Qualitative behavior of weak solutions of the drift diffusion model for semiconductor devices coupled with Maxwell s equations

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Abstract:
The transient drift-diffusion model describing the charge transport in semiconductors is considered. Poisson's equation, which is usually used, is replaced by Maxwell's equations. The diffusion- and mobility-coefficients and the dielectric and magnetic susceptibilities may depend on the space-variables. Global existence and convergence to the thermal equilibrium is shown.
Key words:
Drift-diffusion-model for Semiconductors, Maxwell's equations, parabolic PDE nonlinearely coupled with hyperbolic system, global existence, asymptotic behavior.
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1 Introduction

The subject of this paper are the transient drift diffusion equations describing the charge transport in semiconductors coupled with Maxwell's equations

$$\partial_t \rho_k(t,x) = -\nabla \cdot \mathbf{j}_k(t,x) - R(x, \rho_1(t,x), \rho_2(t,x)), \tag{1.1}$$

$$\mathbf{j}_k(t,x) = -D_k(x)\nabla \rho_k(t,x) - (-1)^k \mu_k(x)\rho_k(t,x)\mathbf{E}(t,x) \tag{1.2}$$

for $k \in \{1,2\}$

$$\varepsilon(x)\partial_t \mathbf{E}(t,x) = \text{ curl } \mathbf{H}(t,x) + \mathbf{j}_2(t,x) - \mathbf{j}_1(t,x) \tag{1.3}$$

$$\mu(x)\partial_t \mathbf{H}(t,x) = - \text{ curl } \mathbf{E}(t,x) \tag{1.4}$$

$$\text{div } (\varepsilon(x)\mathbf{E}(t,x)) = \rho_1(t,x) - \rho_2(t,x) + C(x), \text{ div } (\mu(x)\mathbf{H}(t,x)) = 0 \tag{1.5}$$

Here ρ_1, ρ_2 denote the charge densities and $\mathbf{j_1}, \mathbf{j_2}$ denote the current densities of the holes and electrons respectively.
The self-consistent electromagnetic field $(\mathbf{E}, \mathbf{H})$ obeys Maxwell's equations 1.3, 1.4 and 1.5.
The unknown functions $\rho_1, \rho_2, \mathbf{E}, \mathbf{H}$ depend on the time $t \in I\!\!R$ and space variable $x \in \Omega$
$\Omega \subset I\!\!R^3$ is a bounded Lipschitz-domain with $\partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ_D, Γ_N are disjoint subsets of $\partial\Omega$.

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Γ_D represents the perfectly conducting Ohmic contacts and Γ_N represents the insulating boundary of the semiconductor device. The mobilities μ_1, μ_2 of the holes and electrons resp. are assumed to be positive constants. The diffusion coefficients $D_1, D_2$and the recombination generation rate R are functions of the densities ρ_1, ρ_2 and the space variable x.
C is a bounded function of x, which describes the doping profile of the device. The system 1.2 - 1.5 is supplemented by the following initial boundary conditions

$$\rho = U^D \quad \text{on} \quad I\!R \times \Gamma_D \tag{1.6}$$

$$\vec{n} \cdot \mathbf{j}_k = 0 \quad \text{on} \quad I\!R \times \Gamma_N \quad \text{for } k \in \{1, 2\} \tag{1.7}$$

$$\vec{n} \wedge \mathbf{E} = 0 \quad \text{on} \quad I\!R \times \Gamma_D \tag{1.8}$$

$$\varepsilon \vec{n} \cdot \mathbf{E} = \sigma \quad \text{on} \quad I\!R \times \Gamma_N \tag{1.9}$$

$$\vec{n} \wedge \mathbf{H} = \vec{n} \wedge \mathbf{H}_\Gamma \quad \text{on} \quad I\!R \times \Gamma_N \tag{1.10}$$

$$\rho(0, x) = \rho_0(x) \tag{1.11}$$

$$\mathbf{E}(0, x) = \mathbf{E}_0(x), \quad \mathbf{H}(0, x) = \mathbf{H}_0(x) \tag{1.12}$$

On the perfectly conducting boundary Γ_D the tangential component of the electric field must vanish, which is expressed by boundary condition 1.8, while on Γ_N the tangential component of $\mathbf{H}$ and the normal component of $\mathbf{E}$ are prescribed by 1.9 and 1.10.
The initial-boundary data must satisfy the following compatibility conditions

$$\text{div } (\mu \mathbf{H}_0) = 0 \tag{1.13}$$

$$\text{div } (\varepsilon \mathbf{E}_0) = \rho_{1,0} - \rho_{2,0} + C \tag{1.14}$$

$$\varepsilon \vec{n} \cdot \mathbf{E}_0 = \sigma(0) \quad \text{on} \quad \Gamma_N \tag{1.15}$$

Note that the boundary-conditions 1.7,1.9 and 1.10 are not independent, since 1.3 implies (at least formally)

$$\partial_t \sigma = \varepsilon \partial_t (\vec{n} \cdot \mathbf{E}) = \vec{n} \cdot (\text{ curl } \mathbf{H} - \mathbf{j}_1 + \mathbf{j}_2) = \vec{n} \cdot \text{ curl } \mathbf{H}_\Gamma \quad \text{on} \quad I\!R \times \Gamma_N \tag{1.16}$$

Therefore, the compatibility condition

$$\partial_t \sigma = \vec{n} \cdot \text{ curl } \mathbf{H}_\Gamma \quad \text{on} \quad I\!R \times \Gamma_N \tag{1.17}$$

has to be required. Now, if $\rho, (\mathbf{E}, \mathbf{H})$ is a solution of the equations 1.2, 1.3, 1.4 and the initial boundary conditions 1.7, 1.8 and 1.10 - 1.12, the equations 1.5 and the boundary condition 1.9 are automatically fulfilled for all times $t \geq 0$. Therefore, the following problem will be considered

$$\partial_t \rho_t = - \text{ div } \mathbf{j_k} - R(x, \rho_1, \rho_2), k \in \{1, 2\} \tag{1.18}$$

$$\mathbf{j}_k = -D_k(x, \rho_1, \rho_2) \nabla \rho_k - (-1)^k \mu_k \rho_k \mathbf{E}, k \in \{1, 2\} \tag{1.19}$$

$$\varepsilon \partial_t \mathbf{E} = \text{ curl } \mathbf{h} + \mathbf{j_2} - \mathbf{j_1} - \text{ curl } \mathbf{H}_\Gamma \tag{1.20}$$

$$\mu \partial_t \mathbf{h} = - \text{ curl } \mathbf{E} - \mu \partial_t \mathbf{H}_\Gamma \tag{1.21}$$

$$\rho = U^D \quad \text{on} \quad I\!R \times \Gamma_D \tag{1.22}$$

$$\vec{n} \cdot \mathbf{j_k} = 0 \quad \text{on} \quad I\!R \times \Gamma_N \quad \text{for } k \in \{1, 2\} \tag{1.23}$$

$$\vec{n} \wedge \mathbf{E} = 0 \quad \text{on} \quad I\!R \times \Gamma_D \tag{1.24}$$

$$\vec{n} \wedge \mathbf{h} = 0 \quad \text{on} \quad I\!R \times \Gamma_N \tag{1.25}$$

$$\rho(0, x) = \rho_0(x) \tag{1.26}$$

$$\mathbf{E}(0, x) = \mathbf{E}_0(x) \ , \ \mathbf{h}(0, x) = \mathbf{h}_0(x) \tag{1.27}$$

where $\mathbf{h} \stackrel{\text{def}}{=} \mathbf{H} - \mathbf{H}_\Gamma$.
Analysis of the drift diffusion model for semiconductors has been presented in [5], [6], [8], [14], [15] and [16] in the case that Maxwell's equations 1.3 - 1.5 are replaced by Poisson's equation $-\text{ div } (\varepsilon \nabla V) = \rho_1 - \rho_2 - C$ for an electrostatic field $\mathbf{E} = -\nabla V$.
However, at very high frequencies the time dependent magnetic field generates a non curl-free electric field, which cannot be written as the gradient of an electrostatic potential.
Therefore, Poisson's equation has to be replaced by Maxwell's equations 1.3, 1.4 - 1.5 for the electromagnetic field.
Global existence of weak solutions $(\rho, \mathbf{E}, \mathbf{H})$ to problem 1.18 - 1.27, with

$$\rho \in L^2_{loc}([0, \infty), H^1(\Omega)) \cap L^\infty_{loc}([0, \infty), L^\infty(\Omega)) \cap C([0, \infty), L^2(\Omega))$$

and $(\mathbf{E}, \mathbf{H}) \in C([0, \infty), L^2(\Omega))$ has been shown in [9] under the assumption that the mobility coefficents μ_k, the dielectric and magnetic susceptibilities are constant. Moreover, only the homogeneous boundary condition $\vec{n} \cdot \mathbf{E} = 0$ has been considered in [9].
Uniqueness of weak solutions in the two-dimensional case is proved in [10].
In this paper it is assumed that $D_k, \mu_k, \varepsilon, \mu$ belong to $L^\infty(\Omega)$ and are uniformly positive and that $\frac{\varepsilon}{\mu_k}$ is Lipschitz-continuous. $\nu \stackrel{\text{def}}{=} \frac{\mu_1}{D_1} = \frac{\mu_2}{D_2}$ is assumed to be constant, which is motivated physically by the so-called Einstein-relations. The recombination term has the form

$$R(x, u) = r(x, u)(u_1 u_2 - n_i^2),$$

for $x \in \Omega, u \in [0, \infty)^2$ with a nonnegative function r defined on $x \in \Omega \times [0, \infty)^2$.
In the case of the nonconstant coefficients and the inhomogeneous boundary-condition 1.9 the L^∞-estimates for the densities ρ_k are obtained by estimating successively L^p-norms. This iteration-technique has also been used in [7] for the standard drift-diffusion model involving Poisson's equation. These estimates yield global existence of weak solutions to 1.18 - 1.27 in the third section of this paper in connection with a-priori-estimates for the densities in $L^\infty((0, T), L^1(\Omega))$ and for the electromagnetic field in $L^\infty((0, T), L^2(\Omega))$, which are derived by using the energy-functional

$$\mathcal{E}(t) = \int_\Omega \sum_{k=1}^{2} \int_{U_k(x)}^{\rho_k(t,x)} \ln \frac{s}{U_k(x)} ds dx + \frac{\nu}{2} \int_\Omega \varepsilon(x) |\mathbf{E}(t, x)|^2 + \mu(x) |\mathbf{h}(t, x)|^2 dx \tag{1.28}$$

The main part of this paper is the investigation of the asymptotic behavior of weak solutions for $t \to \infty$ in the fourth section. For this purpose it is assumed that the device is in thermal

eqilibrium on the ohmic contacts Γ_D, that means that $U_1^D U_2^D = n_i^2$, such that the recombination-rate $R(x, U^D(x))$ vanishes and that U^D is constant on every connected component of Γ_D, which implies by 1.19 and 1.24 that $\vec{n} \wedge \mathbf{j}_k(t,x) = 0$ on Γ_D.
For $\sigma_0 \in L^{p_0}(\partial\Omega)$, $C \in L^\infty(\Omega)$ with $p_0 > 2$ it has been shown in [7] under suitable conditions on R that there exist a solution for the stationary drift-diffusion model in thermal equilibrium, that means $\varphi_e \in H^1(\Omega)$ and strictly positive $U_1, U_2 \in L^\infty(\Omega) \cap H^1(\Omega)$ with

$$\nabla(\ln U_k + (-1)^k \nu \varphi_e) = 0, \quad U_1(x)U_2(x) = n_i^2,$$

where φ_e solves Poisson's equation

$$\operatorname{div}\,(\varepsilon \nabla \varphi_e) = U_1 - U_2 + C$$

supplemented by the boundary-conditions

$$\varphi_e = \nu^{-1} \ln U_1$$

on Γ_D,

$$\varepsilon \vec{n} \nabla \varphi_e = \sigma_0$$

on Γ_N.
In the fourth section it is shown that the densities ρ_k are globally bounded and uniformly positive on $(0,\infty) \times \Omega$ and that the electromagnetic field $\mathbf{E}, \mathbf{h}$ belongs to $L^\infty((0,\infty), L^2(\Omega))$.
Moreover, it is shown that if $\mathbf{H}_\Gamma$, $\sigma(t) - \sigma_0$ satisfy suitable decay assumptions for $t \to \infty$ the densities ρ_k converge to the carrier distributions U_k in the thermal equilibrium,
i.e. $\lim_{t\to\infty} ||\rho(t) - U||_{\mathrm{L}^p(\Omega)} = 0$ for all $p \in [0,\infty)$. In particular exponential convergence to the equilibrium is shown, i.e. there exist some $K \in (0,\infty)$ and $\gamma > 0$, such that

$$||\rho(t) - U||_{\mathrm{L}^p(\Omega)} + ||\mathbf{E}(t) + \varphi_e||_{\mathrm{L}^2(\Omega)} \leq K \exp(-\gamma t) \text{ for all } t \in [0,\infty),$$

if $\mathbf{H}_\Gamma(t) = 0$, $\sigma(t) = \sigma_0$ for sufficienctly large $t \geq 0$.

