A Generalization of the Fundamental Estimates for $W^{m,p}$- Solutions of Linear Systems with Constant Coefficients (the case $1 < p < 2$)

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Abstract

The aim of the present paper is to extent the well known fundamental estimates (w.r.t. the L^2-norm) for weak solutions of a linear elliptic system with constant coefficients:

$$(1) \qquad \sum_{j=1}^{N} \sum_{|\alpha|,|\beta|=m} D^\alpha (A_{ij}^{\alpha\beta} D^\beta u^j) = 0 \quad \text{in } \Omega \quad (i = 1, \ldots, N),$$

where $\nu_\circ \, \|\xi\|^2 \leq A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \leq c_\circ \, \|\xi\|^2 \;\; \forall \xi \in I\!R^{nN}, (\Omega \subset I\!R^n$ is open and bounded).

Based on a generalization of the "Cacciopoli - inequality" we are able to establish the extended fundamental estimates w.r.t. the L^p- norm of $W^{m,p}$- solutions $(1 < p < 2)$ of the linear system (1).

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1 Introduction and statement of the result

Let $\Omega \subset I\!R^n (n \geq 2)$ be an open and bounded set with points $x = (x_1, \ldots, x_n)$. Further, let N, m be integers ≥ 1.

First of all we introduce some notations which are used frequently in the present paper. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \, (\alpha_j \in I\!N; j = 1, \ldots, n)$ the length of α is defined by $|\alpha| = \sum_{j=1}^n \alpha_j$. Then we recall the usual differential operator

$$D^\alpha = D^{\alpha_1} \cdots D^{\alpha_n}, \qquad \text{where} \quad D^{\alpha_j} = \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}.$$

If $u : \Omega \longrightarrow I\!R^N$ then $D^k u$ denotes the matrix of spatial derivative of order $k \, (k \in I\!N \geq 1)$, i.e. $D^k u = \{D^\alpha u^i\}_{|\alpha|=k}^{i=1,\ldots,N}$.

$W^{k,q}(\Omega)$ ($\overset{\circ}{W}{}^{k,q}(\Omega)$ resp.) ($1 \le q \le \infty; k = 1,2,\dots$) denote the usual Sobolev spaces equipped with the following norm

$$\|u\|_{k,q,\Omega} = \left(\sum_{l=0}^{k} |u|^q_{l,q,\Omega} \right)^{1/q}, \quad \text{where}$$

$$|u|^q_{l,q,\Omega} = \int_\Omega \|D^l u\|^q \, dx = \sum_{|\alpha|=l} \int_\Omega |D^\alpha u|^q \qquad (l = 0,\dots,k).$$ [1]

$W^{k,q}(\Omega;\mathbb{R}^N)$, $\overset{\circ}{W}{}^{k,q}(\Omega;\mathbb{R}^N), \dots$ etc. denote the spaces $[W^{m,q}(\Omega)]^N, [\overset{\circ}{W}{}^{k,q}(\Omega)]^N, \dots$ etc. of functions with values in $\mathbb{R}^N$.

In the present paper we shall consider the following linear system of PDE's of order $2m$ with constant coefficients:

$$(1.1) \qquad \sum_{j=1}^{N} \sum_{|\alpha|,|\beta|=m} D^\alpha (A^{\alpha\beta}_{ij} D^\beta u^j) = 0 \qquad \text{in} \quad \Omega \qquad (i = 1,\dots,N),$$

where the coefficients $A^{\alpha\beta}_{ij}$ ($|\alpha|, |\beta| = m; i,j = 1,\dots,N$) are constants satisfying the condition

$$(1.2) \qquad \nu_\circ \|\xi\|^2 \le \sum_{i,j=1}^{N} \sum_{|\alpha|,|\beta|=m} A^{\alpha\beta}_{ij} \xi^i_\alpha \xi^j_\beta \le c_\circ \|\xi\|^2 \quad \forall \xi \in \mathbb{R}^{\mu N}.$$

(here μ denotes the number card$\{\alpha = (\alpha_1,\dots\alpha_n) \in \mathbb{N}^n | \; |\alpha| = m\}$) ($\nu_\circ, c_\circ = $ const. > 0).

