On Modifications of the Standard Embedding in Nonlinear Optimization

R. Fandom Noubiap*

Abstract

This paper deals with pathfollowing methods in nonlinear optimization. We study the so called "standard embedding" and show its limits. Then, we modify this embedding from several points of view and obtain modified standard embeddings having some advantages. Singularity theory developed by Jongen-Jonker-Twilt plays a great role in our investigation. In some cases, we have to jump from one connected component to another one in the set of local minimizers and in the set of generalized critical points, respectively. In the worst case, we have to find all connected components and that is still an open problem. Computationary results are presented.

Keywords: Nonlinear optimization, pathfollowing methods, singularities, Linear Independence Constraint Qualification, Mangasarian-Fromovitz Constraint Qualification

1 Introduction

Let $C^k(\mathbb{R}^n, \mathbb{R})$, $k \geq 1$ be the space of k-times continuously differentiable realvalued functions on the n-dimensional Euclidean space $\mathbb{R}^n$. We consider the nonlinear optimization problem

(P) $\quad \min\{f(x) | x \in M\}$

where

$M := \{x \in \mathbb{R}^n | h_i(x) = 0, i \in I,\ g_j(x) \leq 0, j \in J\}$

$I := \{1, \ldots, m\}, m < n$, $J := \{1, \ldots, s\}$, and $f, h_i, g_j \in C^2(\mathbb{R}^n, \mathbb{R})$, $i \in I$, $j \in J$.

* Humboldt-Universität, FB Mathematik, PSF 1297, D-10099 Berlin
fandom@mathematik.hu-berlin.de

We follow the concept of the investigations of the pathfollowing methods in [11] (Penalty Embedding), [3] (Exact Penalty Embedding) and [4] (Multiplier Embedding). In this paper we investigate why the so called "standard embedding" is not successful in some cases. Then we modify this embedding from several points of view and show some advantages of the modified embeddings. For our investigation we distinguish between two kinds of regular problems: KH-regular problems (regular problems in the sense of Kojima and Hirabayashi) and JJT-regular problems (regular problems in the sense of Jongen, Jonker and Twilt) (cf. Definition 2.4). For the analysis with respect to JJT-regular problems, we will assume a higher degree of differentiability of the problemfunctions.
Let us recall now the well-known concept of embedding (cf. e.g. [1],[2], [4], [9],[10], [11], [13], [21], [22], [24]): Construct a one-parametric optimization problem

$$\bar{P}(t) \quad \min\{f(y,t) | y \in \bar{M}(t)\}, \quad t \in [0,1],$$

where

$$\bar{M}(t) := \{y \in \mathbb{R}^{\bar{n}} | h_i(y,t) = 0, i \in I, \; g_j(y,t) \leq 0, j \in \bar{J}\}$$

$n \leq \bar{n}$, $\bar{J}$ is a finite index set with $J \subseteq \bar{J}$, with at least the following properties:

(A1) A generalized critical point of $\bar{P}(0)$ is known (and the corresponding Lagrange multipliers are known or easy to compute);

(A2) $\bar{P}(t)$ has a generalized critical point for all $t \in [0,1]$;

(A3) $\bar{P}(1)$ is equivalent to (P).

For the following standard embedding we have $\bar{n} = n$ and $\bar{J} = J$ and we put for the sake of simplicity $y := x$ and $\bar{P}(t) := P(t)$.

$$\mathrm{P(t)} \quad \min\{tf(x) + (1-t)\|x - x^0\|^2 \mid x \in M(t)\}$$

where

$$M(t) := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} h_i(x) + (t-1)h_i(x^0) = 0, \quad i \in I, \\ g_j(x) + (t-1)|g_j(x^0)| \leq 0, \; j \in J \end{array} \right\}$$

and $x^0 \in \mathbb{R}^n$ is arbitrarily chosen. We observe that for this embedding x^0 is even a global minimizer of $P(0)$. Furthermore the problems $P(1)$ and (P) are identical.

(A1) and (A2) are a minimum of assumptions for determining a discretization of the intervall $[0,1]$:

$$0 = t_0 < \cdots < t_i < \cdots < t_N = 1$$

and corresponding

(A) local minimizers $x(t_i)$ of $P(t_i), i = 1, \ldots, N$

or

(B) stationary points $x(t_i)$ of $P(t_i), i = 1, \ldots, N$

or

(C) generalized critical points (g.c. points) $x(t_i)$ of $P(t_i), i = 1, \ldots, N$

Now, we recall some basic definitions (cf. e.g. [13], [18]). A point $x \in I\!R^n$ is called **generalized critical point** (g.c. point) for the problem (P) if $x \in M$ and the set

$$\{Df(x), Dh_i(x), i \in I, Dg_j(x), j \in J_0(x)\}$$

is linearly dependent, where $J_0(x) := \{j \in J | g_j(x) = 0\}$ is the index set of active constraints and Df(x) denotes the row vector of partial derivatives. A **critical point** x for the problem (P) is a point $x \in M$ for which there exist real numbers $\alpha_i, \beta_j, i \in I, j \in J_0(x)$ such that

$$Df(x) + \sum_{i \in I} \alpha_i Dh_i(x) + \sum_{j \in J_0(x)} \beta_j Dg_j(x) = 0.$$

Obviously each critical point x ist a g.c. point. Under the additional condition $\beta_j \geq 0, j \in J_0(x)$, the critical point x is called **stationary point** and (x, α, β) is called **KKT-point**(Karush-Kuhn-Tucker point). These notations are not uniform in the literature.

Pathfollowing methods are a good tool for finding the above discretization of the intervall $[0,1]$, but we cannot expect to be successful in general. Using the standard embedding P(t), we will provide a deeper insight into appearing difficulties and give some possible improvements.

First, we deal with problems without equalities constraints, i.e. $I = \emptyset$. In this case if the feasible set M of the problem (P) is not empty , then for all $t \in [0,1]$, the set M(t) is also not empty. However, the assumption $(A2)$ can be violated. In fact, it may happen that for a $\bar{t} \in (0,1)$, the problem $P(\bar{t})$ has no g.c. point, although a global minimizer of the original problem (P) exists. We give an example which illustrates this phenomenon. Since the assumption $(A2)$ could be violated for the standard embedding P(t), we propose two modifications $\tilde{P}(t)$ and $P^*(t)$ of P(t) for which the assumptions $(A1)$, $(A2)$ and $(A3)$ are fulfilled.

$$\tilde{P}(t): \qquad min\{tf(x) + (1-t)\|x - x^0\|^2 \mid x \in \tilde{M}(t)\}, t \in [0,1],$$

where

$$\tilde{M}(t) := \left\{ x \in I\!R^n \;\middle|\; \begin{array}{c} g_j(x) + (t-1)|g_j(x^0)| \leq 0, j = 1, \ldots, s \\ \|x\|^2 - q \leq 0 \end{array} \right\}$$

(q is a "sufficiently large" real constant, which will be specified later.)

resp.

$$P^*(t): \quad min\{tx_{n+1} + (1-t)\|(x, x_{n+1}) - (x^0, x^0_{n+1})\|^2 \mid (x, x_{n+1}) \in M^*(t)\}, t \in [0,1],$$

where

$$M^*(t) := \left\{ (x, x_{n+1}) \in I\!R^{n+1} \;\middle|\; \begin{array}{l} g_j(x) + (t-1)|g_j(x^0)| \leq 0, j = 1, \ldots, s \\ f(x) - x_{n+1} + (t-1)|f(x^0) - x^0_{n+1}| \leq 0 \end{array} \right\}$$

The problems $\tilde{P}(t)$ and P(t) are in the ball $B_q(0) := \{x \in I\!R^n \mid \|x\|^2 < q\}$ equivalent. Between the problems $P^*(t)$ and P(t) we only have the relation $M(t) = \pi_x(M^*(t))$, where π_x denotes the projection onto the x-space. However, for $t = 1$ the problems $P^*(1)$ and $P(1)$ are equivalent in the sense of theorem 3.1 in this article.

Secondly, we investigate the case where equalities constraints are present. We show that in that case the feasible set M(t) can become empty for certain values of $t \in (0,1)$ although the set M is not empty (and obviously the assumption $(A2)$ is violated). If this phenomenon occurs, then starting the pathfollowing process at $t = 0$ we cannot achieve $t = 1$, i.e. we cannot find a solution of the original problem (P). This disadvantage can be eliminated by the following modifications:

$$\approx\!\!\!\!\!\!P(t): \quad \min\{tf(x) + (1-t)\|x - x^0\|^2 \mid x \in \overset{\approx}{M}(t)\}$$

$$\overset{\approx}{M}(t) = \{x \in I\!R^n \left| \begin{array}{r} h_i(x) + (t-1)|h_i(x^0)| \leq 0,\ i \in I \\ g_j(x) + (t-1)|g_j(x^0)| \leq 0,\ j \in J \\ \|x\|^2 - q \leq 0 \\ -\sum_{i \in I} h_i(x) + (t-1)|\sum_{i \in I} h_i(x^0)| \leq 0 \end{array} \right\}$$

resp.

$$P^{**}(t): \quad \min\{tx_{n+1} + (1-t)\|(x, x_{n+1}) - (x^0, x^0_{n+1})\|^2 \mid (x, x_{n+1}) \in M^{**}(t)\}$$

$$M^{**}(t) = \{(x, x_{n+1}) \in I\!R^{n+1} \left| \begin{array}{r} h_i(x) + (t-1)|h_i(x^0)| \leq 0,\ i \in I \\ g_j(x) + (t-1)|g_j(x^0)| \leq 0,\ j \in J \\ f(x) - x_{n+1} + (t-1)|f(x^0) - x^0_{n+1}| \leq 0 \\ -\sum_{i \in I} h_i(x) + (t-1)|\sum_{i \in I} h_i(x^0)| \leq 0 \end{array} \right\}$$

The problems $\overset{\approx}{P}(t)$ and $P^{**}(t)$ are equivalent to P(t) for $t = 1$ in the sense of Theorem 3.2 in this paper. Moreover, the assumptions $(A1)$, $(A2)$ and $(A3)$ are fulfilled.

Next, we discuss the JJT and the KH -regularisations of the modified embeddings as well as the role of the Enlarged Mangasarian-Fromovitz Constrain Qualification.

Chapter 2 gives an overview on the theoretical background and pathfollowing algorithms. In Chapter 3 we study the Standard Embedding and its modifications. The appearing difficulties are illustrated with numerical examples. The equivalence of the problems (P) and

$P^*(1)$, (P) and $P^{**}(1)$ as well as the equivalence of (P) and $\stackrel{\approx}{P}(1)$ are proved. We show that we can reduce the number of connected components by treating the equalities constraints as inequalities constraints. In Chapter 4 we propose JJT and KH-regular versions of the above embeddings. Furthermore, the role of the EnMFCQ is cleared. Chapter 5 gives a summary of conclusions and further remarks.
All illustrative examples were generated on the computer by PAFO ([25], [12]).

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2 Theoretical Background

We consider the one-parametric optimization problem

P(t): $$min\{f(x,t)/x \in M(t)\}, \quad t \in I\!R,$$

where

$$M(t) = \{x \in I\!R^n / h_i(x,t) = 0, i \in I, g_j(x,t) \leq 0, j \in J\}$$

with $I = \{1, \ldots, m\}, m \leq n, J = \{1, \ldots, s\}, f, h_i, g_j, \in C^k(I\!R^n \times I\!R, I\!R), k \geq 2$.

Definition 2.1 *The Linear Independence Constraint Qualification (***LICQ***) is satisfied at $\bar{x} \in M(\bar{t})$, if the vectors $D_x h_i(\bar{x},\bar{t}), i \in I$, $D_x g_j(\bar{x},\bar{t}), j \in J_0(\bar{x},\bar{t})$ are linearly independent ($J_0(x,t) := \{j \in J / g_j(x,t) = 0\}$).*

Definition 2.2 *The Mangasarian-Fromovitz Constraint Qualification (***MFCQ***) is satisfied at $\bar{x} \in M(\bar{t})$, if the following two conditions hold:*

$(MFCQ1)$ *The vectors $D_x h_i(\bar{x},\bar{t}), i \in I$ are linearly independent.*

$$(MFCQ2) \qquad \exists \eta \in I\!R^n \text{ such that } \begin{cases} D_x h_i(\bar{x},\bar{t})\eta = 0, i \in I & (a) \\ D_x g_j(\bar{x},\bar{t})\eta < 0, j \in J_0(\bar{x},\bar{t}) & (b) \end{cases}$$

The vector $\eta \in I\!R^n$ with the properties (a) and (b) is called **MF-Vektor** *in point $(\bar{x},\bar{t})$.*

Let $t \in I\!R$ be fixed. If LICQ is satisfied at all point $x \in M(t)$, then locally around every $x \in M(t)$, the set M(t) has a nice structure in new differentiable coordinates. In fact, in this case, the set M(t) is a topological C^k-manifold of dimension $n-m$. In case when the MFCQ holds at each $x \in M(t)$, but not the LICQ, the local structure of M(t) is more complicated. The set M(t) is a manifold of dimension $n-m$, but not a differentiable one. We only have a continuous change of coordinates, even if we assume $k > 1$ (cf. [13],[15]). The following theorem is a direct consequence of Theorem B in [15]:

Theorem 2.3 *Assume for all* $t \in [0,1]$ *that* $M(t)$ *is compact and the MFCQ is satisfied at all* $x \in M(t)$. *Then* $M(t_1)$ *and* $M(t_2)$ *are homeomorphic for all* $t_i \in [0,1]$, $i = 1,2$.

Now we introduce the following notations:

$$\sum\nolimits_{loc} := \{(x,t) \in I\!R^n \times I\!R / x \text{ is a local minimizer of } P(t)\ \}$$

$$\sum\nolimits_{stat} := \{(x,t) \in I\!R^n \times I\!R / x \text{ is a stationary point of } P(t)\ \}$$

$$\sum\nolimits_{kkt} := \{(x,\alpha,\beta,t) \in I\!R^n \times I\!R^m \times I\!R^s \times I\!R / (x,\alpha,\beta) \text{ is a KKT-point of } P(t)\ \}$$

$$\sum\nolimits_{gc} := \{(x,t) \in I\!R^n \times I\!R / x \text{ is a generalized critical point of } P(t)\ \}$$

The following mapping $\mathcal{H}$ characterizes the set $\sum_{kkt}$:

$$\mathcal{H} : I\!R^{n+m+s+1} \longrightarrow I\!R^{n+m+s}$$

$$\begin{pmatrix} x \\ \alpha \\ \beta \\ t \end{pmatrix} \longmapsto \begin{bmatrix} D_x f(x,t) + \sum\limits_{i\in I} \alpha_i D_x h_i(x,t) + \sum\limits_{j\in J} \beta_j^+ D_x g_j(x,t) \\ -h_i(x,t), i \in I \\ \beta_j^- - g_j(x,t), j \in J \end{bmatrix}$$

where $\beta_j^+ = max\{\beta_j, 0\}$ and $\beta_j^- = min\{\beta_j, 0\}$. The mapping $\mathcal{H}$ is called Kojima-mapping. Obviously $\mathcal{H}$ is piecewise continuously differentiable. In [21] the classical definition of a regular value of a continuously differentiable function is generalized for piecewise continuously differentiable functions. Furthermore, it is shown that if $0 \in R^{n+m+s}$ is a regular value of $\mathcal{H}$, then the set $\mathcal{H}^{-1}(0)$ is a piecewise one-dimensional C^1-manifold.

The local structure of $\sum_{gc}$ is completely described for (f,H,G) belonging to an open and dense subset $\mathcal{F}$ of $(C_s^3(I\!R^n \times I\!R, I\!R))^{1+m+s}$ with respect to the strong C^3-topology (Whitney-topology). We assume $k \geq 3$ and $(f,H,G) \in \mathcal{F}$ and give a short characterization of this class (for details see [18], [13]).
The points of $\sum_{gc}$ can be divided into 5 types:
Type 1: A point $\bar{z} = (\bar{x}, \bar{t}) \in \sum_{gc}$ is of Type 1 if the following 4 conditions are satisfied:

$$(D_x f + \sum_{i\in I} \alpha_i D_x h_i + \sum_{j\in J_0(\bar{z})} \beta_j D_x g_j)|_{\bar{z}} = 0, \tag{2.1}$$

$$LICQ \text{ is satisfied at } \bar{x} \in M(\bar{t}), \tag{2.2}$$

$$\beta_j \neq 0, \quad j \in J_0(\bar{z}), \tag{2.3}$$

$$D_x^2 L(\bar{z})|_{T(\bar{z})} \text{ is non} - \text{singular}, \tag{2.4}$$

where $D_x^2 L$ is the Hessian for the Lagrangian

$$L = f + \sum_{i\in I} \alpha_i h_i + \sum_{j\in J_0(z)} \beta_j g_j,$$

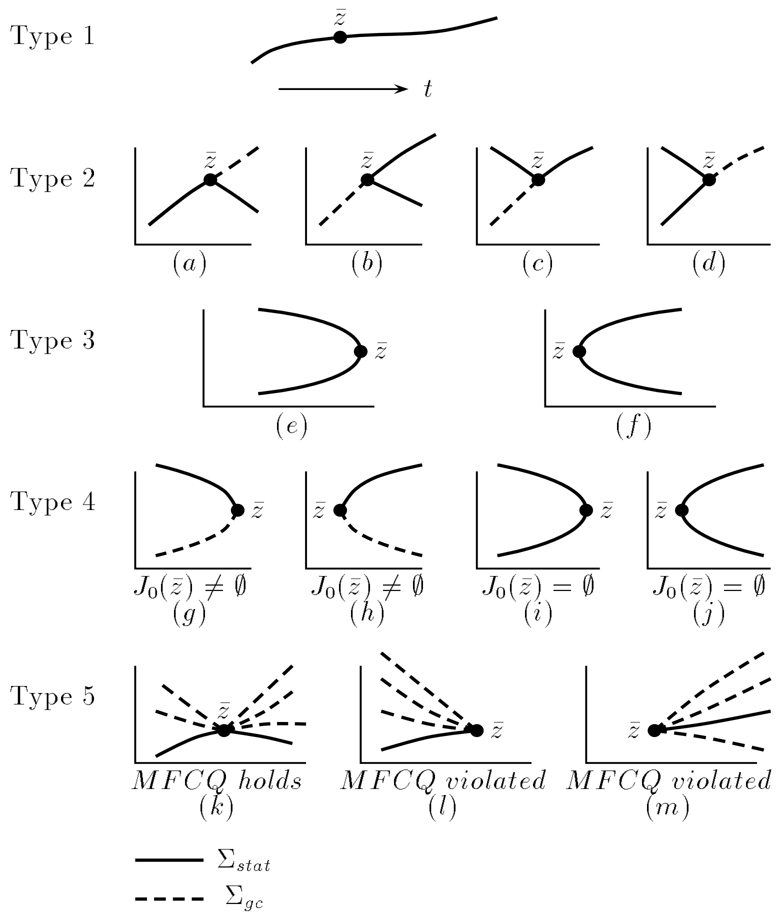

Figure 1:

the numbers α_i, β_j taken from (2.1). Furthermore,

$$T(z) = \{w \in \mathbb{R}^n | D_x h_i(z) w = 0,\ i \in I,\ \ D_x g_j(z) w = 0,\ \ j \in J_0(z)\}$$

is the tangent space at z. $D_x^2 L(z)|_{T(z)}$ represents $V^T D_x^2 L V$, where V is a matrix whose columns form a basis for $T(z)$.
A point of Type 1 is called a nondegenerate critical point. The set $\sum_{gc}$ is the closure of the set of all points of Type 1, the points of Type 2-5 consitute a discrete subset of $\sum_{gc}$. The points of Type 2-5 represent three basic degeneracies:

Type 2 – the violation of (2.3)
Type 3 – the violation of (2.4)
Type 4 – the violation of (2.2) and $|I| + |J_0(\bar z)| - 1 < n$
Type 5 – the violation of (2.2) and $|I| + |J_0(\bar z)| = n + 1$.
Figure 1 illustrates the different possibilities of the local structure of $\sum_{gc}$.