2 Notation, assumptions and preliminary results

Let $\Omega \subset I\!R^3$ be a bounded Lipschitz-domain with $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$, where Γ_D is closed and has finitely many connected components.
The diffusion and mobility coefficients D_k, μ_k, the dielectric and magnetic susceptibilities ε, μ belong to $L^\infty(\Omega)$ and are assumed to be uniformly positive, i.e. there exist $\delta > 0$ and $d \in (0,\infty)$, such that

$$\delta \leq D_k(x), \mu_k(x), \varepsilon(x), \mu(x) \leq d \text{ for all } x \in \Omega, k \in \{1,2\} \tag{2.29}$$

Moreover,

$$\nu \stackrel{\text{def}}{=} \frac{\mu_1}{D_1} = \frac{\mu_2}{D_2} \tag{2.30}$$

is a positive constant and the following regularity assumption is required

$$\frac{\varepsilon}{\mu_k} \in W^{1,\infty}(\Omega). \tag{2.31}$$

The recombination term R has the form

$$R(x,u) = r(x,u)(u_1 u_2 - n_i^2) \text{ for all } x \in \Omega, u \in [0,\infty)^2, \tag{2.32}$$

where the nonnegative function $r : \Omega \times [0,\infty)^2 \to [0,\infty)$ obeys

$$r(\cdot,u) \in L^\infty(\Omega) \text{ for fixed } u \in [0,\infty)^2 \tag{2.33}$$

and

$$|r(x,u) - r(x,v)| \leq K|u-v| \text{ for all } x \in \Omega, u \in [0,\infty)^2 \tag{2.34}$$

with some $K \in (0,\infty)$ independent of $x \in \Omega, u, v \in [0,\infty)^2$.
As in [9] the following function spaces are used.
Let Y be the closure of $C_0^\infty(\mathbb{R}^3 \setminus \Gamma_D)$ in $H^1(\Omega)$, where $H^1(\Omega)$ is the usual first order Sobolev space vof L^2type and $C_0^\infty(\mathbb{R}^3 \setminus \Gamma_D)$ denotes the space of all infinetely differentiable functions with compact support contained in $\mathbb{R}^3 \setminus \Gamma_D$.
$H^{1,\Gamma}$ denotes the space of all functions belonging to $H^1(\Omega)$, whose trace is constant on every connected component of Γ_D.
Let W_0 be the space of all $\mathbf{w} \in L^2(\Omega, \mathbb{C}^3)$ with curl $\mathbf{w} \in L^2(\Omega, \mathbb{R}^3)$ in the sense of distributions. By W_E we denote the space of all $\mathbf{E} \in W_0$, such that the tangential component of $\mathbf{E}$ vanishes on Γ_D in a generalized sense,i.e.

$$\int_\Omega (\mathbf{E} \cdot \text{ curl } \mathbf{h} - \mathbf{h} \cdot \text{ curl } \mathbf{E}) dx = 0 \text{ for all } \mathbf{h} \in C_0^\infty(\mathbb{R}^3 \setminus \overline{\Gamma_N}, \mathbb{C}^3).$$

W_H denotes the set of all $\mathbf{h} \in W_0$, such that

$$\int_\Omega (\mathbf{h} \cdot \text{ curl } \mathbf{F} - \mathbf{F} \cdot \text{ curl } \mathbf{h}) dx = 0 \text{ for all } \mathbf{F} \in W_E,$$

which is a weak formulation of the boundary condition $\mathbf{u} \wedge \vec{n} = 0$ on Γ_N.
The assumptions on the initial boundary data for the densities are

$$\rho_0 \in L^\infty(\Omega), U^D \in W^{1,\infty}(\partial\Omega), \tag{2.35}$$

i.e. U^D is Lipschitz-continuous. Moreover,

$$\rho_{0,k} \geq a, \quad U_k^D \geq a \text{ with some } a > 0, \tag{2.36}$$

$$C \in L^\infty(\Omega), \quad \mathbf{E}_0, \mathbf{H}_0 \in L^2(\Omega) \tag{2.37}$$

and

$$\mathbf{H}_\Gamma \in W_{loc}^{1,2}(\mathbb{R}, L^2(\Omega)) \text{ with } \text{ curl } \mathbf{H}_\Gamma \in L_{loc}^2(\mathbb{R}, L^2(\Omega)). \tag{2.38}$$

Let

$$\sigma \in W_{loc}^{1,2}(\mathbb{R}, L^2(\partial\Omega)) \cap L^\infty(\mathbb{R}, L^{p_0}(\partial\Omega)) \text{ with some } p_0 > 2. \tag{2.39}$$

It is assumed that 1.13 is fulfilled in the sense of distributions and that 1.14 is satified weakly in the sense that

$$\int_\Omega \varepsilon \mathbf{E}_0 \nabla \varphi dx = \int_{\partial\Omega} \sigma(0) \varphi dS - \int_\Omega (\rho_{0,1} - \rho_{0,2} + C) \varphi dx \text{ for all } \varphi \in Y. \tag{2.40}$$

By the compatibility-condition 1.17 σ and $\mathbf{H}_\Gamma$ must satisfy

$$\frac{d}{dt}\int_{\partial\Omega}\sigma(t)\varphi dx = \int_\Omega(\text{ curl } \mathbf{H}_\Gamma(t))\cdot\nabla\varphi dx \text{ for all } \varphi\in Y \text{ and } t\in I\!R. \tag{2.41}$$

In the Hilbert-space $X_0 = L^2(\Omega, \mathbb{C}^6)$ endowed with the scalar-product

$$<\mathbf{w},\tilde{\mathbf{w}}>_{X_0}=\int_\Omega(\varepsilon\mathbf{w}_1\overline{\tilde{\mathbf{w}}_1}+\mu\mathbf{w}_2\overline{\tilde{\mathbf{w}}_2}dx \text{ for } \mathbf{w},\tilde{\mathbf{w}}\in X_0$$

the following operator is defined

$$B(\mathbf{E},\mathbf{h}) = (\varepsilon^{-1}\text{ curl } \mathbf{h}, -\mu^{-1}\text{ curl } \mathbf{E}) \text{ for } (\mathbf{E},\mathbf{h})\in D(B) = W_E\times W_H. \tag{2.42}$$

Lemma 1 *B is skew self-adjoint in X_0, i.e. $B^* = -B$.*
Moreover, $(\nabla\varphi, 0, 0, 0)\in kerB$ for all $\varphi\in H^{1,\Gamma}$.

Proof: Define the operator $B^{(0)}$ in $L^2(\Omega,\mathbb{C}^6)$ endowed with the usual scalar-product $<\cdot,\cdot>$ by

$$B^{(0)}(\mathbf{E},\mathbf{h}) \stackrel{\text{def}}{=} (\text{ curl } \mathbf{h}, -\text{ curl } \mathbf{E})$$

for $(\mathbf{E},\mathbf{h})\in D(B^{(0)}) = W_E\times W_H = D(B)$.
Then one has for $\mathbf{v}\in X_0, \mathbf{w}\in D(B)$

$$<B\mathbf{w},\mathbf{v}>_{X_0}=<B^{(0)}\mathbf{w},\mathbf{v}>$$

This yields the skew-self-adjointness of B in $(X_0, <\cdot,\cdot>_{X_0}$, since $<\cdot,\cdot>$ and $<\cdot,\cdot>_{X_0}$ are equivalent onX_0 and $B^{(0)}$ is skew-self-adjoint in $L^2(\Omega,\mathbb{C}^6), <\cdot,\cdot>$, see [9].
To prove the second assertion let $\Gamma_{D,1},..,\Gamma_{D,N}$ be the connected components of Γ_D and
$\alpha_k \stackrel{\text{def}}{=} \varphi|_{\Gamma_{D,k}}$.
Let $\psi\in C_0^\infty(I\!R^3)$ with $\Gamma_N\cap\text{supp}\psi=\emptyset$.
Then

$$\int_\Omega\nabla\varphi\cdot\text{ curl }\psi dx = \int_\Omega\text{ div }[\varphi\text{ curl }\psi]dx$$

$$= \int_{\partial\Omega}\varphi\vec{n}\cdot\text{ curl }\psi dS = \sum_{k=1}^N\alpha_k\int_{\Gamma_{D,k}}\vec{n}\cdot\text{ curl }\psi dS$$

Choose $\chi_k\in C_0^\infty(I\!R^3)$ with $\chi_k = 1$ on a neighbourhood of $\Gamma_{D,k}$ and $\chi_k(x) = 0$ if $x\in\Gamma_{D,j}$, $j\neq k$.
Then

$$\int_{\Gamma_{D,k}}\vec{n}\cdot\text{ curl }\psi dS = \int_{\partial\Omega}\vec{n}\cdot\text{ curl }(\chi_k\psi)dS = \int_\Omega\text{ div }(\text{ curl }(\chi_k\psi))dx = 0$$

Hence $\int_\Omega\nabla\varphi\cdot\text{ curl }\psi dx = 0$ for all $\psi\in C_0^\infty(I\!R^3\setminus\Gamma_N)$ and therefore

$$\nabla\varphi\in W_E,\quad B(\nabla\varphi,0) = 0$$

□

In the sequel $\mathbf{w}\stackrel{\text{def}}{=}(\mathbf{E},\mathbf{h})\in C([0,T],X_0)$ is called for $\mathbf{f}\in L^2((0,T),X_0)$ a weak solution to

$$\partial\mathbf{E} = \varepsilon^{-1}\text{ curl }\mathbf{h}+\mathbf{f}_1,\quad \partial\mathbf{h} = -\mu^{-1}\text{ curl }\mathbf{E}+\mathbf{f}_2,$$

$$\vec{n} \wedge \mathbf{E} = 0 \text{ on } \Gamma_D, \text{ and } \vec{n} \wedge \mathbf{h} = 0 \text{ on } \Gamma_N,$$

if

$$\frac{d}{dt} < \mathbf{w}(t), \mathbf{u} >_{X_0} = - < \mathbf{w}(t), B\mathbf{u} >_{X_0} + < \mathbf{f}(t), \mathbf{u} >_{X_0} \text{ for all } \mathbf{u} \in D(B). \tag{2.43}$$

This is equivalent to the variation of constants-formula

$$(\mathbf{E}(t), \mathbf{h}(t)) = exp(tB)(\mathbf{E}(0), \mathbf{h}(0)) + \int_0^t exp((t-s)B)\mathbf{f}(s)ds \tag{2.44}$$

where $exp(tB), t \in \mathbb{R}$ is the unitary group in X_0 generated by B.
By $W_{E,0}$, $W_{H,0}$ we denote the spaces of all $\mathbf{E} \in W_E$, $\mathbf{h} \in W_H$, $\operatorname{curl} \mathbf{E} = 0$, $\operatorname{curl} \mathbf{h} = 0$, respectively.
X_E, denotes the spaces of all $\mathbf{E} \in L^2(\Omega, \mathbb{C}^3)$ with

$$\int_\Omega \varepsilon \mathbf{E} \cdot \mathbf{F} dx = 0 \text{ for all } \mathbf{F} \in W_{E,0} \tag{2.45}$$

Analogously X_H, denotes the spaces of all $\mathbf{h} \in L^2(\Omega, \mathbb{C}^3)$ with

$$\int_\Omega \mu \mathbf{h} \cdot \mathbf{g} dx = 0 \text{ for all } \mathbf{g} \in W_{H,0} \tag{2.46}$$

The orthogonal-projection on X_E with respect to the scalarproduct

$$< \mathbf{E}, \tilde{\mathbf{F}} >_\varepsilon \stackrel{\text{def}}{=} \int_\Omega \varepsilon \mathbf{E} \overline{\mathbf{F}} dx \text{ for } \mathbf{E}, \mathbf{F} \in L^2(\Omega, \mathbb{C}^3)$$

is denoted by P_E. P_H denotes the orthogonal-projection on X_H with respect to the scalarproduct

$$< \mathbf{h}, \tilde{\mathbf{g}} >_\mu \stackrel{\text{def}}{=} \int_\Omega \mu \mathbf{h} \overline{\mathbf{g}} dx \text{ for } \mathbf{h}, \mathbf{g} \in L^2(\Omega, \mathbb{C}^3).$$

From the above definitions it follows immediately that $\varepsilon^{-1} \operatorname{curl} \mathbf{h} \in X_E$ for $\mathbf{h} \in W_H$ and that $1 - P_H$ is the orthogonal projection on $W_{H,0}$, which is a closed subspace of $L^2(\Omega)$. Hence

$$P_E \varepsilon^{-1} \operatorname{curl} \mathbf{h} = \varepsilon^{-1} \operatorname{curl} \mathbf{h} = \varepsilon^{-1} \operatorname{curl} (P_H \mathbf{h}) \text{ for all } \mathbf{h} \in W_H \tag{2.47}$$

and similarely

$$P_H \mu^{-1} \operatorname{curl} \mathbf{E} = \mu^{-1} \operatorname{curl} \mathbf{E} = \mu^{-1} \operatorname{curl} (P_E \mathbf{E}) \text{ for all } \mathbf{E} \in W_E \tag{2.48}$$

Lemma 2 *i) $W_E \cap X_E$ and $W_H \cap X_H$ are compactly imbedded in $L^2(\Omega, \mathbb{C}^3)$.*
ii) There exists a constant $K \in (0, \infty)$ depending only on ε, μ and Ω, such that

$$||\mathbf{h}||_{L^2} \leq K|| \; curl\, \mathbf{h}||_{L^2} \; and \; ||\mathbf{E}||_{L^2} \leq C|| \; curl\, \mathbf{E}||_{L^2}$$

for all $\mathbf{h} \in W_H \cap X_H, \mathbf{E} \in W_E \cap X_E$.

$$iii) X_E = \{\varepsilon^{-1} \; curl\, \mathbf{h} : \mathbf{h} \in W_H\} = \{\varepsilon^{-1} \; curl\, \mathbf{h} : \mathbf{h} \in W_H \cap X_H\}$$

$$and \; X_H = \{\mu^{-1} \; curl\, \mathbf{E} : \mathbf{E} \in W_E\} = \{\mu^{-1} \; curl\, \mathbf{E} : \mathbf{E} \in W_E \cap X_E\}$$

Proof: It has been shown in [9] that $\nabla\varphi \in W_{E,0}$ for all $\varphi \in Y$. Hence

$$\int_\Omega \varepsilon \mathbf{E} \nabla \varphi dx = 0 \text{ for all } \mathbf{E} \in X_E \text{ and } \varphi \in Y. \tag{2.49}$$

This is the weak formulation for div $(\varepsilon\mathbf{E}) = 0$ and $\varepsilon\vec{n}\cdot\mathbf{E} = 0$ on Γ_N. By the result in [11] that the space of all $\mathbf{E} \in W_E$, which obey 2.49 is compactly imbedded in $L^2(\Omega, \mathbb{C}^3)$. Therefore, the embedding $W_E \cap X_E \hookrightarrow L^2(\Omega, \mathbb{C}^3)$ is compact. The compactness of the embedding $W_H \cap X_H \hookrightarrow L^2(\Omega, \mathbb{C}^3)$ is obtained from the same argument.
ii) can be proved indierctly. Assume there were a sequence $(\mathbf{h}_n)_{n\in\mathbb{N}}$ in $W_H \cap X_H$, such that

$$1 = ||\mathbf{h}_n||_{L^2} \geq n|| \text{ curl } \mathbf{h}_n||_{L^2} \text{ for all } n \in \mathbb{N}. \tag{2.50}$$

Then $(\mathbf{h}_n)_{n\in\mathbb{N}}$ is bounded in $W_H \cap X_H$and therefore precompact in $L^2(\Omega)$ by i), i.e. there is a subsequence $(\mathbf{h}_{n_k})_{k\in\mathbb{N}}$ and some $\mathbf{h} \in X_H$ with

$$||\mathbf{h}_{n_k} - \mathbf{h}||_{L^2} \stackrel{\text{k}\to\infty}{\longrightarrow} 0 \text{ in particular } ||\mathbf{h}||_{L^2} = 1. \tag{2.51}$$

From 2.50 and 2.51 it follows easily that $\mathbf{h} \in W_{H,0}$ and hence $\mathbf{h} \in W_{H,0} \cap X_H = \{0\}$, which contradicts 2.51.
Now, iii) can be proved. 2.47 yields

$$R \stackrel{\text{def}}{=} \{\varepsilon^{-1} \text{ curl } \mathbf{h} : \mathbf{h} \in W_H\} = \{\varepsilon^{-1} \text{ curl } \mathbf{h} : \mathbf{h} \in W_H \cap X_H\} \tag{2.52}$$

and it follows easily from ii) that this space is closed in $L^2(\Omega)$. By the definition $W_{E,0}$ coincides with the orthogonal complement $R^\perp$ of R with respect to $< \cdot, \cdot >_\varepsilon$. Hence

$$X_E = W_{E,0}^\perp = R^{\perp\perp} = \overline{R} = R$$

and the lemma is proved.