In what follows let $1 < p < 2$ be some fixed real number.

DEFINITION 1.1. *A vector function $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is said to be a weak solution of the system* (1.1) *if the following integral identity is valid for each $\varphi \in C^\infty(\Omega;\mathbb{R}^N)$ with supp$(\varphi) \subset \Omega$*

$$(1.3) \qquad \sum_{i,j=1}^{N} \sum_{|\alpha|,|\beta|=m} \int_\Omega A^{\alpha\beta}_{ij} D^\beta u^j D^\alpha \varphi^i dx = 0. \quad \Box$$

Let $B_R = B_R(x_\circ)$ [2] denote the open ball in $\mathbb{R}^n$ with the centre $x_\circ$ and the radius $R > 0$. Let $v \in W^{k,1}(B_R)$ ($k \in \mathbb{N} \ge 0$). Then there exists exactly one polynomial of degree $\le k$ denoted by $P^{x_\circ,R}_k$ such that

$$(1.4) \qquad \int_{B_R} D^\alpha (v - P^{x_\circ,R}_k) dx = 0 \qquad \forall \, |\alpha| \le k.$$

[1] We write $\|u\|_{q,\Omega}$ instead of $|u|_{0,q,\Omega}$.
[2] If no confusion can arise we write shortly B_R instead of $B_R(x_\circ)$.

The polynomial $P_k^{x_\circ,R}$ is called the "*mean value polynomial*" of v of order k for the ball B_R (cf. appendix; pp. 12).

The main result of the present paper is the following general form of the fundamental estimate with respect to the L^p- norm.

THEOREM 1.1. *Assume* (1.2). *Let* $u \in \mathrm{W}^{m,p}(\Omega; \mathbb{R}^N)$ *be a weak solution of the system* (1.1). *Then for each integers* $0 \le k \le l+1 \le m$ *there exists a positive constant* $A_{k,l}$ *depending only on* n, m, N, p *and* $c_\circ/\nu_\circ$ *such that for all concentric balls* $B_\sigma \subset\subset B_R \subset \Omega$,

$$\int_{B_\sigma} \|D^k(u - P_l^{x_\circ,\sigma})\|^p \, dx \;\le\; A_{k,l} \left(\frac{\sigma}{R}\right)^{m+p(l+1-k)} \int_{B_R} \|D^k(u - P_l^{x_\circ,R})\|^p \, dx, \tag{1.5}$$

where $P_l^{x_\circ,R}$ $(P_l^{x_\circ,\sigma}$ *resp.) denotes the "mean value polynomial" of* u *for the ball* B_R $(B_\sigma$ *resp.).*

The definition of $P_k^{x_\circ,R}$ shows that

$$D^k(P_k^{x_\circ,R}) = (D^k u)_{B_R} \quad \text{[3]} \qquad \text{for} \quad k = 0, \ldots, m \quad \text{and}$$

$$D^k(P_{k-1}^{x_\circ,R}) = 0 \qquad \text{for} \quad k = 1, \ldots, m.$$

Hence, from (1.5) we easily deduce that for all concentric balls $B_\sigma \subset\subset B_R \subset \Omega$,

$$\int_{B_\sigma} \|D^k u - (D^k u)_{B_\sigma}\|^p \, dx \;\le\; A_k \left(\frac{\sigma}{R}\right)^{n+p} \int_{B_R} \|D^k u - (D^k u)_{B_R}\|^p \, dx \tag{1.6}$$

$(k = 0, \ldots, m-1)$,

$$\int_{B_\sigma} \|D^k u\|^p \, dx \;\le\; A'_k \left(\frac{\sigma}{R}\right)^{n} \int_{B_R} \|D^k u\|^p \, dx \quad \text{[4]} \tag{1.7}$$

$(k = 1, \ldots, m)$. $\square$

In recent time, the fundamental estimates (cf. [1]) has frequently been used for study of regularity or partial regularity of weak solutions of elliptic systems (for instance see in [6], [8]). Some generalizations of these estimates are known for the case $p \ge 2$. For instance, harmonic vector functions and weak solutions of quasilinear systems have been investigated in [3]. Further results have been obtained in [2] upon investigating parabolic systems.