The class $\mathcal{F}$ is defined by

$$\mathcal{F} := \left\{(f,H,G) \in C^3(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})^{1+m+s} \;\middle|\; \begin{array}{l} \textit{each point of } \sum_{gc} \textit{ belongs to one} \\ \textit{of the Types } 1,2,3,4 \textit{ and } 5 \end{array}\right\}$$

If $(f,H,G) \in \mathcal{F}$, then the set $\sum_{gc}$ is the union of 1-dimensional C^2-manifolds. We denote by $\sum_{gc}^k$ the set of g.c. points of Type k, $k = 1,2,3,4,5$ and we define $\sum_{stat}^k := \sum_{stat} \cap \sum_{gc}^k$.

We note that the condition $(f,H,G) \in \mathcal{F}$ implies that zero is a regular value of the Kojima-mapping $\mathcal{H}$.

Definition 2.4 *Let T be a subset of $\mathbb{R}$.*

(i) The problem $P(t)$ is called regular in the sense of Kojima-Hirabayashi (KH-regular) for $t \in T$ if the condition

(RKH) $\quad 0 \in R^{n+m+s}$ *is a regular value of* $\mathcal{H}|_{t \in T}$

is fulfilled.

(ii) The problem $P(t)$ is called regular in the sense of Jongen-Jonker-Twilt (JJT-regular) for $t \in T$ if the condition

(RJJT) $\quad (f,H,G)|_{t\in T} \in \mathcal{F}$

is fulfilled.

Now we present a brief survey of pathfollowing algorithms. (For a full description of these algorithms see [13].) The algorithm PATH III computes a numerical description of a compact connected component of the set $\sum_{gc}$. The corresponding computer program is called PAFO. The algorithm JUMP I works in $cl\sum_{loc}$ and creates jumps from one connected component in $cl\sum_{loc}$ to another one if a turning point $(\bar{x},\bar{t})$ of the set $\sum_{stat}$ appears. A jump at $\bar{z}$ is the computation of a feasible descent direction at $\bar{x}$ with respect to $P(\bar{t})$ and of another local minimizer of $P(\bar{t})$ by using a descent algorithm. Such jumps are possible at points of Type 2, Type 3 and in the Case Ia resp. Case IIa of Type 4 (cf. [13], pp. 128-130). In Case Ib resp. Case IIb of Type 4 as presented there, we cannot jump to another connected component, the current component of the feasible set becomes empty and there is no feasible descent direction at $\bar{x}$ with respect to $P(\bar{t})$. The algorithm JUMP II works in $\sum_{gc}$ and uses in the first step the algorithm PATH III to find a numerical description of a compact connected component of $\sum_{gc}$. JUMP II creates jumps in $\sum_{gc}$ and try to find as many connected components of $\sum_{gc}$ as possible. Clearly, if we can follow all connected components of $\sum_{gc}$ numerically, we will surely find the discretization

$$0 = t^0 < t^1 \ldots < t^k < t^{k+1} < \ldots < t^N = 1$$

of the intervall $[0, 1]$ and the corresponding points of $\sum_{gc}$. But since the problem of the determination of the number of connected components of $\sum_{gc}$ is still open, we cannot be sure that we have found all of them.

We recall the following theorem, which gives conditions under which a continuous curve in $\sum_{stat}$ exists, connecting the starting point of the parametric problem P(t) with a stationary point of the original problem (P).

Theorem 2.5 *(cf. Theorem* 6.2.2 *in* [13]*)*
Assume that the following conditions are satisfied:

$(B0)$ *A local minimizer for $P(0)$ is known (and the corresponding Lagrange multipliers are easy to compute).*

$(B1)$ *For each t in some neighbourhood of zero, there exists a unique KKT-point $(x(t), \alpha(t), \beta(t))$ of P(t) where x(t) is the only stationary point of P(t).*

$(B2)$ *The MFCQ is satisfied for all $x \in M(t)$ and for all $t \in [0, 1]$.*

$(B3)$ *P(t) is KH-regular.*

$(B4)$ *For all $t \in [0, 1]$, M(t) is non-empty and there exists a compact set C such that $M(t) \subset C$.*

$(B5)$ *$P(1)$ is equivalent to (P).*

Then there exists a continuous curve in $\sum_{stat}$ connecting $(x^0, 0)$ and $(\hat{x}, 1)$ where $\hat{x}$ is a stationary point of (P) and x^0 is the known starting point for $P(0)$.

3 On modifications of the standard embedding

3.1 Preliminary outline

We consider the problem

$$(P): \quad \min\{f(x) \mid x \in M\},$$

$$M := \{x \in \mathbb{R}^n \mid h_i(x) = 0,\ i \in I,\ g_j(x) \leq 0,\ j \in J\}$$
$$I := \{1, \ldots, m\},\ m \leq n,\ J := \{1, \ldots, s\},\ f, h_i, g_j \in C^k(\mathbb{R}^n, \mathbb{R}),\ i \in I,\ j \in J,\ k \geq 2,$$

and the standard embedding

$$P(t): \quad \min\{tf(x) + (1-t)\|x - x^0\|^2 \mid x \in M(t)\}, \quad t \in [0, 1],$$

$$M(t) := \{x \in \mathbb{R}^n \mid h_i(x) + (t-1)h_i(x^0) = 0,\ i \in I,\ g_j(x) + (t-1)|g_j(x^0)| \leq 0,\ j \in J\},$$

where $x^0 \in \mathbb{R}^n$ arbitrarily chosen. The problems $P(1)$ and (P) are identical, i.e. the condition (A3) introduced in Section 1 is satisfied. Furthermore x^0 is a global minimizer for $P(0)$.

The corresponding Lagrange multiplier vector (α^0, β^0) satisfies the following homogeneous linear system

$$\sum_{i\in I} \alpha_i Dh_i(x^0) + \sum_{j\in J_0(x^0,0)} \beta_j Dg_j(x^0) = 0, \quad \beta_j = 0,\ j \in J \setminus J_0(x^0, 0).$$

and $(\alpha^0, \beta^0) = (0_m, 0_s)$ is a solution. Then the assumption (A1) is fulfilled. Furthermore $(\alpha^0, \beta^0) = (0_m, 0_s)$ is the unique solution if the LICQ is satisfied.

In the following we consider the condition (A2). We will show that (A2) could be violated for $I = \emptyset$ and for $I \neq \emptyset$. For this reason we modify the embedding $P(t)$ in such a way that (A2) is satisfied. Furthermore we give an answer to the question which kind of singularities may appear for the considered embeddings.

For a modification of $P(t)$ we reformulate (P) in an equivalent form

$$(P^*): \quad \min\{x_{n+1} \mid (x, x_{n+1}) \in M^*\}$$

where

$$M^* := \left\{ (x, x_{n+1}) \in I\!R^{n+1} \left| \begin{array}{l} h_i^*(x, x_{n+1}) := h_i(x) = 0,\ i \in I \\ g_j^*(x, x_{n+1}) := g_j(x) \leq 0,\ j \in J \\ g_{s+1}^*(x, x_{n+1}) := f(x) - x_{n+1} \leq 0 \end{array} \right. \right\}.$$

We introduce the following notations

$$\begin{aligned} \psi_{\text{gc}}(P(t)) &:= \{x \in I\!R^n \mid x \text{ is a g.c. point for } P(t)\} \\ \psi_{\text{stat}}(P(t)) &:= \{x \in I\!R^n \mid x \text{ is a stationary point for } P(t)\} \\ \psi_{\text{loc}}(P(t)) &:= \{x \in I\!R^n \mid x \text{ is a local minimizer for } P(t)\} \\ \psi_{\text{glob}}(P(t)) &:= \{x \in I\!R^n \mid x \text{ is a global minimizer for } P(t)\}. \end{aligned}$$

Then we have $x \in \psi_{\text{gc}}(P(t))$ ($\psi_{\text{stat}}(P(t))$ and $\psi_{\text{loc}}(P(t))$ resp.) if and only if $(x, t) \in \sum_{gc}$ (Σ_{stat} and $\sum_{loc}$ resp.).

Analogously is $\psi_{\text{glob}}(P)$ and $\psi_{\text{glob}}(P^*)$ etc. defined. Furthermore let $\psi_{\text{reg}}(P)$ and $\psi_{\text{reg}}(P^*)$ respectively be the set of all nondegenerated local minimizers of (P) and (P^*), respectively.

The following theorem shows the equivalence between (P) and (P^*).

Theorem 3.1 *The problems (P) and (P^*) are equivalent in the following sense:*

a) $(\bar{x}, \bar{x}_{n+1}) \in \psi_{\text{glob}}(P^*) \iff \bar{x} \in \psi_{\text{glob}}(P)$ *and* $f(\bar{x}) = \bar{x}_{n+1}$.

b) (i) $(\bar{x}, \bar{x}_{n+1}) \in \psi_{\text{reg}}(P^*)$ *and* $f(\bar{x}) = \bar{x}_{n+1} \Longrightarrow \bar{x} \in \psi_{\text{reg}}(P)$,
(ii) $(\bar{x}, \bar{x}_{n+1}) \in \psi_{\text{stat}}(P^*)$ *and* $f(\bar{x}) = \bar{x}_{n+1} \Longrightarrow \bar{x} \in \psi_{\text{stat}}(P)$,
(iii) $(\bar{x}, \bar{x}_{n+1}) \in \psi_{\text{gc}}(P^*) \Longrightarrow \bar{x} \in \psi_{\text{gc}}(P)$.

c) $\bar{x} \in \psi_{\text{loc}}(P)$ ($\psi_{\text{stat}}(P)$ *and* $\psi_{\text{gc}}(P)$ *respectively*)
$\Longrightarrow (\bar{x}, f(\bar{x})) \in \psi_{\text{loc}}(P^*)$ ($\psi_{\text{stat}}(P^*)$ *and* $\psi_{\text{gc}}(P^*)$ *respectively*).

Proof. a) "$\Rightarrow$" Let $(\bar{x}, \bar{x}_{n+1}) \in \psi_{\text{glob}}(P^*)$. Since $(\bar{x}, \bar{x}_{n+1}) \in M^*$ we have $f(\bar{x}) \leq \bar{x}_{n+1}$. From $(\bar{x}, \bar{x}_{n+1}) \in M^*$ follows $(\bar{x}, f(\bar{x})) \in M^*$ and since $(\bar{x}, \bar{x}_{n+1})$ is a global minimizer of (P^*), it holds: $\bar{x}_{n+1} \leq f(\bar{x})$. Then, we have $f(\bar{x}) = \bar{x}_{n+1}$. We assume that $\bar{x} \notin \psi_{\text{glob}}(P)$. Then there exists $\hat{x} \in M$ with $f(\hat{x}) < f(\bar{x})$. Then we have that $(\hat{x}, f(\hat{x})) \in M^*$ and $f(\hat{x}) < f(\bar{x}) = \bar{x}_{n+1}$. This contradicts to $(\bar{x}, \bar{x}_{n+1}) \in \psi_{\text{glob}}(P^*)$.
"$\Leftarrow$" Let $\bar{x} \in \psi_{\text{glob}}(P)$ and $f(\bar{x}) = \bar{x}_{n+1}$. We assume that $(\bar{x}, \bar{x}_{n+1}) \notin \psi_{\text{glob}}(P^*)$, then there exists $(\hat{x}, \hat{x}_{n+1}) \in M^*$ with $\hat{x}_{n+1} < \bar{x}_{n+1}$, i.e., $\hat{x} \in M$ and $f(\hat{x}) \leq \hat{x}_{n+1} < \bar{x}_{n+1} = f(\bar{x})$. This is a contradiction that $\bar{x}$ is a global minimizer of (P).

b) We put $J^* := J \cup \{s+1\}$ and $J_0^*(x, x_{n+1}) := \{j \in J^* \mid g_j^*(x, x_{n+1}) = 0\}$.

(i) Let $(\bar{x}, \bar{x}_{n+1}) \in \psi_{\text{reg}}(P^*)$ with $f(\bar{x}) = \bar{x}_{n+1}$. W.l.o.g. let $J_0^*(\bar{x}, \bar{x}_{n+1}) = \{1, \ldots, p\} \cup \{s+1\}$. Let $\bar{\alpha}_1, \ldots, \bar{\alpha}_m, \bar{\beta}_1, \ldots, \bar{\beta}_p, \bar{\beta}_{s+1}$ be the Lagrange multipliers corresponding to $(\bar{x}, \bar{x}_{n+1})$. Let $L^*(x, x_{n+1}) = x_{n+1} + \sum\limits_{i=1}^{m} \bar{\alpha}_i h_i(x) + \sum\limits_{j=1}^{p} \bar{\beta}_j g_j(x) + \bar{\beta}_{s+1}(f(x) - x_{n+1}))$.
$(\bar{x}, \bar{x}_{n+1}) \in \psi_{\text{reg}}(P^*)$ means:

(a) LICQ is satisfied at $(\bar{x}, \bar{x}_{n+1})$,
(b) $D_{(x, x_{n+1})} L^*(\bar{x}, \bar{x}_{n+1}) = 0$,
(c) $\bar{\beta}_j > 0, \ j \in J_0^*(\bar{x}, \bar{x}_{n+1})$,
(d) $D^2_{(x, x_{n+1})} L^*(\bar{x}, \bar{x}_{n+1})$ positiv definite on $T_{(\bar{x}, \bar{x}_{n+1})} M^*$,

where

$$T_{(\bar{x}, \bar{x}_{n+1})} M^* := \left\{ \begin{pmatrix} w \\ w_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} (Dh_i(\bar{x}), 0) \begin{pmatrix} w \\ w_{n+1} \end{pmatrix} = 0, \ i = 1, \ldots, m \\ (Dg_j(\bar{x}), 0) \begin{pmatrix} w \\ w_{n+1} \end{pmatrix} = 0, \ j = 1, \ldots, p \\ (Df(\bar{x}), -1) \begin{pmatrix} w \\ w_{n+1} \end{pmatrix} = 0 \end{array} \right\}.$$

From (a) follows that LICQ is satisfied at $\bar{x} \in M$. (b) means: $\bar{\beta}_{s+1} = 1$ and $Df(\bar{x}) + \sum\limits_{i=1}^{m} \bar{\alpha}_i Dh_i(\bar{x}) + \sum\limits_{j=1}^{p} \bar{\beta}_j Dg_j(\bar{x}) = 0$. From (c) follows $\bar{\beta}_j > 0$, for all $j \in J_0(\bar{x})$ $(= \{1, \ldots, p\})$. (d) means:

$$\begin{pmatrix} w \\ w_{n+1} \end{pmatrix}^T D^2_{(x, x_{n+1})} L^*(\bar{x}, \bar{x}_{n+1}) \begin{pmatrix} w \\ w_{n+1} \end{pmatrix} > 0 \ \text{ for all } \begin{pmatrix} w \\ w_{n+1} \end{pmatrix} \in T_{(\bar{x}, \bar{x}_{n+1})} M^*,$$

By two-times differentiation of the function L^* we obtain using $\bar{\beta}_{s+1} = 1$

$$D^2_{(x, x_{n+1})} L^*(\bar{x}, \bar{x}_{n+1}) = \begin{pmatrix} D^2 L(\bar{x}) & 0 \\ 0 & 0 \end{pmatrix}$$

where $L(x) := f(x) + \sum_{i=1}^{m} \bar{\alpha}_i Dh_i(x) + \sum_{j=1}^{p} \bar{\beta}_j Dg_j(x)$. Then we have:

$$\text{(d) and (b)} \Longrightarrow w^T D^2 L(\bar{x}) w > 0 \ \text{ for all } \begin{pmatrix} w \\ w_{n+1} \end{pmatrix} \in T_{(\bar{x}, \bar{x}_{n+1})} M^*. \tag{3.1}$$

For all $w \in T_{\bar{x}} M := \left\{ w \in \mathbb{R}^n \;\middle|\; \begin{array}{l} Dh_i(\bar{x})w = 0, \ i = 1, \ldots, m \\ Dg_j(\bar{x})w = 0, \ j = 1, \ldots, p \end{array} \right\}$ holds $\begin{pmatrix} w \\ Df(\bar{x})w \end{pmatrix} \in T_{(\bar{x}, \bar{x}_{n+1})} M^*$ and using (3.1) $w^T D^2 L(\bar{x}) w > 0$. Therefore, $D^2 L(\bar{x})|_{T_{\bar{x}} M}$ is positive definite. Summarized it holds from (a), (b), (c) and (d) $\bar{x} \in \psi_{\text{reg}}(P)$.