□

In order to prove global existence to 1.18-1.27 the following truncated system is considered for $M \in (0, \infty)$.

$$\partial_t \rho_k = - \text{ div } \mathbf{j}_{k,M} - R_M(x, \rho_1, \rho_2), k \in \{1, 2\} \tag{2.53}$$

$$\mathbf{j}_{k,M} = -D_k(x)\nabla\rho_k - (-1)^k \mu_k \text{ min } \{M, \rho_k\}\mathbf{E}, k \in \{1, 2\} \tag{2.54}$$

$$\varepsilon\partial_t\mathbf{E} = \text{ curl } \mathbf{h} + \mathbf{j}_{2,M} - \mathbf{j}_{1,M} - \text{ curl } \mathbf{H}_\Gamma \tag{2.55}$$

$$\mu\partial_t\mathbf{h} = - \text{ curl } \mathbf{E} - \mu\partial_t\mathbf{H}_\Gamma \tag{2.56}$$

$$\rho = U^D \quad \text{on} \quad \mathbb{R} \times \Gamma_D \tag{2.57}$$

$$\vec{n}\cdot\mathbf{j}_{k,M} = 0 \quad \text{on} \quad \mathbb{R} \times \Gamma_N \quad \text{for } k \in \{1, 2\} \tag{2.58}$$

$$\vec{n} \wedge \mathbf{E} = 0 \quad \text{on} \quad \mathbb{R} \times \Gamma_D \tag{2.59}$$

$$\vec{n} \wedge \mathbf{h} = 0 \quad \text{on} \quad \mathbb{R} \times \Gamma_N \tag{2.60}$$

$$\rho(0,x) = \rho_0(x) \tag{2.61}$$

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), \qquad \mathbf{h}(0,x) = \mathbf{h}_0(x) \tag{2.62}$$

with

$$R_M(x,u_1,u_2) = R(x,\ \min\{M,u_1\},\ \min\{M,u_2\}) \text{ for } x\in\Omega, u\in[0,\infty)^2$$

A modification of the proof given in [9] shows that 2.53-2.62 admits a weak solution $(\rho,\mathbf{E},\mathbf{h})$ with

$$\rho - U^D \in L^2_{loc}([0,\infty),Y)\cap C([0,\infty),L^2(\Omega)), \quad (\mathbf{E},\mathbf{h})\in C([0,\infty),X_0).$$

For example as in [9] existence of a weak solution to 2.53 - 2.62 can be proved by using the regularization operators $R_n : L^2(\Omega)\to C^\infty(\bar{\Omega})$ in [9] with the property $R_n u\to u$ for $n\to\infty$ in $L^2(\Omega)$ strongly for all $u\in L^2(\Omega)$. Note that in [9] the assumption that μ,ε,μ_k are constant is only used for the proof of the M-independent L^∞-estimates for the densities.
This restrictive assumption is removed in this paper by using an iteration-techniqe similar to [7].

Lemma 3 *Let $(\rho,\mathbf{E},\mathbf{h})$ be a weak solution to 2.53 - 2.62.*
Then one has for all $\varphi\in H^{1,\Gamma}$

$$\frac{d}{dt}\int_\Omega \varepsilon\mathbf{E}(t)\cdot\nabla\varphi dx = \int_\Omega(\mathbf{j}_{2,M}(t)-\mathbf{j}_{1,M}(t)+\ curl\ \mathbf{H}_\Gamma(t))\cdot\nabla\varphi dx$$

If $\varphi\in Y$, then

$$\int_\Omega \varepsilon\mathbf{E}(t)\cdot\nabla\varphi dx = \int_{\partial\Omega}\sigma(t)\varphi dS - \int_\Omega(\rho_1(t)-\rho_2(t)+C)\varphi dx$$

(This is the weak formulation for $div\ (\varepsilon\mathbf{E}) = \rho_1-\rho_2+C$ and $\varepsilon\vec{n}\cdot\mathbf{E}=\sigma$ for $x\in\Gamma_N$).

Proof: Let $\varphi\in H^{1,\Gamma}$.
$\mathbf{w}\stackrel{\text{def}}{=}(\mathbf{E},\mathbf{h})$ is a weak solution of the Maxwell-system, i.e. it obeys

$$\mathbf{w}(t) = \exp(tB)(\mathbf{E}_0,\mathbf{h}_0) + \int_0^t \exp((t-s)B)\mathbf{f}ds \tag{2.63}$$

with

$$\mathbf{f}\stackrel{\text{def}}{=}(\varepsilon^{-1}[\mathbf{j}_{2,M}-\mathbf{j}_{1,M}+\ \text{curl}\ \mathbf{H}_\Gamma], -\partial_t\mathbf{H}_\Gamma)$$

By lemma 1 we have $\psi = (\nabla\varphi,0,0,0)\in\ \ker B$ and therefore 2.63 yields

$$\int_\Omega \varepsilon\mathbf{E}(t)\nabla\varphi dx = <\mathbf{w}(t),\psi>_{X_0} \tag{2.64}$$

$$= <(\mathbf{E}_0,\mathbf{h}_0),\exp(-tB)\psi>_{X_0} + \int_0^t <\mathbf{f}(s),\exp((s-t)B)\psi>_{X_0} ds$$

$$= <(\mathbf{E}_0,\mathbf{h}_0),\psi>_{X_0} + \int_0^t <\mathbf{f}(s),\psi>_{X_0} ds$$

$$= \int_\Omega \varepsilon \mathbf{E}_0 \nabla\varphi dx + \int_0^t \int_\Omega (\mathbf{j}_{2,M} - \mathbf{j}_{1,M} + \text{ curl } \mathbf{H}_\Gamma) \cdot \nabla\varphi dx ds$$

Hence

$$\frac{d}{dt}\int_\Omega \varepsilon \mathbf{E}(t) \cdot \nabla\varphi dx = \int_\Omega (\mathbf{j}_{2,M}(t) - \mathbf{j}_{1,M}(t) + \text{ curl } \mathbf{H}_\Gamma(t)) \cdot \nabla\varphi dx$$

In order to prove the second assertion let $\varphi \in Y \subset H^{1,\Gamma}$.
Then 2.64, 2.40, 2.41 and 2.53, 2.54 yield

$$\int_\Omega \varepsilon \mathbf{E}(t)\nabla\varphi dx = \int_\Omega \varepsilon \mathbf{E}_0 \nabla\varphi dx - \int_0^t \int_\Omega (\mathbf{j}_{2,M} - \mathbf{j}_{1,M}) \cdot \nabla\varphi dx ds$$

$$+ \int_{\partial\Omega} \sigma(t)\varphi dS - \int_{\partial\Omega} \sigma(0)\varphi dS$$

$$= - \int_\Omega (\rho_{0,1} - \rho_{0,2} + C)\varphi dx - \int_\Omega [\rho_1(t) - \rho_2(t) - \rho_1(0) + \rho_2(0)]\varphi dx$$

$$+ \int_{\partial\Omega} \sigma(t)\varphi dS = - \int_\Omega (\rho_1(t) - \rho_2(t) + C)\varphi dx + \int_{\partial\Omega} \sigma(t)\varphi dS$$

3 A-priori-estimates

In this section estimates for solutions to 2.53 - 2.62 are given.
Estimates for ρ in $L^\infty((0,T), L^\infty(\Omega))$ and for $\mathbf{E}, \mathbf{h}$ in $L^\infty((0,T), L^2(\Omega))$, $T > 0$ are proved, which yield in particular the global existence of solutions to 1.18 -1.27,
since they are independent of M.

Theorem 1 *Let $T \in (0, \infty)$.*
Then there exists a constant $C_T \in (0, \infty)$, such that for all $M > 0$, every weak solution to 2.53 - 2.62 the esimate

$$||\rho(t)||_{L^1(\Omega)} + ||(\mathbf{E}(t), \mathbf{h}(t))||^2_{X_0} \leq C_T$$

holds for all $t \in [0, T]$.

Proof:
Let

$$h_M(u) \stackrel{\text{def}}{=} \ln\ \min\{u, M\} + M^{-1}(u - M)^+. \tag{3.65}$$

Then $h'_M(u) \stackrel{\text{def}}{=} \frac{1}{\min\{u,M\}}$.
Let $U_k \in W^{1,\infty}(\Omega)$ be uniformly positive extensions of U_k^D defined on Ω. For $\delta > 0$ the following energy-functional is defined

$$\mathcal{E}_\delta(t) \stackrel{\text{def}}{=} \nu^{-1} \sum_{k=1}^2 \int_\Omega \int_{U_k(x)}^{\rho_k(t,x)} (h_M(s + \delta) - \ln(U_k(x) + \delta)) ds dx \tag{3.66}$$

$$+\frac{1}{2}||(\mathbf{E}(t), \mathbf{h}(t))||^2_{X_0}$$

Since $(\mathbf{E},\mathbf{h})$ solves the Maxwell system, the variation of constant formula yields

$$(\mathbf{E}(t),\mathbf{h}(t)) = \exp(tB)(\mathbf{E}_0,\mathbf{h}_0) + \int_0^t \exp((t-s)B)\mathbf{f}(s)ds$$

with $\mathbf{f} \stackrel{\text{def}}{=} (\varepsilon^{-1}[\mathbf{j}_2 - \mathbf{j}_1 + \text{ curl } \mathbf{H}_\Gamma], -\partial_t \mathbf{H}_\Gamma)$. Hence, it follws from the fact that $\exp(sB), s \in I\!R$ is a unitary group in X_0

$$||(\mathbf{E}(t),\mathbf{h}(t))||_{X_0}^2 = ||(\mathbf{E}(0),\mathbf{h}(0))||_{X_0}^2 + 2\int_0^t < \mathbf{f}(s), (\mathbf{E}(s),\mathbf{h}(s)) >_{X_0} ds$$

and therefore

$$1/2\frac{d}{dt}||(\mathbf{E}(t),\mathbf{h}(t))||_{X_0}^2 =< \mathbf{f}(t), (\mathbf{E}(t),\mathbf{h}(t)) >_{X_0}$$

$$= \int_\Omega (\mathbf{j}_2(t) - \mathbf{j}_1(t) + \text{ curl } \mathbf{H}_\Gamma(t)) \cdot \mathbf{E}(t) - \mu \mathbf{h}(t) \cdot \partial_t \mathbf{H}_\Gamma(t) dx$$

Since $h_M(\rho_k(t)+\delta) - \ln(U_k+\delta) \in Y$ this yields

$$\mathcal{E}_\delta'(t) = \nu^{-1}\sum_{k=1}^2 < \partial_t \rho_k(t), h_M(\rho_k(t)+\delta) - \ln(U_k+\delta) >_{Y^*,Y} \tag{3.67}$$

$$+ \int_\Omega \mathbf{E}(t)\cdot(\mathbf{j}_{2,M}(t) - \mathbf{j}_{1,M}(t) + \text{ curl } \mathbf{H}_\Gamma(t)) - \mu \mathbf{h}(t)\cdot \partial_t \mathbf{H}_\Gamma(t) dx$$

$$\leq \nu^{-1}\sum_{k=1}^2 \int_\Omega \mathbf{j}_{k,M} \cdot [\frac{\nabla \rho_k}{\min\{\rho_k+\delta, M\}} - \nabla \ln(U_k+\delta)]dx$$

$$+ \int_\Omega \mathbf{E}\cdot(\mathbf{j}_{2,M} - \mathbf{j}_{1,M})dx + f(t)||(\mathbf{E}(t),\mathbf{h}(t))||_{X_0}$$

$$-\nu^{-1}\sum_{k=1}^2 \int_\Omega R_M(x,\rho)(h_M(\rho_k+\delta) - \ln(U_k+\delta))dx$$

$$\leq \nu^{-1}\sum_{k=1}^2 ||\nabla \ln(U_k+\delta)||_{L^\infty}||\mathbf{j}_{k,M}||_{L^1}$$

$$+ \sum_{k=1}^2 \int_\Omega (\min\{\rho_k+\delta, M\})^{-1}\mathbf{j}_{k,M}\cdot[\nu^{-1}\nabla\rho_k + (-1)^k(\min\{\rho_k+\delta,M\})\mathbf{E}]dx$$

$$-\nu^{-1}\sum_{k=1}^2 \int_\Omega R_M(x,\rho)(h_M(\rho_k+\delta) - \ln(U_k+\delta))dx + f(t)\mathcal{E}_\delta(t)^{1/2}$$

with $f(t) \stackrel{\text{def}}{=} ||\text{ curl } \mathbf{H}_\Gamma(t)||_{L^2(\Omega)} + ||\partial_t \mathbf{H}_\Gamma(t)||_{L^2(\Omega)}$. Therefore

$$\mathcal{E}_\delta'(t) \leq C_1 \sum_{k=1}^2 ||\min\{\rho_k+\delta, M\}||_{L^1}^{1/2} ||(\min\{\rho_k+\delta,M\})^{-1/2}\mathbf{j}_{k,M}||_{L^2} \tag{3.68}$$

$$-c_0 \sum_{k=1}^{2} ||(\min \{\rho_k + \delta, M\})^{-1/2} \mathbf{j}_{k,M}||_{L^2}^2$$

$$-\nu^{-1} \sum_{k=1}^{2} \int_\Omega R_M(x,\rho)(h_M(\rho_k+\delta) - \ln(U_k+\delta))dx$$

$$+\delta^{1/2} \sum_{k=1}^{2} \int_\Omega |(\min \{\rho_k + \delta, M\})^{-1/2} \mathbf{j}_{k,M} \mathbf{E}|dx + f(t)\mathcal{E}_\delta(t)^{1/2}$$

$$\leq (-c_0/2 + \delta^{1/2}) \sum_{k=1}^{2} ||(\min \{\rho_k + \delta, M\})^{-1/2} \mathbf{j}_{k,M}||_{L^2}^2$$

$$-\nu^{-1} \sum_{k=1}^{2} \int_\Omega R_M(x,\rho)(h_M(\rho_k+\delta) - \ln(U_k+\delta))dx$$

$$+C_2 \sum_{k=1}^{2} || \min \{\rho_k + \delta, M\}||_{L^1} + \delta^{1/2}||\mathbf{E}||_{L^2}^2 + f(t)\mathcal{E}_\delta(t)^{1/2}$$