For the prove of the fundamental estimates we need an appropriate CACCIOPPOLI - inequality with respect to the L^p- norm. In section 3 we will show that such an inequality

[3] For $v \in \mathrm{L}^1(B_R)$ we put $(v)_{B_R} = \frac{1}{meas(B_R)} \int_{B_R} v dx$.

[4] Here $A_k = A_{k,k}$ and $A'_k = A_{k,k-1}$.

follows from the CACCIOPPOLI - inequality with respect to the L^2- norm. Based on this general result, in section 4 we will derive a corresponding fundamental estimate for weak solutions u of system (1.1) belonging to the space $W^{m,p}(\Omega; I\!R^N)$.

2 Preliminary lemmas

Here we recall some known results, which we are going to apply in the following sections. We start with a well known GAGLIARDO - NIRENBERG interpolation inequality (cf. [7]).

LEMMA 2.1. *Let $B_\sigma \subset I\!R^n (\sigma > 0)$ be any open ball. Then for each $u \in W^{m,q}(B_\sigma)$ $(1 \leq q < \infty; k \in I\!N \leq m-1)$ the multiplicative inequality*

$$(2.1) \qquad \|D^k u\|_{r,B_\sigma} \leq c \left[\sigma^{-\theta m} \|u\|_{q,B_\sigma} + \|D^m u\|^{\theta}_{q,B_\sigma} \|u\|^{1-\theta}_{q,B_\sigma}\right]$$

holds true for all $k/m \leq \theta < 1$, *where* $\dfrac{1}{r} = \dfrac{1}{q} - \dfrac{\theta m - k}{n}$

(*here $c = const. > 0$ depending only on n, q, θ and k*).

PROOF. – 1°) Firstly, let us consider the case $\sigma = 1$ and $x_\circ = 0$. Then the assertion (2.1) is proved by using the GAGLIARDO - NIRENBERG interpolation inequality provided in [7].

2°) Secondely, let $\sigma > 0$ and $x_\circ \in I\!R^n$ be arbitrarily chosen. Then let us consider the coordinate transformation $y = (x - x_\circ)/\sigma$. Now the desired inequality (2.1) follows from 1°) having applied the transformation formula of the Lebesgue-integral. ■

LEMMA 2.2. *Let $D \subset I\!R^n$ be a bounded domain with lipschitz boundery ∂D. Moreover, let $l \in I\!N \geq 1$ and $1 \leq q < \infty$ with $q > n/l$. Then the Sobolev space $W^{l,q}(D)$ is continuous imbetted in $C^0(\overline{D})$. Further there exists a positive constant c depending only on n, q and ∂D such that*

$$(2.2) \qquad \sup_D |w| \leq c \ \|w\|_{l,q,D} \qquad \forall w \in W^{l,q}(D)$$

(cf. [4]). ■

Next, we mention a general form of the POINCARÉ - inequality (cf. also [5]). For the sake of completion we give a short prove of this result in the appendix.

LEMMA 2.3. *Let $v \in W^{k,q}(B_R)$ $(k \in I\!N \geq 1; 1 \leq q < \infty)$. Then we have the following general form of the* POINCARÉ - *inequality*

$$(2.3)\qquad \sum_{|\alpha|=s}\int_{B_R} |D^\alpha(v-P_{k-1}^{x_\circ,R})|^q\,dx \;\le\; cR^{q(t-s)}\sum_{|\alpha|=t}\int_{B_R} |D^\alpha(v-P_{k-1}^{x_\circ,R})|^q\,dx$$

($s,t\in I\!N; 0\le s<t\le k$), where the constant c depends only on n,q,k,t and s. □

Finally, we prove a technical lemma which can also be found in [5], chapter 5.3.

LEMMA 2.4. *Let f be a non-negative and bounded function defined on the interval $[a,b]$ ($-\infty<a<b<+\infty$). Furthermore, let A,B,α and $0<\varepsilon<1$ are positive constants such that for all $a\le\sigma<R\le b$,*

$$(2.4)\qquad f(\sigma)\;\le\; A(R-\sigma)^{-\alpha}+B+\varepsilon f(R).$$

Then there exists a positive constant $c=c(\alpha,\varepsilon)$ such that for all $a\le\sigma<R\le b$,

$$(2.5)\qquad f(\sigma)\;\le\; c\,\Big(A(R-\sigma)^{-\alpha}+B\Big).$$

(Remark that $c=c(\alpha,\varepsilon)=(1-\varepsilon^{1/(\alpha+1)})^{-(\alpha+1)}$ is the best constant in (2.5)).