(ii) is obvious.

(iii) Let $(\bar{x}, \bar{x}_{n+1}) \in \psi_{\text{gc}}(P^*)$. We distinguish two cases: $f(\bar{x}) = \bar{x}_{n+1}$ and $f(\bar{x}) < \bar{x}_{n+1}$.
Case I $f(\bar{x}) = \bar{x}_{n+1}$. Then we have: $s+1 \in J_0^*(\bar{x}, \bar{x}_{n+1})$. W.l.o.g. let $J_0^*(\bar{x}, \bar{x}_{n+1}) = \{1, \ldots, p\} \cup \{s+1\}$ be. Since $(\bar{x}, \bar{x}_{n+1}) \in \psi_{\text{gc}}(P^*)$, there exists numbers $\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_m, \bar{\beta}_1, \ldots, \bar{\beta}_p, \bar{\beta}_{s+1} \in \mathbb{R}$ such that

$$|\bar{\alpha}_0| + \sum_{i=1}^{m} |\bar{\alpha}_i| + \sum_{j=1}^{p} |\bar{\beta}_j| + |\bar{\beta}_{s+1}| > 0 \quad \text{and}$$
$$\bar{\alpha}_0 \begin{pmatrix} 0_n \\ 1 \end{pmatrix} + \sum_{i=1}^{m} \bar{\alpha}_i \begin{pmatrix} D^T h_i(\bar{x}) \\ 0 \end{pmatrix} + \sum_{j=1}^{p} \bar{\beta}_j \begin{pmatrix} D^T g_j(\bar{x}) \\ 0 \end{pmatrix} + \bar{\beta}_{s+1} \begin{pmatrix} D^T f(\bar{x}) \\ -1 \end{pmatrix} = 0_{n+1}.$$

Then it holds: $|\bar{\alpha}_0| + \sum_{i=1}^{m} |\bar{\alpha}_i| + \sum_{j=1}^{p} |\bar{\beta}_j| > 0$ and $\bar{\alpha}_0 Df(\bar{x}) + \sum_{i=1}^{m} \bar{\alpha}_i Dh_i(\bar{x}) + \sum_{j=1}^{p} \bar{\beta}_j Dg_j(\bar{x}) = 0$. Therefore $\bar{x} \in \psi_{\text{gc}}(P)$.

Case II $f(\bar{x}) < \bar{x}_{n+1}$. Then we have: $s+1 \notin J_0^*(\bar{x}, \bar{x}_{n+1})$. Let $J_0^*(\bar{x}, \bar{x}_{n+1}) = \{1, \ldots, p\}$ be. We get $\bar{x} \in \psi_{\text{gc}}(P)$ by an analog argumentation as in Case I.

c) c_1) Let $\bar{x} \in \psi_{\text{loc}}(P)$ be. Then

$$\exists \varepsilon > 0 \text{ such that } \forall x \in M \text{ with } \|x - \bar{x}\|^2 < \varepsilon^2 \text{ holds } f(\bar{x}) \leq f(x). \tag{3.2}$$

Let $\varepsilon' := \varepsilon$ be. $\forall (x, x_{n+1}) \in M^*$ with $\|(x, x_{n+1}) - (\bar{x}, f(\bar{x}))\| < \varepsilon'$ holds: $x \in M$, $f(x) \leq x_{n+1}$ and $\|x - \bar{x}\|^2 + (x_{n+1} - f(\bar{x}))^2 < \varepsilon^2$, i.e. $\|x - \bar{x}\| < \varepsilon$. Then we have using (3.2) $f(\bar{x}) \leq f(x) \leq x_{n+1}$. Therefore $(\bar{x}, f(\bar{x})) \in \psi_{\text{loc}}(P^*)$.

c_2) Let $\bar{x} \in \psi_{\text{stat}}(P)$ be. Then $\bar{x} \in M$ and there exist numbers $\bar{\alpha}_1, \ldots, \bar{\alpha}_m, \bar{\beta}_1, \ldots, \bar{\beta}_s \in \mathbb{R}$ such that

$$\begin{bmatrix} Df(\bar{x}) + \sum_{i=1}^{m} \bar{\alpha}_i Dh(\bar{x}) + \sum_{j=1}^{s} \bar{\beta}_j Dg_j(\bar{x}) = 0 \\ \bar{\beta}_j g_j(\bar{x}) = 0, \ j = 1, \ldots, s \\ \bar{\beta}_j \geq 0, \ j = 1, \ldots, s \end{bmatrix} \tag{3.3}$$

From $\bar{x} \in M$ follows $(\bar{x}, f(\bar{x})) \in M^*$ and using (3.3) and $\bar{\beta}_{s+1} := 1$ we have

$$\left[\begin{array}{c} \begin{pmatrix} 0_n \\ 1 \end{pmatrix} + \sum_{i=1}^{m} \bar{\alpha}_i \begin{pmatrix} D^T h_i(\bar{x}) \\ 0 \end{pmatrix} + \sum_{j=1}^{s} \bar{\beta}_j \begin{pmatrix} D^T g_j(\bar{x}) \\ 0 \end{pmatrix} + \bar{\beta}_{s+1} \begin{pmatrix} D^T f(\bar{x}) \\ -1 \end{pmatrix} = 0_{n+1} \\ \bar{\beta}_j g_j^*(\bar{x}, f(\bar{x})) = 0, \ j = 1, \ldots, s+1 \\ \bar{\beta}_j \geq 0, \ j = 1, \ldots, s+1 \end{array} \right] \tag{3.4}$$

Therefore $(\bar{x}, f(\bar{x})) \in \psi_{\text{stat}}(P^*)$.

c$_3$) Let $\bar{x} \in \psi_{\text{gc}}(P)$ be. Furthermore let $J_0(\bar{x}) = \{1, \ldots, p\}$ $(= \{j \in J \mid g_j(\bar{x}) = 0\})$ be. From $\bar{x} \in \psi_{\text{gc}}(P)$ follows $x \in M$ and there exist numbers $\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_m, \bar{\beta}_1, \ldots, \bar{\beta}_p \in I\!R$ such that

$$\left. \begin{array}{c} |\bar{\alpha}_0| + \sum_{i=1}^{m} |\bar{\alpha}_i| + \sum_{j=1}^{p} |\bar{\beta}_j| > 0 \\ \bar{\alpha}_0 Df(\bar{x}) + \sum_{i=1}^{m} \bar{\alpha}_i Dh_i(\bar{x}) + \sum_{j=1}^{p} \bar{\beta}_j Dg_j(\bar{x}) = 0 \end{array} \right\} \tag{3.5}$$

$\bar{x} \in M \Rightarrow (\bar{x}, f(\bar{x})) \in M^*$ and $J_0^*(\bar{x}, f(\bar{x})) = J_0(\bar{x}) \cup \{s+1\}$. From (3.5) follows $2|\bar{\alpha}_0| + \sum_{i=1}^{m} |\bar{\alpha}_i| + \sum_{j=1}^{p} |\bar{\beta}_j| > 0$ and $\bar{\alpha}_0 \begin{pmatrix} 0_n \\ 1 \end{pmatrix} + \sum_{i=1}^{m} \bar{\alpha}_i \begin{pmatrix} D^T h_i(\bar{x}) \\ 0 \end{pmatrix} + \sum_{j=1}^{p} \bar{\beta}_j \begin{pmatrix} D^T g_j(\bar{x}) \\ 0 \end{pmatrix} + \bar{\alpha}_0 \begin{pmatrix} D^T f(\bar{x}) \\ -1 \end{pmatrix} = 0_{n+1}$, i.e. $(\bar{x}, f(\bar{x})) \in \psi_{\text{gc}}(P^*)$.

This theorem will be useful for our further investigations.

3.2 Problems without equality constraints

In this section we consider the problem (P) with $I = \emptyset$, i.e.

$$(P): \quad \min\{f(x) \mid x \in M\}, \quad M := \{x \in I\!R^n \mid g_j(x) \leq 0, \ j \in J\}, \quad J := \{1, \ldots, s\}$$

and the corresponding standard embedding

$$P(t): \quad \min\{tf(x) + (1-t)\|x - x^0\|^2 \mid x \in M(t)\},$$

$$M(t) := \{x \in I\!R^n \mid g_j(x,t) = g_j(x) + (t-1)|g_j(x^0)| \leq 0, \ j \in J\}$$

Remark 3.2 The following examples show that

(a) from the compactness of M not follows generally the compactness of $M(t)$ for all $t \in [0,1]$,

(b) from $\psi_{\text{glob}}(P(1)) \neq \emptyset$ and $\psi_{\text{loc}}(P(1)) \neq \emptyset$ respectively not follows generally that $\psi_{\text{glob}}(P(t)) \neq \emptyset$ and $\psi_{\text{loc}}(P(t)) \neq \emptyset$ for all $t \in [0,1]$.

(c) from $\psi_{\mathrm{gc}}(P(1)) \neq \emptyset$ not follows generally that $\psi_{\mathrm{gc}}(P(t)) \neq \emptyset$ for all $t \in [0,1]$.

To a) we consider the set

$$M = \left\{ (x_1,x_2) \in I\!R^2 \;\middle|\; \begin{array}{ll} -\exp x_1 + \frac{1}{2} \leq 0, & x_1 \leq 0 \\ -\exp(-x_2) + \frac{1}{2} \leq 0, & -x_2 \leq 0 \end{array} \right\} = \left[\ln\left(\frac{1}{2}\right), 0\right] \times \left[0, -\ln\left(\frac{1}{2}\right)\right].$$

M is a compact set. We choose $x^0 = (1,-1)^T$. Then $x^0 \notin M$. We construct

$$M(t) = \left\{ (x_1,x_2) \in I\!R^2 \;\middle|\; \begin{array}{ll} -\exp x_1 + \frac{1}{2} + (t-1)(e-\frac{1}{2}) \leq 0, & x_1 + t - 1 \leq 0 \\ -\exp(-x_2) + \frac{1}{2} + (t-1)(e-\frac{1}{2}) \leq 0, & -x_2 + t - 1 \leq 0 \end{array} \right\}.$$

For $0 \leq t \leq \dfrac{e-1}{e-\frac{1}{2}}$ (≈ 0.77) is $\frac{1}{2} + (t-1)(e-\frac{1}{2}) \leq 0$ and $M(t) = (-\infty, 1-t] \times [t-1, \infty)$ is not compact.

To b) let M, x^0 and $M(t)$ as above. Let $f(x) := x_1 + x_2 - (x_1-1)^2 - (x_2+1)^2$ be. Since f is continuous and M is compact then we have $\psi_{\mathrm{glob}}(P(1)) \neq \emptyset$ and $\psi_{\mathrm{loc}}(P(1)) \neq \emptyset$.
We construct $f(x,t) = t[x_1 + x_2 - (x_1-1)^2 - (x_2+1)^2] + (1-t)[(x_1-1)^2 + (x_2+1)^2]$ and consider

$$P\left(\frac{1}{2}\right): \quad \min\left\{ \frac{x_1+x_2}{2} \mid (x_1,x_2) \in \left(-\infty, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \infty\right) \right\}.$$

$P(\frac{1}{2})$ has not a local minimizer (and therefore $P(\frac{1}{2})$ has not a global minimizer), i.e. $\psi_{\mathrm{glob}}(P(\frac{1}{2}))$ and $\psi_{\mathrm{loc}}(P(\frac{1}{2}))$ are empty. Then it holds b).

To c) We consider the problem (P) $\quad min\{\sum_{i=1}^{n}(x_i - x_i^2) \mid x \in M\}$, where

$$M := \{x \in I\!R^n \mid -e^{x_i} + \frac{1}{4} \leq 0, \; -e^{-x_i} + \frac{1}{4} \leq 0, \; i = 1,\dots,n\} = \left[ln(\frac{1}{4}), -ln(\frac{1}{4})\right]^n$$

Then (P) has a global minimizer and a g.c. point. If we choose $x^0 = 0_n$ and construct the problem P(t), it is easy to see that $P(\frac{1}{2}):\; min\{\frac{1}{2}\sum_{i=1}^{n} x_i \mid x \in I\!R^n\}$ has no g.c. point.

Since $P(t)$ has not a generalized critical point for all $t \in [0,1)$, we consider instead of the embedding $P(t)$ the following modified embedding

$$P^*(t): \quad \min\{f^*(x,x_{n+1},t) \mid (x,x_{n+1}) \in M^*(t)\}, \quad t \in [0,1], \quad where$$

$$f^*(x,x_{n+1},t) := t x_{n+1} + (1-t)\|(x,x_{n+1}) - (x^0,x^0_{n+1})\|^2 \qquad ,$$

$$M^*(t) := \{(x,x_{n+1}) \in I\!R^{n+1} \mid \begin{array}{rcl} g_j^*(x,x_{n+1},t) & := & g_j(x) + (t-1)|g_j(x^0)| \leq 0, \; j \in J \\ g_{s+1}^*(x,x_{n+1},t) & := & f(x) - x_{n+1} + (t-1)|f(x^0) - x^0_{n+1}| \leq 0 \end{array} \}$$

and

$$\tilde{P}(t): \quad \min\{\tilde{f}(x,t) \mid x \in \tilde{M}(t)\}, \; t \in [0,1], \quad where$$

$$\tilde{f}(x,t) := t f(x) + (1-t)\|x - x^0\|^2 \quad ,$$

$$\tilde{M}(t) := \left\{ x \in I\!R^n \;\middle|\; \begin{array}{rcl} \tilde{g}_j(x,t) & := & g_j(x) + (t-1)|g_j(x^0)| \leq 0 \quad j \in J \\ \tilde{g}_{s+1}(x,t) & := & \|x\|^2 - q \leq 0 \end{array} \right\}$$

q is a positive real number such that $M \subset B_q(0) := \{x \in I\!R^n | \; \|x\|^2 < q\}$ and $x^0 \in B_q(0)$.

Theorem 3.3 *Let $M = \{x \in I\!R^n \mid g_j(x) \leq 0,\ j \in J\}$ be non-empty and compact, f and g_j, $j \in J$ continuous. Then we have:*

A) (a) *$M^* = M^*(1)$ is non-empty, $M^*(1)$ is not compact,*

(b) *$M^*(t^2) \subset M^*(t^1)$ for $t^2 > t^1$, $t^1, t^2 \in [0,1]$, (especially it holds: $M^*(t) \neq \emptyset$, for all $t \in [0,1]$),*

(c) *The problems $P^*(1)$ and (P^*) are identical,*

(d) *$\psi_{\text{glob}}(P^*(t)) \neq \emptyset$ for all $t \in [0,1]$, $\psi_{\text{glob}}(P^*(1))$ is compact.*

(e) *$\psi_{\text{glob}}(P^*(0)) = \{(x^0, x^0_{n+1})\}$.*

B) (a) *$\tilde{M}(1)$ is compact and non-empty.*

(b) *$\tilde{M}(t^2) \subset \tilde{M}(t^1)$ for $t^2 > t^1$, $t^1, t^2 \in [0,1]$ (especially it holds: $\tilde{M}(t) \neq \emptyset$ for all $t \in [0,1]$),*

(c) *$x \in \psi_{\text{gc}}(P(t))$ ($\psi_{\text{stat}}(P(t))$ and $\psi_{\text{loc}}(P(t))$ respectively) and $\|x\|^2 < q$ if and only if $x \in \psi_{\text{gc}}(\tilde{P}(t))$ ($\psi_{\text{stat}}(\tilde{P}(t))$ and $\psi_{\text{loc}}(\tilde{P}(t))$ respectively) and $\|x\|^2 < q$,*

(d) *$\psi_{\text{glob}}(\tilde{P}(t))$ is compact and non-empty for all $t \in [0,1]$,*

(e) *$\psi_{\text{glob}}(\tilde{P}(0)) = \{x^0\}$.*

Proof: A)

(a) $M \neq \emptyset \Rightarrow \exists\, \bar{x} \in M$. Then it holds $(\bar{x}, f(\bar{x})) \in M^*(1)$ and therefore $M^*(1) \neq \emptyset$. We show that for all $K > 0$ there exists a $(x, x_{n+1}) \in M^*(1)$ such that $\|(x, x_{n+1})\|^2 > K$ (then it holds that $M^*(1)$ is not compact). Let $K > 0$ be arbitrarily. Let $\bar{x}_{n+1} := sup\{\max_{x \in M} f(x), \sqrt{K+1}\}$. From $\bar{x} \in M$ and $\bar{x}_{n+1} \geq \max_{x \in M} f(x) \geq f(\bar{x})$ follows that $(\bar{x}, \bar{x}_{n+1}) \in M^*(1)$. Furthermore it holds $\|(\bar{x}, \bar{x}_{n+1})\|^2 = \|\bar{x}\|^2 + \bar{x}^2_{n+1} \geq \bar{x}^2_{n+1} \geq K + 1 > K$.