For $M > 2||U||_{L^\infty}$ one has by 3.65

$$\mathcal{E}_\delta(t) = \nu^{-1} \sum_{k=1}^{2} \int_\Omega \int_{U_k(x)}^{\rho_k(t,x)} (h_M(s+\delta) - \ln(U_k(x)+\delta))dsdx + 1/2||(\mathbf{E}(t), \mathbf{h}(t))||_{X_0}^2 \quad (3.69)$$

$$\geq c_0||\rho_k(t)||_{L^1} + 1/2||(\mathbf{E}(t), \mathbf{h}(t))||_{X_0}^2 - C_3$$

In the previous estimates $c_0 > 0, C_j \in (0,\infty)$ are independent of $\rho, \mathbf{E}, \mathbf{h}, M, \delta, t$. Now, 3.68, 3.69 yield

$$\mathcal{E}_\delta'(t) \leq C_4(1 + f(t)^2 + \mathcal{E}_\delta(t)) \quad (3.70)$$

$$-\nu^{-1} \sum_{k=1}^{2} \int_\Omega R_M(x,\rho)(h_M(\rho_k+\delta) - \ln(U_k+\delta))dx$$

for all $t \in (0,T)$, with some $C_4 \in (0,\infty)$ independent of $\rho, \mathbf{E}, \mathbf{h}, \delta$. Next, we pass to the limit $\delta \to 0$. Let $T > 0$.
By 3.70 there exists a constant $C_{M,T} \in (0,\infty)$ independent of $\rho, \mathbf{E}, \mathbf{h}, \delta$, such that

$$\mathcal{E}_\delta'(t) \leq C_{M,T}(1 + f(t)^2 + \mathcal{E}_\delta(t)) \quad (3.71)$$

Let

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \nu^{-1} \sum_{k=1}^{2} \int_\Omega \int_{U_k(x)}^{\rho_k(t,x)} (h_M(s) - \ln U_k(x))dsdx + \frac{1}{2}||(\mathbf{E}(t), \mathbf{h}(t))||_{X_0}^2 \quad (3.72)$$

Since $\rho \in C([0,\infty), L^2(\Omega))$ if follows

$$\mathcal{E}_\delta(t) \to \mathcal{E}(t) \quad (3.73)$$

for $\delta \to 0$ locally uniformly on $[0,\infty)$. Now, it follows from 3.70, 3.71, 3.73 and 3.72

$$\mathcal{E}'(t) \le C_4(1+f(t)^2+\mathcal{E}(t)) \tag{3.74}$$
$$-\nu^{-1}\sum_{k=1}^{2}\int_\Omega R_M(x,\rho)(h_M(\rho_k)-\ln U_k)dx$$

for all $t \in (0,T)$ with $f^2 \in L^1(0,T)$.
Next, the last term in 3.74 is estimated.

$$\begin{aligned}
&-R_M(x,u)[h_M(u_1)+h_M(u_2)-\ln(U_1U_2)] \qquad (3.75)\\
&= -r_M(x,u)(\min\{M,u_1\}\min\{M,u_2\}-n_i^2)[h_M(u_1)+h_M(u_2)-\ln(U_1U_2)]\\
&= -r_M(x,u)(\min\{M,u_1\}\min\{M,u_2\}-n_i^2)\\
&\quad [M^{-1}(u_1-M)^+ + M^{-1}(u_2-M)^+ + \ln(\min\{M,u_1\}\min\{M,u_2\})\\
&\qquad -\ln(U_1U_2)]\\
&\le -r_M(x,u)(\min\{M,u_1\}\min\{M,u_2\}-n_i^2)\\
&\quad [\ln(\min\{M,u_1\}\min\{M,u_2\})-\ln(U_1U_2)]\\
&\quad + r_M(x,u)n_i^2M^{-1}[(u_1-M)^+ + (u_2-M)^+]\\
&\le C_4(1+|u|)M^{-1}[(u_1-M)^+ + (u_2-M)^+] + C_4 r_M(x,u)\\
&\le C_5(1+u_1+u_2+M^{-1}[((u_1-M)^+)^2 + ((u_2-M)^+)^2])\\
&\le C_6\left(1+u_1+u_2+\int_{U_1(x)}^{u_1}(h_M(s)-\ln U_1(x))ds + \int_{U_2(x)}^{u_2}(h_M(s)-\ln U_2(x))ds\right)
\end{aligned}$$

for all $x \in \Omega, u \in [0,\infty)^2$. Now, it follows from 3.69, 3.74 and 3.75

$$\mathcal{E}'(t) \le C_4(1+f(t)^2+\mathcal{E}(t)+C_5||\rho(t)||_{L^1}) \le C_6(1+f(t)^2+\mathcal{E}(t)) \tag{3.76}$$

with some $C_6 \in (0,\infty)$ independent of $M,\rho,\mathbf{E},\mathbf{h},t$. Finally, the assertion follows from Gronwall's lemma, 3.69 and 3.76.

□

Next, L^∞ a-priori-estimates for the densities are given.

Theorem 2 *Every solution $(\rho,\mathbf{E},\mathbf{h})$ of 2.53 - 2.62 obeys*

$$\rho \in L^\infty_{loc}([0,\infty),L^\infty(\Omega))$$

and

$$||\rho||_{L^\infty((0,T)\times\Omega)} \le C(1+||\mathbf{E}||_{L^\infty((0,T),L^2(\Omega))})^\lambda(1+||\rho||_{L^\infty((0,T),L^1(\Omega))})$$

with some $C \in (0,\infty)$ independent of $M,T,\rho,\mathbf{E},\mathbf{h}$ and with $\lambda = \max\{10,\frac{5p_0^*}{2-p_0^*}\}$

Proof: It is shown by induction that for all $n \in I\!N$

$$\rho_k^{(2^n)} \in L^\infty_{loc}([0,\infty), L^1(\Omega)), \quad \rho_k^{(2^{n-1})} \in L^2_{loc}([0,\infty), H^1(\Omega)) \tag{3.77}$$

and the following recursive estimate

$$\sum_{k=1}^{2} ||\rho_k^{(2^n)}||_{L^\infty((0,T),L^1(\Omega))} \tag{3.78}$$

$$\leq C_1^{(2^n)} + C_1^n(1+||\mathbf{E}||_{L^\infty((0,T),L^2(\Omega))})^\lambda \sum_{k=1}^{2} ||\rho_k^{(2^{n-1})}||^2_{L^\infty((0,T),L^1(\Omega))}$$

with some $C_1 \in (0,\infty)$ independent of $n \in I\!N, T > 0$.
Suppose

$$\rho_k^{p/2} \in L^\infty_{loc}([0,\infty), L^1(\Omega)) \text{ and } \rho_k^{p/4} \in L^2_{loc}([0,\infty), H^1(\Omega)) \tag{3.79}$$

for $p \stackrel{\text{def}}{=} 2^n$ with some $n \in I\!N$.
For $N > A \stackrel{\text{def}}{=} ||U||_\infty$ the following functions are defined

$$h_N(u) \stackrel{\text{def}}{=} p(\max\{u, A\}^{p-1} - A^{p-1}) \text{ if } u \leq N \tag{3.80}$$

$$h_N(u) \stackrel{\text{def}}{=} p(N^{\frac{3p}{4}-1} u^{p/4} - A^{p-1}) \text{ if } u \geq N$$

$$\text{and } g_N(u) \stackrel{\text{def}}{=} \int_0^u h_N(s)ds.$$

Moreover,

$$G_N(u) \stackrel{\text{def}}{=} \max\{u, A\}^{p/2} \text{ if } u \leq N, \quad G_N(u) \stackrel{\text{def}}{=} N^{\frac{3p}{8}-1/2} u^{p/8+1/2} \text{ if } u \geq N$$

It follows easily that these functions obey the following estimates

$$G_N'(u)^2 \leq C_2 h_N'(u), \quad u h_N(u) \leq C_2 p G_N(u)^2 \tag{3.81}$$

$$\text{and } h_N(u) \leq C_2 G_N(u) G_N'(u)$$

with some $C_2 \in (0,\infty)$ independent of $p = 2^n$ and $u \in [0,\infty)$. It follows from 3.79 and $h_N = 0$ for $u \leq A$ that $h_N(\rho_k) \in L^2_{loc}([0,\infty), Y)$ is an admissible testing-function for 2.53, 2.54. Moreover $\frac{p}{4} + 1 \leq \max\{2, p/2\}$, 3.79 and $\rho \in L^\infty_{loc}([0,\infty), L^2(\Omega))$ imply $g_N(\rho_k) \in L^\infty_{loc}([0,\infty), L^1(\Omega))$.
Now, 2.53, 2.54 yield by using $h_N(\rho_k)$ as testing-function

$$\frac{d}{dt}\sum_{k=1}^{2}\int_\Omega \frac{\varepsilon}{\mu_k} g_N(\rho_k(t))dx = \sum_{k=1}^{2} < \partial_t \rho_k, \frac{\varepsilon}{\mu_k} h_N(\rho_k) >_{Y^*,Y} \tag{3.82}$$

$$= \sum_{k=1}^{2}\int_\Omega \mathbf{j}_{k,M} \cdot \nabla[\frac{\varepsilon}{\mu_k} h_N(\rho_k)]dx - \sum_{k=1}^{2}\int_\Omega R_M(x,\rho)\frac{\varepsilon}{\mu_k} h_N(\rho_k)dx$$

$$\leq -c_0 \sum_{k=1}^{2} \int_\Omega h'_N(\rho_k)|\nabla\rho_k|^2 dx + C_3 \sum_{k=1}^{2} \int_\Omega |\mathbf{j}_{k,M}||h_N(\rho_k)|dx$$

$$- \sum_{k=1}^{2} (-1)^k \int_\Omega \varepsilon \min\{M,\rho_k\} \mathbf{E} h'_N(\rho_k)\nabla\rho_k dx$$

$$+C_3 \int_\Omega (1+\rho_1+\rho_2)[h_N(\rho_1)+h_N(\rho_2)]dx$$

$$\leq -c_0 \sum_{k=1}^{2} \int_\Omega h'_N(\rho_k)|\nabla\rho_k|^2 dx + C_3 \sum_{k=1}^{2} \int_\Omega (|\nabla\rho_k| + |\mathbf{E}|\rho_k)|h_N(\rho_k)|dx$$

$$- \sum_{k=1}^{2} (-1)^k \int_\Omega \varepsilon \mathbf{E}\cdot\nabla H_{N,M}(\rho_k)dx + 4C_3 \sum_{k=1}^{2} \int_\Omega (1+\rho_k)h_N(\rho_k)dx$$

with $H_{N,M}(u) \stackrel{\text{def}}{=} \int_0^u \min\{M,s\}h'_N(s)ds, \quad u>0.$
3.81 and 3.82 yield

$$\frac{d}{dt} \sum_{k=1}^{2} \int_\Omega \frac{\varepsilon}{\mu_k} g_N(\rho_k(t))dx \tag{3.83}$$

$$\leq -c_0 \sum_{k=1}^{2} \int_\Omega |G'_N(\rho_k)\nabla\rho_k|^2 dx + C_4 \sum_{k=1}^{2} \int_\Omega (G_N(\rho_k)|G'_N(\rho_k)\nabla\rho_k| + p|\mathbf{E}|G_N(\rho_k)^2 dx$$

$$- \sum_{k=1}^{2} (-1)^k \int_\Omega \varepsilon \mathbf{E}\cdot\nabla H_{N,M}(\rho_k)dx + C_4 p \sum_{k=1}^{2} \int_\Omega G_N(\rho_k)^2 dx + pC_4^p$$

$$\leq -c_0/2 \sum_{k=1}^{2} ||G_N(\rho_k)||^2_{H^1} dx + C_5 p(1+||\mathbf{E}||_{L^2}) \sum_{k=1}^{2} ||G_N(\rho_k)||^2_{L^4} + pC_4^p$$

$$- \sum_{k=1}^{2} (-1)^k \int_\Omega \varepsilon \mathbf{E}\cdot\nabla H_{N,M}(\rho_k)dx,$$

where $C_j \in (0,\infty), c_0 > 0$ independent of $M, T, \rho, \mathbf{E}, \mathbf{h}$.
The aim of the following considerations is to estimate the last term in 3.83. For $u \leq A = ||U||_{L^\infty}$ one has $h'_N(s) = 0$ on $[0,u]$ and therefore $H_{N,M}(u) = 0$. Hence $H_{N,M}(\rho_k(t)) \in Y$ and lemma 3 yields

$$- \sum_{k=1}^{2} (-1)^k \int_\Omega \varepsilon \mathbf{E}\cdot\nabla H_{N,M}(\rho_k)dx$$

$$= - \sum_{k=1}^{2} (-1)^k [\int_{\partial\Omega} \sigma H_{N,M}(\rho_k)dS - \int_\Omega (\rho_1-\rho_2+C)H_{N,M}(\rho_k)dx]$$

$$\leq -\int_\Omega (\rho_1 - \rho_2)[H_{N,M}(\rho_1) - H_{N,M}(\rho_2)]dx$$

$$+||C||_{L^\infty} \sum_{k=1}^{2} ||H_{N,M}(\rho_k)||_{L^1} + ||\sigma||_{L^\infty(I\!R, L^{p_0}(\partial\Omega))} \sum_{k=1}^{2} ||H_{N,M}(\rho_k)||_{L^{p_0^*}(\partial\Omega)}$$

$$\leq C_6 \sum_{k=1}^{2} (||H_{N,M}(\rho_k)||_{L^1} + ||H_{N,M}(\rho_k)||_{L^{p_0^*}(\partial\Omega)})$$

With $|H_{N,M}(u)| \leq u \int_0^u h_N'(s)ds = uh_N(u) \leq C_1 p G_N(u)^2$ it follows

$$-\sum_{k=1}^{2} (-1)^k \int_\Omega \varepsilon \mathbf{E} \cdot \nabla H_{N,M}(\rho_k) dx \tag{3.84}$$

$$\leq C_6 p \sum_{k=1}^{2} (||G_N(\rho_k)||^2_{L^2} + ||G_N(\rho_k)||^2_{L^{2p_0^*}(\partial\Omega)})$$

