in \mathbb{R}^{0.5n(n+1)} \times \mathbb{R}^n \times \mathbb{R}^{pn} \times \mathbb{R}^p$$

Writing the one-parameric optimization problem with the data $T_{(x^0,q)}(\bar{f} + 0.5x^TAx + b^Tx, \bar{g}_1 + c_1^Tx + d_1, \ldots, \bar{g}_p + c_p^Tx + d_p)$, we obtain the following one-parametric problem (which depends on the vector $\mathcal{A}$);

$$\begin{array}{ll} P_{\mathcal{A}}(t) & \\ \quad\min & t\bar{f} + \|x - x^0\|^2 + t(0.5x^TAx + b^Tx) \\ subject\ to: & \bar{g}_j - (1-t)q_j + c_j^Tx + d_j \geq 0, \ \forall j \in J \end{array}$$

The similarity of the resulting perturbed problems here and in Theorem 2 allows us, with the same proof as for Theorem 2, to state the following fact: Each measurable subset of the set

$$\left\{ \mathcal{A} \in I\!\!R^{0.5n(n+1)+n+np+p} \;\middle|\; \begin{array}{r} T_{(x^0,q)}(\bar{f} + 0.5x^TAx + b^Tx, \bar{g}_1 + c_1^Tx + d_1, .. \\ \ldots, \bar{g}_p + c_p^Tx + d_p) \notin \mathcal{F}|_{I\!\!R\setminus\{0\}} \end{array} \right\}$$

has Lebesgue measure zero.

We have to exclude the value $t = 0$, because, in the objective function, the perturbation is multiplied by t. For the analysis in the point $t = 0$ we have to prove the following fact, which is a direct consequence of the well-known parametrized Sard's theorem.

For almost all $(c^1, \ldots, c^p, d) \in I\!\!R^{pn} \times I\!\!R^p$ it holds that each generalized critical point of the problem

$$\min \{\|x - x^0\| \mid \bar{g}_j(x) - q_j + c_j^Tx + d_j \geq 0\}$$

is nondegenerated.

The statement of the Claim is obtained taking into account that the last statement produces a measurable subset of $I\!\!R^{0.5n(n+1)+n+np+p}$ and then the statement obtained about the set

$$\left\{ \mathcal{A} \in I\!\!R^{0.5n(n+1)+n+np+p} \;\middle|\; \begin{array}{r} T_{(x^0,q)}(\bar{f} + 0.5x^TAx + b^Tx, \bar{g}_1 + c_1^Tx + d_1, .. \\ \ldots, \bar{g}_p + c_p^Tx + d_p) \notin \mathcal{F}|_{I\!\!R\setminus\{0\}} \end{array} \right\}$$

can be extended without difficulties to the case $\mathcal{F}$.

Proof of Claim II:

The idea is to use again Proposition 4.

Let us suppose that Claim II is false for fixed functions $(\bar{f}, \bar{g}_1, \ldots, \bar{g}_p) \in C^3(I\!\!R^n, I\!\!R)$ and a fixed point $\bar{x} \in I\!\!R^n$, with $T_{(x^0,q)}(\bar{f}, \bar{g}_1, \ldots, \bar{g}_p) \in \mathcal{F}|_{\{\bar{x}\}\times[-n,n]}$. Then there exist sequences $x_k \in I\!\!R^n$ and $f^k, g_1^k, \ldots, g_p^k \in C^3(I\!\!R^n, I\!\!R)$ such that:

- $x_k \to \bar{x}$.
- $D^s_{(\alpha_1,\ldots,\alpha_s)} f^k(x_k) \to D^s_{(\alpha_1,\ldots,\alpha_s)} \bar{f}(\bar{x})$ for all $s \in \{0,1,2,3\}$ and $\alpha_i \in \{x_1,\ldots,x_n\}$.
- $D^s_{(\alpha_1,\ldots,\alpha_s)} g^k_j(x_k) \to D^s_{(\alpha_1,\ldots,\alpha_s)} \bar{g}_j(\bar{x})$ for all $j \in J$, $s \in \{0,1,2,3\}$ and $\alpha_i \in \{x_1,\ldots,x_n\}$.

and $T_{(x^0,q)}(f^k, g^k_1,\ldots,g^k_p) \notin \mathcal{F}|_{\{x_k\}\times\{[-n,n]\}}$. This implies the existence of a sequence $t_k \in [-n,n]$, such that $T_{(x^0,q)}(f^k, g^k_1,\ldots,g^k_p) \notin \mathcal{F}|_{\{x_k\}\times\{t_k\}}$.

