Densely defined selections of set-valued mappings and applications to the geometry of Banach spaces and optimization*

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Introduction

Some time ago when studying, in [ČK1, ČK2, ČKR1], different genericity properties in geometry of Banach spaces, optimization and topology, involving set-valued maps, we were getting more and more aware that in many situations there had been a common thread passing through all investigations. This common thing has turned out to be the question whether a given set-valued mapping possesses a densely defined continuous (usually single-valued, but sometimes also set-valued) selection. Let us use an example and briefly explain how the idea for such a study has emerged.

Given a completely regular topological space X, consider $C(X)$–the space of all continuous bounded real-valued functions in X, equipped with the usual sup-norm $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$, $f \in C(X)$. From the point of view of geometry of Banach spaces, it has been always of interest to study the nature of the set of functions in $C(X)$ at which the norm is Gâteaux (or Fréchet) differentiable. It is well-known that for a compact X the Gâteaux differentiability of the norm at a particular function $f \in C(X)$ is closely related to the properties of the *maximization problem* (X, f) determined by f:

find $x_0 \in X$ so that $f(x_0) = \sup\{f(x) : x \in X\} =: \sup(X, f)$.

Indeed, if X is compact then the sup-norm $\|\cdot\|_\infty$ is Gâteaux differentiable at $f \in C(X)$, $f \not\equiv 0$, if and only if the maximization problem $(X, |f|)$ has unique solution. This result is extended ([ČKR1], see also Section 4 below) to the case of an arbitrary X through the following notion: the maximization problem (X, f), $f \in C(X)$, is called *well-posed* (in

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the sense of Tykhonov, see [DZ, T]) if it has unique maximizer $x_0 \in X$ and moreover any sequence $\{x_n\} \subset X$ with the property $f(x_n) \to \sup(X, f)$ (called *maximizing sequence*) converges to x_0. This, and similar other concepts, reflect the original idea for continuous dependence of the unique solution of the problem on the data. For a detailed study of these notions (which have an important independent role in the optimization) the interested reader is refereed to the monograph [DZ].

Now the above result is extended as follows (see Proposition 4.3): the sup-norm $\|\cdot\|_\infty$ is Gâteaux differentiable at $f \in C(X)$, $f \not\equiv 0$, if and only if the maximization problem $(X, |f|)$ is well-posed.

One can go further by clarifying the corresponding question for Fréchet differentiability of the norm as it is done in [ČKR1] (see also below Section 4). But one observation is obvious: the study of differentiability of the norm in $C(X)$ could be achieved by studying the well-posedness of the maximization problems generated by the functions from $C(X)$. This in turn is related to the study of certain properties of the corresponding *solution mapping* $M : C(X) \to X$ assigning to each $f \in C(X)$ the set of its maximizers $M(f) := \{x \in X : f(x) = \sup(X, f)\}$. It is true ([ČKR3]) that the problem (X, f) is well-posed if and only if the set-valued mapping M is single-valued and upper semicontinuous at f (see below Section 4 for the precise definitions and results).

Summarizing, we see that the study of the set of points where the sup-norm in $C(X)$ is differentiable, or the set of functions in $C(X)$ which generate well-posed maximization problems, can be restricted to the study of the set of functions at which the set-valued mapping M is single-valued and upper semicontinuous. The idea was to find out conditions (necessary and sufficient) under which the set $D := \{f \in C(X) : \|\cdot\|_\infty$ is Gâteaux (or Fréchet) differentable at $f\}$ and the set $W := \{f \in C(X) :$ the maximization problem (X, f) is well-posed$\}$ are as bigger as possible from the point of view of Baire category in $C(X)$. More precisely, the question was: under which assumptions are the sets D and W *residual* in $C(X)$.

Let us remind that a subset Z in the topological space Y is *residual* if its complement in Y is from *the first Baire category in* Y, i.e. $Y \setminus Z$ can be represented as a countable union of sets F_i which are *nowhere dense* in Y. The latter means that the interior of the closure of F_i in Y is empty for every i. The sets that are not from the first Baire category in Y are called sets of *the second Baire category in* Y. The space Y is a *Baire space* if it satisfies the classical theorem of Baire that the intersection of countably many dense and open subsets of Y is a dense (and G_δ) subset of Y. Every complete metric space (as $C(X)$ above for example) and every (locally) compact Hausdorff topological space Y are Baire spaces. Evidently in Baire spaces Y the residual subsets of Y contain a dense G_δ-subset of Y and are considered to be big in Y from the Baire category sense and their complements–small from the same point of view. Roughly, it could be said that the residual subsets of a Baire space Y play analogous role in Y as the sets with complements of Lebesgues measure zero in finite dimensional spaces. Sometimes when a property P is fulfilled at the points of a residual subset of the Baire space Y we say that the property P is fulfilled almost everywhere (from the point of view of the Baire category).

Coming back to our setting above, the question is whether the sets D or W contain dense G_δ-subsets of the space $C(X)$. This will mean that the sup-norm in $C(X)$ is differentiable at most of the functions in $C(X)$ or that most of the functions in $C(X)$ attain their maximum at a unique point of X (even more, are well-posed). In particular every function in $C(X)$ can be approximated by a function where the sup-norm is differentiable and which attains its maximum at exactly one point. It turns out (see below Theorem 4.4 and 4.6) that the above is true exactly when the space X contains a dense subspace whose inherited topology is metrizable with a complete metric–we call such spaces *dense completely metrizable subspaces of* X.

Investigating this (and similar) question(s) we have observed that, in fact, what underlines the things is that we need to have a single-valued selections both to the mapping M and its converse M^{-1}, which are defined at the points of some residual subsets of the domains of the mappings. This led us to study the question of the existence of densely defined selections of set-valued mappings (sometimes not necessarily single-valued). It has turned out that the existence of such selections is applicable also in many other situations from Banach space theory, topology and optimization theory (see the results further and also the cited papers below).

The rest of these notes is organized as follows. In the next short section we give some standard notation and terminology related to set-valued mappings. In Section 2 we discuss a relaxation of the notion of lower semicontinuity and a relation between a set-valued mapping and a subset of the range space. Both these notions will be important for our investigations. Then in Section 3 we prove the main two selection theorems and give their immediate corollaries. In Section 4 we apply the results to give answers to the questions that were posed above, related to differentiability of the sup-norm in $C(X)$ and well-posedness of maximization problems generated by the functions from $C(X)$. In Section 5 we see further applications in geometry of Banach spaces: to the best approximation theory. The selection theorems have also different applications in topology: for example one can prove a Lavrentiev type theorem (see [ČKR2]) or can get as consequences some results of E. Michael from [M2], related to single-valued mappings (see again [ČKR2]). But instead of doing this, we prefer in the final Section 6 to present another application of the general approach which we use for the above results–the approach of maximal disjoint families of suitable sets and properties of set-valued mappings–to the study of the existence of winning strategies in the famous Banach-Mazur game in topological spaces.

The notes have been made self-contained presenting all (non trivial) proofs and using only well-know facts from functional analysis and topology. Most of the results below were jointly obtained by M.M. Čoban from University of Tiraspol, Moldova, P.S. Kenderov from the Institute of Mathematics and Informatics, BAS, Sofia, and the author. Almost all of them have been already published (some of them in a more general form) in a series of papers [ČKR1, ČKR2, ČKR3, ČKR4, KR1, KR2, KR3].

1 Some preliminaries

Let $\Phi : T \to X$ be a set-valued mapping from a topological space T, into the subsets of a topological space X. We will use sometimes the equivalent term multivalued mapping. In view of the fact that in many cases we have mappings with eventually empty values (as for example the mapping M above in the general case) we will consider also mappings $\Phi : T \to X$ that can have empty values at some points. Then the *domain* of Φ is

$\mathrm{Dom}(\Phi) := \{t \in T : \Phi(t) \neq \emptyset\}$.

For $A \subset T$, let $\Phi(A) := \bigcup\{\Phi(t) \colon t \in A\}$. Further, for $B \subset X$ put $\Phi^{-1}(B) := \{t \in T \colon \Phi(t) \bigcap B \neq \emptyset\}$ and $\Phi^{\#}(B) := \{t \in T \colon \Phi(t) \subset B\}$. Observe that $\Phi^{\#}(B)$ contains each point $t \in T$ with $\Phi(t) = \emptyset$. Let us mention also that $\Phi^{-1}(X) = \mathrm{Dom}(\Phi)$ and $\Phi^{\#}(X) = T$.

The mapping Φ is called *upper* (resp. *lower*) *semicontinuous* at a point $t_0 \in T$ if for every open $V \subset X$ with $\Phi(t_0) \subset V$ (resp. $\Phi(t_0) \bigcap V \neq \emptyset$) there is an open set $U \subset T$ with $t_0 \in U$ such that $\Phi(t) \subset V$ (resp. $\Phi(t) \bigcap V \neq \emptyset$) whenever $t \in U$. In this case we write Φ is usc (resp. lsc) at t_0. Φ is usc (resp. lsc) in T if it is usc (resp. lsc) at any point of T. Equivalently, Φ is usc (resp. lsc) in T if for every open $V \subset X$ the set $\Phi^{\#}(V)$ (resp. the set $\Phi^{-1}(V)$) is open in T.