Since $p_0^* < 2$ the trace-theorem yields for $\varphi \in H^1(\Omega)$

$$\varphi|_{\partial\Omega} \in L^4(\partial\Omega) \subset L^{p_0^*}(\partial\Omega)$$

and

$$||\varphi||_{L^{2p_0^*}(\partial\Omega)} \leq K_1 ||\varphi||_{H^1(\Omega)}^{\frac{1}{2p_0^*}} ||\varphi||_{L^{4p_0^*-2}(\Omega)}^{1-\frac{1}{2p_0^*}}$$

Since $4p_0^* - 2 < 6$, the L^p-interpolation-inequality and the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ yield

$$||\varphi||_{L^{4p_0^*-2}(\Omega)} \leq ||\varphi||^\kappa_{L^6(\Omega)} ||\varphi||^{1-\kappa}_{L^1(\Omega)} \leq K_2 ||\varphi||^\kappa_{H^1(\Omega)} ||\varphi||^{1-\kappa}_{L^1(\Omega)}$$

with $\kappa \in [0,1)$ with $\frac{1}{4p_0^*-2} = \frac{\kappa}{6} + 1 - \kappa$. Hence

$$||\varphi||_{L^{2p_0^*}(\partial\Omega)} \leq K_3 ||\varphi||_{H^1(\Omega)}^{\frac{1}{2p_0^*}+\kappa(1-\frac{1}{2p_0^*})} ||\varphi||_{L^1(\Omega)}^{(1-\kappa)(1-\frac{1}{2p_0^*})}$$

By using the inequality $x^a y^b \leq ax + by$ for $x, y, a, b > 0, a + b = 1$ it follows that for all $\alpha > 0, \varphi \in H^1(\Omega)$

$$\alpha ||\varphi||^2_{L^{2p_0^*}(\partial\Omega)} \leq \frac{c_0}{4} ||\varphi||^2_{H^1(\Omega)} + k\alpha^\lambda ||\varphi||^2_{L^1(\Omega)} \tag{3.85}$$

with some $k \in (0, \infty)$ independent of α, φ and $c_0 > 0$ as in 3.83, since $\lambda \geq \frac{5p_0^*}{2-p_0^*}$.
Now, 3.83 - 3.85 imply

$$\frac{d}{dt} \sum_{k=1}^{2} \int_\Omega \frac{\varepsilon}{\mu_k} g_N(\rho_k(t)) dx \tag{3.86}$$

$$\leq -c_0/2 \sum_{k=1}^{2} ||G_N(\rho_k)||^2_{H^1} + C_5 p(1 + ||\mathbf{E}||_{L^2}) \sum_{k=1}^{2} ||G_N(\rho_k)||^2_{L^4} + pC_4^p$$

$$+C_6 p\sum_{k=1}^{2}(||G_N(\rho_k)||^2_{L^2}+||G_N(\rho_k)||^2_{L^{2p^*_0}(\partial\Omega)})$$

$$\leq C_7 p(1+||\mathbf{E}||_{L^2})\sum_{k=1}^{2}||G_N(\rho_k)||^2_{L^4}+pC_4^p$$

$$-c_0/4\sum_{k=1}^{2}||G_N(\rho_k)||^2_{H^1}+C_7p^\lambda\sum_{k=1}^{2}||G_N(\rho_k)||^2_{L^1}$$

By using the L^p-interpolation-inequality and the embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ again it follows from 3.86, since $\lambda\geq 10$

$$\frac{d}{dt}\sum_{k=1}^{2}\int_\Omega \frac{\varepsilon}{\mu_k}g_N(\rho_k(t))dx \tag{3.87}$$

$$\leq c_0/8\sum_{k=1}^{2}||G_N(\rho_k)||^2_{H^1}\quad +C_8p^{10}(1+||\mathbf{E}||_{L^2})^{10}\sum_{k=1}^{2}||G_N(\rho_k)||^2_{L^1}$$

$$-c_0/4\sum_{k=1}^{2}||G_N(\rho_k)||^2_{H^1}+C_7p^\lambda\sum_{k=1}^{2}||G_N(\rho_k)||^2_{L^1}+pC_4^p$$

$$\leq -c_0/8\sum_{k=1}^{2}||G_N(\rho_k)||^2_{H^1}+C_9p^\lambda(1+||\mathbf{E}||_{L^2})^\lambda\sum_{k=1}^{2}||G_N(\rho_k)||^2_{L^1}+pC_4^p$$

Since

$$\sum_{k=1}^{2}\int_\Omega \frac{\varepsilon}{\mu_k}g_N(\rho_k(0))dx\leq C_9^p \tag{3.88}$$

$$g_N(u)\rightarrow\ \max\{u,A\}^p-pA^{p-1}u,\quad G_N(u)\rightarrow\ \max\{u,A\}^{p/2} \tag{3.89}$$

for $N\to\infty$, 3.79 and 3.87 yield by letting $N\to\infty$

$$\rho_k^p\in L^\infty_{loc}([0,\infty),L^1(\Omega)),\quad \rho_k^{p/2}\in L^2_{loc}([0,\infty),H^1(\Omega)) \tag{3.90}$$

Now, it follows from 3.87 - 3.89 and 3.90 with $m\stackrel{\text{def}}{=}c_0/8$

$$\exp(mt)\sum_{k=1}^{2}\int_\Omega\frac{\varepsilon}{\mu_k}g_N(\rho_k(t))dx-m\int_0^t\exp(ms)\sum_{k=1}^{2}\int_\Omega\frac{\varepsilon}{\mu_k}g_N(\rho_k)dxds$$

$$\leq -c_0/8\int_0^t\exp(ms)\sum_{k=1}^{2}||\rho_k^{p/2}||^2_{H^1}ds$$

$$+\exp(ms)m^{-1}[C_{10}^p+C_{10}p^\lambda(1+\sup_{s\in[0,t]}||\mathbf{E}(\mathbf{s})||_{L^2})^\lambda\sum_{k=1}^{2}\sup_{s\in[0,t]}||\rho_k(s)^{p/2}||^2_{L^1}]$$

and hence by letting $N \to \infty$

$$\sum_{k=1}^{2}(||\rho_k(t)^p||_{L^1} - pA^{p-1}||\rho_k(t)||_{L^1}) \tag{3.91}$$

$$\leq m^{-1}[C_{10}^p + C_{10}p^\lambda(1 + \sup_{s\in[0,t]} ||\mathbf{E}(\mathbf{s})||_{L^2})^\lambda \sum_{k=1}^{2} \sup_{s\in[0,t]} ||\rho_k(s)^{p/2}||_{L^1}^2]$$

By Hölders inequality there exists a constant $C \in (0,\infty)$ with

$$qA^{q-1}||u||_{L^1} \leq C^q + ||u^{q/2}||_{L^1}^2$$

for all $q \geq 2, u \in L^q(\Omega)$.
Hence, 3.91 yields

$$\sum_{k=1}^{2} ||\rho_k(t)^p||_{L^1} \leq C_{11}^p + C_{11}p^\lambda(1 + \sup_{s\in[0,t]} ||\mathbf{E}(\mathbf{s})||_{L^2})^\lambda \sum_{k=1}^{2} \sup_{s\in[0,t]} ||\rho_k(s)^{p/2}||_{L^1}^2 \tag{3.92}$$

with some constant $C_{11} \in (0,\infty)$ independent of $\rho, \mathbf{E}, \mathbf{h}, p$.
Recall that $p = 2^n$.
Let $T > 0$ and

$$\beta_n \overset{\text{def}}{=} (1 + \sup_{t\in[0,T]} ||\mathbf{E}(t)||_{L^2})^{\lambda(1-2^n)} \sum_{k=1}^{2} \sup_{t\in[0,T]} ||\rho_k(t)^{(2^n)}||_{L^1}$$

Then 3.92 yields

$$\beta_n \leq (1 + \sup_{t\in[0,T]} ||\mathbf{E}(t)||_{L^2})^{\lambda(1-2^n)}[C_{11}^{(2^n)} + C_{11}2^{\lambda n}(1 + \sup_{t\in[0,T]} ||\mathbf{E}(t)||_{L^2})^\lambda \tag{3.93}$$

$$\sum_{k=1}^{2} \sup_{t\in[0,T]} ||\rho_k(t)^{(2^{n-1})}||_{L^1}^2]$$

$$\leq C_{11}^{(2^n)} + C_{11}2^{\lambda n}(1 + \sup_{t\in[0,T]} ||\mathbf{E}(t)||_{L^2})^{2\lambda(1-2^{n-1})} \sum_{k=1}^{2} \sup_{t\in[0,T]} ||\rho_k(t)^{(2^{n-1})}||_{L^1}^2]$$

$$\leq C_{11}^{(2^n)} + C_{11}2^{\lambda n}\beta_{n-1}^2 \leq C_{12}^{(2^n)} + C_{12}^n\beta_{n-1}^2$$

Now, $\alpha_n \overset{\text{def}}{=} \max\{1, C_{12}^{-1}\beta_n^{(2^{-n})}\}$ obey the recursive estimate

$$\alpha_n \leq (1 + C_{12})^{(n2^{-n})}\alpha_{n-1}$$

and hence

$$\alpha_n \leq C_{13}\alpha_0 \tag{3.94}$$

with some constant $C_{13} \in (0,\infty)$ independent of $\rho, \mathbf{E}, \mathbf{h}, n$.
By the definition of α_n, β_n 3.94 implies

$$\overline{\lim_{n\to\infty}} \sup_{t\in[0,T]} ||\rho_k(t)||_{L^{(2^n)}} \leq (1 + \sup_{t\in[0,T]} ||\mathbf{E}(t)||_{L^2})^\lambda \overline{\lim_{n\to\infty}} \beta_n^{(2^{-n})}$$

$$\leq C_{12}(1+\sup_{t\in[0,T]}||\mathbf{E}(t)||_{L^2})^\lambda \overline{\lim_{n\to\infty}}\alpha_n \leq C_{12}C_{13}\alpha_0(1+\sup_{t\in[0,T]}||\mathbf{E}(t)||_{L^2})^\lambda$$

$$\leq C_{13}(1+\sup_{t\in[0,T]}||\mathbf{E}(t)||_{L^2})^\lambda \sup_{t\in[0,T]}||\rho(t)||_{L^1}$$

Therefore, the estimate is proved.

□

Now, it follows from the previous theorems that for arbitrary large $T>0$ there exists some $M_T\in(0,\infty)$, such that every weak solution $\rho,\mathbf{E},\mathbf{h}$ to 2.53 - 2.62 obeys

$$\rho\in L^\infty((0,T),L^\infty(\Omega)) \text{ and } ||\rho||_{L^\infty((0,T),L^\infty(\Omega))}\leq M_T \tag{3.95}$$

Hence, if $\rho,\mathbf{E},\mathbf{h}$ is a weak solution to 2.53 - 2.62, whose existence is shown in section 1, with $M>M_T$, it follows from 3.95 that it is actually a solution to 1.18 - 1.27.
Therefore, the following theorem is proved.

Theorem 3 *Problem 1.18 - 1.27 admits a weak solution* $\rho,\mathbf{E},\mathbf{h}$ *with* $\rho\in L^\infty_{loc}([0,\infty),L^\infty(\Omega))$ *and* $(\mathbf{E},\mathbf{h})\in C([0,\infty),X_0)$

Next, it is shown that the densities are strictly positive.
For this purpose it is assumed in the sequel that in addition to 2.36 $\rho_{k,0}$ is uniformly positive, i. e.

$$\rho_{k,0}\geq a \quad \text{with some } a>0. \tag{3.96}$$

Theorem 4 *Suppose* $\rho,\mathbf{E},\mathbf{h}$ *is a solution to 1.18 - 1.27 and* $T\in(0,\infty]$*, such that* $\rho\in L^\infty((0,T),L^\infty(\Omega))$ *and* $(\mathbf{E},\mathbf{h})\in L^\infty((0,T),X_0)$.
Then $\quad ess\inf_{(0,T)\times\Omega}\rho(t,x)>0$.
(Here $T=+\infty$ *is allowed.)*

Proof:
The assertion will be proved by estimating sucessively L^p-norms ofρ_k^{-1} in a similiar way as in the previous theorem. With $a>0$ as in 3.96 the following functions are defined for $\alpha>0,p>1$

$$h_{\alpha,p}(u)=-p(\min\{u+\alpha,a\})^{-p-1}+pa^{-p-1} \text{ and} \tag{3.97}$$

$$g_{\alpha,p}(u)=(\min\{u+\alpha,a\})^{-p}+pa^{-p-1}\min\{u+\alpha,a\}.$$

The estimates hold

$$h'_{\alpha,p}(u)=4\frac{p+1}{p}(\frac{d}{du}(\min\{u+\alpha,a\})^{-p/2})^2 \tag{3.98}$$

and $g_{\alpha,p}(u)\geq(\min\{u+\alpha,a\})^{-p}$ hold.
Now, let $(\rho,\mathbf{E},\mathbf{h})$ be a solution to 1.18 - 1.27 and $T>0$, such that

$$\rho\in L^\infty((0,T),L^\infty(\Omega)),\quad (\mathbf{E},\mathbf{h})\in L^\infty((0,T),X_0) \tag{3.99}$$

Since $\rho_k = U_k^D \geq a$ on $(0,T) \times \Gamma_D$ one has $h_{\alpha,p}(\rho_k) \in L^2((0,T),Y)$. Hence, the parabolic equations 1.18 and 1.19 yield

$$\frac{d}{dt}\sum_{k=1}^{2}\int_\Omega \frac{\varepsilon}{\mu_k} g_{\alpha,p}(\rho_k(t))dx = \sum_{k=1}^{2} < \partial_t \rho_k, \frac{\varepsilon}{\mu_k} h_{\alpha,p}(\rho_k) >_{Y^*,Y} \tag{3.100}$$

$$= \sum_{k=1}^{2}\int_\Omega \mathbf{j}_k \cdot \nabla[\frac{\varepsilon}{\mu_k} h_{\alpha,p}(\rho_k)]dx - \sum_{k=1}^{2}\int_\Omega R(x,\rho)\frac{\varepsilon}{\mu_k} h_{\alpha,p}(\rho_k)dx$$

$$\leq C_1 \sum_{k=1}^{2}\int_\Omega |\mathbf{j}_k||h_{\alpha,p}(\rho_k)|dx - c_0 \sum_{k=1}^{2}\int_\Omega h'_{\alpha,p}(\rho_k)|\nabla \rho_k|^2 dx$$

$$-\sum_{k=1}^{2}(-1)^k \int_\Omega \varepsilon \rho_k \mathbf{E} h'_{\alpha,p}(\rho_k)\nabla\rho_k dx + \sum_{k=1}^{2}\int_\Omega R(x,\rho)\frac{\varepsilon}{\mu_k}|h_{\alpha,p}(\rho_k)|dx$$