From the compacity of $[-n,n]$ we can suppose that t_k converges to some $\bar{t} \in [-n,n]$. From the definition of the functions $f^{int}(f^k, x^0, t) \in C^3(I\!R^{n+1}, I\!R)$ and $g^{int}_j(g^k_1, x^0, t) \in C^3(I\!R^{n+1}, I\!R)$, for $j \in J$, and the convergence of t_k to $\bar{t}$ we derive the following conditions:

- $(x_k, t_k) \to (\bar{x}, \bar{t})$.
- $D^s_{(\alpha_1,\ldots,\alpha_s)} f^{int}(f^k, x_k, t_k) \to D^s_{(\alpha_1,\ldots,\alpha_s)} \bar{f}^{int}(\bar{f}, \bar{x}, \bar{t})$ for all $s \in \{0,1,2,3\}$ and $\alpha_i \in \{x_1,\ldots,x_n,t\}$.
- $D^s_{(\alpha_1,\ldots,\alpha_s)} g^{int}_j(g^k, x_k, t_k) \to D^s_{(\alpha_1,\ldots,\alpha_s)} \bar{g}^{int}_j(\bar{g}, \bar{x}, \bar{t})$ for all $s \in \{0,1,2,3\}$, $j \in J$ and $\alpha_i \in \{x_1,\ldots,x_n,t\}$.

The above conditions together with the assumptions

$$\begin{array}{rcl} T_{(x^0,q)}(\bar{f}, \bar{g}_1,\ldots,\bar{g}_p) & \in & \mathcal{F}|_{\{\bar{x}\}\times[-n,n]} \\ T_{(x^0,q)}(f^k, g^k_1,\ldots,g^k_p) & \notin & \mathcal{F}|_{\{x_k\}\times\{t_k\}} \end{array}$$

contradict Proposition 4, since the following two statements are true:

- The assumption $T_{(x^0,q)}(\bar{f}, \bar{g}_1,\ldots,\bar{g}_p) \in \mathcal{F}|_{\{\bar{x}\}\times[-n,n]}$ implies that, if the point $(\bar{x},\bar{t})$ is a generalized critical point of the problem $P^{int}_{(x^0,q)}(\bar{f},\bar{g},t)$, it is of Type 1,2,3,4 or 5.
- The condition $T_{(x^0,q)}(f^k, g^k_1,\ldots,g^k_p) \notin \mathcal{F}|_{\{x_k\}\times\{t_k\}}$ means that the point (x_k,t_k) is a generalized critical point of the problem $P^{int}_{(x^0,q)}(f^k,g^k,t)$, that is not of Type 1,2,3,4 or 5.

This contradiction proves the Claim II and finishes the proof of the theorem.
$\square$

5 Concluding remarks

We have introduced a condition (Condition B) for a one-parametric optimization problem. Under this condition the pathfollowing procedures with jumps, applied to JJT-regular problems, are always successful (attain the value $t = 1$). The decisive relation between the Condition B and the position of singularities without jumps (see Theorem 1) is proved in a directed way (different as the results presented in [5]), without use of the normal form of the singularities.

Normally, the well-known condition MFCQ is used as sufficient condition for the success of pathfollowing procedures. The introduced Condition B is another type of assumption that is useful in this respect.

A specific parametrization generating critical curves in the interior of a given optimization problem is presented. This interior embedding is an example of a parametric problem where Condition B is satisfied. For the practical use of this parametrization an interior point of the feasible set should be known. Hence the presented embedding is only applicable to such optimization problems where an interior point of the feasible set can be calculated. This is the case for some classes of optimization problems (for example linear problems). It is important to note that the assumption about the feasible set does not imply any requirement on the objective function. That means, the embedding is applicable independently of the form of the objective function.

Finally, the Assumption of JJT-regularity is discussed for the presented interior embedding. The basic regularity assumption of belonging to the class $\mathcal{F}$ provides a generic assumption, if it is studied in terms of the data of the original optimization problem belonging to $C^3(\mathbb{R}^n, \mathbb{R})$. This idea of studying the JJT-regularity of specific embeddings in terms of the data used for their construction can be used in other embeddings, which represent well-known methods of the nonlinear programming. For example, the embeddings studied in the papers [2, 3, 4, 6].

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