Everywhere below we will consider only mappings Φ with domain $\mathrm{Dom}(\Phi)$ which is **dense** in the domain space T. The reason is obvious, since if t_0 is a point outside the closure (in T) of $\mathrm{Dom}(\Phi)$ then for some open set U of T containing t_0 we have $\Phi(t) = \emptyset$ for every $t \in U$ (i.e. Φ is usc and lsc at any such point). When we say that Φ is non-empty valued we mean that $\mathrm{Dom}(\Phi) = T$. Observe that in our setting, when $\mathrm{Dom}(\Phi)$ is dense in T, if Φ is usc at some $t_0 \in T$ then $\Phi(t_0) \neq \emptyset$. Hence, if Φ is usc in T then it is necessarily non-empty valued in T.

The mapping Φ is called *usco* in T if it is usc and compact-valued in T. For every usco mapping $\Phi : T \to X$ its *graph* $\mathrm{Gr}(\Phi)$, which is the set $\mathrm{Gr}(\Phi) := \{(t, x) \in T \times X : x \in \Phi(t)\}$, is a closed subset in the product topology of $T \times X$. In this case we say that the mapping Φ has a *closed graph*. Sometimes the closedness of $\mathrm{Gr}(\Phi)$ entails that Φ is usco. For example, if $\Phi : T \to X$ has a closed graph and X is compact then Φ is usco.

An usco $\Phi : T \to X$ is *minimal* if its graph does not contain properly the graph of any other usco $G : T \to X$. We mention that every non-empty valued mapping $G : T \to X$ with closed graph which is contained in an usco mapping $\Phi : T \to X$ (that is $G(t) \subset \Phi(t)$ for every $t \in T$) is usco itself. Hence, by the Kuratowski-Zorn Lemma every usco mapping $\Phi : T \to X$ contains a minimal usco $G : T \to X$. We remind an important characterization of the minimal usco mappings:

Proposition 1.1 (see [Chr, ChrK]) *Let $\Phi : T \to X$ be an usco set-valued mapping. Then the following are equivalent:*

(a) Φ *is a minimal usco mapping from T into X;*

(b) *for every two open sets $U \subset T$ and $V \subset X$ such that $U \cap \Phi^{-1}(V) \neq \emptyset$, there exists a non-empty open set $U' \subset U$ so that $\Phi(U') \subset V$.*

Let us mention that the minimal usco maps play an important role in the study of differentiability properties of convex functions in Banach spaces (see [Chr, ChrK, ČK1, ČK2, ČKR1, ČKR2, ČKR3, S1, S2]). The mapping M introduced above is a minimal usco mapping between $C(X)$ and X when X is compact. But even without the compactness of X, the mapping M still have the property (b) from the above proposition (see Proposition 4.1 below) and moreover, is minimal in a class of mappings (for the latter see [ČKR3, ČKR4]). Another type of minimal mappings could be found in [DrLa].

Before finish this section, let us introduce a piece of notation. For a subset A of the topological space Y we will denote by $\mathrm{int}_Y(A)$ the interior of the set A in Y, and by $\overline{A}^Y$ its closure in Y. If there is no danger of ambiguity the sub- (or super-) script Y will be omitted. Finally, in a metric space (X,d) for a subset $A \subset X$ the symbol $\mathrm{diam}(A)$ has the usual meaning of the diameter of the set A.

2 Lower demicontinuous mappings

Let us start this section with a notion of continuity for set-valued mappings. The mapping $\Phi : T \to X$ between the topological spaces T and X is said to be *lower demicontinuous* in T (see [ČKR2]) if for every open set $V \subset X$ the set $\mathrm{int}\overline{\Phi^{-1}(V)}$ is dense in $\overline{\Phi^{-1}(V)}$. An equivalent local definition is the following: Φ is lower demicontinuous in T if, and only if, it is *lower demicontinuous at any* $t_0 \in T$ by which we mean that for every open V with $\Phi(t_0) \cap V \neq \emptyset$ there exists an open set U of T such that $t_0 \in \overline{U}$ and the set $\{t \in U : \Phi(t) \cap V \neq \emptyset\}$ is dense in U. Obviously, every lower semicontinuous mapping is lower demicontinuous. The converse is not true.

The mapping Φ is called *demi-open* in T (see [HS]) if for every open set $U \subset T$ the set $\mathrm{int}\overline{\Phi(U)}$ is dense in $\overline{\Phi(U)}$. Obviously every mapping Φ which is open (i.e. maps open sets of T into open sets of X) is demi-open, the converse being not true in general. The following proposition could be proved using standard arguments. Therefore the proof is omitted.

Proposition 2.1 ([ČKR2], Proposition 3.2) *The set-valued mapping $\Phi : T \to X$ is demi-open if, and only if, the mapping $\Phi^{-1} : \overline{\Phi(T)}^X \to T$ is lower demicontinuous.*

Further, let us introduce a relation between the mapping Φ and a subspace $X_1 \subset X$ which will be important for our next considerations. Namely (see [ČKR2]), the mapping $\Phi : T \to X$ is said to *embrace* $X_1 \subset X$ if for every open set $W \subset X$ which contains X_1 the set $\{(t,x) \in \mathrm{Gr}(\Phi) : x \in W\}$ is dense in $\mathrm{Gr}(\Phi)$. We have

Proposition 2.2 ([ČKR2], Proposition 3.3) *Let $\Phi : T \to X$ be a set-valued mapping and X be regular. Then Φ embraces $X_1 \subset X$ if and only if for all open sets $V \subset X$ and $V_\lambda \subset V, \lambda \in \Lambda$, for which $V \cap X_1 = \cup\{V_\lambda \cap X_1 : \lambda \in \Lambda\}$ the set $\cup\{\Phi^{-1}(V_\lambda) : \lambda \in \Lambda\}$ is dense in $\Phi^{-1}(V)$.*

Proof: The proof is obviously reduced to the case when the family Λ consist of one element. So let Φ embrace X_1. Let V_1 and V_2 be open subsets of X such that $V_1 \subset V_2$ and $V_1 \cap X_1 = V_2 \cap X_1$. Suppose that $\Phi^{-1}(V_2) \setminus \overline{\Phi^{-1}(V_1)} \neq \emptyset$. Put $H := T \setminus \overline{\Phi^{-1}(V_1)}$ and take $t_0 \in H \cap \Phi^{-1}(V_2)$. Therefore there is $x_0 \in V_2 \cap \Phi(t_0)$. Take disjoint open sets W_1 and W_2 of X so that $x_0 \in W_1 \subset V_2$ and $X \setminus V_2 \subset W_2$. Let $W := V_1 \cup W_2$. Then $X_1 \subset W$. Consider the set $B := \{(t,x) \in \mathrm{Gr}(\Phi) : x \in W\}$. We have $(H \times W_1) \cap B = \emptyset$, while $(t_0, x_0) \in (H \times W_1) \cap \mathrm{Gr}(\Phi)$. The last is a contradiction.

Conversely, let the property from the proposition be fulfilled and take some open set $W \subset X$ so that $X_1 \subset W$. Let further U and V be open subsets of T and X respectively with $(U \times V) \cap \mathrm{Gr}(\Phi) \neq \emptyset$. Put $V_1 := W \cap V$. Evidently $V_1 \cap X_1 = V \cap X_1$ and hence $\Phi^{-1}(V_1)$ is dense in $\Phi^{-1}(V)$. Since $U \cap \Phi^{-1}(V) \neq \emptyset$ we get $U \cap \Phi^{-1}(V_1) \neq \emptyset$. The proof is completed. ■

Another simple (and easily proved) property related to embracing is the following fact:

Proposition 2.3 *Let $\Phi : T \to X$ embrace the subspace $X_1 \subset X$ and X is regular. Then $\Phi(T) \subset \overline{X_1}$.*

At the end of this section we give three sufficient conditions for a mapping $\Phi : T \to X$ to embrace a subspace X_1 of X. The proof is again straightforward and uses also Proposition 2.1 above.

Proposition 2.4 *Each one of the following conditions ensure that the mapping $\Phi : T \to X$ embraces $X_1 \subset X$:*
(i) $\Phi(T) \subset X_1$*;*
(ii) X_1 *is dense in* X *and the mapping* Φ *is demi-open;*
(iii) X_1 *is dense in* X *and the mapping* Φ^{-1} *is lower demicontinuous.*

3 Densely defined selections

In this section we are interested in the existence of continuous single-valued selections of a given set-valued mapping $\Phi : T \to X$ which are defined on a residual subset of the domain of Φ. Precisely, we are looking for a residual subset A of X and a single-valued continuous mapping $\phi : A \to X$ such that $A \subset \mathrm{Dom}(\Phi)$ and $\phi(t) \in \Phi(t)$ for every $t \in A$. For results asserting that the selection ϕ is in general set-valued and upper semicontinuous the reader may consult [ČKR2, ČKR3, KR2].