Since ρ is asumed to be uniformly bounded by 3.99 there exists a constant $c \in (0,\infty)$ independent of α, p, such that $R(x,\rho) \leq 0$ if $\rho_1 \leq c$ or $\rho_2 \leq c$.
Hence, $\sum_{k=1}^2 \int_\Omega R(x,\rho)\frac{\varepsilon}{\mu_k}|h_{\alpha,p}(\rho_k)|dx \leq C_2^{p+1}$. Therefore 3.98 and 3.100 yield

$$\frac{d}{dt}\sum_{k=1}^{2}\int_\Omega \frac{\varepsilon}{\mu_k} g_{\alpha,p}(\rho_k(t))dx \tag{3.101}$$

$$\leq C_2^{p+1} + C_2 \sum_{k=1}^{2}\int_\Omega |\nabla\rho_k||h_{\alpha,p}(\rho_k)|dx + C_2||\mathbf{E}(t)||_{L^2}\sum_{k=1}^{2}||\rho_k h_{\alpha,p}(\rho_k)||_{L^2}$$

$$-4c_0(1+1/p)\sum_{k=1}^{2}||\nabla(\min\{\rho_k+\alpha,a\})^{-p/2}||_{L^2}^2$$

$$-\sum_{k=1}^{2}(-1)^k \int_\Omega \varepsilon\rho_k \mathbf{E} h'_{\alpha,p}(\rho_k)\nabla\rho_k dx$$

with some $C_2 \in (0,\infty)$ independent of α, p, t.
Since $h_{\alpha,p}(u) = 0$ if $u \geq a$, one has

$$\int_\Omega |\nabla\rho_k||h_{\alpha,p}(\rho_k)|dx = \int_\Omega |h_{\alpha,p}(\rho_k)\nabla(\min\{\rho_k+\alpha,a\})|dx \tag{3.102}$$

$$\leq p\int_\Omega |(\min\{\rho_k+\alpha,a\})^{-p-1}\nabla(\min\{\rho_k+\alpha,a\})|dx$$

$$= 2\int_\Omega |(\min\{\rho_k+\alpha,a\})^{-p/2}\nabla(\min\{\rho_k+\alpha,a\})^{-p/2}|dx$$

$$\leq \frac{c_0}{2}||(\min\{\rho_k+\alpha,a\})^{-p/2}||_{H^1}^2 + \frac{1}{2c_0}||(\min\{\rho_k+\alpha,a\})^{-p/2}||_{L^2}^2$$

$$\leq c_0||(\min\{\rho_k+\alpha,a\})^{-p/2}||_{H^1}^2 + C_3||(\min\{\rho_k+\alpha,a\})^{-p/2}||_{L^1}^2$$

and

$$||\rho_k h_{\alpha,p}(\rho_k)||_{L^2} \leq p||(\min\{\rho_k+\alpha, a\})^{-p}||_{L^2} = p||(\min\{\rho_k+\alpha, a\})^{-p/2}||^2_{L^4} \tag{3.103}$$

$$\leq (||(\min\{\rho_k+\alpha, a\})^{-p/2}||^{9/10}_{L^6}||(\min\{\rho_k+\alpha, a\})^{-p/2}||^{1/10}_{L^1})^2$$

$$\leq c_0||(\min\{\rho_k+\alpha, a\})^{-p/2}||^2_{H^1} + C_3||(\min\{\rho_k+\alpha, a\})^{-p/2}||^2_{L^1}$$

where the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ has been used.
Next, the term $-\sum_{k=1}^2(-1)^k\int_\Omega \varepsilon\rho_k\mathbf{E}h'_{\alpha,p}(\rho_k)\nabla\rho_k dx$ can be estimated in a similar way as in theorem 2 . Let

$$H_{\alpha,p}(u) \stackrel{\text{def}}{=} uh_{\alpha,p}(u) - g_{\alpha,p}(u), \quad \text{for } u \geq 0$$

Then it follows from $H_{\alpha,p}(\rho_k(t)) \in Y$

$$-\sum_{k=1}^2(-1)^k\int_\Omega \varepsilon\mathbf{E}\cdot\rho_k h'_{\alpha,p}(\rho_k)\nabla\rho_k dx = -\sum_{k=1}^2(-1)^k\int_\Omega \varepsilon\mathbf{E}\nabla H_{\alpha,p}(\rho_k)dx \tag{3.104}$$

$$\leq C_4\sum_{k=1}^2(||H_{\alpha,p}(\rho_k)||_{L^1} + ||H_{\alpha,p}(\rho_k)||_{L^{p^*_0}(\partial\Omega)})$$

$$\leq C_5^{1+p} + C_5 p\sum_{k=1}^2(||(\min\{\rho_k+\alpha, a\})^{-p}||_{L^1} + ||(\min\{\rho_k+\alpha, a\})^{-p}||_{L^{p^*_0}(\partial\Omega)})$$

$$= C_5^{1+p} + C_5 p\sum_{k=1}^2(||(\min\{\rho_k+\alpha, a\})^{-p/2}||^2_{L^2} + ||(\min\{\rho_k+\alpha, a\})^{-p/2}||^2_{L^{2p^*_0}(\partial\Omega)})$$

$$\leq C_5^{1+p} + c_0||(\min\{\rho_k+\alpha, a\})^{-p/2}||^2_{H^1(\Omega)} + C_6 p^\lambda||(\min\{\rho_k+\alpha, a\})^{-p/2}||^2_{L^1(\Omega)}$$

with some $C_4 \in (0,\infty)$ independent of α, p and $c_0 > 0$ as in 3.101.
Now. 3.101 - 3.104 yield

$$\frac{d}{dt}\sum_{k=1}^2\int_\Omega \frac{\varepsilon}{\mu_k}g_{\alpha,p}(\rho_k(t))dx \leq -c_0(1+1/p)\sum_{k=1}^2||(\min\{\rho_k+\alpha, a\})^{-p/2}||^2_{H^1} \tag{3.105}$$

$$+C_5^{1+p} + C_6 p^\lambda||(\min\{\rho_k+\alpha, a\})^{-p/2}||^2_{L^1(\Omega)}$$

Next, it is shown that there exists some $p_1 > 0$, such that

$$\rho_k^{-p_1} \in L^\infty((0,T), L^1(\Omega)) \tag{3.106}$$

Choose $p_1 \in (0,1)$ with $p_1 < \frac{c_0}{C_6 2^\lambda}$. Then it follows from 3.105

$$\frac{d}{dt}\sum_{k=1}^2\int_\Omega \frac{\varepsilon}{\mu_k}g_{\alpha,p_1}(\rho_k(t))dx$$

$$\le C_5^2 - c_0 \sum_{k=1}^{2} ||(\min\ \{\rho_k + \alpha, a\})^{-p_1}||_{L^1} \le C_5^2 - \gamma \sum_{k=1}^{2} \int_\Omega \frac{\varepsilon}{\mu_k} g_{\alpha,p_1}(\rho_k(t))dx$$

with some $\gamma > 0$. This yields by Gronwall's lemma

$$\sup_{\alpha>0,t>0} \sum_{k=1}^{2} \int_\Omega g_{\alpha,p_1}(\rho_k(t))dx < \infty \tag{3.107}$$

Since $(\min\ \{u,a\})^{-p} \le \overline{\lim}_{\alpha\to 0} g_{\alpha,p}(u)$, 3.106 follows from 3.107.
Next, it is shown inductively by using 3.105, 3.106 that

$$\rho_k^{-2^n p_1} \in L^\infty((0,T), L^1(\Omega)) \tag{3.108}$$

Suppose 3.108 holds for some $n \in I\!N_0$. Then from 3.105 with $p = 2^{n+1}p_1$ follows

$$\frac{d}{dt} \sum_{k=1}^{2} \int_\Omega \frac{\varepsilon}{\mu_k} g_{\alpha,2^{n+1}p_1}(\rho_k(t))dx \le -c_0 \sum_{k=1}^{2} ||(\min\ \{\rho_k + \alpha, a\})^{-2^n p_1}||_{L^2}^2 \tag{3.109}$$

$$+C_7^{(2^n)} + C_7 2^{n\lambda} ||(\min\ \{\rho_k + \alpha, a\})^{-2^n p_1}||_{L^1(\Omega)}^2$$

Let

$$\beta_{n,\alpha} \stackrel{\text{def}}{=} \sup_{t\in(0,T)} \sum_{k=1}^{2} ||(\min\ \{\rho_k(t) + \alpha, a\})^{-2^n p_1}||_{L^1},$$

$$\beta_n \stackrel{\text{def}}{=} \sup_{t\in(0,T)} \sum_{k=1}^{2} ||(\min\ \{\rho_k(t), a\})^{-2^n p_1}||_{L^1}$$

and

$$G_{n,\alpha}(t) \stackrel{\text{def}}{=} \sum_{k=1}^{2} \int_\Omega \frac{\varepsilon}{\mu_k} g_{\alpha,2^{n+1}p_1}(\rho_k(t))dx$$

Then it follows from 3.109

$$\frac{d}{dt}(\exp{(c_0 t)} G_{n,\alpha}(t)) \le \exp{(c_0 t)}(c_8^{n+1}\beta_n^2 + c_8^{(2^n)}).$$

Since $\sup_{t\in(0,T)} G_{n,\alpha}(t) \ge \beta_{n+1,\alpha}$ this yields

$$\beta_{n+1,\alpha} \le c_9^{n+1}\beta_n^2 + c_9^{(2^n)} \tag{3.110}$$

with some $c_9 \in (0,\infty)$ independent of n. By letting $\alpha \to 0$ 3.110 implies

$$(\min\ \{\rho_k(t), a\})^{-2^{n+1}p_1} \in L^\infty((0,T), L^1(\Omega))$$

and

$$\beta_{n+1} \le c_9^{n+1}\beta_n^2 + c_9^{(2^n)} \tag{3.111}$$

By the same argument as in the proof of the previous theorem it follows from 3.111 that

$$\sup_{t\in(0,T),n\in I\!N} \sum_{k=1}^{2} ||(\min\ \{\rho_k(t), a\})^{-1}||_{L^{2^n p_1}} \le \sup_{n\in I\!N} \beta_n^{2^{-n}/p_1} \le \infty.$$

Hence $(\min\ \{\rho_k(t), a\})^{-1} \in L^\infty((0,T), L^\infty(\Omega))$ and the assertion is proved.

Corollary 1 *Every solution* $\rho, \mathbf{E}, \mathbf{h}$ *1.18 - 1.27 obeys*

$$ess \inf_{(0,T)\times\Omega} \rho(t,x) > 0 \textit{ for all } T \in (0,\infty].$$

4 Convergence to thermal equilibrium for $t \to \infty$

In this section it is assumed in addition to 2.38, 2.39, 2.40 and 2.41 that

$$\text{curl } \mathbf{H}_\Gamma, \partial_t \mathbf{H}_\Gamma \in (L^1 \cap L^2)((0,\infty), L^2(\Omega)) \tag{4.112}$$

$$\text{and } ||\sigma(\cdot) - \sigma_0||_{L^{4/3}(\partial\Omega)} \in L^2(0,\infty) \tag{4.113}$$

with some $\sigma_0 \in L^{p_0}(\partial\Omega)$.
Moreover, $U^D \in H^1_\Gamma$, i.e. U_D is constant on every conncected component of Γ_D. This condition is reasoable from the physical point of view, since in thermal equilibrium the Fermi-potential is constant on Γ_D and the electric potential φ is constant on every conncected component of Γ_D.
Let $\varphi_e \in H^1_\Gamma$, $U_1, U_2 \in H^1(\Omega) \cap L^\infty(\Omega)$ be solutions of the following system

$$\nabla(\ln U_k - (-1)^k \nu \varphi_e) = 0, \quad U_1 U_2 = n_i^2 \tag{4.114}$$

and for all $\varphi \in Y$

$$-\int_\Omega \varepsilon \nabla\varphi_e \nabla\varphi dx = \int_{\Gamma_N} \sigma_0 \varphi dS - \int_\Omega (U_1 - U_2 + C)\varphi dx \tag{4.115}$$

These are the stationary drift-diffusion-equations in thermal equilibrium. Existence of solutions to the system 4.114, 4.115 has been shown in [7].
The aim of the following considerations is to show that $\lim_{t\to\infty} ||\rho(t) - U||_{L^p(\Omega)} = 0$ for every solution $(\rho, \mathbf{E}, \mathbf{h})$ to the system 1.18 -1.27 and every $p \in [1,\infty)$. Firsty, some global in time estimates are given. In the sequel $(\rho, \mathbf{E}, \mathbf{h})$ is an arbitrary solution to the system 1.18 -1.27 with the properties
$\rho - U^D \in L^\infty_{loc}([0,\infty), L^\infty(\Omega)) \cap L^2_{loc}([0,\infty), Y)$ and $(\mathbf{E}, \mathbf{h}) \in C([0,\infty), X_0)$.