(b) Let $t^2 > t^1$, $t^1, t^2 \in [0,1]$ be and let $(x, x_{n+1}) \in M^*(t^2)$.

$$(x, x_{n+1}) \in M^*(t^2) \Longrightarrow g^*_j(x, x_{n+1}, t^2) \leq 0,\ j = 1, \ldots, s+1$$
$$t^1 < t^2 \Longrightarrow g^*_j(x, x_{n+1}, t^1) \leq g^*_j(x, x_{n+1}, t^2),\ j = 1, \ldots, s+1.$$

Then it follows $(x, x_{n+1}) \in M^*(t^1)$. Therefore it holds: $M^*(t^2) \subset M^*(t^1)$.

(c) is obvious.

(d) Firstly, we consider $t \in [0,1)$. For all $t \in [0,1)$ is $M^*(t)$ non-empty, since $M^*(t)$ contains the non-empty subset $M^*(1)$. The objective $f^* = f^*(x, x_{n+1}, t)$ is strictly convex and quadratic $\forall t \in [0,1)$. Therefore $\psi_{\text{glob}}(P^*(t))$ is non-empty $\forall t \in [0,1)$. Let $t = 1$ be. By assumption M is compact and f is continuous. Therefore $\psi_{\text{glob}}(P) \neq \emptyset$. Let $\bar{x} \in \psi_{\text{glob}}(P)$ be. Using Theorem 3.1 we have $(\bar{x}, f(\bar{x})) \in \psi_{\text{glob}}(P^*)$. Since P^* and $P^*(1)$ are identical we have: $\psi_{\text{glob}}(P^*(1)) \neq \emptyset$. We show that $\psi_{\text{glob}}(P^*(1))$ is compact. Let $(\bar{x}, \bar{x}_{n+1}) \in \psi_{\text{glob}}(P^*(1))$ be. Using Theorem 3.1 it holds $\bar{x} \in \psi_{\text{glob}}(P)$ and $f(\bar{x}) = \bar{x}_{n+1}$. Since $\bar{x}$ belongs to the compact set M, there exist a positive number c such that $\|\bar{x}\|^2 + (f(\bar{x}))^2 < c$. Therefore $\psi_{\text{glob}}(P^*(1))$ is bounded. The closeness of $\psi_{\text{glob}}(P^*(1))$ follows from the continuity of the functions f and g_j, $j \in J$.

(e) is obvious.

B) (a) Since M is non-empty and $M \subset B_q(0)$ then it holds $\tilde{M}(1) \neq \emptyset$. $\tilde{M}(1)$ is bounded and closed by standard arguments.

(b) analogously to (b) in Part A).

(c) is obvious.

(d) For all $t \in [0,1]$ $\tilde{M}(t)$ is compact and $\tilde{f}(x,t)$ continuous. Then it holds $\psi_{\text{glob}}(\tilde{P}(t)) \neq \emptyset$ for all $t \in [0,1]$. $\psi_{\text{glob}}(\tilde{P}(t))$ is bounded as a subset of the bounded $\tilde{M}(t)$. The proof of the closeness of $\psi_{\text{glob}}(\tilde{P}(t))$ is analogously as for $\psi_{\text{glob}}(P^*(1))$ in (d), Part A).

(e) is obvious.

We assume that $P^*(t)$ and $\tilde{P}(t)$ are JJT-regular for $t \in [0,1]$ and want to know which kind of singularities may occur for convex and non-convex problems. We refer the reader to Appendix I. We will see that we have nice properties for optimization problems with convexity conditions.

Theorem 3.4 *Let M be non-empty and compact, $f, g_j \in C^3(\mathbb{R}^n, \mathbb{R})$, $j = 1, \ldots, s$; f convex and g_j quasiconvex, $j = 1, \ldots, s$. Assume that $P(t) \in \{P^*(t), \tilde{P}(t)\}$ is JJT-regular with respect to $[0,1]$. Then the following assertions are true:*

(A) a) *$\psi_{\text{glob}}(P(t))$ has only one element $\forall\, t \in [0,1)$,*

b) *At each point of $(\text{cl} \sum_{loc} \cap \sum_{gc}^2)|_{[0,1)}$, there is a continuation in $\sum_{loc}$ for increasing t,*

c) $(\text{cl} \sum_{loc} \cap \sum_{gc}^3)|_{[0,1)} = \emptyset$.

(B) *If additional the functions g_j are pseudoconvex, $j \in J$, there is a $\hat{x} \in M$ such that $g_j(\hat{x}) < 0$, $j \in J$, then it holds:*

d) $(\text{cl} \sum_{loc} \cap \sum_{gc}^4)|_{[0,1]} = \emptyset$,

e) *At each point of* $(\mathrm{cl}\sum_{loc}\cap\Sigma^5_{\mathrm{gc}})|_{[0,1]}$ *there is a continuation in* $\sum_{loc}$ *for increasing* t.

Proof:

a) for $P^*(t)$: For $j \in J$: $M_j^*(t) := \{(x, x_{n+1}) \in I\!R^{n+1} \mid g_j(x) \leq (1-t)|g_j(x^0)|\}$ is convex $\forall\, t \in I\!R$, since g_j are quasiconvex. Since f convex, $g_{s+1}(x, x_{n+1}) := f(x) - x_{n+1}$ is convex, too. Then we have that $M^*_{s+1}(t) := \{(x, x_{n+1}) \in I\!R^{n+1} \mid f(x) - x_{n+1} \leq (1-t)|f(x^0) - x^0_{n+1}|\}$ is convex $\forall\, t \in I\!R$. Therefore $M^*(t) = \bigcap\limits_{j=1}^{s+1} M_j^*(t)$ is convex. The objective of the problem $P^*(t)$ is strictly convex for all $t \in [0,1)$ and thefeasible set $M^*(t)$ is convex and non-empty for all $t \in I\!R$. Therefore, $\psi_{\mathrm{glob}}(P^*(t))$ has only one element for all $t \in [0,1)$.

for $\tilde{P}(t)$: The objective of the problem $\tilde{P}(t)$ is strictly convex and the feasible set $\tilde{M}(t)$ is convex, too. Therefore $\psi_{\mathrm{glob}}(\tilde{P}(t))$ has only one element.

b) Let $P(t) \in \{P^*(t), \tilde{P}(t)\}$ be and let $(\bar{x}, \bar{x}_{n+1}, \bar{t})$ and $(\bar{x}, \bar{t})$ respectively an element of $(\mathrm{cl}\Sigma_{\mathrm{loc}} \cap \sum^2_{gc})|_{[0,1)}$. Assume that there is no continuation in $\sum_{loc}$ for increasing t, then the problem $P(\hat{t})$ with $\hat{t}$ near by $\bar{t}$ has two stationary points, i.e. two global minimizers (since $P(\hat{t})$ is a convex problem) which is excluded by a).

c) follows with the same arguments described in b).

d) and e) It is sufficient to show that the MFCQ is satisfied at each point of $M^*(t)$ and $\tilde{M}(t)$ for all $t \in [0,1]$, i.e. we show the existence of a MF-vector at each point of $M^*(t)$ and $\tilde{M}(t)$ for all $t \in [0,1]$.

for $M^*(t)$: Let $\hat{x} \in M$ be with $g_j(\hat{x}) < 0$, $j = 1, \ldots, s$. We put $\hat{x}_{n+1} := 1 + \max\limits_{x\in M} f(x)$ (M is compact). Then it holds: $f(\hat{x}) < \hat{x}_{n+1}$. Let

$$
\begin{aligned}
g_j^*(x, x_{n+1}, t) &:= g_j(x) + (t-1)|g_j(x^0)|, \quad j = 1, \ldots, s \\
g^*_{s+1}(x, x_{n+1}, t) &:= f(x) - x_{n+1} + (t-1)|f(x^0) - x^0_{n+1}|
\end{aligned}
$$

of course, the functions $g_j^*(\cdot, t)$, $j = 1, \ldots, s+1$ are pseudoconvex. Let $\bar{t} \in [0,1]$ be arbitrarily fixed and $(\bar{x}, \bar{x}_{n+1}) \in M^*(\bar{t})$. Then, $\eta := (\hat{x}, \hat{x}_{n+1}) - (\bar{x}, \bar{x}_{n+1})$ is a MF-vector at $(\bar{x}, \bar{x}_{n+1})$ using the pseudoconvexity.

for $\tilde{M}(t)$: Let $\bar{t} \in [0,1]$ and $\bar{x} \in \tilde{M}(\bar{t})$. Then, $\eta := \hat{x} - \bar{x}$ is a MF-vector at $\bar{x}$ using the pseudoconvexity.

Remark 3.5 (i) We note that we do not know in this moment whether a point of Type 4 and turning points of Type 5 may appear for the problem (P) with a convex function f and quasiconvex functions $g_j(x)$, $j = 1, \ldots, s$.
(ii) If f is non-convex or one g_j, $j \in J$ is non-quasiconvex, then all singularities may occur.

Remark 3.6 If points of Type 4 appear, then we are interested to know whether a jump to another connected component is possible or not.

We want to discuss this question now for the embedding $P(t)$. We consider at a point $(\bar{x},\bar{t})$ of Type 4, $J_0(\bar{x},\bar{t}) = \{1,\ldots,p\}$ and

$$\delta = \text{sign}(-D_t L(\bar{x},\bar{t}))$$

where

$$L(x,t) = g_p(x,t) + \sum_{j=1}^{p-1} \frac{\bar{\beta}_j}{\bar{\beta}_p} g_j(x,t)$$

$\bar{\beta}_j$, $j \in J_0(\bar{x},\bar{t})$ are numbers, all not vanishing, such that

$$\sum_{j=1}^{p} \bar{\beta}_j D_x g_j(\bar{x},\bar{t}) = 0.$$

The numbers $\bar{\beta}_j$ are unique up to a common multiple and $\bar{\beta}_j \neq 0$, $j = 1,\ldots,p$. Now we calculate δ for the concrete embedding $P(t)$, i.e.

$$\delta = \text{sign}\Big(-|g_p(x^0)| - \sum_{j=1}^{p-1} \frac{\bar{\beta}_j}{\bar{\beta}_p} |g_j(x^0)| \Big).$$

From [13] we know that for $\delta = 1$, a jump to another connected component at this point is possible and for $\delta = -1$ we do not have a jump. The worst case ($\delta = -1$) will appear, if all $\bar{\beta}_j$, $j = 1,\ldots,p$ have the same sign.

Illustrationexamples

Firstly, we illustrate the different structures of $\sum_{gc}$ for the embeddings $P(t)$, $\tilde{P}(t)$ and $P^*(t)$.

Example 3.1

$$(P): \quad \min\{\sin x - x^2 \mid x \in M\}$$
$$M = \left\{ x \in I\!R \,\middle|\, \begin{array}{c} -e^{-x} + 0.25 \leq 0 \\ -e^{x} + 0.25 \leq 0 \end{array} \right\} = [\ln(0.25), -\ln(0.25)] \approx [-1.38, 1.38]$$

(P) has a global maximizer at $x = 0.450184$ and two local minimizers at $x = -1.38$ and $x = 1.35$. The global maximum is

$$\sin(0.450184) - 0.450184 * 0.450184 = 0.435131209 - 0.202665633 = 0.232465575.$$

We choose $x^0 = 0$ and construct $P(t)$:

$$P(t): \quad \min\{f(x,t) = t(\sin x - x^2) + (1-t)x^2 \mid x \in M(t)\}$$

$$M(t) = \left\{ x \in I\!R \,\middle|\, \begin{array}{r} -e^{-x} + 0.25 + (t-1)0.75 \le 0 \\ -e^{x} + 0.25 + (t-1)0.75 \le 0 \end{array} \right\}.$$

The objective $f(x,t)$ is strictly convex for $t < 0.4$ and strictly concave for $t > \frac{2}{3}$. For $0.4 < t < \frac{2}{3}$ f(x,t) is non-convex and non-concave. Thefeasible set $M(t)$ is empty for $t > 2$. For $t \in (\frac{2}{3}, 2]$ $M(t)$ is compact and for $0 < t < \frac{2}{3}$ we have $M(t) = I\!R$. The Figure 2 shows the set $\sum_{gc}|_{[0.2]}$. $P(\frac{1}{2}) : \min\{\frac{1}{2}\sin x \mid x \in I\!R\}$ has a infinite number of local minimizers and local maximizers. A infinite number of points of Type 3 appear. The two curves of local minimizers containing the two local minimizers of (P) have an asymptote at $t = \frac{2}{3}$. There is not a curve "connecting $t = 0$ and $t = 1$". Table 1 shows the singularities. We are not successful with PATH III but with JUMP I we achieve $t = 1$. We jump at the first point of Type 3 from the first connected component to the second one and we achieve 3 critical points of the problem $P(1)$. We have another possibility with the embedding $\tilde{P}(t)$ (with compactification):

$$\tilde{P}(t): \quad \min\{t(\sin x - x^2) + (1-t)x^2 \mid x \in \tilde{M}(t)\}$$

$$\tilde{M}(t) = \left\{ x \in I\!R \,\middle|\, \begin{array}{r} -e^{-x} + 0.25 + (t-1)0.75 \le 0 \\ -e^{x} + 0.25 + (t-1)0.75 \le 0 \\ x^2 \le q = 60 \end{array} \right\}.$$

Then there appear a connection between the both connected components and we obtain the 3 critical points of $\tilde{P}(1)$ (cf. Fig. 3). For the singularities we refer to Table 2.

t	x	Type
0.545401	-2.857614	3
0.481294	-6.338081	3
0.509872	-9.311866	3
0.495375	-17.64180	3
...	...	...
2.00000	0.00000	5 (MFCQ violated)
1.61448	0.341275	2
0.461144	2.837158	3
0.518523	6.262758	3
0.488207	9.326819	3
0.504624	17.843014	3
...	...	...

Table 1

t	x	Type
0.545489	-2.93286	3
0.480967	-5.93209	3
0.498266	-7.74596	2
$\frac{2}{3}$	-7.74596	5 (MFCQ satisfied)
2.0000	0.00000	5 (MFCQ violated)
1.61448	0.341275	2
$\frac{2}{3}$	7.74596	5 (MFCQ satisfied)
0.501746	7.74596	2
0.463175	2.47889	3
0.519675	6.47814	3

Table 2

We consider the embedding $P^*(t)$ with the strictly convex quadratic objective and the starting point $(x_1^0, x_2^0) = (0, 0)$

$$P^*(t): \quad \min\{tx_2 + (1-t)(x_1^2 + x_2^2) \mid (x_1, x_2) \in M^*(t)\}$$

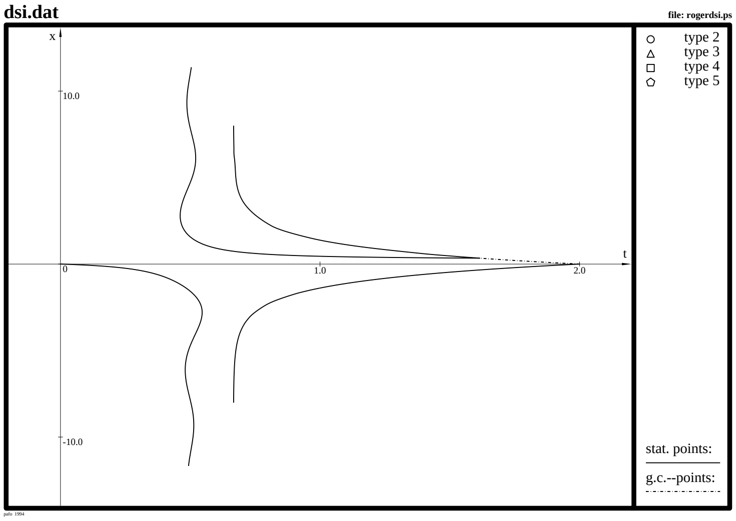

Figure 2:

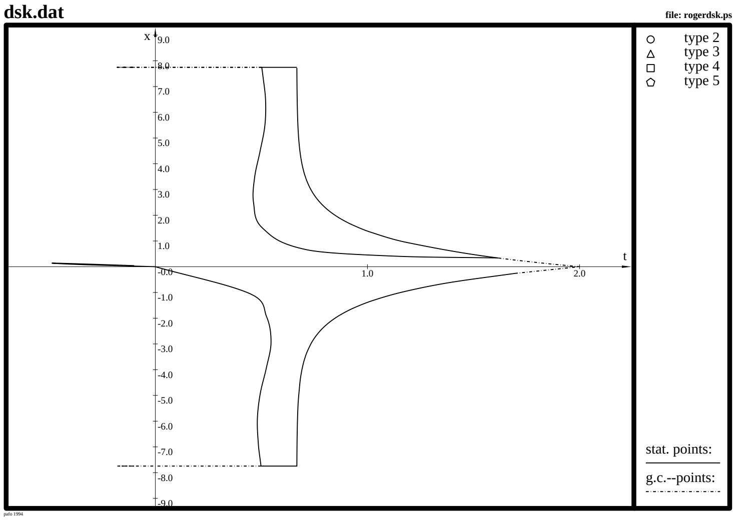

Figure 3:

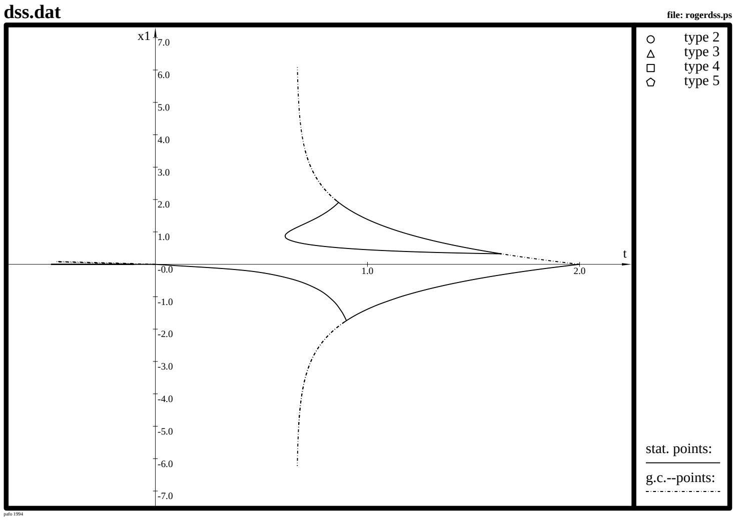

Figure 4:

$$M^*(t) = \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{r} -e^{-x_1} + 0.25 + (t-1)0.75 \le 0 \\ -e^{x_1} + 0.25 + (t-1)0.75 \le 0 \\ \sin(x_1) - x_1^2 - x_2 \le 0 \end{array} \right\}.$$

Table 3 contains the singularities and Fig. 4 shows the connected component with the 3 critical points for $P^*(1)$.

t	x_1	x_2	Type
0.892	-1.777	-4.137	2
2.0000	0.00000	0.00000	5 (MFCQ violated)
1.630	0.324	0.213	2
0.612	0.890	0.00000	3
0.854	1.962	-2.926	2

Table 3

Example 3.2 Minimize $f(x) = (x+4)^2$ subject to $g(x) \le 0$, where

$$\begin{aligned} g(x) &= 0.0265073509x^8 + 0.211505207x^7 + 0.25753848x^6 + 1.34579642x^5 \\ &\quad -2.34222067x^4 - 2.65029635x^3 + 3.45664738x^2 + 0.91447716x + 5.0. \end{aligned}$$

g is a non-convex function with 4 local minimizers and 3 local maximizers. The problem

$$(P): \quad \min\{(x+4)^2 \mid g(x) \le 0\}$$

has the global minimizer at $x = 3,670154$ and the global maximizer at $x = 4.0216$. We choose $x^0 = -4$ and $x^0 = -1$ respectively and consider the parametric optimization problem

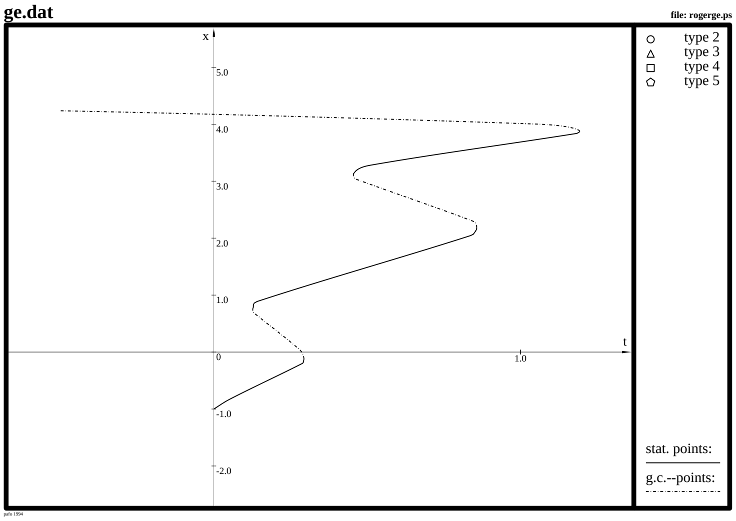

Figure 5:

$$\tilde{P}(x^0,t): \quad \min\{t(x+4)^2+(1-t)(x-x^0)^2 \mid g(x)+(t-1)|g(x^0)| \leq 0,$$

It holds: $g(-4) = 4.505928$ and $g(-1) = 7$. We find with the algorithm PATH III the global minimizer and maximizer for (P). The Table 4 contains the singularities for $x^0 = -4$ and Table 5 for $x^0 = -1$. Fig. 5 shows the curve of stationary points for $x^0 = -1$. We observe that with both starting points the additional compactification constraint $x^2 \leq 1000$ is not active along the curve in $\sum_{stat}|_{[0,1]}$. That means we can cancel this constraint. The standard embedding is successful while the penalty embedding ([11]), the exact penalty embedding ([3]) and the multiplier embedding ([4]) are not. In section 4, we will show that the so called "Enlarged Mangasarian-Fromovitz Constraint Qualification" is not satisfied for this problem.

t	x	Type
0.0999004	-1.73801	4
0.998438	-1.07132	4
0.998903	-0.119176	4
0.998644	0.741932	4
0.999778	2.18657	4
0.999153	3.10390	4
1.00000	3.670154	minimizer
1.00030	3.87355	4
1.00000	4.0216	maximizer

Table 4

t	x	Type
0.293	-0.119176	4
0.127	0.741932	4
0.8573	2.18657	4
0.454	3.10390	4
1.00000	3.670154	minimizer
1.192	3.87355	4
1.00000	4.0216	maximizer

Table 5

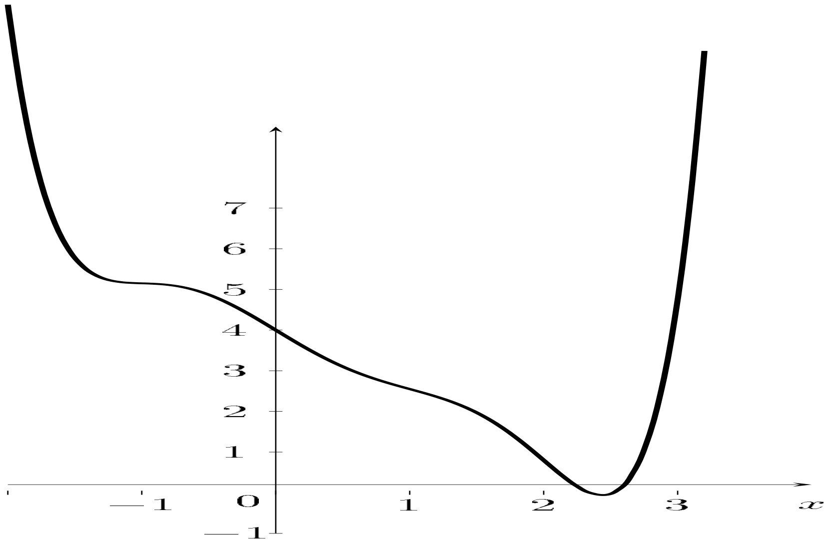

Figure 6:

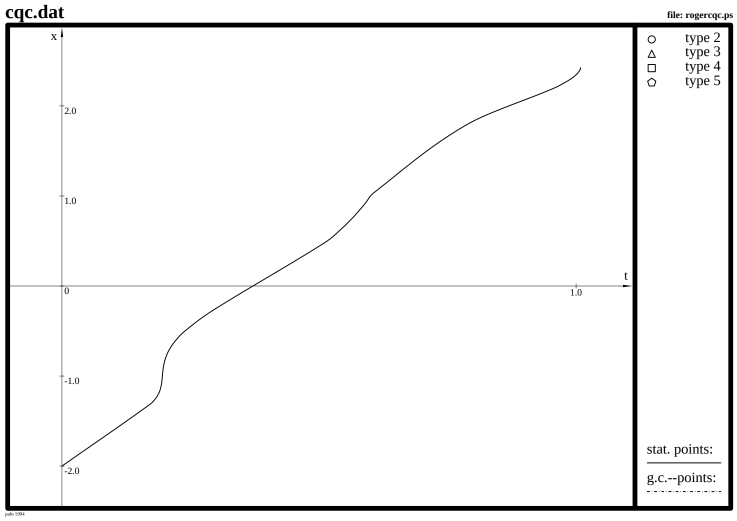

Figure 7:

Example 3.3 We consider the following problem with the strict convex objective $f(x) = (x+2)^2$ and the quasiconvex constraint function

$$g(x) = 0.1x^6 - 0.3x^5 - 0.25x^4 + x^3 - 2x + 4.$$

Fig. 6 illustrates this function. Furthermore, let $x^0 = -2$ be. With the embedding

$$\tilde{P}(t): \quad \min\left\{(x+2)^2 \,\middle|\, \begin{array}{l} g(x) + (t-1)6.4 \le 0 \\ x^2 - 100 \le 0 \end{array}\right\}$$

we obtain a curve of stationary point of Type 1 connecting points at $t = 0$ and $t = 1$ (cf. Fig. 7).

3.3 Problems with equality constraints

We consider the following problem

$$(P): \quad \min\{f(x) \mid x \in M\}$$

$$M = \{x \in I\!R^n \mid h_i(x) = 0,\ i \in I,\ g_j(x) \le 0,\ j \in J\} \text{ with } I \neq \emptyset$$

and the corresponding standard embedding

$$P(x^0, t): \quad \min\{tf(x) + (1-t)\|x - x^0\|^2 \mid x \in M(x^0, t)\}$$

$$M(x^0, t) = \left\{x \in I\!R^n \,\middle|\, \begin{array}{r} h_i(x) + (t-1)h_i(x^0) = 0,\ i \in I \\ g_j(x) + (t-1)|g_j(x^0)| \le 0,\ j \in J \end{array}\right\}.$$

We will show that $M(t) := M(x^0, t)$ could be empty for certain $t \in [0, 1)$ also in the case that M is non-empty. Then, there is no connection in $\sum_{gc}$ between points at $t = 0$ and $t = 1$. For this reason we have to modify the embedding in such a way that the following two conditions are satisfied:

$$M \neq \emptyset \Longrightarrow M(t) \neq \emptyset \text{ for all } t \in [0, 1]$$

and (A2) for arbitrarily chosen starting points $x^0 \in I\!R^n$.
Now we consider $h(x) = \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2 + 4$ and $g(x) = (x + \frac{3}{2})^2 - \frac{9}{4}$ (cf. Fig. 8).
The point $x = -1.43523$ is the unique solution of the system $\left\{\begin{array}{c} h(x) = 0 \\ g(x) \le 0 \end{array}\right\}$. It holds

$$\begin{aligned} h(2.40858) &= g(2.40858) = 13 \\ h(-4.15524) &= g(-4.15524) \\ h(-1.805) &= g(-1.805) \\ h(0.887) &= g(0.887). \end{aligned}$$

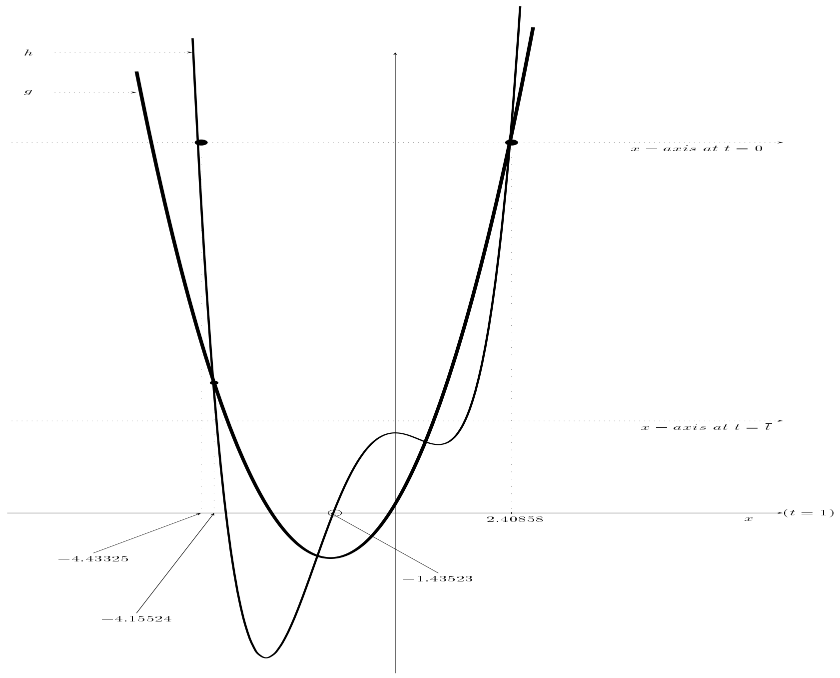

Figure 8:

We consider the set

$$M = \left\{ (x,y) \in I\!R^2 \;\middle|\; \begin{array}{r} h(x,y) = \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2 + 4 = 0 \\ g_1(x,y) = (x+\frac{3}{2})^2 - \frac{9}{4} \leq 0 \\ g_2(x,y) = -x - y + 5 \leq 0 \end{array} \right\} \qquad (3.6)$$

$$= \{-1.43523\} \times [6.43523, \infty)$$

We choose $(x^0, y^0) = (2.40858, 0)$ and consider

$$M(x^0,y^0,t) = \left\{ (x,y) \in I\!R^2 \;\middle|\; \begin{array}{r} h(x,y,t) = h(x,y) + (t-1)13 = 0 \\ g_1(x,y,t) = g_1(x,y) + (t-1)13 \leq 0 \\ g_2(x,y,t) = g_2(x,y) + (t-1)2.59142 \leq 0 \end{array} \right\}.$$

For $t = 0$ it holds:

$$\pi_x(M(x^0,y^0,0)) = \{2.40858, -4.43325\}$$

where π_x denotes the projection onto the x-axis. The perturbation of the functions $h(x,y)$ and $g_1(x,y)$ by $13(t-1)$ means a translation of the x-axis (cf. Fig. 8). Then we see that for certain $\bar{t} \in (0,1)$ the zeros of $h(x,y,\bar{t})$ are outside of $\{(x,y) \in I\!R^2 \mid g_1(x,y,\bar{t}) \leq 0\}$. Then $M(x^0,y^0,t) = \emptyset$ for such $t = \bar{t}$. Let $f(x,y) = x + y^4$ be. We consider

$$P(x^0,y^0,t): \quad \min\{t(x+y^4) + (1-t)[(x-2.40858)^2 + y^2] \mid x \in M(x^0,y^0,t)\}$$

with $(x^0,y^0) = (2.40858, 0)$.

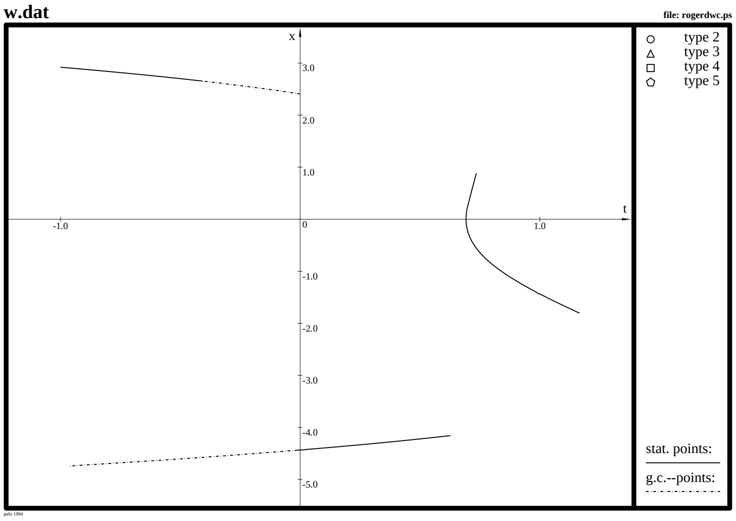

Figure 9:

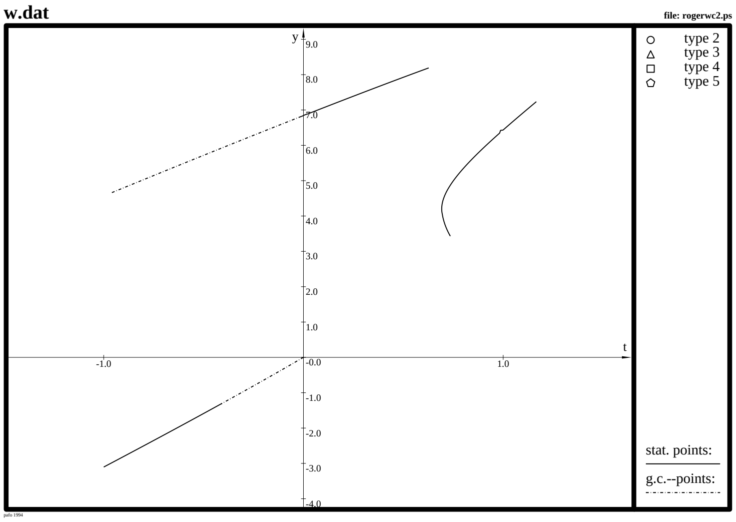

Figure 10:

The Figures 9 and 10 respectively illustrate the curves of stationary points in the $(t-x)$-space and $(t-y)$-space respectively. We obtain 3 connected components. Table 6 contains the singularities. The starting (x^0, y^0) is a point of Type 5 where the MFCQ is violated. $M(t, x^0, y^0)$ is empty for all $t \in (0.6923, 0.7353)$. We have all together 4 points of Type 5. Now we choose $(x^0, y^0) = (4, 0)$ and $(x^0, y^0) = (2, 0)$ respectively. We consider $(P):\ \min\{x + y^4 \mid (x, y) \in M\}$, M introduced by (3.6) and the corresponding embedding with these new starting points

$$P(x^0, y^0, t): \quad \min\{t(x+y^4) + (1-t)[(x-x^0)^2 + (y-y^0)^2] \mid (x, y) \in M(x^0, y^0, t)\}$$

$$M(x^0, y^0, t) = \left\{ (x, y) \in I\!R^2 \;\middle|\; \begin{array}{c} \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2 + 4 + (t-1)h(x^0) = 0 \\ (x+\frac{3}{2})^2 - \frac{9}{4} + (t-1)|(x^0+\frac{3}{2})^2 - \frac{9}{4}| \leq 0 \\ -x - y + 5 + (t-1)|-x^0 - y^0 + 5| \leq 0 \end{array} \right\}.$$

The Fig. 11 shows the connected components of stationary points in the $(t-x)$-space for $(x^0, y^0) = (4, 0)$. Table 7 contains the corresponding singularities. $M(x^0, y^0, t)$ is empty for all $t \in (0.705197, 0.953850)$.
For $(x^0, y^0) = (2, 0)$, Table 8 describes the singularities. We have once more 3 connected components (cf. Fig. 12 for the illustration in the $(t-x)$-space). In difference to the starting point $(x^0, y^0) = (4, 0)$ the feasible set is non-empty for all $t \in [0, 1]$ and the curves overcover the interval $[0, 1]$.