The above setting differs form the original one when one looks for a selection defined on the whole domain $\mathrm{Dom}(\Phi)$ of Φ (see e.g. the classical results of Michael [M1]). But the selections, which we will get, have the additional property that they take their values in a priori chosen subset of X. And this is important in the applications (see Remark 4.5 below).

Let us start with the first theorem for existence of densely defined continuous selections of set-valued mappings.

Theorem 3.1 ([ČKR2], Theorem 4.7) *Let $\Phi : T \to X$ be a lower demicontinuous mapping with closed graph and dense domain from the Baire space T into the regular space X. Suppose in addition that X contains a completely metrizable subspace X_1 which is embraced by Φ. Then there exist a dense G_δ-subset T_1 of T and a continuous single-valued mapping $\phi : T_1 \to X_1$ such that $T_1 \subset \mathrm{Dom}(\Phi)$ and ϕ is a selection of Φ on T_1.*

Proof: Let d be a complete metric in X_1 which is compatible with the inherited topology from X. Since Φ embraces X_1 then $\Phi(T) \subset \overline{X_1}$ (see Proposition 2.3 above). Hence, it is no loss of generality to assume that X_1 is dense in X.

The pair (U, V) will be called *admissible* if:

1) $U \subset T$ and $V \subset X$ are non-empty open subsets of T and X respectively;
2) the set $\{t \in U : \Phi(t) \cap V \neq \emptyset\}$ is dense in U.

Let $\{\gamma_n\}_{n \geq 0}$, where $\gamma_0 = \{(T, X)\}$, be a sequence of families of admissible pairs which is maximal with respect to the following properties:

a) for every n the family $\{U : (U, V) \in \gamma_n \text{ for some } V\}$ is pair-wise disjoint;

b) if $(U,V) \in \gamma_n$ then $\operatorname{diam}(V \cap X_1) < 1/n$;

c) for every $(U,V) \in \gamma_{n+1}$ there exists $(U',V') \in \gamma_n$ such that $U \subset U'$ and $\overline{V}^X \subset V'$.

We claim that for every n the set $H_n := \cup\{U : (U,V) \in \gamma_n \text{ for some } V\}$ is dense (and open) in T. To prove this we proceed by induction. For $n = 0$ this is obviously true. Suppose this is true for some $k \geq 0$ but H_{k+1} is not dense in T. Hence, there is an open set $U_0 \subset T$ such that $U_0 \cap H_{k+1} = \emptyset$. On the other hand, $U_0 \cap H_k \neq \emptyset$. Therefore, there exists some $(U_k, V_k) \in \gamma_k$ such that $U_0 \cap U_k \neq \emptyset$.

Consider the family $\Delta := \{V \subset X : V \text{ is open}, \operatorname{diam}(V \cap X_1) < 1/(k+1) \text{ and } \overline{V}^X \subset V_k\}$. It is easily seen that $V_k \cap X_1 = \cup\{V \cap X_1 : V \in \Delta\}$. Hence, by Proposition 2.2, we have that $\cup\{\Phi^{-1}(V) : V \in \Delta\}$ is dense in $\Phi^{-1}(V_k)$. Consequently, $U_0 \cap U_k \cap \Phi^{-1}(V_{k+1}) \neq \emptyset$ for some $V_{k+1} \in \Delta$. By the fact that Φ is lower demicontinuous it follows that for some $U_{k+1} \subset U_0 \cap U_k$ the pair (U_{k+1}, V_{k+1}) is admissible. Now, the family $\gamma_{k+1} \cup \{(U_{k+1}, V_{k+1})\}$ is strictly larger than γ_{k+1} and still satisfies a)-c). This is a contradiction showing that the sets H_n are dense (and open) subsets of T.

Put now $T_1 := \cap_{n=0}^{\infty} H_n$. Since T is a Baire space then T_1 is a dense G_δ-subset of T. By a) above, each $t \in T_1$ uniquely determines a sequence of admissible pairs $\{(U_n(t), V_n(t))\}_{n=0}^{\infty}$ such that $(U_n(t), V_n(t)) \in \gamma_n$ for every n and $t \in \cap_{n=0}^{\infty} U_n(t)$. Hence, the following mapping (which will turn out to be single-valued) $\phi : T_1 \to X$:

$$\phi(t) := \cap_{n=0}^{\infty} V_n(t), \quad t \in T_1,$$

is well-defined.

Fix $t \in T_1$. By b) and c) above and the fact that (X_1, d) is a complete metric space, it follows that $\cap_{n=0}^{\infty} V_n(t) \cap X_1$ is a one-point set in X_1, say x, and that the family $\{V_n(t) \cap X_1\}_{n=0}^{\infty}$ is a local base for x in X_1. Since X_1 is dense in X and X is regular, routine considerations show that $\cap_{n=0}^{\infty} V_n(t) = \{x\}$ and that again $\{V_n(t)\}_{n=0}^{\infty}$ is a local base, this time in X, for x. Hence the mapping ϕ is single-valued and takes its values in X_1. Moreover, ϕ is continuous. To this end, let $t_0 \in T_1$ and V be an open subset of X with $\phi(t_0) \in V$. Since $\{V_n(t_0)\}_{n=0}^{\infty}$ is a local base for $\phi(t_0)$ in X we have $V_n(t_0) \subset V$ for some n. Let now $t \in T_1 \cap U_n(t_0)$. Then by a) above, $U_n(t) = U_n(t_0)$, and hence $V_n(t) = V_n(t_0)$. Therefore, $\phi(t) \in V_n(t) = V_n(t_0) \subset V$. Consequently, ϕ is continuous in T_1.

We show finally that $\phi(t) \in \Phi(t)$ for every $t \in T_1$. Suppose the contrary and let $t_0 \in T_1$ be such that $\phi(t_0) \notin \Phi(t_0)$. Since Φ has a closed graph and $(t_0, \phi(t_0)) \notin \operatorname{Gr}(\Phi)$ there are open sets $U \subset T$ and V of X such that $t_0 \in U$, $\phi(t_0) \in V$ and $\Phi(U) \cap V = \emptyset$. As above, we have $V_n(t_0) \subset V$ for some n. But the couple $(U_n(t_0), V_n(t_0))$ is admissible, hence, the set $\{t \in U_n(t_0) : \Phi(t) \cap V_n(t_0) \neq \emptyset\}$ is dense in $U_n(t_0)$. Hence, in particular, there is a point $t' \in U \cap U_n(t_0)$ so that $\Phi(t') \cap V_n(t_0) \neq \emptyset$. This is a contradiction. Therefore, $\phi(t) \in \Phi(t)$ for every $t \in T_1$. The proof of the theorem is completed. ∎

In view of Proposition 2.4 the following theorem is an immediate corollary:

Theorem 3.2 ([ČKR2], Theorem 4.8) *Let $\Phi : T \to X$ be a lower demicontinuous and demi-open mapping with closed graph and dense domain from the Baire space T into the regular space X. Suppose in addition that X contains a dense completely metrizable subspace X_1. Then there exist a dense G_δ-subset T_1 of T and a continuous single-valued mapping $\phi : T_1 \to X_1$ such that $T_1 \subset \operatorname{Dom}(\Phi)$ and ϕ is a selection of Φ on T_1.*

The above results could be sharpened if we consider a smaller class of mappings. Remember that a set-valued mapping $\Phi : T \to X$ is minimal usco if its graph $\mathrm{Gr}(\Phi)$ does not contain properly the graph of any other usco from T into X. The minimal usco maps are characterized by property (b) in Proposition 1.1: for every open $V \subset X$ and every open $U \subset T$ with $\Phi(U) \cap V \neq \emptyset$ there is a non-empty open $U' \subset U$ such that $\Phi(U') \subset V$. Sometimes mappings that have this property are called *minimal* (even in the case when they are not usco (see [KO])). Let us mention that mappings with this last property may not be minimal in the usual sense of being with minimal graph in some class of mappings. For a class of mappings that are minimal in the usual sense and are characterized exactly as the minimal usco maps, the interested reader is refereed to [ČKR3, ČKR4].

So adopting the above idea we call the mapping $\Phi : T \to X$ minimal if for every open sets $U \subset T$ and $V \subset X$ with $U \cap \Phi^{-1}(V) \neq \emptyset$ there exits a non-empty open set $U' \subset U$ so that $\Phi(U') \subset V$. It is a routine matter to see that an equivalent way to say that $\Phi : T \to X$ is a minimal mapping is the following: if $\Phi(t_0) \cap V \neq \emptyset$ for some $t_0 \in T$ and some open $V \subset X$, it follows that there exists a non-empty open U in T such that $t_0 \in \overline{U}$ and $\Phi(U) \subset V$. An immediate consequence of this observation is that every minimal mapping $\Phi : T \to X$ with dense domain $\mathrm{Dom}(\Phi)$ is lower demicontinuous.