Theorem 5 $\rho \in L^\infty((0,\infty), L^\infty(\Omega))$, $\quad (\mathbf{E}, \mathbf{h}) \in L^\infty((0,\infty), X_0)$,

$$ess \inf_{(0,\infty)\times\Omega} \rho_k > 0 \textit{ and } \mathbf{j}_k \in L^2((0,\infty), L^2(\Omega)).$$

Proof:
Suppose that $(\rho, \mathbf{E}, \mathbf{h})$ is a solution to the system 1.18 -1.27. The following energy-functional is considered

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \nu^{-1} \sum_{k=1}^{2} \int_\Omega \int_{U_k(x)}^{\rho_k(t,x)} [\ln(s) - \ln(U_k(x))] ds dx + \frac{1}{2} ||(\mathbf{E}(t) + \nabla\varphi_e, \mathbf{h}(t))||^2_{X_0} \tag{4.116}$$

By corollary 1 $\rho_k(t)$ is bounded and uniformly positive for fixed $t \in [0,\infty)$.
Since $\varphi \in H^{1,\Gamma}$, one has $(\nabla\varphi_e, 0, 0, 0) \in \ker B$. Therefore, by 2.43, $(\mathbf{E}(\cdot) + \nabla\varphi_e, \mathbf{h}(\cdot))$ is a mild solution of
$\partial_t \mathbf{u} = B\mathbf{u} + \mathbf{f}$ with $\mathbf{f} \stackrel{\text{def}}{=} (\varepsilon^{-1}[\mathbf{j}_2 - \mathbf{j}_1 + \text{ curl } \mathbf{H}_\Gamma], -\partial_t \mathbf{H}_\Gamma)$. This provides the energy-conservation

$$\frac{1}{2}\frac{d}{dt} ||(\mathbf{E}(t) + \nabla\varphi_e, \mathbf{h}(t))||^2_{X_0} \tag{4.117}$$

$$= \int_\Omega (\mathbf{E}(t) + \nabla\varphi_e) \cdot (\mathbf{j}_2(t) - \mathbf{j}_1(t) + \text{ curl } \mathbf{H}_\Gamma(t)) - \mu \mathbf{h}(t) \cdot \partial_t \mathbf{H}_\Gamma(t) dx.$$

Since $\ln(\rho_k(t)) - \ln U_k \in Y$ and $\varphi \in H^{1,\Gamma}$, one obtains from 1.18 and 4.117

$$\mathcal{E}'(t) = \nu^{-1} \sum_{k=1}^{2} < \partial_t \rho_k(t), \ln \rho_k(t) - \ln U_k >_{Y^*,Y}$$

$$+ \int_\Omega (\mathbf{E}(t) + \nabla \varphi_e) \cdot (\mathbf{j}_2(t) - \mathbf{j}_1(t) + \text{ curl } \mathbf{H}_\Gamma(t)) - \mu \mathbf{h}(t) \cdot \partial_t \mathbf{H}_\Gamma(t) dx$$

$$\leq \nu^{-1} \sum_{k=1}^{2} \int_\Omega \mathbf{j}_k [\rho_k^{-1} \nabla \rho_k + (-1)^k \nu \mathbf{E} - \nabla \ln U_k + (-1)^k \nu \nabla \varphi_e] dx$$

$$+ f(t) ||(\mathbf{E}(t) + \nabla \varphi_e, \mathbf{h}(t))||_{X_0} - \nu^{-1} \sum_{k=1}^{2} \int_\Omega R(x, \rho)(\ln(\rho_1 \rho_2) - \ln(U_1 U_2)) dx$$

where $f(t) \stackrel{\text{def}}{=} ||\text{ curl } \mathbf{H}_\Gamma(t)||_{L^2(\Omega)} + ||\partial_t \mathbf{H}_\Gamma(t)||_{L^2(\Omega)}$ obeys $f \in L^1(0, \infty)$. Now, it follows from 4.114 and the assumptions on the recombination term

$$\mathcal{E}'(t) \leq -\sum_{k=1}^{2} \int_\Omega \mu_k^{-1} \rho_k^{-1} |\mathbf{j}_k|^2 dx + f(t) ||(\mathbf{E}(t) + \nabla \varphi_e, \mathbf{h}(t))||_{X_0} \tag{4.118}$$

$$-\nu^{-1} \sum_{k=1}^{2} \int_\Omega r(x, \rho)(\rho_1 \rho_2 - n_i^2)(\ln(\rho_1 \rho_2) - \ln(n_i^2)) dx$$

$$\leq -\sum_{k=1}^{2} \int_\Omega \mu_k^{-1} \rho_k^{-1} |\mathbf{j}_k|^2 dx + f(t) \mathcal{E}(t)^{1/2}$$

Since $f \in L^1(0, \infty)$, this implies

$$\mathcal{E} \in L^\infty(0, \infty) \text{ and } \rho_k^{-1/2} \mathbf{j}_k \in L^2((0, \infty), L^2(\Omega)) \tag{4.119}$$

It follows immediately from the definition of $\mathcal{E}$ that

$$\mathcal{E}(t) \geq \frac{1}{2} ||(\mathbf{E}(t) + \nabla \varphi_e, \mathbf{h}(t))||_{X_0}^2 + c_0 ||\rho_k(t)||_{L^1} - C_1 \tag{4.120}$$

with constants $c_0 > 0, C_1 \in (0, \infty)$ independent of $\rho, \mathbf{E}, \mathbf{h}, t$. From 4.119 and 4.120 one obtains

$$(\mathbf{E}, \mathbf{h}) \in L^\infty((0, \infty), L^2(\Omega)), \text{ and } \rho \in L^\infty((0, \infty), L^1(\Omega)),$$

which implies by theorem 3 ii) and theorem 4

$$\rho \in L^\infty((0, \infty), L^\infty(\Omega)) \text{ and } ess \inf_{(0,\infty) \times \Omega} \rho_k > 0. \tag{4.121}$$

Finally, the last assertion follows from the first one and 4.119.

□

Lemma 4 *There exists a constant $K \in (0,\infty)$, such that*

$$||\rho(t) - U||^2_{L^2(\Omega)} + ||\sqrt{\varepsilon}(\mathbf{E}(t) + \nabla\varphi_e)||^2_{L^2(\Omega)}$$

$$\leq K(||\sigma(t) - \sigma_0||^2_{L^{4/3}(\partial\Omega)} + \sum_{k=1}^{2} ||\mathbf{j}_k(t)||^2_{L^2(\Omega)}) \text{ for all } t \in (0,\infty).$$

Proof:
Let P_0 denote the orthogonal projection on the closed subspace $Z \stackrel{\text{def}}{=} \{\nabla\varphi : \varphi \in Y\}$ of $L^2(\Omega, \mathbb{C}^3)$ with respect to the scalar product

$$< \mathbf{f}, \mathbf{g} >_\varepsilon = \int_\Omega \varepsilon \mathbf{f}\overline{\mathbf{g}} dx, \quad \mathbf{f}, \mathbf{g} \in L^2(\Omega, \mathbb{C}^3).$$

Moreover, $\varphi \in C([0,\infty), Y)$ is defined by

$$-\nabla\varphi(t) = P_0(\mathbf{E}(t) + \nabla\varphi_e) \tag{4.122}$$

By theorem 5 one has

$$\phi_k \stackrel{\text{def}}{=} \nu^{-1} \ln \frac{\rho_k}{U_k} - (-1)^k \varphi \in L^2_{loc}([0,\infty), Y).$$

It follows from 4.114, theorem 5 that

$$||\mathbf{j}||^2_{L^2(\Omega)} \geq c_0 ||\sqrt{\varepsilon}(\nu^{-1}\nabla \ln \rho_k + (-1)^k \mathbf{E})||^2_{L^2(\Omega)} \tag{4.123}$$

$$= c_0 ||\sqrt{\varepsilon}(\nu^{-1}\nabla \ln \frac{\rho_k}{U_k} + (-1)^k(\mathbf{E} + \nabla\varphi_e))||^2_{L^2(\Omega)}$$

$$= c_0 ||\sqrt{\varepsilon} P_0[\nu^{-1}\nabla \ln \frac{\rho_k}{U_k} + (-1)^k(\mathbf{E} + \nabla\varphi_e)]||^2_{L^2(\Omega)}$$

$$+ c_0 ||\sqrt{\varepsilon}(1 - P_0)(\mathbf{E} + \nabla\varphi_e)||^2_{L^2(\Omega)}$$

$$= c_0 ||\sqrt{\varepsilon}\nabla\phi_k||^2_{L^2(\Omega)} + c_0 ||\sqrt{\varepsilon}(1 - P_0)(\mathbf{E} + \nabla\varphi_e)||^2_{L^2(\Omega)}$$

$$\geq c||\phi_k||^2_{L^2(\Omega)} + c_0 ||\sqrt{\varepsilon}(1 - P_0)(\mathbf{E} + \nabla\varphi_e)||^2_{L^2(\Omega)}$$

with $c, c_0 > 0$ independent of t. In the last inequality Poincare's inequality for the space Y is used.
Now, 4.113, 4.122, lemma 3 and lemma 5 yield

$$\sum_{k=1}^{2} ||\phi_k(t)||_{L^2(\Omega)} ||\rho_k(t) - U_k||_{L^2(\Omega)} \geq \sum_{k=1}^{2} \int_\Omega \phi_k(t)(\rho_k(t) - U_k) dx$$

$$= \sum_{k=1}^{2} \int_\Omega [\nu^{-1}(\ln \rho_k(t) - \ln U_k)(\rho_k(t) - U_k) + (\rho_1(t) - U_1 - \rho_2(t) + U_2)\varphi(t)] dx$$

$$\geq c||\rho(t) - U||^2_{L^2(\Omega)} + \int_\Omega (\rho_1(t) - \rho_2(t) + C)\varphi(t) dx - \int_\Omega (U_1 - U_2 + C)\varphi(t) dx$$

$$= c||\rho(t) - U||^2_{L^2(\Omega)} - \int_\Omega \varepsilon \mathbf{E}(t)\nabla\varphi(t)dx + \int_{\partial\Omega} \sigma(t)\varphi(t)dS$$

$$- \int_\Omega \varepsilon\nabla\varphi_e\nabla\varphi(t)dx - \int_{\partial\Omega} \sigma_0\varphi(t)dS$$

$$\geq c||\rho(t) - U||^2_{L^2(\Omega)} + ||\sqrt{\varepsilon}\nabla\varphi(t)||^2_{L^2(\Omega)} - ||\sigma(t) - \sigma_0||_{L^{4/3}(\partial\Omega)}||\varphi(t)||_{H^1(\Omega)}$$

$$\geq c||\rho(t) - U||^2_{L^2(\Omega)} + 1/2||\sqrt{\varepsilon}P_0(\mathbf{E} + \nabla\varphi_e)||^2_{L^2(\Omega)} - f_2(t)$$

with $f_2(t) \stackrel{\text{def}}{=} ||\sigma(t) - \sigma_0||^2_{L^{4/3}(\partial\Omega)}$. Hence

$$c/2||\rho(t) - U||^2_{L^2(\Omega)} + 1/2||\sqrt{\varepsilon}P_0(\mathbf{E}(t) + \nabla\varphi_e)||^2_{L^2(\Omega)} \leq \sum_{k=1}^{2} ||\phi_k(t)||^2_{L^2(\Omega)} + f_2(t) \quad (4.124)$$

with $c > 0$ independent of t and $f \in L^1(0,\infty)$ by assumption 4.113.
4.123 and 4.124 imply

$$||\rho(t) - U||^2_{L^2(\Omega)} + ||\sqrt{\varepsilon}(\mathbf{E}(t) + \nabla\varphi_e)||^2_{L^2(\Omega)} \quad (4.125)$$

$$= ||\rho(t) - U||^2_{L^2(\Omega)} + ||\sqrt{\varepsilon}P_0(\mathbf{E}(t) + \nabla\varphi_e)||^2_{L^2(\Omega)} + ||\sqrt{\varepsilon}(1 - P_0)(\mathbf{E}(t) + \nabla\varphi_e)||^2_{L^2(\Omega)}$$

$$\leq K(f_2(t) + \sum_{k=1}^{2} ||\mathbf{j}_k(t)||^2_{L^2(\Omega)})$$

and the proof is complete.

□

Now, it can be shown that for all $p \in [1,\infty)$

$$\lim_{t\to\infty} ||\rho(t) - U||_{L^p(\Omega)} + ||\mathbf{E}(t) + \nabla\varphi_e||_{L^2(\Omega)} = 0 \quad (4.126)$$

For this purpose a vector-potential $\mathbf{A}$ is introduced. By lemma 2 there exists a unique $\mathbf{A}(t) \in W_E \cap X_E$ with

$$\text{curl } \mathbf{A}(t) = \mu P_H \mathbf{h}(t) \quad (4.127)$$

There exists a constant $C_0 \in (0,\infty)$ with

$$||\mathbf{A}(t)||_{L^2(\Omega)} \leq C_0||P_H\mathbf{h}(t)||_{L^2(\Omega)} \quad (4.128)$$

Recall that $\mathbf{w} \stackrel{\text{def}}{=} (\mathbf{E}, \mathbf{h})$ as a solution of the Maxwell-system obeys

$$\frac{d}{dt} < \mathbf{w}(t), \mathbf{u} >_{X_0} = - < \mathbf{w}(t), B\mathbf{u} >_{X_0} + < \mathbf{f}(t), \mathbf{u} >_{X_0} \text{ for all } \mathbf{u} \in D(B), \quad (4.129)$$

where $\mathbf{f} \stackrel{\text{def}}{=} (\varepsilon^{-1}[\mathbf{j}_2 - \mathbf{j}_1 + \text{ curl } \mathbf{H}_\Gamma], -\partial_t\mathbf{H}_\Gamma)$.
Let $\mathbf{g} \in W_H$ and define $\tilde{\mathbf{g}} \stackrel{\text{def}}{=} (0,0,0,P_H\mathbf{g}) \in D(B)$. Then it follows from 4.127 and 4.129 that

$$< \mathbf{A}(t), \varepsilon^{-1} \text{ curl } \mathbf{g} >_\varepsilon = \int_\Omega \mathbf{A}(t) \text{ curl } \mathbf{g}dx$$

$$= \int_\Omega \mu P_H \mathbf{h}(t)\mathbf{g} dx =< \mathbf{w}(t), \tilde{\mathbf{g}} >_{X_0}$$

$$=< \mathbf{w}_0, \tilde{\mathbf{g}} >_{X_0} + \int_0^t [< \mathbf{f}(s), \tilde{\mathbf{g}} >_{X_0} - < \mathbf{w}(s), B\tilde{\mathbf{g}} >_{X_0}] ds$$

$$= \int_\Omega \mu \mathbf{h}_0 P_H \mathbf{g} dx - \int_0^t [\int_\Omega \mu \partial_s \mathbf{H}_\Gamma(s) P_H \mathbf{g} dx + \int_\Omega \mathbf{E}(s) \text{ curl } (P_H \mathbf{g}) dx] ds$$

$$= \int_\Omega \mu P_H [\mathbf{h}_0 + \mathbf{H}_\Gamma(0) - \mathbf{H}_\Gamma(t)] \mathbf{g} dx - \int_0^t \int_\Omega \varepsilon \mathbf{E} P_E (\varepsilon^{-1} \text{ curl } \mathbf{g}) dx ds$$

$$= \int_\Omega [\mathbf{A}_\Gamma(t) - \int_0^t P_E \mathbf{E}(s) ds] \text{ curl } \mathbf{g} dx,$$

where $\mathbf{A}_\Gamma(t) \in W_E \cap X_E$ is uniquely determined by

$$\text{curl } \mathbf{A}_\Gamma(t) = \mu P_H [\mathbf{h}_0 + \mathbf{H}_\Gamma(0) - \mathbf{H}_\Gamma(t)]. \tag{4.130}$$

Hence

$$< \mathbf{A}(t) - \mathbf{A}_\Gamma(t) + \int_0^t P_E \mathbf{E}(s) ds, \varepsilon^{-1} \text{ curl } \mathbf{g} >_\varepsilon = 0 \tag{4.131}$$