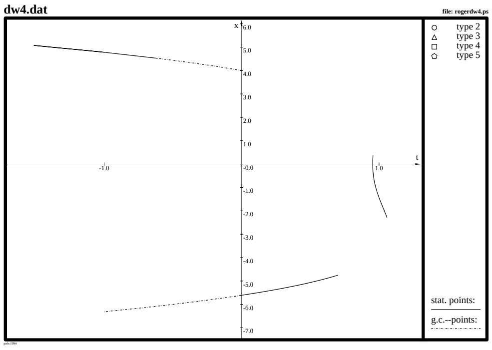

Figure 11

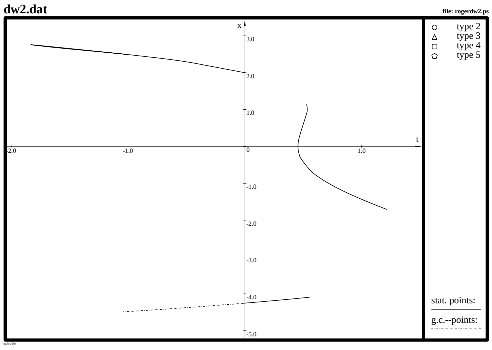

Figure 12

t	x	y	Type
0	2.40858	0	5 (MFCQ violated)
0.735290	0.883482	3.43054	5 (MFCQ violated)
0.692309	0.00000	4.19870	4
1.16590	-1.80548	7.23541	5 (MFCQ violated)
0.627170	-4.15712	8.19096	5 (MFCQ violated)

Table 6

t	x	y	Type
0	4.00000	0.00000	5 (MFCQ violated)
0.955753	0.367872	4.58788	5 (MFCQ violated)
0.953850	0.00000	4.96928	4
1.05787	-2.29349	7.35136	5 (MFCQ violated)
0.700614	-4.74941	9.45002	5 (MFCQ violated)

Table 7

t	x	y	Type
0	2.00000	0.00000	5 (MFCQ violated)
0.528129	1.13977	2.44462	5 (MFCQ violated)
0.534087	1.00000	2.6	4
0.454555	0.00000	3.36	4
1.22035	-1.71567	7.37673	5 (MFCQ violated)
0.553752	-4.09085	7.75210	5 (MFCQ violated)

Table 8

Now we are looking for the condition

$$M \neq \emptyset \Longrightarrow M(x^0,t) \neq \emptyset \quad \forall\, t \in [0,1] \tag{3.7}$$

for arbitrarily chosen $x^0 \in \mathbb{R}^n$. We modify the set

$$M(x^0,t) = \{x \in \mathbb{R}^n \mid h_i(x) + (t-1)h_i x^0 = 0,\ i \in I,\ g_j(x) + (t-1)|g_j(x^0)| \leq 0,\ j \in J\}$$

to the new set

$$\breve{M}(x^0,t) = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{r} h_i(x) + (t-1)|h_i(x^0)| \leq 0,\ i \in I \\ g_j(x) + (t-1)|g_j(x^0)| \leq 0,\ j \in J \\ -\sum\limits_{i\in I} h_i(x) + (t-1)|\sum\limits_{i\in I} h_i(x^0)| \leq 0 \end{array} \right\}.$$

We have for $t = 1$:

$$\breve{M}(x^0,1) = M(x^0,1) = \{x \in \mathbb{R}^n \mid h_i(x) = 0,\ i \in I,\ g_j(x) \leq 0,\ j \in J\}.$$

Since the condition (3.7) together with the condition (A2) is violated for certain starting points $x^0 \in \mathbb{R}^n$, we consider instead of $P(x^0,t)$ the following problems $P^{**}(x^0,t)$ und $\tilde{\tilde{P}}(x^0,t)$ respectively

$$P^{**}(x^0,t): \quad \min\{t x_{n+1} + (1-t)\|(x,x_{n+1}) - (x^0, x^0_{n+1})\|^2 \mid (x,x_{n+1}) \in M^{**}(t)\}$$

$$M^{**}(x^0,t) = \left\{ (x,x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{r} h_i(x) + (t-1)|h_i(x^0)| \leq 0,\ i \in I \\ g_j(x) + (t-1)|g_j(x^0)| \leq 0,\ j \in J \\ f(x) - x_{n+1} + (t-1)|f(x^0) - x^0_{n+1}| \leq 0 \\ -\sum\limits_{i\in I} h_i(x) + (t-1)|\sum\limits_{i\in I} h_i(x^0)| \leq 0 \end{array} \right\}$$

$$\tilde{\tilde{P}}(x^0,t): \quad \min\{tf(x)+(1-t)\|x-x^0\|^2 \mid x \in \tilde{\tilde{M}}(t)\}$$

$$\tilde{\tilde{M}}(x^0,t) = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{r} h_i(x)+(t-1)|h_i(x^0)| \le 0,\ i \in I \\ g_j(x)+(t-1)|g_j(x^0)| \le 0,\ j \in J \\ \|x\|^2 - q \le 0 \\ -\sum_{i\in I} h_i(x) + (t-1)|\sum_{i \in I} h_i(x^0)| \le 0 \end{array} \right\}$$

Theorem 3.29 *Let M be non-empty and compact. Then there exists for a fixed x^0 a $\tilde{q}>0$ such that*

$$M \subseteq \{x \in \mathbb{R}^n \mid \|x\|^2 \le q\} \quad \text{and} \quad \|x^0\|^2 < q \ \text{ for all } \ q \ge \tilde{q} \tag{3.8}$$

where $q>0$. Let $f, h_i, g_j \in C^1(\mathbb{R}^n, \mathbb{R})$, $i \in I$, $j \in J$ and M non-empty and compact. We assume for $\tilde{\tilde{P}}(x^0,t)$ that $q \ge \tilde{q}$.

(i) *The problems $\tilde{\tilde{P}}(x^0,1)$ and (P) are equivalent in the following sense:*

  (a$_1$) *$x \in \psi_{\text{stat}}(\tilde{\tilde{P}}(x^0,1))$ and $\psi_{\text{loc}}(\tilde{\tilde{P}}(x^0,1))$ respectively and $\|x\|^2 < q \Longrightarrow x \in \psi_{\text{stat}}(P)$ and $\psi_{\text{loc}}(P)$ respectively,*

  (a$_2$) *$\psi_{gc}(\tilde{\tilde{P}}(x^0,1)) = \tilde{\tilde{M}}(x^0,1) = M \cap \{x \in \mathbb{R}^n \mid \|x\|^2 \le q\}$,*

  (a$_3$) *$x \in \psi_{\text{gc}}(P)$ and $x \in \psi_{\text{loc}}(P)$ respectively $\Longrightarrow x \in \psi_{\text{gc}}(\tilde{\tilde{P}}(x^0,1))$ and $x \in \psi_{\text{loc}}(\tilde{\tilde{P}}(x^0,1))$ respectively.*

  (a$_4$) *If x is a stationary point for (P) with $\|x\|^2 \le q$, then x is a critical point for $(\tilde{\tilde{P}}(x^0,1))$.*

  The problems $P^{**}(x^0,1))$ and (P) are equivalent in the following sense:

  (b$_1$) *$(x, x_{n+1}) \in \psi_{\text{stat}}(P^{**}(x^0,1))$ and $f(x) = x_{n+1} \Longrightarrow x \in \psi_{\text{stat}}(P)$,*

  (b$_2$) *$\psi_{\text{gc}}(P^{**}(x^0,1)) = M^{**}(x^0,1)$, $\pi_x(M^{**}(x^0,1)) = M$, where π_x is the projection onto the x-space.*

  (b$_3$) *If x is a stationary point for (P), then $(x, f(x))$ is a critical point for $P^{**}(x^0,1)$.*

  (b$_4$) *$x \in \psi_{\text{gc}}(P)$ and $\psi_{\text{loc}}(P)$ respectively $\Longrightarrow (x, f(x)) \in \psi_{\text{gc}}(P^{**}(x^0,1))$ and $\psi_{\text{loc}}(P^{**}(x^0,1))$ respectively.*

  (c) *The MFCQ is violated at each point $x \in \psi_{\text{gc}}(\tilde{\tilde{P}}(x^0,1))$ and $(x, x_{n+1}) \in \psi_{\text{gc}}(P^{**}(x^0,1))$ respectively.*

(ii) *x^0 and (x^0, x^0_{n+1}) respectively is the unique global minimizer for $\tilde{\tilde{P}}(x^0,0)$ and $P^{**}(x^0,0)$ respectively.*

(iii) *$\tilde{\tilde{P}}(x^0,t)$ and $P^{**}(x^0,t)$ has a global minimizer for all $t \in [0,1]$ if $q > \tilde{q}$ (cf. (3.8)).*

Proof:

(a$_1$) This statement follows from $\tilde{\tilde{M}}(x^0,1) = M \cap \{x \in I\!R^n \mid \|x\|^2 \leq q\}$.

(a$_2$) $\psi_{\text{gc}}(\tilde{\tilde{P}}(x^0,1)) \subset M(x^0,1)$ per definition. We show that $M(x^0,1) \subset \psi_{\text{gc}}(\tilde{P}(x^0,1))$. Let $x \in \tilde{\tilde{M}}(x^0,1)$ be. Then it holds: $h_i(x) = 0$, $i \in I$. The set of vectors

$$\Big\{Dh_i(x),\ i \in I,\ D\Big(-\sum_{i=1}^{m} h_i(x)\Big)\Big\}$$

is linearly dependent. Then we have $x \in \psi_{\text{gc}}(\tilde{\tilde{P}}(x^0,1))$.

(a$_3$) follows with the same argument as in (a$_1$).

(a$_4$) is obvious.

(b$_1$) Let $(x, x_{n+1}) \in \psi_{\text{stat}}(P^{**}(x^0,1))$ with $f(x) = x_{n+1}$. Let $g_j(x) = 0$, $j = 1,\dots,p$. Then there exist non-negative numbers $\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_p,\beta_{s+1},\beta_{s+2}$ such that

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} + \sum_{i=1}^{m} \alpha_i \begin{pmatrix} D^T h_i(x) \\ 0 \end{pmatrix} + \sum_{j=1}^{p} \beta_j \begin{pmatrix} D^T g_j(x) \\ 0 \end{pmatrix} + \beta_{s+1} \begin{pmatrix} D^T f(x) \\ -1 \end{pmatrix} +$$
$$\beta_{s+2} \begin{pmatrix} D^T(-\sum\limits_{i=1}^{m} h_i(x)) \\ 0 \end{pmatrix} = 0_{n+1}.$$

Therefore, $\beta_{s+1} = 1$ and $Df(x) + \sum\limits_{i=1}^{m} (\alpha_i - \beta_{s+2}) Dh_i(x) + \sum\limits_{j=1}^{p} \beta_j Dg_j(x) = 0$ with $\beta_j \geq 0$, $j = 1,\dots,p$, i.e. $x \in \psi_{\text{stat}}(P)$.

(b$_2$) This statement is obvious using analog arguments as in (a$_2$).

(b$_3$) is obvious.

(b$_4$) Assume $x \in \psi_{\text{gc}}(P)$ and let $J_0(x) = \{1,\dots,p\}$ be. Then $x \in M$ and there exist numbers $\alpha_0,\alpha_1,\dots,\alpha_m,\beta_0,\dots,\beta_p \in I\!R$ such that $\sum\limits_{i=0}^{m} |\alpha_i| + \sum\limits_{j=1}^{p} |\beta_j| > 0$ and

$$\alpha_0 Df(x) + \sum_{i=1}^{m} \alpha_i Dh_i(x) + \sum_{j=1}^{p} \beta_j Dg_j(x) = 0.$$

Then we have

$$\alpha_0 \begin{pmatrix} 0_n \\ 1 \end{pmatrix} + \sum_{i=1}^{m} \alpha_i \begin{pmatrix} D^T h_i(x) \\ 0 \end{pmatrix} + \sum_{j=1}^{p} \beta_j \begin{pmatrix} D^T g_j(x) \\ 0 \end{pmatrix} + \alpha_0 \begin{pmatrix} D^T f(x) \\ -1 \end{pmatrix} = 0_{n+1}$$

and from $x \in M$ follows that $(x, f(x)) \in M^{**}(1)$, i.e. $(x, f(x)) \in \psi_{\text{gc}}(P^{**}(x^0,1))$. Let $x \in \psi_{\text{loc}}(P)$ be. Then $x \in M$ and there exists an $\varepsilon > 0$ such that it holds

$\forall\, y \in M$ with $\sum\limits_{i=1}^{n}(y_i - x_i)^2 < \varepsilon$: $f(x) \leq f(y)$. For all $(y, y_{n+1}) \in M^{**}(x^0, 1)$ with $\sum\limits_{i=1}^{n}(y_i - x_i)^2 + (y_{n+1} - f(x))^2 < \varepsilon$ it holds $y \in M$, $f(y) \leq y_{n+1}$ and $\sum\limits_{i=1}^{n}(y_i - x_i)^2 < \varepsilon$ and therefore $f(x) \leq f(y) \leq y_{n+1}$, i.e. $(x, f(x)) \in \psi_{\text{loc}}(P^{**}(x^0, 1))$.

(c) Let $x \in \psi_{\text{gc}}(\stackrel{\approx}{P}(x^0, 1))$ be. Then it holds: $h_i(x) = 0$, $i \in I$. We assume that the MFCQ is satisfied at x. Then there exists a vector $\eta \in I\!R^n$ such that

$$\left.\begin{array}{l} Dh_i(x)\eta < 0,\ i = 1, \ldots, n \\ D(-\sum\limits_{i=1}^{m} h_i(x))\eta < 0 \end{array}\right\}.$$

Then it holds: $\sum\limits_{i=1}^{m} Dh_i(x)\eta < 0$ and $-\sum\limits_{i=1}^{m} Dh_i(x)\eta < 0$. This is a contradiction. For $(x, x_{n+1}) \in \psi_{\text{gc}}(P^{**}(x^0, 1))$ it holds an analog argumentation.

(ii) is obvious.

(iii) is obvious for $\stackrel{\approx}{P}(t)$, since $\stackrel{\approx}{M}(t)$ is non-empty and compact for all $t \in [0, 1]$. The objective of $P^{**}(t)$ is a strictly convex quadratic function for all $t \in [0, 1)$ and the feasible set $M^{**}(t)$ is non-empty for all $t \in [0, 1]$. Then there exists a global minimizer for all $t \in [0, 1)$. Since M is compact, the situation for $t = 1$ is also obvious.

Remark 3.30 (i) Consider $\stackrel{\approx}{P}(x^0, t)$ and $P^{**}(x^0, t)$ respectively. Then there exist gc-points of Type 1 and gc-points of Type 2, 3, 4, 5 could appear (see illustration examples).
(ii) Let $x \in \psi_{\text{gc}}(\stackrel{\approx}{P}(x^0, 1))$ and $(x, x_{n+1}) \in \psi_{\text{gc}}(P^{**}(x^0, 1))$ respectively. Then $(x, 1)$ and $(x, x_{n+1}, 1)$ belongs to $\sum_{gc}^{4} \cup \sum_{gc}^{5}$ (and consequently a boundary point of cl $\sum_{stat}$) if $\stackrel{\approx}{P}(x^0, t)|_{\{1\}}$ and $P^{**}(x^0, t)|_{\{1\}}$ respectively is JJT-regular.