Now we have the following result:

Theorem 3.3 ([ČKR2], Theorem 5.3) *Let Φ be a minimal closed graph mapping between the Baire space T and the regular space X with dense domain* $\mathrm{Dom}(\Phi)$. *Suppose X contains a completely metrizable subspace X_1 which is embraced by Φ. Then there exist a dense G_δ-subset T_1 of T at the points of which Φ is single-valued and upper semicontinuous. Moreover, $\Phi(t) \in X_1$ whenever $t \in T_1$.*

Proof: By Theorem 3.1 there are a dense G_δ-subset T_1 of T and a continuous single-valued mapping $\phi : T_1 \to X_1$ such that $T_1 \subset \mathrm{Dom}(\Phi)$ and ϕ is a selection of Φ on T_1. Using the minimality of Φ we will show next that $\Phi(t) = \phi(t)$ for every $t \in T_1$.

Indeed, suppose for some $t_0 \in T_1$ there exists $x_0 \in \Phi(t_0)$ with $x_0 \neq \phi(t_0)$. Take non-empty open subsets V_1, V_2 of X such that $x_0 \in V_1$, $\phi(t_0) \in V_2$ but $V_1 \cap V_2 = \emptyset$. By the minimality of Φ there exists an open $U_1 \subset T$ such that $t_0 \in \overline{U}_1$ and $\Phi(U_1) \subset V_1$. On the other hand, the continuity of ϕ gives the existence of a non-empty open U_2 such that $t_0 \in U_2$ and $\phi(U_2 \cap T_1) \subset V_2$. Obviously, for $t^* \in U_1 \cap U_2 \cap T_1 \neq \emptyset$ we have $\phi(t^*) \in \Phi(t^*) \cap V_2 \subset V_1 \cap V_2 = \emptyset$. This is a contradiction.

To prove that Φ is upper semicontinuous at the points of T_1 take some arbitrary $t_0 \in T_1$ and let V be an open subset of X such that $\Phi(t_0) = \phi(t_0) \in V$. Since X is regular there is an open set $W \subset X$ with $\phi(t_0) \in W \in \overline{W} \subset V$. By the continuity of ϕ there exists an open set U which contains t_0 and $\phi(U \cap T_1) \subset W$. We claim that $\Phi(U) \subset \overline{W}$ (i.e. Φ is upper semicontinuous at t_0). To see this we assume that there exists $x_0 \in \Phi(U) \setminus \overline{W}$ and proceed as above to get a contradiction. The proof is completed. ∎

The proof of the above theorem shows that if a minimal mapping $\Phi : T \to X$ possesses a continuous selection defined on a dense subset of the domain of the mapping, then Φ coincides with this selection at the points where the latter is defined. It seems, for the first time phenomenon like this has been observed for the subdifferential mapping related to a convex function in a Banach space (see e.g. Phelps [Ph]).

Here, again having in mind Proposition 2.4, we get an immediate corollary:

Theorem 3.4 *Let Φ be a demi-open minimal closed graph mapping between the Baire space T and the regular space X with dense domain* $\mathrm{Dom}(\Phi)$*. Suppose X contains a dense completely metrizable subspace X_1. Then there exist a dense G_δ-subset T_1 of T at the points of which Φ is single-valued and upper semicontinuous. Moreover, $\Phi(t) \in X_1$ whenever $t \in T_1$.*

4 Applications to the geometry of Banach spaces and optimization

In this section we consider applications of the selection theorems related to the study of differentiability properties of the sup-norm in spaces of continuous functions and well-posedness of the corresponding maximization problems. We briefly remind the setting.

Let X be a completely regular topological space and $C(X)$ denotes the space of all continuous and bounded real-valued functions in X. We equip $C(X)$ with the usual sup-norm $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$, $f \in C(X)$, under which $C(X)$ is a Banach space. In $C(X)$ we consider the *solution mapping* $M : C(X) \to X$ defined by

$$M(f) := \{x \in X : f(x) = \sup(X, f)\}.$$

It provides the solutions to every maximization problem (X, f), $f \in C(X)$. Obviously M is onto. The next proposition lists some of the important properties of M:

Proposition 4.1 ([KR1], Proposition 2.1) *Let X be a completely regular topological space. Then the mapping M has the following properties:*

(a) $\mathrm{Gr}(M)$ *is a closed subset of* $C(X) \times X$*;*

(b) $\mathrm{Dom}(M)$ *is dense in* $C(X)$*;*

(c) *M is open;*

(d) *for every non-empty open set U in X the set* $\mathrm{int}M^{\#}(U)$ *is non-empty and dense in* $M^{\#}(U) \bigcap \mathrm{Dom}(M)$*;*

(e) *for every two open sets U in $C(X)$ and W in X with $U \bigcap M^{-1}(W) \neq \emptyset$ there is a non-empty open set $U' \subset U$ such that $M(U') \subset W$.*

(f) *if $f_0 \in C(X)$ is such that $\{f_0\} = \cap_{n=1}^\infty B_n$, $B_n \subset C(X)$ and* $\mathrm{diam}(B_n) \to 0$ *then* $M(f_0) = \cap_{n=1}^\infty M(B_n)$

Proof: (a) is well-known. As to (b), let $f \in C(X)$ and $\varepsilon > 0$ be arbitrary. Then, obviously $M(f_\varepsilon) \neq \emptyset$ for $f_\varepsilon(x) := \inf\{f(x), \sup(X, f) - \varepsilon\}$.

We prove (c). Let U be an open subset of $C(X)$ and $x_0 \in M(f_0)$ for some $f_0 \in U$. Take $\varepsilon > 0$ such that the ball $B(f_0, \varepsilon) := \{f \in C(X) : \|f - f_0\| < \varepsilon\} \subset U$. Then each $x' \in \{x \in X : f_0(x) > \sup(X, f_0) - \varepsilon\}$ is a maximizer of some f from U, e.g. of the function $(f_0)_\varepsilon$ considered above.

Since it is easily seen that (d) is a consequence of (e) we prove (e).

Let $x_0 \in M(f_0) \bigcap W$ for some $f_0 \in U$ where U and W are open subsets of $C(X)$ and X. Since X is completely regular there exists a function $h_0 \in C(X)$ such that $h_0(x_0) = 1$, $h_0(X \setminus W) = 0$ and $\|h_0\|_\infty = 1$. Find $\delta > 0$ such that $f_0 + \delta h_0 \in U$. Let further, $U' \subset U$ be an open set in $C(X)$ containing $f_0 + \delta h_0$ and such that $\mathrm{diam}(U') < \delta/3$.

Take $f \in U'$. Since for $x \in X \setminus W$ one has $f(x) \leq (f_0 + \delta h_0)(x) + \delta/3 = f_0(x) + \delta/3 \leq f_0(x_0) + \delta/3 = (f_0 + \delta h_0)(x_0) - (2\delta)/3 < (f_0 + \delta h_0)(x_0) - \delta/3 \leq f(x_0)$, we see that $M(f) \subset W$.

Finally, let us prove (f). Since obviously $M(f_0) \subset \cap_{n=1}^{\infty}(B_n)$ we prove the converse inclusion. Take $x \in M(B_n)$. Then $x \in M(f_n)$ for some f_n with $||f_n - f_0||_\infty \leq \mathrm{diam}(B_n)$. Hence $x \in \{y \in X : f_0(y) \geq \sup(X, f_0) - 2\mathrm{diam}(B_n)\}$. Therefore $M(B_n) \subset \{y \in X : f_0(y) \geq \sup(X, f_0) - 2\mathrm{diam}(B_n)\}$. This entails $\cap_{n=1}^{\infty} M(B_n) \subset \cap_{n=1}^{\infty}\{y \in X : f_0(y) \geq \sup(X, f_0) - 2\mathrm{diam}(B_n)\} = M(f_0)$.

The proof of the proposition is completed. ■

It is immediately seen from (e) above that the mapping M is minimal wrt the definition given in the previous section.

Remember that the problem (X, f) to maximize a function f over the space X is called well-posed if it has unique maximizer towards which every maximizing sequence converges. One might wonder whether the above requirement is not imposed also on the maximizing nets (because X might not be first countable). There is no need to do this: it can be checked that if the maximization problem (X, f), $f \in C(X)$, is well-posed then every maximizing net also converges to the unique maximizer. Before we prove the following fact, let us recall a well-known notion from general topology. For a given completely regular topological space X its *Stone-Čech compactification* βX is a Hausdorff compact toopological space which contains X as a dense subspace and which is characterized by the fact that every continuous bounded function $f \in C(X)$ has a unique continuous extension $e(f)$ over βX. Observe that through the mapping $f \mapsto e(f)$ the Banach spaces $C(X)$ and $C(\beta X)$ are congruent.

The following proposition relates the well-posedness of a maximization problem (X, f) with properties of the solution mapping and also with the behavior of the maximizers of the continuous extension $e(f)$ of f in βX.

Proposition 4.2 *Let X be completely regular. Then the following are equivalent:*

(a) *the maximization problem (X, f), $f \in C(X)$, is well-posed;*

(b) *the solution mapping $M : C(X) \to X$ is single-valued and usc at f;*

(c) *the unique continuous extension $e(f)$ of f in βX has unique maximizer lying necessarily in X.*

Proof: (a)$\Rightarrow$(c). Let (X, f) be well-posed with unique solution $x_0 \in X$ and suppose that there is $x_1 \in \beta X$ which is a maximizer to $e(f)$ different from x_0. Since X is dense in βX there is a net $\{x_\lambda\} \subset X$ which converges to x_1. By the continuity of $e(f)$ we get $e(f)(x_\lambda) \to e(f)(x_1) = \sup(\beta X, e(f)) = \sup(X, f)$. Thus the net $\{x_\lambda\}$ is maximizing for the problem (X, f). The last is a contradiction with the well-posedness of (X, f) since this net does not converge to x_0.