Since $\mathbf{A}(t)$, $\mathbf{A}_\Gamma(t)$ and $P_E \mathbf{E}(t)$ belong to X_E, 4.131 implies

$$\mathbf{A}(t) = \mathbf{A}_\Gamma(t) - \int_0^t P_E \mathbf{E}(s) ds. \tag{4.132}$$

From the assumptions on $\mathbf{H}_\Gamma$ it follows that $\mathbf{A} \in W^{1,2}_{loc}([0,\infty), X_E) \cap L^2_{loc}([0,\infty), W_E)$ and there exists a constant $K \in (0,\infty)$ with

$$||\partial_t \mathbf{A}(t) + P_E \mathbf{E}(t)||_{L^2(\Omega)} = ||\partial_t \mathbf{A}_\Gamma(t)||_{L^2(\Omega)} \leq K ||\partial_t \mathbf{H}_\Gamma(t)||_{L^2(\Omega)} \tag{4.133}$$

Now, it follows from 4.129 and 4.132 that

$$\frac{d}{dt} \int_\Omega \varepsilon \mathbf{E}(t) \mathbf{A}(t) dx = \frac{d}{dt} < \mathbf{w}(t), (\mathbf{A}(t), 0, 0, 0) >_{X_0} \tag{4.134}$$

$$= - < \mathbf{w}(t), B(\mathbf{A}(t), 0, 0, 0) >_{X_0} + < \mathbf{f}(t), (\mathbf{A}(t), 0, 0, 0) >_{X_0}$$

$$+ < \mathbf{w}(t), (\partial_t \mathbf{A}(t), 0, 0, 0) >_{X_0}$$

$$= \int_\Omega \mathbf{h}(t) \text{ curl } \mathbf{A}(t) dx + \int_\Omega [\mathbf{j}_2(t) - \mathbf{j}_1(t) + \text{ curl } \mathbf{H}_\Gamma(t)] \mathbf{A}(t) dx + \int_\Omega \varepsilon \mathbf{E}(t) \partial_t \mathbf{A}(t) dx$$

$$= \int_\Omega \mu |P_H \mathbf{h}(t)|^2 - \varepsilon |P_E \mathbf{A}(t)|^2 dx + \int_\Omega \varepsilon \mathbf{E}(t) \partial_t \mathbf{A}_\Gamma(t) dx$$

$$+ \int_\Omega [\mathbf{j}_2(t) - \mathbf{j}_1(t) + \text{ curl } \mathbf{H}_\Gamma(t)] \mathbf{A}(t) dx.$$

Theorem 6 *There exist constants $K \in (0,\infty)$, $\gamma > 0$, such that*

$$||\rho(t) - U||^2_{L^2(\Omega)} + ||(\mathbf{E}(t) + \nabla\varphi_e, P_H\mathbf{h}(t))||^2_{L^2(\Omega)}$$

$$\leq K\exp(-\gamma t) + K\int_0^t \exp(-\gamma(t-s))$$

$$(||\partial_t \mathbf{H}_\Gamma(s)||^2_{L^2(\Omega)} + ||\ curl\ \mathbf{H}_\Gamma(s)||^2_{L^2(\Omega)} + ||\sigma(s) - \sigma_0||^2_{L^{4/3}(\partial\Omega)})ds \overset{t\to\infty}{\longrightarrow} 0.$$

Proof:
For $\alpha > 0$ the following energy-functional is introduced

$$\mathcal{E}_\alpha(t) \overset{\text{def}}{=} \nu^{-1}\sum_{k=1}^{2}\int_\Omega\int_{U_k}^{\rho_k(t,x)} \ln\frac{s}{U_k}dsdx + \frac{1}{2}||(\mathbf{E}(t) + \nabla\varphi_e, P_H\mathbf{h}(t))||^2_{X_0} \tag{4.135}$$

$$-\alpha\int_\Omega \varepsilon\mathbf{E}(t)\mathbf{A}(t)dx$$

Let $\tilde{P}\mathbf{u} \overset{\text{def}}{=} (u_1,u_2,u_3,P_H(u_4,u_5,u_6))$ for $\mathbf{u} \in X_0$. For $\mathbf{u} \in D(B)$ one has by 2.47 and 2.48 $\tilde{P}\mathbf{u} \in D(B)$ and

$$B\tilde{P}\mathbf{u} = (\varepsilon^{-1}\text{ curl }(P_H(u_4,u_5,u_6)), -\mu^{-1}\text{ curl }(u_1,u_2,u_3))$$

$$= (\varepsilon^{-1}\text{ curl }(u_4,u_5,u_6), -P_H(\mu^{-1}\text{ curl }(u_1,u_2,u_3))) = \tilde{P}B\mathbf{u}.$$

Thus, it follows easily from 4.129 that $\tilde{\mathbf{w}}(t) \overset{\text{def}}{=} \tilde{P}\mathbf{w}(t)$ is a mild solution of
$\frac{d}{dt}\tilde{\mathbf{w}} = B\tilde{\mathbf{w}} - \tilde{P}\mathbf{f}$. Hence

$$\frac{d}{dt}||(\mathbf{E}(t) + \nabla\varphi_e, P_H\mathbf{h}(t))||^2_{X_0} = \frac{d}{dt}||\tilde{\mathbf{w}}(t)||^2_{X_0} = 2 < \mathbf{f}(t), \tilde{P}\mathbf{w}(t) >_{X_0} \tag{4.136}$$

$$= \int_\Omega \mathbf{E}(t)[\mathbf{j}_2(t) - \mathbf{j}_1(t) + \text{ curl }\mathbf{H}_\Gamma(t)] - \mu P_H\mathbf{h}(t)\cdot\partial_t\mathbf{H}_\Gamma(t))dx$$

Since $\nabla\varphi_e \in W_{E,0}$,i.e. $(\nabla\varphi_e,0,0,0) \in \ker B$, we have by 1.18, 1.19, 4.134 and 4.136

$$\mathcal{E}'_\alpha(t) = \nu^{-1}\sum_{k=1}^{2}\int_\Omega \mathbf{j}_k(t)\nabla\ln\frac{\rho_k(t)}{U_k}dx - \int_\Omega r(x,\rho)(\rho_1\rho_2 - n_i^2)[\ln(\rho_1\rho_2) - \ln(n_i^2)]dx \tag{4.137}$$

$$+\int_\Omega(\mathbf{E}(t) + \nabla\varphi_e)[\mathbf{j}_2(t) - \mathbf{j}_1(t) + \text{ curl }\mathbf{H}_\Gamma(t)] - \mu P_H\mathbf{h}(t)\cdot\partial_t\mathbf{H}_\Gamma(t))dx$$

$$-\alpha\int_\Omega(\mu|P_H\mathbf{h}(t)|^2 - \varepsilon|P_E\mathbf{E}(t)|^2)dx - \alpha\int_\Omega\varepsilon\mathbf{E}(t)\partial_t\mathbf{A}_\Gamma(t)dx$$

$$-\alpha\int_\Omega[\mathbf{j}_2(t) - \mathbf{j}_1(t) + \text{ curl }\mathbf{H}_\Gamma(t)]\mathbf{A}(t)dx.$$

$$\leq \nu^{-1}\sum_{k=1}^{2}\int_\Omega \mathbf{j}_k[\rho_k^{-1}\nabla\rho_k + (-1)^k\nu\mathbf{E} - \nabla\ln U_k + (-1)^k\nu\nabla\varphi_e]dx$$

$$+f(t)||(\mathbf{E}(t)+\nabla\varphi_e,\mathbf{h}(t))||_{X_0}-\alpha\int_\Omega(\mu|P_H\mathbf{h}(t)|^2-\varepsilon|P_E\mathbf{E}(t)|^2)dx$$

$$-\alpha\int_\Omega\varepsilon\mathbf{E}(t)\partial_t\mathbf{A}_\Gamma(t)dx-\alpha\int_\Omega[\mathbf{j}_2(t)-\mathbf{j}_1(t)+\text{ curl }\mathbf{H}_\Gamma(t)]\mathbf{A}(t)dx.$$

$$\leq -c\sum_{k=1}^2||\mathbf{j}_k||_{L^2(\Omega)}+C_1\alpha^2||\mathbf{A}(t)||^2_{L^2(\Omega)}$$

$$+f(t)||(\mathbf{E}(t)+\nabla\varphi_e,\mathbf{h}(t))||_{X_0}-\alpha\int_\Omega\mu|P_H\mathbf{h}(t)|^2dx+\alpha\int_\Omega\varepsilon|P_E\mathbf{E}(t)|^2dx$$

$$-\alpha\int_\Omega\varepsilon\mathbf{E}(t)\partial_t\mathbf{A}_\Gamma(t)dx-\alpha\int_\Omega\text{ curl }\mathbf{H}_\Gamma(t)\mathbf{A}(t)dx.$$

by 4.114 and theorem 5, where $f(t)\stackrel{\text{def}}{=}||\partial_t\mathbf{H}_\Gamma(t)||_{L^2(\Omega)}+||\text{ curl }\mathbf{H}_\Gamma(t)||_{L^2(\Omega)}$. Since $\nabla\varphi_e\in W_{E,0}$, it follows from the definition of P_E that

$$\int_\Omega\varepsilon|P_E\mathbf{E}(t)|^2dx=\int_\Omega\varepsilon|P_E(\mathbf{E}(t)+\nabla\varphi_e)|^2dx\leq\int_\Omega\varepsilon|\mathbf{E}(t)+\nabla\varphi_e|^2dx.$$

It follows from 4.128, 4.133, 4.137 and the previous estimate that

$$\mathcal{E}'_\alpha(t)\leq -c\sum_{k=1}^2||\mathbf{j}_k(t)||^2_{L^2(\Omega)}$$

$$+C_2\alpha\int_\Omega\varepsilon|\mathbf{E}(t)+\nabla\varphi_e|^2dx-c\alpha||(\mathbf{E}(t)+\nabla\varphi_e,P_H\mathbf{h}(t))||^2_{X_0}$$

$$+C_2(1+1/\alpha)(f(t)^2+||\partial_t\mathbf{A}_\Gamma(t)||^2_{L^2(\Omega)})+C_2\alpha^2||\mathbf{A}(t)||^2_{L^2(\Omega)}$$

$$\leq -c\sum_{k=1}^2||\mathbf{j}_k(t)||^2_{L^2(\Omega)}+C_3\alpha||\sqrt{\varepsilon}(\mathbf{E}(t)+\nabla\varphi_e)||^2_{L^2(\Omega)}$$

$$-c(1-C_3\alpha)\alpha||(\mathbf{E}(t)+\nabla\varphi_e,P_H\mathbf{h}(t))||^2_{X_0}+C_3(1+1/\alpha)f(t)^2$$

According to lemma 4 one obtains

$$\mathcal{E}'_\alpha(t)\leq -c_0(||\rho(t)-U||^2_{L^2(\Omega)}+||\sqrt{\varepsilon}(\mathbf{E}(t)+\nabla\varphi_e)||^2_{L^2(\Omega)}) \tag{4.138}$$

$$-c_0(1-C_3\alpha)\alpha||(\mathbf{E}(t)+\nabla\varphi_e,P_H\mathbf{h}(t))||^2_{X_0}$$

$$+C_3\alpha||\sqrt{\varepsilon}(\mathbf{E}(t)+\nabla\varphi_e)||^2_{L^2(\Omega)}+C_3(1+1/\alpha)F(t)^2$$

with $c_0>0$ and $C_3\in(0,\infty)$ independent of t,α.
Here $F(t)\stackrel{\text{def}}{=}||\partial_t\mathbf{H}_\Gamma(t)||_{L^2(\Omega)}+||\text{ curl }\mathbf{H}_\Gamma(t)||_{L^2(\Omega)}+||\sigma(t)-\sigma_0||_{L^{4/3}(\partial\Omega)}$. Since $\mathbf{A}(t)\in X_E$ and $\nabla\varphi_e\in W_{E,0}$, 4.128 yields

$$|\int_\Omega\varepsilon\mathbf{E}(t)\mathbf{A}(t)dx|=|\int_\Omega\varepsilon(\mathbf{E}(t)+\nabla\varphi_e)\mathbf{A}(t)dx| \tag{4.139}$$

$$\leq C_0||\sqrt{\varepsilon}(\mathbf{E}(t)+\nabla\varphi_e)||_{L^2(\Omega)}||\sqrt{\mu}P_H\mathbf{h}(t)||_{L^2(\Omega)} \leq C_0||(\mathbf{E}(t)+\nabla\varphi_e, P_H\mathbf{h}(t))||^2_{X_0}.$$

By theorem 5, 4.135 and 4.139 there exist $\alpha_0 > 0$, and constants $c > 0$ and $C_5 \in (0,\infty)$, such that for all $\alpha \in (0,\alpha_0)$ the estimate

$$c(||\rho(t)-U||^2_{L^2(\Omega)} + ||(\mathbf{E}(t)+\nabla\varphi_e, P_H\mathbf{h}(t))||^2_{X_0}) \leq \mathcal{E}_\alpha(t) \tag{4.140}$$

$$\leq C_5(||\rho(t)-U||^2_{L^2(\Omega)} + ||(\mathbf{E}(t)+\nabla\varphi_e, P_H\mathbf{h}(t))||^2_{X_0})$$

holds. Now, it follows from 4.138 and 4.139 that $\alpha > 0$ can be chosen so small such that there exist constants $\gamma > 0$ and $C_6 \in (0,\infty)$ with

$$\mathcal{E}'_\alpha(t) \leq -\gamma\mathcal{E}_\alpha(t) + C_3(1+1/\alpha)F(t)^2 \tag{4.141}$$

By 4.140 and 4.141 the proof of the estimate in theorem 6 is complete. Since $F \in L^1(0,\infty)$ by the assumptions on $\mathbf{H}_\Gamma$ and σ , it follows

$$\lim_{t\to\infty}(||\rho(t)-U||_{L^2(\Omega)} + ||(\mathbf{E}(t)+\nabla\varphi_e, P_H\mathbf{h}(t))||_{X_0}) = 0.$$

□

Finally, exponential convergence the thermal equilibrium for $t \to \infty$ is obtained from lemma 5 and the previous theorem.

Corollary 2 *Let $p \in [1,\infty)$ and suppose that $\mathbf{H}_\Gamma(t) = 0$ and $\sigma(t) = \sigma_0$ for all $t \geq T_0$ with some $T_0 \in (0,\infty)$. There exist constants $K \in (0,\infty)$, $\gamma > 0$, such that*

$$||\rho(t)-U||^2_{L^p(\Omega)} + ||(\mathbf{E}(t)+\nabla\varphi_e, P_H\mathbf{h}(t))||^2_{L^2(\Omega)} \leq K\exp(-\gamma t)$$

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