PROOF. – Let $a\le\sigma<R\le b$ be arbitrarily fixed. Then we define a decreasing sequence $\{t_j\}_{j=0,1,\ldots,}$ of real numbers by

(*i*) $\quad t_0=\sigma$

(*ii*) $\quad t_{j+1}-t_j=(1-\tau)\tau^j(R-\sigma)\qquad(j=0,1,\ldots).$ [5]

For simplicity we set $\Phi(t-s)=A(t-s)^{-\alpha}+B\quad(a\le s<t\le b)$. Then, applying assumption (2.4) iteratively to the sequence $\{t_j\}_{j=0,1,\ldots}$ we obtain for each $k\in I\!N\ge1$

$$\begin{aligned} f(\sigma)=f(t_0)\;&\le\;\Phi(t_1-t_0)+\varepsilon f(t_1)\;\le\\ &\le\;\Phi(t_1-t_0)+\varepsilon\Phi(t_2-t_1)+\varepsilon^2 f(t_2)\le\cdots\\ &\cdots\le\;\sum_{j=0}^{k-1}\varepsilon^j\Phi(t_{j+1}-t_j)+\varepsilon^k f(t_k). \end{aligned}$$

Because of $\varepsilon^k f(t_k)\le\varepsilon^k\sup\limits_{t\in[a,b]} f(t)$ the right hand side of the latter inequality converges to $\sum_{j=0}^\infty\varepsilon^j\Phi(t_{j+1}-t_j)$ as $k\to\infty$ in case that this sum is finite. Thus, we obtain

$$(2.6)\qquad f(\sigma)\le\sum_{j=0}^{\infty}\varepsilon^j(A(1-\tau)^{-\alpha}\tau^{-j\alpha}(R-\sigma)^{-\alpha}+B).$$

[5] This definition immediately implies $t_j\to R$ as $j\to\infty$.

Now, let us choose τ such that

(2.7) $$\varepsilon\tau^{-\alpha} < 1.$$

Condition (2.7) guarantees that the right hand side of (2.6) is finite, which entails the assertion (2.5) with $c = c(\alpha,\varepsilon,\tau) = (1-\tau)^{-\alpha}(1-\varepsilon\tau^{-\alpha})^{-1}$.

Finally, the fact that $c(\alpha,\varepsilon,\bullet)$ attains its minimum at $\tau_\circ = \varepsilon^{1/(\alpha+1)}$ implies that $c = c(\alpha,\varepsilon) = (1-\varepsilon^{1/(\alpha+1)})^{-(\alpha+1)}$ is the best constant in (2.5). ■

3 A Generalization of the Caccioppoli Inequality with respect to the L^p-norm

In this section we are going to prove an extension of the so called "CACCIOPPOLI type" inequality. We have the following

THEOREM 3.1. *Let $u \in \mathrm{W}^{m,p}(\Omega;\mathbb{R}^N)$ be a weak solution of the linear system* (1.1). *Let condition* (1.2) *be fulfilled. Then there exists a constant $\gamma \geq m^2$ depending only on n, m and p such that for all concentric balls $B_\sigma \subset\subset B_R \subset \Omega$,*

(3.1) $$\Big(\int_{B_\sigma} \|D^m u\|^p\,dx\Big)^{1/p} \leq c\Big(\frac{R}{R-\sigma}\Big)^{\gamma} R^{-m}\Big(\int_{B_R}\|u\|^p\,dx\Big)^{1/p},$$

where $c = const. > 0$ depending only on n, m, N, p and $c_\circ/\nu_\circ$.

PROOF. - Let $x_\circ \in \Omega$ be arbitrarily fixed. The proof of the theorem will be devided into two steps.