(c)$\Rightarrow$(b) Observe that if we consider the solution mapping $M' : C(\beta X) \to \beta X$, then this solution mapping is usco (since by Proposition 4.1 it has a closed graph and the range space is compact). Having in mind that if M is the solution mapping from $C(X)$ into X then $M(f) = M'(e(f)) \cap X$, the only thing we need is to show that every time when a problem in βX is with unique solution this unique solution lies in X. Any unique maximizer of a continuous function h in a topological space Y is a G_δ-point in Y since the set of maximizers of h is the set: $\cap_{n=1}^{\infty}\{y \in Y : h(y) > \sup(Y,h) - 1/n\}$. On the other hand it is a well-known fact from the topology (see e.g. [ArP]) that there are no G_δ-points in $\beta X \setminus X$. Hence, any unique maximizer of a function $e(f)$, $f \in C(X)$, automatically lies in X, which proves the implication.

(b)$\Rightarrow$(a) Let M be single-valued and usc at $f \in C(X)$. Take a maximizing sequence $\{x_n\}$ for the problem (X, f) and consider the continuous (in X) and bounded real-valued functions $f_n(x) := \inf\{f(x), f(x_n)\}$, $x \in X$. Observe that $x_n \in M(f_n)$ for every n. Since $f(x_n) \to \sup(X, f)$ we get $f_n \to f$ in $C(X)$. Hence, by the upper semicontinuity of M at f for every open V in X containing $M(f)$ we have $M(f_n) \subset V$ for large n. Thus $x_n \in V$ for large n. The proof is completed. ∎

We are going further by formulating the connection between well-posedness and differentiability properties of the sup-norm in $C(X)$. Let us denote by $J : X \to C^*(X)$ the natural embedding of X into $C^*(X)$–the dual of $C(X)$ defined by: $\langle f, Jx\rangle = f(x)$, $x \in X$, $f \in C(X)$. Here $\langle\cdot,\cdot\rangle$ is the usual pairing between $C(X)$ and $C^*(X)$. It is well-known that if X is a compact space then the sup-norm in $C(X)$ is Gâteaux differentiable at $f \in C(X)$, $f \not\equiv 0$, if and only if the maximization problem $(X, |f|)$ has unique solution. And if the unique solution is $x_0 \in X$ then the derivative of the norm is $\nabla\|f\|_\infty = Jx_0$ if $f(x_0) > 0$ and $\nabla\|f\|_\infty = -Jx_0$ if $f(x_0) < 0$.

This result can be extended to non-compact X in the following way:

Proposition 4.3 *Let X be a completely regular topological space. Then the sup-norm in $C(X)$ is Gâteaux differentiable at $f \in C(X)$, $f \not\equiv 0$, if and only if the maximization problem $(X, |f|)$ is well-posed with unique solution x_0. And again $\nabla\|f\|_\infty = Jx_0$ if $f(x_0) > 0$ and $\nabla\|f\|_\infty = -Jx_0$ if $f(x_0) < 0$.*

The interested reader may try to prove this fact directly. Another proof, which uses the known fact for compact spaces X, follows almost immediately by Proposition 4.2 above and the fact that $C(X)$ and $C(\beta X)$ are congruent (i.e. undistinguishable as Banach spaces).

Now we turn back to the questions posed in the introduction of these notes. Namely, when is the set $W := \{f \in C(X)$: the maximization problem (X, f) is well-posed$\}$ (or the set $D := \{f \in C(X)$: the sup-norm is Gâteaux differentiable at $f\}$ residual in $C(X)$? In other words when the sets W and D contain a dense G_δ-subset of $C(X)$? To prove these results we use the selection theorems from the previous section.

Theorem 4.4 ([ČKR1], Theorem 3.5) *For the completely regular topological space X the following statements are equivalent:*

(a) *X contains a dense completely metrizable subspace X_1;*

(b) *$W = \{f \in C(X) : (X, f)$ is well-posed$\}$ contains a dense G_δ-subset of $C(X)$.*

I.e., in other words, most of maximization problems in $C(X)$ are well-posed exactly when the space X contains a dense completely metrizable subspace.

Proof: Suppose that (a) is fulfilled, i.e. there exists a dense completely metrizable subspace X_1 of X. Consider the mapping $M : C(X) \to X$. By Proposition 4.1 it is open and minimal. Therefore by Theorem 3.4 (applied to $T := C(X)$ and $\Phi := M$) we find some dense G_δ-subset T_1 of $C(X)$ at the points of which M is single-valued and usc. Hence, by Proposition 4.2, (b) takes place.

Conversely, suppose now that (b) is fulfilled. Let T_1 be a dense G_δ-subset of $C(X)$ which is contained in W. Let us point out that being a G_δ-subset of a complete metric space the set T_1 is itself completely metrizable.

Consider now the Stone-Čech compactification βX of X and the corresponding solution mapping $M' : C(\beta X) \to \beta X$. Since $C(X)$ and $C(\beta X)$ are congruent we can consider M' as a mapping from $C(X)$ to βX assigning to each $f \in C(X)$ the maximizers of the unique continuous extension $e(f)$ of f in βX.

By Proposition 4.1 the mapping M' is open and minimal usco. Consider the converse mapping $\Phi := M'^{-1} : \beta X \to C(X)$. Its domain $\mathrm{Dom}(M'^{-1})$ is the whole space βX since M' is onto. It has a closed graph since the same does M'. And since M' is open M'^{-1} is lower demicontinuous (even lower semicontinuous). Finally, the mapping M'^{-1} is demi-open since M' is minimal and hence lower demicontinuous (cf. Proposition 2.1). The last means that M'^{-1} embraces T_1 (see Proposition 2.4). Hence we can apply Theorem 3.2 and conclude that there exist a dense G_δ-subset X_1 of βX and a continuous single-valued mapping $\phi : X_1 \to T_1$ such that $\phi(x) \in M'^{-1}(x)$ for every $x \in X_1$. Since $T_1 \subset W$ and by Proposition 4.2, the mapping M' coincides with the original solution mapping $M : C(X) \to X$ on T_1, the last being single-valued, we get that at the points of T_1 the mapping M' is single-valued as well. Hence we conclude that ϕ is a one-to-one mapping between X_1 and $\phi(X_1) \subset T_1$. Moreover, taking into account that M' is upper semicontinuous we see that the inverse map ϕ^{-1} is also continuous. Therefore ϕ is a homeomorphism between X_1 and $\phi(X_1)$. Another conclusion is the following: let $x \in X_1$, hence $\phi(x) \in T_1$ and by the fact that $M'(\phi(x)) = M(\phi(x))$ we get $x \in X$. Thus $X_1 \subset X$.

So, ϕ homeomorphically embeds X_1 into a complete metric space. In particular, X_1 is metrizable. As a G_δ-subset of the compact space βX the space X_1 is a *Čech complete* metrizable space (for some origins of the latter notion see e.g. [Fro1, Fro2]). But it is well-known that every Čech complete metrizable space is completely metrizable. Thus X_1 is completely metrizable. The proof is completed. ∎

Remark 4.5 Observe that the classical Michael selection theorem ensures the existence of an everywhere defined continuous selection of the mapping M'^{-1} above since for any $x \in \beta X$ the set $M^{-1}(x)$ is closed convex in $C(X)$ and M'^{-1} is lower semicontinuous. But the values of this selection are not obliged to lie in T_1–a fact which is of vital importance for the conclusion that ϕ is a homeomorphism.

Analogous result to Theorem 4.4 is true when we investigate the set of functions at which the sup-norm is Gâteaux differentiable.

Theorem 4.6 ([ČKR1]) *For the completely regular topological space X the following statements are equivalent:*

(a) *X contains a dense completely metrizable subspace X_1;*

(b) *the sup-norm in $C(X)$ is Gâteaux differentiable at the points of a dense G_δ-subset of $C(X)$.*

Proof: The proof is quite similar to that one of the above theorem. Put:

$$U_1 := \{f \in C(X) : \sup(X, f) > \sup(X, -f)\}$$

and

$$U_2 := \{f \in C(X) : \sup(X, f) < \sup(X, -f)\}.$$

Then the set $U_1 \cup U_2$ is open and dense in $C(X)$. Moreover, if $f \in U_1$ then the maximization problem $(X, |f|)$ is well-posed if and only if the problem (X, f) is well-posed (and they have the same solution). Symmetrically, if $f \in U_2$ then the maximization problem $(X, |f|)$ is well-posed if and only if the problem $(X, -f)$ is well-posed (and again they have the same solution).