We consider once more the problem

$$(P): \quad \min\{x + y^4 \mid (x, y) \in M\}, \quad M \text{ described in (3.6)}$$

and we use the embedding

$$\stackrel{\approx}{P}(x^0, y^0, t): \quad \min\{t(x + y^4) + (1 - t)[(x - x^0)^2 + (y - y^0)^2] \mid (x, y) \in \stackrel{\approx}{M}(x^0, y^0, t)\}$$

$$\stackrel{\approx}{M}(x^0, y^0, t) = \left\{ (x, y) \in I\!R^n \;\middle|\; \begin{array}{r} h(x, y) + (t - 1)|h(x^0, y^0)| \leq 0 \\ g_1(x, y) + (t - 1)|g_1(x^0, y^0)| \leq 0 \\ g_2(x, y) + (t - 1)|g_2(x^0, y^0)| \leq 0 \\ x^2 + y^2 - q \leq 0 \\ -h(x, y) + (t - 1)|h(x^0, y^0)| \leq 0 \end{array} \right\}$$

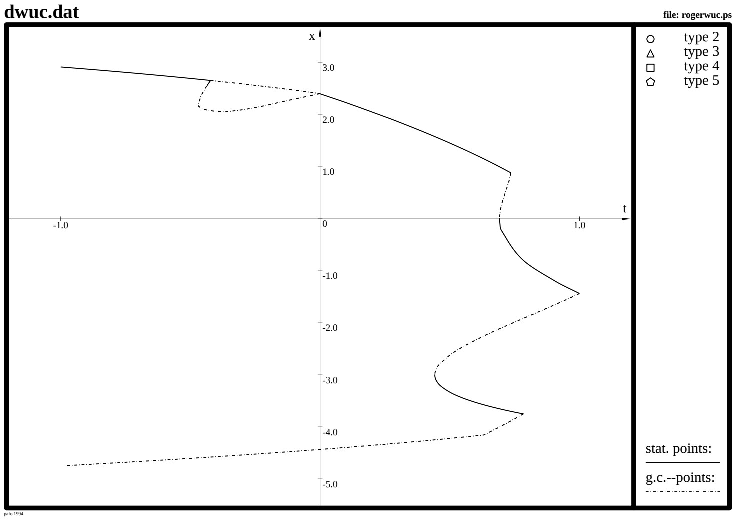

Figure 13:

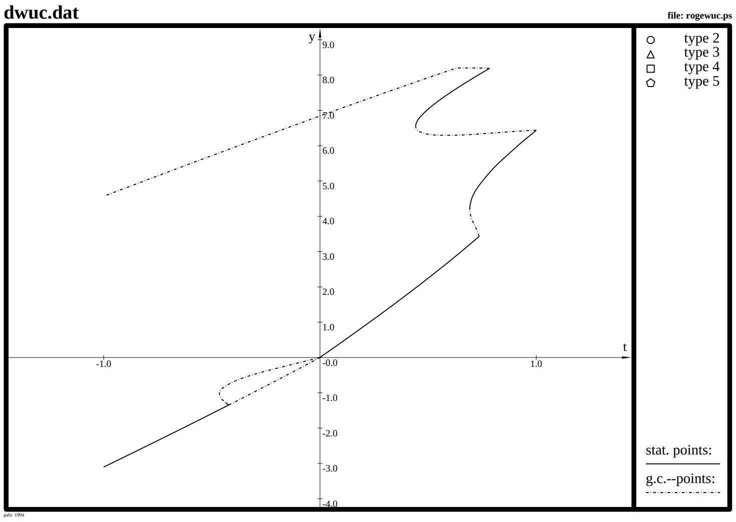

Figure 14:

where

$$\begin{aligned} h(x,y) &= \tfrac{1}{4}x^4 + \tfrac{2}{3}x^3 + \tfrac{3}{2}x^2 + 4 \\ g_1(x,y) &= (x+\tfrac{3}{2})^2 - \tfrac{9}{4} \\ g_2(x,y) &= -x-y+5 \\ q &= 1000. \end{aligned}$$

For $(x^0, y^0) = (2.40858, 0)$ we achieve $t = 1$ with the point $(x^*, y^*) \in \psi_{\mathrm{gc}}(\tilde{\tilde{P}}\,(x^0, 1))$ of Type 5 and (x^*, y^*) is the global minimizer of (P). Table 9 contains the singularities and the Figures 13 and 14 show the connected components in the $(t-x)$-space and the $(t-y)$-space.

t	x	y	Type
0	2.40858	0	5 (MFCQ satisfied)
0.7353	0.8834	3.430	5 (MFCQ violated)
0.692310	0.00000	4.20713	4
1.00000	-1.43523	6.43523	5 (MFCQ violated) minimizer
0.442337	-3.00000	6.56276	4
0.783654	-3.74993	8.18928	5 (MFCQ violated)
0.630744	-4.15525	8.19835	5 (MFCQ violated)

Table 9

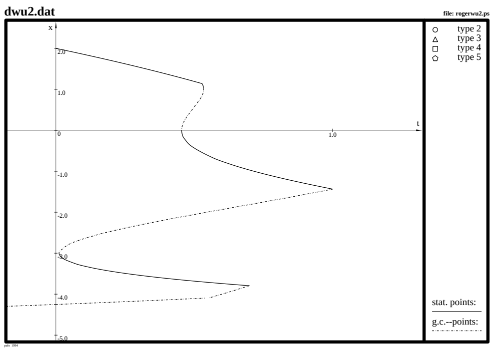

Figure 15

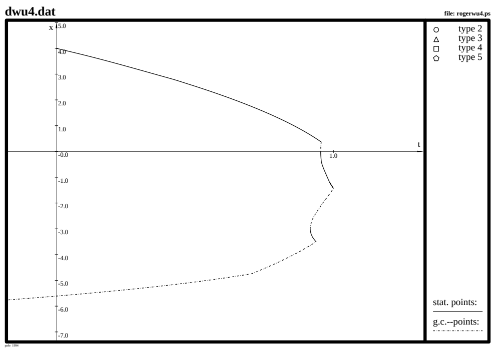

Figure 16

For $(x^0, y^0) = (2, 0)$ and $(x^0, y^0) = (4, 0)$ respectively we achieve $t = 1$, too. The corresponding singularities are in the Tables 10 and 11. Fig. 15 shows the connected component for $(x^0, y^0) = (2.0)$ and Fig. 16 the connected component for $(x^0, y^0) = (4, 0)$ in the $(t-x)$-space. Fig. 17 contains the components for the 3 different starting points. We observe that the changing of the starting points has not an influence of the structure of $\sum_{gc}$.

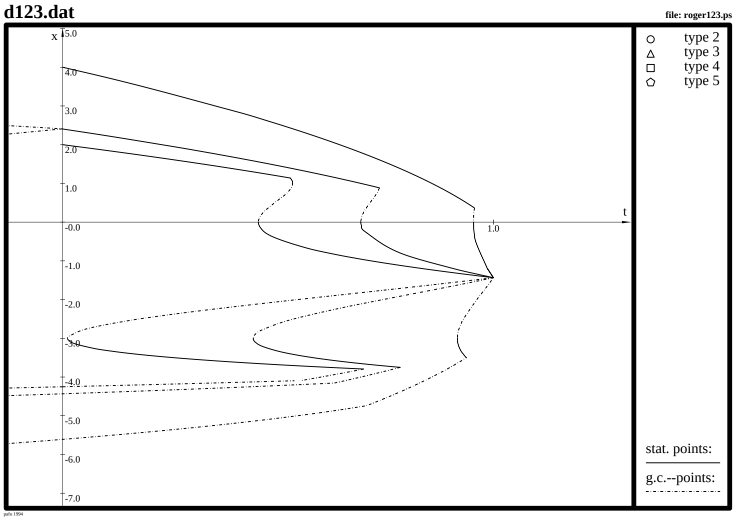

Figure 17:

t	x	y	Type
0	2.00000	0.00000	5 (MFCQ satisfied)
0.52812	1.1397	2.4446	5 (MFCQ violated)
0.5340	1.00000	2.6	4
0.454553	0.00000	3.36	4
1.00000	-1.43523	6.43523	5 (MFCQ violated) minimizer
0.0113669	-3.00000	5.03	4
0.698957	-3.79347	7.89034	5 (MFCQ violated)
0.55375	-4.090	7.752	5 (MFCQ satisfied)

Table 10

t	x	y	Type
0	4.00000	0.00000	5 (MFCQ satisfied)
0.955753	0.367872	4.58788	5 (MFCQ violated)
0.953850	0.00000	4.96928	4
1.00000	-1.43523	6.43523	5 (MFCQ violated) minimizer
0.916357	-3.01262	7.92898	4
0.933916	-3.46208	8.39599	5 (MFCQ violated)
0.705197	-4.74106	9.44626	5 (MFCQ satisfied)

Table 11

4 The regularisations of the modified embeddings and the Enlarged Mangasarian-Fromovitz Constraint Qualification

First we discuss the JJT-regularity and the KH-regularity.(See Definition 2.4). We consider the problem $\tilde{\tilde{P}}(t)$. The investigations with respect to $P^{**}(t)$ run analogously. Looking to the starting point $(x^0, 0)$ of the problem $\tilde{\tilde{P}}(0)$, there is a disadvantage. The corresponding Lagrangemultipliers $\beta_j, j \in J_0(x^0, 0)$, where

$$\begin{aligned} J_0(x^0,0) := \ & \{j \in I | h_i(x^0) + (t-1)|h_i(x^0)| = 0\} \cup \{j \in J | g_j(x^0) + (t-1)|g_j(x^0)| = 0\} \cup \\ & \cup \{s+1 | g_{s+1}(x^0) := -\sum_{i \in I} h_i(x^0) + (t-1)|\sum_{i \in I} h_i(x^0)| = 0\}, \end{aligned}$$

are zero for all $j \in J_0(x^0, 0)$. That means $(x^0, 0)$ is strongly degenerated. Therefore we consider the following modification of the problem $\tilde{\tilde{P}}(t)$

$$\tilde{\tilde{P}}_{(A,x^0,b,q)}(t): \quad \min\Big\{tf(x) + (1-t)\frac{1}{2}(x-x^0)^T A(x-x^0) \mid x \in \tilde{\tilde{M}}_{(x^0,b,q)}(t)\Big\},\ t \in [0,1)$$

where

$$\tilde{\tilde{M}}_{(x^0,b,q)}(t) := \left\{ x \in I\!R^n \ \middle| \ \begin{array}{r} h_i(x) + (t-1)|h_i(x^0)| + (t-1)b_i^T x \le 0 \\ i = 1, \dots, m \\ g_j(x) + (t-1)|g_j(x^0)| + (t-1)b_{m+j}^T x \le 0 \\ j = 1, \dots, s \\ \frac{1}{2}\|x\|^2 + b_{m+s+1}^T x \le q \\ -\sum_{i=1}^m h_i(x) + (t-1)|\sum_{i=1}^m h_i(x^0)| + (t-1)b_{m+s+2}^T x \le 0 \end{array} \right\},$$

$A \in I\!R^{\frac{1}{2}n(n+1)}$ symmetric (n,n)-matrix,
$b^T := (b_1^T, \dots, b_m^T, b_{m+1}^T, \dots, b_{m+s}^T, b_{m+s+1}^T, b_{m+s+2}^T)$ belongs to $\mathcal{B}$,
$\mathcal{B} := \{b = (b_1, \dots, b_{m+s+2}) \in I\!R^{n(m+s+2)} \mid b_i^T x^0$ is "sufficiently large", $i = 1, \dots, m+s+2\}$.

Here "sufficiently large" means such that $b_i^T x^0 > 0$ and $x^0 \in \tilde{\tilde{M}}_{(x^0,b,q)}(t)$ for all $t \in [0,1)$. If we assume that M is non-empty and compact, then there exists a $\tilde{q} > 0$ such that

$$M \subset \{x \in I\!R^n | \frac{1}{2}\|x\|^2 + b_{m+s+1}^T x \le q\}, \quad \frac{1}{2}\|x^0\|^2 + b_{m+s+1}^T x^0 \le q \ for\ all\ q \ge \tilde{q}. \tag{4.1}$$

Let $\mathcal{A} \subset I\!R^{\frac{1}{2}n(n+1)}$ be the set of all non-singular symmetric (n,n)-Matrices. Then $\mathcal{A}$ is open in $I\!R^{\frac{1}{2}n(n+1)}$ and $I\!R^{\frac{1}{2}n(n+1)} \backslash \mathcal{A}$ has the Lebesgue-mesure 0 ([23]).

In the sequel 'almost all' is always meant in the sense of the Lebesgue measure of the corresponding dimension.

Theorem 4.1 *Assume that $f, h_i, g_j \in C^3(\mathbb{R}^n, \mathbb{R})$, $i \in I$, $j \in J$. Then, for almost all $(A, x^0, b, q) \in \mathcal{A} \times \mathbb{R}^n \times \mathcal{B} \times \{q \in \mathbb{R} | q > \tilde{q}\}$,*

(i) $\overset{\approx}{P}_{(A,x^0,b,q)}(t)$ is JJT-regular with respect to $(-\infty, 1) \cup (1, \infty)$, where $\tilde{q}$ is defined by (4.1).

(ii) $(x^0, 0)$ is a g.c. point of type 1.
Furthermore x^0 is a global minimizer for $\overset{\approx}{P}_{(A,x^0,b)}(0)$ if A is positiv definite.

(iii) $\overset{\approx}{P}_{(A,x^0,b,q)}(t)$ has a global minimizer for all $t \in [0, 1)$

Proof:

(i) Here we just recall the main idea of the proof. A complete proof is in [6]. In "step 1" we show that for almost all (A, x^0) - and hence for almost all (A, x^0, b, q) - each critical point of $\overset{\approx}{P}_{(A,x^0,b,q)}(t)$ that satisfies (LICQ) is either a point of Type 1 or Type 2 or Type 3. In "step 2" we show that for almost all (b,q) the set of those feasible points that do not satisfy (LICQ) is a zero-dimensional manifold. Then, for any fixed point of these (b,q) we show that for almost all (A, x^0) each of these "without-(LICQ)-points" is either of Type 4 or of Type 5. After using Fubini's Theorem ([16]) the assertion (i) is proved.

(ii) From the structure of the problem $\overset{\approx}{P}_{(A,x^0,b,q)}(0)$ and the special choice of the vector b , we have $J_0(x^0, 0) = \emptyset$. Hence $(x^0, 0) \notin \sum_{gc}^2$. From the proof of part (i) we know that for almost all (x^0, b, q) the LICQ is satisfied at $(x^0, 0)$. Then we have $(x^0, 0) \notin \sum_{gc}^4 \cup \sum_{gc}^5$. Moreover, the Hessian of the Lagrangian at $(x^0, 0)$ is

$$D_x^2 L(x^0, 0) = (1 - t)A.$$

$D_x^2 L(x^0, 0)$ is nonsingular since $A \in \mathcal{A}$. Then we have $(x^0, 0) \notin \sum_{gc}^3$. Then $(x^0, 0) \in \sum_{gc}^1$ for almost all (A, x^0, b, q).

(iii) By the choice of b and q the set $\overset{\approx}{M}_{(x^0,b,q)}(t)$ is non-empty and compact for all $t \in [0, 1)$.

From the proof of theorem 4.1 we obtain the answer with respect to the KH-regularity. We put $A = I^n, b = 0$ and obtain the problem

$$\overset{\approx}{P}_{(x^0,q)}(t): \quad \min\{tf(x) + (1-t)\|x - x^0\|^2 \mid x \in \overset{\approx}{M}_{(x^0,q)}(t)\}, \quad t \in [0, 1]$$

where

$$\overset{\approx}{M}_{(x^0,q)}(t) := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{r} h_i(x) + (t-1)|h_i(x^0)| \leq 0, \ i \in I \\ g_j(x) + (t-1)|g_j(x^0)| \leq 0, \ j \in J \\ \|x\|^2 \leq q \\ -\sum\limits_{i \in I} h_i(x) + (t-1)|\sum\limits_{i \in I} h_i(x^0)| \leq 0 \end{array} \right\}.$$

Theorem 4.2 *Assume that* $f, h_i, g_j \in C^3(\mathbb{R}^n, \mathbb{R})$, $i \in I$, $j \in J$. *Then, for almost all* $(x^0, q) \in \mathbb{R}^n \times \{q \in \mathbb{R} | q > \tilde{q}\}$, $\tilde{\tilde{P}}_{(x^0,q)}(t)$ *is KH-regular with respect to* $(0,1)$, *where* $\tilde{q}$ *is defined by* (4.1).

Now we consider the question whether a continuous curve in $\sum_{stat}$ exists connecting $(x^0, 0)$ and $(\hat{x}, 1)$ where $\hat{x}$ is a stationary point of (P) and x^0 is the known starting point for $P(0)$. Theorem 2.5 gives sufficient conditions for the existence of this curve. There are two conditions in this theorem which we want to point out:

$(B1)$ For each t in some neighbourhoud of zero there is a unique KKT-point $(x(t), \alpha(t), \beta(t))$ of P(t) where x(t) is the only stationary point of P(t).

$(B2)$ The MFCQ is satisfied for all $x \in M(t)$ for all $t \in [0, 1]$.

Furthermore we need at least the KH-regularity. Now we are looking for the starting situation for $\tilde{\tilde{P}}_{(A,x^0,b,q)}(t)$. The condition $(B1)$ is not satisfied in general, namely considering

$$\tilde{\tilde{M}}_{(x^0,b,q)}(0) = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{r} h_i(x) - |h_i(x^0)| - b_i^T x \leq 0, \ i = 1, \ldots, m \\ g_j(x) - |g_j(x^0)| - b_{m+j}^T x \leq 0, \ j = 1, \ldots, s \\ \|x\|^2 + b_{m+s+1}^T x \leq q \\ -\sum\limits_{i=1}^{m} h_i(x) - |\sum\limits_{i=1}^{m} h_i(x^0)| - b_{m+s+2}^T x \leq 0 \end{array} \right\}$$

it is easy to construct examples that more than one stationary points exist for $\tilde{\tilde{P}}_{(A,x^0,b,q)}(0)$. Therefore, we can not use Theorem 2.5. From this point of view we consider once more a modification of the embeddings which we considered until here. Having the equivalence

$$h_i(x) = 0, \ i \in I \Longleftrightarrow \left\{ \begin{array}{l} h_i(x) \leq 0, \ i \in I \\ -\sum\limits_{i \in I} h_i(x) \leq 0 \end{array} \right.$$

in mind, we consider the problem (P) for $I = \emptyset$, i.e.

(P) $\quad min\{f(x) | x \in M\}, \quad M := \{x \in \mathbb{R}^n | g_j(x) \leq 0, j \in J\}$

and the following standard embedding

$\hat{P}(t):$ $\quad min\{tf(x) + (1-t)\|x - x^0\|^2 | x \in \hat{M}(t)\}, \ t \in [0, 1]$

where $\quad \hat{M}(t) := \{x \in \mathbb{R}^n | t g_j(x) + (t-1)|g_j(x^0)| \leq 0, \ j \in J, \ \|x\|^2 \leq q\}.$

Then the structure of $\hat{M}(0)$ is quite different:

$\hat{M}(0) = \{x \in \mathbb{R}^n | \ \|x\|^2 \leq q\}.$

If we choose q sufficienly large (see the discussion with respect to the former embeddings), then x^0 is the global minimizer for $\hat{P}(0)$ and the only stationary point. Furthermore x^0 is a non-degenerated stationary point if $J_0(x^0) = \emptyset$. Therefore, $(B1)$ is satisfied if $J_0(x^0) = \emptyset$. Using the fact that $\hat{M}(t)$ is non empty and compact for all $t \in [0, 1]$, we obtain:

Theorem 4.3 *Assume that $\hat{P}(t)$ is KH-regular with respect to $[0,1]$, $J_0(x^0)=\emptyset$ and the condition $(B2)$ holds. Then there exists a path in $\sum_{stat}$ connecting $(x^0,0)$ and $(\hat{x},1)$, where $\hat{x}$ is a stationary point of (P).*

Proof: Under these conditions, the assumptions of theorem 2.5 are satisfied.