1°) In the first part we will obtain a CACCIOPPOLI -inequality with respect to the L^2-norm. Clearly, any weak solution $u \in \mathrm{W}^{m,p}(\Omega;\mathbb{R}^N)$ of system (1.1) assuming (1.2) is continuous differentiable on each open set $\Omega' \subset\subset \Omega$. In particular, u belongs to $\mathrm{W}^{m,2}_{loc}(\Omega;\mathbb{R}^N)$. Therefore, the following CACCIOPPOLI-inequality is valid,

(3.2) $$\begin{cases} \|D^m u\|_{2,B_\sigma} \leq c\Big(\dfrac{R}{R-\sigma}\Big)^m \displaystyle\sum_{k=1}^{m-1} R^{k-1}\,\|D^k u\|_{2,B_R} \quad {}^{6} \\ \forall\, 0<\sigma<R<dist(x_\circ,\partial\Omega). \end{cases}$$

[6] In what follows the letter c denotes some positive constant which may change their numerical value from line to line, but depends neither on the radius of the chosen balls nor on the solution u.

The inequality (3.2) is easily obtained by inserting into (1.3) the addmissible testfunction

$$\varphi^i = u^i \zeta^{2m} \qquad (i = 1, \ldots, N), \tag{3.3}$$

where $\zeta \in \mathrm{C}_c^m(I\!R^n)$ is an appropriate "cut-off function" such that: $\zeta \equiv 1$ on B_σ, $\zeta \equiv 0$ on $R^n \setminus B_R$, $0 \leq \zeta \leq 1$ and $|D^\alpha \zeta| \leq c(R-\sigma)^{-|\alpha|}$ $\forall\, |\alpha| \leq m$ (Here $c = const. > 0$ depending only on n and m) (cf. also [5], [6]).

Applying lemma 2.1 with $q = 2$ and $\theta = k/m$ $(k = 1, \ldots, m-1)$ proves that for all $B_t \subset\subset \Omega$,

$$\|D^k u\|_{2,B_t} \leq c \left[t^{-k} \, \|u\|_{2,B_t} + \|D^m u\|_{2,B_t}^{k/m} \|u\|_{2,B_t}^{1-k/m} \right]. \tag{3.4}$$

Next, let $0 < s < t \leq R < dist(x_\circ, \partial\Omega)$ be arbitrarily chosen. Having set $\phi(t) = \|D^m u\|_{2,B_t}$ combining (3.2) and (3.4) gives

$$\phi(s) \leq c \left(\frac{t}{t-s} \right)^m t^{-m} \, \|u\|_{2,B_t} + c \sum_{k=1}^{m-1} \left(\frac{t}{t-s} \right)^m t^{k-m} \, \|u\|_{2,B_t}^{1-k/m} \, \phi(t)^{k/m}. \tag{3.5}$$

With the help of the YOUNG's inequality from (3.5) we obtain,

$$\begin{aligned} \phi(s) \; &\leq \; c \left(\frac{t}{t-s} \right)^m t^{-m} \, \|u\|_{2,B_t} \; + \\ &\quad + c \sum_{k=1}^{m-1} \left(\frac{t}{t-s} \right)^{m^2/(m-k)} t^{-m} \, \|u\|_{2,B_t} \; + \frac{1}{2}\phi(t) \; \leq \\ &\leq \; c \left(\frac{t}{t-s} \right)^{m^2} t^{-m} \, \|u\|_{2,B_t} \; + \frac{1}{2}\phi(t) = \frac{A}{(t-s)^{m^2}} + \frac{1}{2}\phi(t), \end{aligned} \tag{3.6}$$

where $A = c\, R^{m^2-m} \, \|u\|_{2,B_R}$.