Now, if the set $D = \{f \in C(X)$: the sup-norm is Gâteaux differentiable at $f\}$ contains a dense G_δ-subset T_1 of $C(X)$ then $D \cap U_1$ contains a dense G_δ-subset of U_1 (the latter being completely metrizable). Now one acts as in the proof of (b)$\Rightarrow$(a) in the above theorem restricting the mapping M' to U_1 which restriction is again a minimal usco mapping (since U_1 is open: this is an interesting property to be done as an exercise).

Conversely, let X contain a dense completely metrizable subspace X_1. Then one gets, using Theorem 4.4 above, that there exist dense G_δ-subsets T_1 and T_2 of $C(X)$ so that for every $f \in T_1$ the maximization problem (X, f) is well-posed and for every $f \in T_2$ the maximization problem $(X, -f)$ is well-posed. To finish just take $T_1 \cap T_2 \cap (U_1 \cup U_2)$ and remember that $U_1 \cup U_2$ was open and dense subset of $C(X)$. ■

Finally, let us formulate the corresponding facts about Fréchet differentiability of the norm in $C(X)$.

Proposition 4.7 ([ČKR1]) *Let X be a completely regular topological space. Then the sup-norm in $C(X)$ is Fréchet differentiable at $f \in C(X)$, $f \not\equiv 0$, if and only if the maximization problem $(X, |f|)$ is well-posed with unique solution x_0 and the point x_0 is an isolated point of X. Moreover, $\nabla\|f\|_\infty = Jx_0$ if $f(x_0) > 0$ and $\nabla\|f\|_\infty = -Jx_0$ if $f(x_0) < 0$.*

Proof: In view of Proposition 4.2 and the remarks before and after Proposition 4.3, we may think that X is compact. Let the norm be Fréchet differentiable at f, $f \not\equiv 0$. We may think (having in mind Proposition 4.3) that f belongs to the set U_1 defined above. The situation when $f \in U_2$ is symmetric. Then, again by Proposition 4.3, the maximization problem (X, f) has unique solution at x_0. We will show that x_0 is isolated in X. Observe that the point x_0 is a G_δ-point in a compact space, hence first countable. Consequently, there exists a sequence $\{x_n\} \subset X \setminus \{x_0\}$ which converges to x_0 and such that $f(x_0) - f(x_n) < 2^{-n}$. Let $\varepsilon_n = 2^{-n} + f(x_0) - f(x_n) > 0$. Obviously we have $\varepsilon_n < 2^{-n+1}$. Construct now, for every n, functions $\phi_n : X \to [0, \varepsilon_n]$ such that $\phi_n(x_0) = 0$ and $\phi(x_n) = \varepsilon_n$. Therefore,

1) $(\phi_n + f)(x_n) = f(x_0) + 2^{-n}$

2) $\|\phi_n\|_\infty = \varepsilon_n$.

In particular, $\|\phi\|_\infty < 2^{-n+1}$. Let n be so large that $f + \phi_n \in U_1$. Consequently,

$$\lim_n \frac{|\, \|f + \phi_n\|_\infty - \|f\|_\infty - \langle \phi_n, Jx_0 \rangle \,|}{\|\phi_n\|_\infty} \geq$$

$$\lim_n \frac{(f+\phi_n)(x_n)-f(x_0)}{\|\phi_n\|_\infty} =$$

$$\lim_n \frac{2^{-n}}{\|\phi_n\|_\infty} \geq \frac{2^{-n}}{2^{-n+1}} = 2^{-1}.$$

The last contradicts Fréchet differentiability of the norm at f.

Conversely, let $(X,|f|)$ have unique solution at x_0 and the point x_0 is isolated in X. Again we may think that $f \in U_1$. Put $Y := X \setminus \{x_0\}$. We may suppose that Y is non-empty since otherwise the proof is completed. Since x_0 is isolated and the maximization problem (X,f) is well-posed then $b = \sup(X,f) - \sup(Z,f) > 0$. Now if $\|\phi\|_\infty < b/2$ then $\sup(X, f+\phi) = f(x_0) + \phi(x_0)$ and hence (if in addition $f + \phi \in U_1$)

$$\lim_{\|\phi\|_\infty \to 0} \frac{|\|f+\phi\|_\infty - \|f\|_\infty - \langle \phi, Jx_0\rangle|}{\|\phi\|_\infty} = 0.$$

Therefore, the norm is Fréchet differentiable at f. The proof is completed. ■

Now using the fact that for the functions $f \in C(X)$ at which M is single-valued and usc the maximization problem (X,f) is well-posed, one can easily deduce the following fact from the results that have been already presented here.

Theorem 4.8 **([ČKR1])** *Let X be a completely regular topological space. Then the sup-norm in $C(X)$ is Fréchet differentiable at the points of a dense and (necessarily) open subset of $C(X)$ if and only if the set of isolated points of X is dense in X.*

5 Further applications to geometry of Banach spaces: best approximation problems

In this section we consider an application of the selection theorems which is related to another problem in geometry of Banach spaces: the best approximation problem.

Let $(X, \|\cdot\|)$ be a real Banach space and denote by S the unit sphere in X, i.e. the set $\{x \in X : \|x\| = 1\}$. Recall that the norm in X is called *locally uniformly rotund* or equivalently *locally uniformly convex* if for every $x_0, x_n \in S$ such that $(1/2)\|x_0 + x_n\| \to 1$, it follows that $x_n \to x_0$. The norm in X is said to be *strictly convex* (or equivalently, *rotund*) if the sphere S does not contain line segments. An equivalent way to say that the norm is strictly convex is the following: for any $x, y \in X$, $x, y \neq 0$, $x \neq y$, it follows that $\|x+y\| < \|x\| + \|y\|$. Obviously every locally uniformly rotund norm is strictly convex, the converse being not true in general.

Let further A be a non-empty closed subset of X. The *best approximation problem* in X generated by A is the following one: for any $x \in X$ find the closest to x (with respect to the norm) element in A, i.e. find $a \in A$ so that $\|x - a\| = d(x, A)$, where $d(x,A) := \inf\{\|x - a'\| : a' \in A\}$ is the distance function in X generated by A. The corresponding solution mapping $P_A : X \to A$, defined by $P_A(x) := \{a \in A : \|x - a\| := d(x, A)\}$ is called *metric projection* generated by the set A. We are interested in the question of the uniqueness of the best approximation. Since, in general, the metric projection may be empty at some points, precisely we are interested what is the set $\{x \in X : P_A(x) = \emptyset$ or $P_A(x)$

is a singleton}. The conjecture that this last set is residual in X (i.e. contains a dense G_δ-subset of X) provided the norm in X is strictly convex was formulated almost 30 years ago by Stečkin [St]. So far there have been a number of results proving this conjecture in many situations, but still not in its full generality–see e.g [FaZh, Ko, L, St, Za, Zh].

We will show below how the result which confirms the above conjecture in the setting of locally uniformly rotund Banach spaces can be deduced from our selection theorems. The result originally belongs to Stečkin [St] but we give it here in a slightly general form which is due to Zhivkov [Zh].

Theorem 5.1 *Let X have locally uniformly rotund norm and A be its non-empty closed subset. Then there exists a dense G_δ-subset X_1 of X such that at any point of X_1 the metric projection P_A is no more than single-valued and usc.*

Proof: Put $Y := \overline{\mathrm{Dom}(P_A)}$. The set $X \setminus Y$ is open and if it is non-empty then obviously for any $x \in X \setminus Y$ we have $P_A(x) = \emptyset$ and P_A is usc at x. So consider $Y_1 := \mathrm{int}\overline{\mathrm{Dom}(P_A)}$. Observe that the set $Y \setminus Y_1$ is nowhere dense in X. So if $Y_1 = \emptyset$ we are done. Hence let us suppose that $Y_1 \neq \emptyset$. Now, if we prove that there is a dense G_δ-subset Y_1 of Y such that at any point of Y_1 the metric projection mapping P_A is single-valued and usc in Y, then we get the conclusion of the theorem by simply putting $X_1 := (X \setminus Y) \cup Y_1$.

Therefore, consider P_A in Y. The set Y is closed in $C(X)$, hence it is a complete metric space. Moreover, the mapping P_A obviously has a closed graph and by definition $\mathrm{Dom}(P_A)$ is dense in Y. We will see further that the mapping P_A is minimal. For, let $x_0 \in Y$ and $P_A(x_0) \neq \emptyset$. Take any $y \in P_A(x_0)$. We will show below that at any point x from the left-open line segment $(x_0, y]$ we have that the metric projection P_A is single-valued with $P_A(x) = y$ and P_A is usc at x (this fact is contained in Lemma 1.7 from the paper of Zhivkov [Zh]).

First, we may think, without loss of generality that the left-open segment $(x_0, y]$ is non-empty. If not, then $x_0 = y \in A$. Then $P_A(x_0) = y$. Take some sequence $\{x_n\}$ converging to $x_0 = y$ and let $y_n \in P_A(x_n)$. Then, because the distance function is continuous, we get

$$\|y_n - y\| = \|y_n - x_0\| \leq \|y_n - x_n\| + \|x_n - x_0\| = d(x_n, A) + \|x_n - x_0\| \to d(x_0, A) = 0.$$

Hence P_A is usc at x_0. From here we easily deduce that P_A is minimal at x_0.