Remark 4.4 *If we assume that $\hat{P}(t)$ is JJT-regular with respect to $[0,1]$ and the condition $(B2)$ holds, then for the set of stationary points of $\hat{P}(t)$ it holds*

$$\sum_{stat} = \sum_{stat}^{1} \cup \sum_{stat}^{2} \cup \sum_{stat}^{3} \cup \sum_{stat}^{5}.$$

Furthermore, if $(x,t) \in \sum_{stat}^{5}$, then (x,t) is no turning point.

This follows from the fact that at every point of $\sum_{stat}^{4}$ as well as at every turning point of $\sum_{stat}^{5}$ the MFCQ is violated. We recall that $\sum_{stat}^{i} = \{(x,t) \in \sum_{gc}^{i} \mid \beta_j \geq 0, j \in J_0(x,t)\}$

Looking to the condition $(B2)$, it is not satisfied in general. Now we are looking for an assumption on the original problem (P), which ensures that condition $(B2)$ is satisfied for $\hat{P}(t)$. We assume that M is non-empty and compact. We put

$$B_q(0) := \{x \in I\!R^n \mid \|x\|^2 \leq q\}.$$

Then there exists a $\tilde{q} > 0$ such that $M \subset B_q(0)$ for all $q \geq \tilde{q}$.

Definition 4.5 *The Enlarged Mangasarian-Fromovitz Constraint Qualification (shortly* **EnMFCQ***) is satisfied for the problem (P) if for all $q \geq \tilde{q}$ and for all $x \in B_q(0)$ there exists a vector $\eta \in I\!R^n$ such that:*

$$\begin{aligned} g_j(x) + Dg_j(x)\eta &< 0,\ j \in J,\ \textit{with}\ \ g_j(x) \geq 0, \\ 2x^T\eta &< 0,\ \textit{if}\ \ \|x\|^2 = q. \end{aligned}$$

Remark 4.6 *In the literature (cf. e.g.[10],[13],[5]) the EnMFCQ is known as a condition for the convergence of the penalty and exact penalty methods.*

Theorem 4.7 *Assume that EnMFCQ is satisfied for the problem (P). Then the MFCQ is fulfilled for all $x \in \hat{M}(t)$ and for all $t \in (0,1]$.*
Furthermore if the set $J_0(x^0) := \{j \in J | g_j(x^0) = 0\}$ is empty, then the MFCQ is also fulfilled for all $x \in \hat{M}(0)$.

Proof: Let $t \in (0,1]$ and $x \in \hat{M}(t)$ be arbitrarily chosen. Then for all $j \in J_0(x,t)$ $(= \{j \in J | tg_j(x) + (t-1)|g_j(x^0)| = 0\})$, it holds

$$g_j(x) = \frac{(1-t)|g_j(x^0)|}{t} \geq 0 \tag{4.2}$$

Since EnMFCQ is satisfied for the problem (P) and $\hat{M}(t) \subset B_q(0)$, there exists a vector $\eta \in \mathbb{R}^n$ such that $g_j(x) + Dg_j(x)\eta < 0$ for all $j \in J_0(x,t)$. Taking the relation (4.2) into account we obtain

$$D_x g_j(x,t)\eta = tDg_j(x)\eta < 0 \textit{ for all } j \in J_0(x,t)$$

and considering the relation $2x^T\eta < 0$ if $\|x\|^2 = q$, we conclude that η is a Mangasarian-Fromovitz vector at the point (x,t).
The assertion for $t = 0$ is obvious since from $J_0(x^0) = \emptyset$ it follows $J_0(x^0, 0) = \emptyset$.

Having Theorem 2.3 in mind we obtain:

Corollary 4.8 *Assume that EnMFCQ is satisfied for the problem (P) and $J_0(x^0) = \emptyset$. Then $\hat{M}(t_1)$ is homeomorphic with $\hat{M}(t_2)$ for all $t_1, t_2 \in [0,1]$.*

Since the set $\hat{M}(0)$ $(= B_q(0))$ is connected, under the assumptions of corollary 4.1 an unconnectedfeasible set $\hat{M}(t)$ for the problem $\hat{P}(t)$, $t \in (0,1]$, is excluded. Using this argument it is easy to check if a given problem is a really non convex one. We consider the function $N(x) :=$ from example 3.2 once more. We consider the set

$$M := \{(x,y) \in \mathbb{R}^2 | g(x,y) = N(x) \leq 0\}.$$

We choose $(x^0, y^0) = (-4, 0)$ and construct the set

$$\hat{M}(t) = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} tN(x) + (t-1)|N(-4)| \leq 0 \\ x^2 + y^2 - 25 \leq 0 \end{array} \right\}$$

We note that $N(-4) = 4505.928$ and for $t \neq 0$ we can write the set $\hat{M}(t)$ in the form

$$\hat{M}(t) = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} N(x) + \frac{t-1}{t}|N(-4)| \leq 0 \\ x^2 + y^2 - 25 \leq 0 \end{array} \right\}$$

The sets $\hat{M}(1)$ and $\hat{M}(0.99933)$ are depicted in figures 18 and 19 resp. The sets $\hat{M}(0)$ $(= B_{25}(0))$ and $\hat{M}(1)$ are connected while the set $\hat{M}(0.99933)$ is not connected, i.e. $\hat{M}(0.99933)$ and $\hat{M}(1)$ are not homeomorphic. Therefore EnMFCQ is not satisfied.
If we consider the quasi-convex function $Q(x) = 0.1x^6 - 0.3x^5 - 0.25x^4 + x^3 - 2x + 4$ from example 3.3 instead of the function N(x), then the EnMFCQ is satisfied.

5 Conclusions and further remarks

(i) Considering the standard embedding

$$\text{P(t)} \quad \min\{tf(x) + (1-t)\|x - x^0\|^2 \mid x \in M(t)\}$$

$$\textit{where } M(t) := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} h_i(x) + (t-1)h_i(x^0) = 0, & i \in I, \\ g_j(x) + (t-1)|g_j(x^0)| \leq 0, & j \in J \end{array} \right\}$$

the assumption

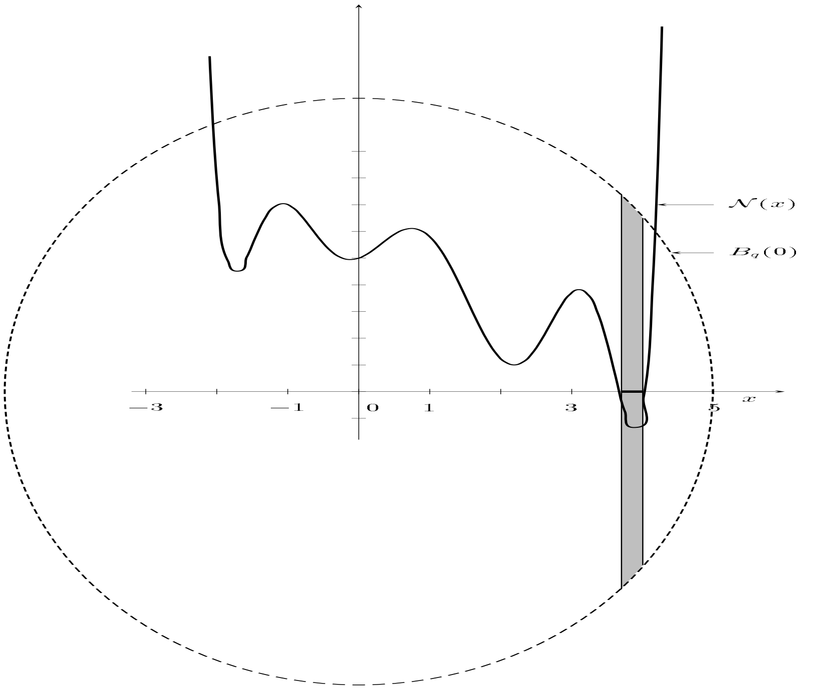

Figure 18:

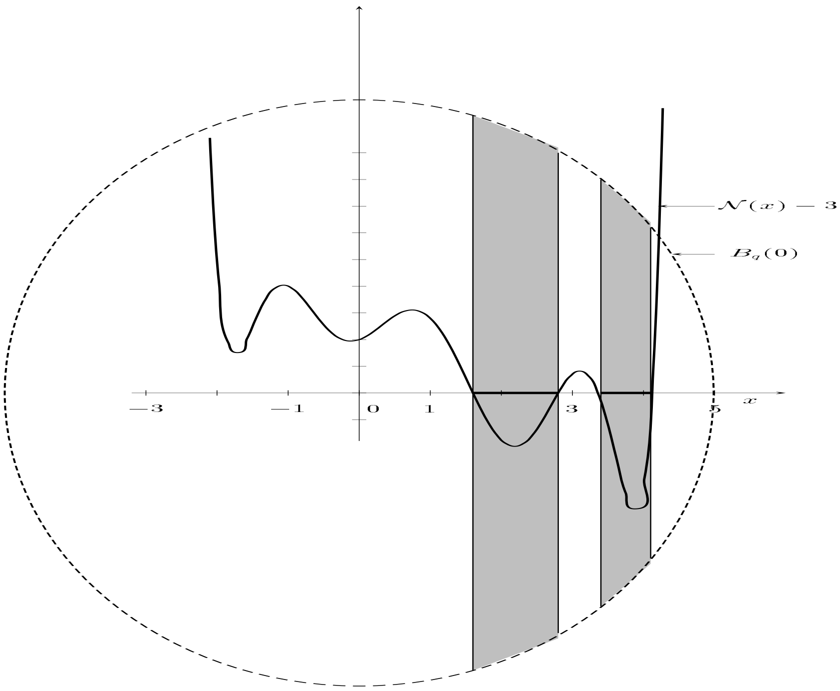

Figure 19:

$(A2)$ P(t) has a generalized critical point for all $t \in [0,1]$

can be violated for $I = \emptyset$ as well as for $I \neq \emptyset$. This disadvantage can be eliminated by the embeddings $\overset{\approx}{P}(t)$ and $P^{**}(t)$. The problems $\overset{\approx}{P}(1)$ and $P^{**}(1)$ are equivalent to the original problem (P) (cf. theorem 3.2).

(ii) Using the embeddings $\overset{\approx}{P}(t)$ and $P^{**}(t)$ resp. we can reduce the number of connected components of $\sum_{gc}$.

(iii) We get KH and JJT-regular problems by pertubations of the problem data.

(iv) Looking to the question whether a continuous curve in $\sum_{stat}$ exists connecting $(x^0, 0)$ and $(\hat{x}, 1)$ where $\hat{x}$ is a stationary point of (P) and x^0 is the known starting point for $P(0)$, the assumption $(B1)$ and $(B2)$ in theorem 2.5 are not satisfied in general. However, if we consider the embedding

$$\hat{P}(t): \qquad min\{tf(x) + (1-t)\|x - x^0\|^2 | x \in \hat{M}(t)\}, \ t \in [0,1]$$

where

$$\hat{M}(t) := \left\{ x \in I\!R^n \;\middle|\; \begin{array}{c} th_i(x) + (t-1)|h_i(x^0)| \leq 0, \ i \in I \\ tg_j(x) + (t-1)|g_j(x^0)| \leq 0, \ j \in J, \\ t(-\sum\limits_{i \in I} h_i(x)) + (t-1)|\sum\limits_{i \in I} h_i(x^0)| \leq 0 \\ \|x\|^2 \leq q \end{array} \right\}$$

the condition $(B1)$ is satisfied if $J_0(x^0) = \emptyset$.

Under the EnMFCQ the condition $(B2)$ is satisfied, but nonconvex constraint functions are excluded.

(v) The standard embedding P(t) has some advantages in comparison to the Penalty Embedding [11], the Exact Penalty Embedding [3] and the Multiplier Embedding [4]: It can be successful for problems where the EnMFCQ is not satisfied. Furthermore, to improve this embedding, we don't need to work in some higher dimensional space by introducing new variables like in [11],[3] and [4].

(vi) If we put

$$\begin{aligned} h_i(x,t) &:= th_i(x) + (t-1)(h_i(x) - h_i(x^0)) \ = \ 0 \ i \in I \\ g_j(x,t) &:= tg_j(x) + (t-1)(g_j(x) - g_j(x^0)) \ \leq \ 0 \ j \in J \end{aligned}$$

we cannot expect some new insights. It suffices to consider the functions in section 3 once more to see that the set M(t) can still be empty for certain $t \in (0,1)$ and certains $x^0 \in I\!R^n$.

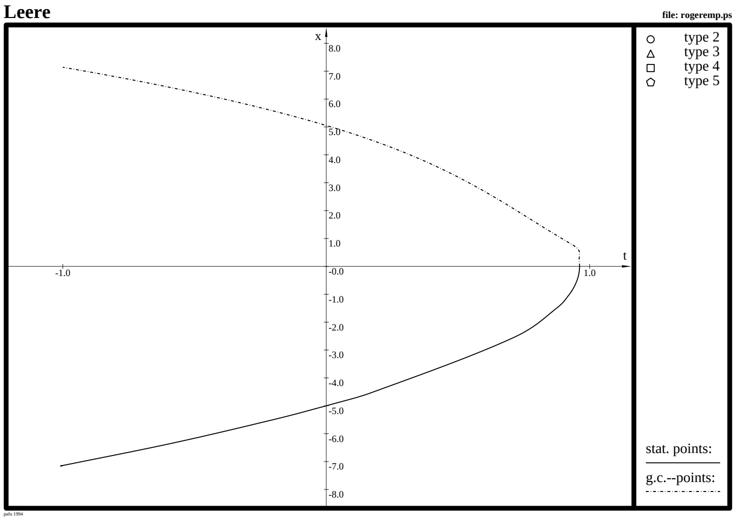

Figure 20:

Remark 5.1 (with the embedded empty feasible set)
We consider the original problem (P) with $M = \emptyset$. Of course, we cannot achieve $t = 1$ for one of the considered embedding, saying $P(t)$. Let $P(t)$ be JJT-regular for $t \in [0, 1]$, then there appears a turning point (of Type 3 or Type 4 or Type 5 where MFCQ is violated) for a $\bar{t} < 1$. This situation will be illustrated by the following example:

$$P(t): \quad \min\{(x+9)^2 \mid x^2 + 1 + (t-1)26 \leq 0\} \text{ with } x^0 = -5$$

(cf. Fig. 20). A turning point of Type 4 appears at $t = 0.96$.

A Appendix I

Definition A.1 *Let $\mathcal{K}$ be an open and convex subset of $\mathbb{R}^n$ and let $\phi : \mathcal{K} \longrightarrow \mathbb{R}$ be a continuously differentiable function.*

ϕ is called quasiconvex on $\mathcal{K}$, if $\forall x^1, x^2 \in \mathcal{K}, \forall \lambda \in (0, 1)$,

$$\phi\left(\lambda x^1 + (1-\lambda)x^2\right) \leq \max\left\{\phi(x^1), \phi(x^2)\right\} \tag{5.1}$$

ϕ is called pseudoconvex on $\mathcal{K}$, if $\forall x^1, x^2 \in \mathcal{K}$

$$\phi(x^2) < \phi(x^1) \Longrightarrow D\phi(x^1)(x^2 - x^1) < 0 \tag{5.2}$$

ϕ is called convex on $\mathcal{K}$, if $\forall x^1, x^2 \in \mathcal{K}$

$$\phi(x^2) - \phi(x^1) \geq D\phi(x^1)(x^2 - x^1) \tag{5.3}$$

Remark A.2 *Let $\mathcal{K} \subset I\!R^n$ be open and convex and let $\phi \in C^1(\mathcal{K}, I\!R)$.*

(A) ϕ is quasiconvex on $\mathcal{K} \Longleftrightarrow \forall \alpha \in I\!R,\ \mathcal{K}_\alpha := \{x \in \mathcal{K} | \phi(x) \leq \alpha\}$ convex .

(B) ϕ is quasiconvex on $\mathcal{K}$ if and only if one of the following conditions is satisfied:

$$(1) \quad \left.\begin{array}{c} x^1, x^2 \in \mathcal{K} \\ \phi(x^2) \leq \phi(x^1) \end{array}\right\} \Longrightarrow D\phi(x^1)(x^2 - x^1) \leq 0.$$

$$(2) \quad \left.\begin{array}{c} x^1, x^2 \in \mathcal{K} \\ \phi(x^2) < \phi(x^1) \end{array}\right\} \Longrightarrow D\phi(x^1)(x^2 - x^1) \leq 0.$$

Proof: (A) obvious.
(B) cf. [8].

Remark A.3 (1) *Let $\mathcal{K} \subset I\!R^n$ be open and convex and $\phi \in C^1(\mathcal{K}, I\!R)$. Then it holds:*

$$\begin{aligned} \phi \text{ convex auf } \mathcal{K} &\Longrightarrow \phi \text{ pseudoconvex auf } \mathcal{K} \\ &\Longrightarrow \phi \text{ quasiconvex auf } \mathcal{K}. \end{aligned}$$

(2) *From the quasiconvexity (and pseudoconvexity respectively) of the funktions ϕ_1 and ϕ_2 not follows generally the quasiconvexity (and pseudoconvexity respectively) of the functions $\phi_1 + \phi_2$. From the konvexity of ϕ_1 and ϕ_2 follows the convexity of $\phi_1 + \phi_2$.*

(3) *If ϕ is quasiconvex (and pseudoconvex respectively) and α is a constant function, then the function $\phi + \alpha$ is quasiconvex (and pseudoconvex respectively).*

Proof: A few calculations show that these assertions are true (cf. e.g. [6]).

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