So fix an element $x \in (x_0, y]$ and take some $y' \in A$. Then obviously $\|y' - x_0\| \geq \|y - x_0\|$ since y is a best approximation of x_0. But $\|y - x_0\| = \|y - x\| + \|x - x_0\|$, while $\|y' - x_0\| \leq \|y' - x\| + \|x - x_0\|$. Hence, $\|y' - x\| \geq \|y - x\|$ showing that $y \in P_A(x)$. Observe that the above arguments show that if y' is outside the closed ball $B[x_0; \|y - x_0\|]$ then it cannot be a best approximation of x since the inequality we got would be strict. Hence, if there are other best approximations of x in A they must be on the surface of the ball $B[x_0, \|y - x_0\|]$. But if y' is such that $y' \neq y$, $\|y' - x_0\| = \|y - x_0\|$ and $y' \in P_A(x)$ then we have:

$$\|y' - x_0\| = \|y - y_0\| = \|y - x\| + \|x - x_0\| = \|y' - x\| + \|x - x_0\|$$

which is a contradiction with the strict convexity of the norm.

Now, let $x_n \to x$ and $y_n \in P_A(x_n)$. Again by the continuity of the distance function we get:

$$\|y - x\| = d(x, A) = \lim d(x_n, A) = \lim \|y_n - x_n\| = \lim \|y_n - x\|$$

the last equality being true since $x_n \to x$. Observe that the points y_n are outside the ball $B(x_0, \|y - x_0\|)$ and are "tending" to the surface of the inner ball $B[x, \|y - x\|]$ which has only one common point with the bigger one–the point y. This together with local uniform rotundity of the space show that $y_n \to y$. Hence P_A is usc at x.

Using the above, we automatically conclude that the mapping P_A is minimal. Then by Theorem 3.3 (observe that the range space is a complete metric space) we get the existence of a dense G_δ-subset of Y at each point of which the mapping P_A is single-valued and usc. The proof of the theorem is completed. ∎

6 The Banach-Mazur game

In this section we consider the famous Banach-Mazur game in a topological space X. The result we will get is not a direct consequence of the selection theorems from Section 3. In fact, we will show, in the particular situation of the solution mapping, how we can get a selection (without any continuity properties required) of the mapping, which is defined on a residual subset of the domain space, using the general approach of maximal disjoint families. This will be done, provided one of the player in the Banach-Mazur game has a winning strategy in the underlying space. In other words, the existence of a winning strategy for one of the players in the game entails that the set of continuous and bounded functions which attain their maximum in the underlying space X contains a dense G_δ-subset of $C(X)$. Moreover, it will turn out that the latter fact is a characterization of the existence of such a winning strategy. For more general results of this type, the reader is refereed to [ČKR4].

Let us be more precise and describe the setting which we study–this is the following well-known modification of the Banach-Mazur game.

Given a topological space X, two players, named α and β, play a game in X in the following way: β chooses first a non-empty open subset U_1 of X. Then α chooses a non-empty open subset V_1 with $V_1 \subset U_1$. Further, β chooses a non-empty open subset U_2 of X with $U_2 \subset V_1$ and α chooses a non-empty open $V_2 \subset U_2$ and so on. The infinite sequence $p = \{U_n, V_n\}_{n=1}^{\infty}$ obtained in this way is called a *play*. Denote by $T(p) := \bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} U_n$ the *"target" set* for this play. The player α wins the play p if $T(p) \neq \emptyset$. Otherwise β wins. Every finite sequence of sets $(U_1, V_1, \ldots, U_n, V_n)$, $n \geq 1$, obtained by the first n steps in this game is called a *partial play* in the game.

Denote this game by $BM(X)$. For notations and terminology we refer to the survey [Tel]. Under a *strategy* for the player α in the game $BM(X)$ we understand a mapping s which assigns to every chain $(U_1, V_1, \ldots, U_n)$ corresponding to the first n legal moves of β and the first $n-1$ moves of α, $n \geq 1$, a non-empty open set $V_n \subset U_n$. The play $p = \{U_n, V_n\}_{n=1}^{\infty}$ obtained by the strategy s (i.e. $V_n = s(U_1, V_1, \ldots, U_n)$ for every $n \geq 1$) is called an *s-play*. The strategy s is called *winning strategy* for the player α (or *α-winning strategy*) if for every s-play $p = \{U_n, V_n\}_{n=1}^{\infty}$ the corresponding target set $T(p)$ is not empty.

A *stationary winning strategy* (called also *α-winning tactic* (see [Ch])) for the player α in the game $BM(X)$ is a winning strategy for α which on each step depends only on the last move of the payer β. Precisely, a stationary winning strategy for the player α is a mapping t from the family of all non-empty open subsets of X into the family of non-empty open subsets of X such that for every non-empty open $U \subset X$ one has $t(U) \subset U$ and, moreover, whenever one has a sequence $\{U_n\}_{n=1}^{\infty}$ such that $U_{n+1} \subset t(U_n)$ for every n, then $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$. In this case the corresponding play $p = \{U_n, t(U_n)\}_{n=1}^{\infty}$ will be called a t-play. Evidently, every α-winning tactic t determines the α-winning strategy $s(U_1, V_1, \ldots, U_n) := t(U_n)$. There are, however, completely regular spaces X with an α-winning strategy which do not admit any α-winning tactic (see [De]).

The space X is called *weakly α-favorable* (see [Wh]) (resp. *α-favorable* [Ch]) if X admits a winning strategy (resp. stationary winning strategy) for the player α in the game $BM(X)$. Every weakly α-favorable space X is a Baire space (see e.g. [Tel]). The converse is not true–as a counterexample can serve any Bernstein subset of the unit interval [0,1]. A Bernstein subset $A \subset [0,1]$ is such set that for any compact set $K \subset [0,1]$ which has the power of the reals, the set A intersects both K and its complement in [0,1]. Any such set is a Baire space but does not admit an α-winning strategy.

There have been different particular results giving sufficient or necessary and sufficient conditions for the player α to have a winning strategy in the Banach-Mazur game. For example if X is a metric space it follows from a result of Oxtoby [Ox] that the player α has a winning strategy in $BM(X)$ if and only if the space X contains a dense completely metrizable subspace. This result was extended for a special class of topological spaces by White [Wh]. Other conditions could be found in [GaTel]. But to the best knowledge of the author, there was not a general characterization of the above fact. The following result which was proved in our joint paper with P. Kenderov gives a characterization of the weak α-favorable topological spaces.

Theorem 6.1 ([KR3], Theorem 3.1) *Let X be a completely regular topological space. Then, the player α has a winning strategy in the Banach-Mazur game $BM(X)$ if and only if the set $\{f \in C(X)$: the maximization problem (X, f) has a solution$\}$ contains a dense G_δ-subset of $C(X)$.*

In other words, the space X is weakly α-favorable exactly when most of the functions from $C(X)$ attain their maximum in X.

Proof: Suppose that s is a winning strategy for the player α in the game $BM(X)$. Consider the solution mapping $M : C(X) \to X$ and remember that (Proposition 4.1) this mapping is open and minimal.

Lemma 6.2 *Let $(U_1, V_1, \ldots, U_n, V_n)$, $n \geq 1$, be a partial play in the game $BM(X)$ and W_n be a non-empty open subset of $C(X)$ such that $M(W_n) \subset V_n$. Then there is a family $\Gamma(W_n)$ of triples $(U_{n+1}, V_{n+1}, W_{n+1})$ such that:*

(i) *U_{n+1} is a non-empty open subset of $M(W_n)$;*

(ii) *$V_{n+1} = s(U_1, V_1, \ldots, U_n, V_n, U_{n+1})$;*

(iii) *W_{n+1} is a non-empty open subset of $C(X)$ such that* $\operatorname{diam}(W_{n+1}) < 1/(n+1)$, *$\overline{W_{n+1}} \subset W_n$ and $M(W_{n+1}) \subset V_{n+1}$;*

(iv) *the family* $\gamma(W_n) := \{W_{n+1} : (U_{n+1}, V_{n+1}, W_{n+1}) \in \Gamma(W_n)$ *for some* U_{n+1}, $V_{n+1}\}$ *is disjoint;*

(v) *the set* $H(W_n) := \bigcup\{W_{n+1} : W_{n+1} \in \gamma(W_n)\}$ *is dense in* W_n.

Proof of the Lemma: Take a maximal family $\Gamma(W_n)$ satisfying the properties (i)-(iv) from the lemma. We prove that it satisfies also the condition (v).

Suppose the contrary. There exists a non-empty open subset G of $C(X)$ with $G \subset W_n$ and $G \bigcap H(W_n) = \emptyset$. Since M is open the set $M(G)$ is open in X. Moreover, $M(G) \subset M(W_n) \subset V_n$. Let $U_{n+1} := M(G)$ and $V_{n+1} := s(U_1, V_1, \ldots, U_n, V_n, U_{n+1})$. By the minimality of M (see Proposition 4.1 (e)) there is a non-empty open subset W_{n+1} of $C(X)$ such that $W_{n+1} \subset G$ and $M(W_{n+1}) \subset V_{n+1}$. We may think, in addition, that $\overline{W_{n+1}} \subset W_n$ and $\operatorname{diam}(W_{n+1}) < 1/(n+1)$. Now, the family $\Gamma(W_n) \bigcup\{(U_{n+1}, V_{n+1}, W_{n+1})\}$ is strictly larger than $\Gamma(W_n)$ and satisfies (i)-(iv). This is a contradiction showing that the maximal family $\Gamma(W_n)$ satisfies also (v). ∎

Let us mention that Lemma 6.2 is true also for $n = 0$ provided we put $U_0 = V_0 = X$. Now, we get back to the proof of the theorem. We proceed in the following way.

Put $\gamma_0 := \{C(X)\}$, $W_0 = C(X)$, $U_0 = V_0 = X$ and apply the Lemma for the triple (U_0, V_0, W_0). We get a family of triples $\Gamma_1 := \Gamma(W_0)$ satisfying conditions (i)-(v) from the Lemma. Let $\gamma_1 := \gamma(W_0)$ and $H_1 := H(W_0)$. By (v) the set H_1 is open and dense in $C(X)$. Further, because of (iv), for every $W_1 \in \gamma_1$ there is a unique couple (U_1, V_1) with $(U_1, V_1, W_1) \in \Gamma_1$. Apply again the Lemma for this triple. As a result, for every $W_1 \in \gamma_1$ we obtain a family of triples $\Gamma(W_1)$ with the properties (i)-(v) fulfilled with respect to the couple (U_1, V_1) corresponding to W_1. Let $\Gamma_2 := \bigcup\{\Gamma(W_1) : W_1 \in \gamma_1\}$, $\gamma_2 := \bigcup\{\gamma(W_1) : W_1 \in \gamma_1\}$ and $H_2 := \bigcup\{H(W_1) : W_1 \in \gamma_1\}$. Since γ_1 is disjoint and each $\gamma(W_1)$ is disjoint too, then the family γ_2 is also disjoint. Moreover, by (v) every $H(W_1)$ is dense in W_1 and since H_1 is dense in $C(X)$, it follows that H_2 is open and dense in $C(X)$ as well.

Proceeding in this way we obtain a sequence of families $(\Gamma_n)_{n\geq 1}$ of triples and a sequence of disjoint families $(\gamma_n)_{n\geq 0}$ of open sets in $C(X)$, with $\gamma_0 = \{C(X)\}$, such that for every $n \geq 1$ we have:

(1) Γ_n is a union of the families $\Gamma(W_{n-1})$, $W_{n-1} \in \gamma_{n-1}$, where $\Gamma(W_{n-1})$ is obtained by the Lemma from some uniquely determined partial s-play $(U_1, V_1, \ldots, U_{n-1}, V_{n-1})$;

(2) γ_n is the union of the families $\gamma(W_{n-1})$ from the condition (iv) of the Lemma;

(3) the set $H_n := \bigcup\{W_n : W_n \in \gamma_n\}$ is open and dense in $C(X)$.

Let $H_0 := \bigcap_{n=1}^{\infty} H_n$. By Baire theorem H_0 is a dense G_δ-subset of $C(X)$. Take $f_0 \in H_0$. By the properties above, this f_0 determines a unique sequence $\{W_n\}_{n=1}^{\infty}$ such that for every $n \geq 1$, $W_n \in \gamma_n$, $f_0 \in W_n$, $\overline{W_{n+1}} \subset W_n$ and $\operatorname{diam}(W_n) < 1/n$. Hence $\{f_0\} = \bigcap_{n=1}^{\infty} W_n$. By the properties (i)-(v) from the Lemma and conditions (1)-(3) above it follows that there is an s-play $p = \{U_n, V_n\}_{n=1}^{\infty}$ such that $U_{n+1} \subset M(W_n) \subset V_n$ for every $n \geq 1$.

Hence, by Proposition 4.1 (f) we have

$$M(f_0) = \bigcap_{n=1}^{\infty} M(W_n) = \bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} U_n = T(p).$$

Since s is a winning strategy, we see that $M(f_0) = T(p) = \bigcap_{n=1}^{\infty} V_n \neq \emptyset$, i.e. f_0 attains its maximum in X. The proof in this direction is completed.

Conversely, suppose that the set of functions in $C(X)$ which attain their maximum in X contains a dense G_δ-subset of $C(X)$. If we consider again the solution mapping $M : C(X) \to X$ this means that its domain $\mathrm{Dom}(M)$ contains a dense G_δ-subset of $C(X)$. I.e. there exist countably many open and dense subsets $\{G_n\}_{n=1}^{\infty}$ of $C(X)$ such that $\bigcap_{n=1}^{\infty} G_n \subset \mathrm{Dom}(M)$. The sets $A_n := C(X) \setminus G_n, n \geq 1$, are closed and nowhere dense in $C(X)$. That is $\mathrm{int}(A_n) = \emptyset$ for every $n \geq 1$.

We show that the player α has a winning strategy s in the game $BM(X)$.

Let U_1 be a non-empty open subset of X. The mapping M is onto, hence the set $\mathrm{int}M^{\#}(U_1)$ is non-empty by Proposition 4.1 (d). Since A_1 is closed and nowhere dense in $C(X)$, the set $\mathrm{int}M^{\#}(U_1) \setminus A_1$ is non-empty and open in $C(X)$. Take an open ball B_1 in $C(X)$ with radius less or equal to 1, such that $B_1 \subset \mathrm{int}M^{\#}(U_1) \setminus A_1$. Define now the value of the strategy s at U_1 by $s(U_1) := M(B_1)$. By Proposition 4.1, $s(U_1)$ is a non-empty open subset of U_1.

Further, let U_2 be an arbitrary non-empty open subset of $V_1 = s(U_1) = M(B_1)$. Since $U_2 \subset M(B_1)$ there is some $f \in B_1$ such that $M(f) \bigcap U_2 \neq \emptyset$. Hence, by Proposition 4.1 (e) there exists a non-empty open $W \subset B_1$ such that $M(W) \subset U_2$. As above the set $W \setminus A_2$ is a non-empty and open subset of $C(X)$. Take an open ball B_2 with radius less or equal to $1/2$ such that $\overline{B_2} \subset W \setminus A_2 \subset B_1$ and put $s(U_1, V_1, U_2) := M(B_2)$. Obviously $s(U_1, V_1, U_2)$ is a non-empty open subset of U_2. Proceeding by induction we define the strategy s for every chain $(U_1, V_1, \ldots, U_n)$, $n \geq 1$, such that $U_k \subset V_{k-1}$ and $V_{k-1} = s(U_1, V_1, \ldots, U_{k-1})$ for every k, $2 \leq k \leq n$.

Let $p = \{U_n, V_n\}_{n=1}^{\infty}$ be an s-play and $\{B_n\}_{n=1}^{\infty}$ be the sequence of open balls in $C(X)$ associated with $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ from the construction of s. Then for every $n \geq 1$:

1) $\overline{B_{n+1}} \subset B_n$ and $B_n \bigcap A_n = \emptyset$;

2) $\mathrm{diam}(B_n) < 1/n$;

3) $V_n = M(B_n)$

The conditions 1) and 2) above guarantee that $\bigcap_{n=1}^{\infty} B_n$ is a one-point set in $C(X)$, say f_0. Moreover, 1) shows in addition that $f_0 \in C(X) \setminus \bigcup_{n=1}^{\infty} A_n \subset \mathrm{Dom}(M)$. Therefore, by 3) and Proposition 4.1 (f) we have

$$\emptyset \neq M(f_0) = \bigcap_{n=1}^{\infty} M(B_n) = \bigcap_{n=1}^{\infty} V_n = T(p).$$

Hence s is a winning strategy for the player α in the Banach-Mazur game $BM(X)$. This completes the proof. ■

Let us mention that the fact that the set $\mathrm{Dom}(M)$ contains a dense G_δ-subset of $C(X)$ was proved by Stegall in [S1], Theorem 5, under a strictly stronger (than the one considered here) condition on X, namely the existence of an α-winning tactic in $BM(X)$. We have already pointed out that there are completely regular topological spaces admitting an α-winning strategy but no α-winning tactic.

To close the circle of generic properties for optimization problems we give a result, concerning the case of generic uniqueness of the solution to the maximization problems generated by the functions from $C(X)$. The following result from [KR1], [KR3] can be proved exactly as Theorem 6.1.

Theorem 6.3 ([KR3], Theorem 3.3). *Let X be completely regular. Then the set $\{f \in C(X) : (X, f)$ has unique solution$\}$ contains a dense G_δ-subset of $C(X)$ if and only if the space X admits an α-winning strategy s such that for every s-play $p = \{U_i, V_i\}_{i=1}^{\infty}$ the target set $T(p)$ is a singleton.*

Finally, let us mention that the existence of other strengthened strategies in the Banach-Mazur game can characterize in another way the generic well-posedness from Theorem 4.3. For this the reader is refereed to [ČKR4]. Another approach to different strengthened strategies in the Banach-Mazur game is proposed and investigated in the paper of Debs and Saint Raymond [DeSRa].

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