Now we are in a position to apply lemma 2.4. According to (3.6),

$$\begin{cases} \|D^m u\|_{2,B_\sigma} \leq \; c \left(\dfrac{R}{R-\sigma} \right)^{m^2} R^{-m} \, \|u\|_{2,B_R} \\ \forall 0 < \sigma < R \leq dist(x_\circ, \partial\Omega). \; \square \end{cases} \tag{3.7}$$

2°) Secondly, let $\overline{k} \geq 0$ denote the uniquely determined integer which satisfies

$$\frac{2n}{n + \overline{k} + 1} \leq p < \frac{2n}{n + \overline{k}}. \tag{3.8}$$

We define for $j=0,\dots,\overline{k}$

$$q_j = \frac{2n}{2n+j}.$$

Next, set $\chi = 2m/(2m-1)$ and suppose that for some $j\in\{0,\dots,m-1\}$ the inequality

$$\|D^m u\|_{q_j,B_\sigma} \le\ c\left(\frac{R}{R-\sigma}\right)^{\chi^j m^2} R^{-m}\ \|u\|_{q_j,B_R} \tag{3.9}$$

holds true for all $0<\sigma<R<dist(x_\circ,\partial\Omega)$.

Let $0<s<t\le R<dist(x_\circ,\partial\Omega)$ be arbitrarily fixed. Setting $\phi(t)=\|D^m u\|_{q_{j+1},B_t}$ using HÖLDER's inequality from the assumption (3.9) we get

$$\phi(s)\ \le\ \omega_n^{1/2n} t^{1/2}\ \|D^m u\|_{q_j,B_s} \le\ c\left(\frac{t}{t-s}\right)^{\chi^j m^2} t^{-m+\frac12}\ \|u\|_{q_j,B_t}\ .\ ^{7} \tag{3.10}$$

On the other hand, by (2.1) (cf. lemma 2.1) putting $r=q_j, q=q_{j+1}$ and $\theta = 1/(2m)$ therein we receive

$$\|u\|_{q_j,B_t} \le\ c\left[t^{-\frac12}\ \|u\|_{q_{j+1},B_t}\ +\ \|D^m u\|^{1/(2m)}_{q_{j+1},B_t}\ \ \|u\|^{1-1/(2m)}_{q_{j+1},B_t}\ \right]. \tag{3.11}$$

Combining (3.10) and (3.11) gives

$$\phi(s)\ \le c\left(\frac{t}{t-s}\right)^{\chi^j m^2} t^{-m}\ \|u\|_{q_{j+1},B_t}\ +\ \left(\frac{t}{t-s}\right)^{\chi^j m^2} t^{-m+\frac12}\ \|u\|^{1-1/(2m)}_{q_{j+1},B_t}\ \phi(t)^{1/(2m)}. \tag{3.12}$$

By YOUNG's inequality,

$$\begin{aligned}\phi(s)\ &\le\ \left(\frac{t}{t-s}\right)^{\chi^{j+1}m^2} t^{-m}\ \|u\|_{q_{j+1},B_t}\ +\frac12\phi(t)\ = \\ &=\ \frac{A}{(t-s)^{\chi^{j+1}m^2}}\ +\ \frac12\phi(t),\end{aligned}$$

where $A = c\ R^{\chi^{j+1}m^2-m}\ \|u\|_{q_{j+1},B_R}$.

Again applying lemma 2.4 we receive

$$\begin{cases} \|D^m u\|_{q_{j+1},B_\sigma} \le\ c\left(\dfrac{R}{R-\sigma}\right)^{\chi^{j+1}m^2} R^{-m}\ \|u\|_{q_{j+1},B_R} \\ \forall 0<\sigma<R<dist(x_\circ,\partial\Omega). \end{cases} \tag{3.13}$$

[7] Here ω_n denotes the measure of the unit ball in $I\!R^n$.

Since (3.9) has been proved for $j=0$ (cf. inequality (3.7)) we may use the implication (3.9) $\Rightarrow$ (3.13) iteratively starting with $j=0$ until $j=\overline{k}-1$ to show that

$$(3.14)\qquad \begin{cases} \|D^m u\|_{q_{\overline{k}},B_\sigma} \le\ c\left(\dfrac{R}{R-\sigma}\right)^{\chi^{\overline{k}}m^2} R^{-m}\ \|u\|_{q_{\overline{k}},B_R} \\ \forall 0<\sigma<R<dist(x_\circ,\partial\Omega). \end{cases}$$

Finally, observing

$$(3.15)\qquad \|u\|_{q_{\overline{k},B_t}} \le\ c\,t^{-\theta m}\left[\|u\|_{p,B_t}\ +t^{\theta m}\ \|D^m u\|^{\theta}_{p,B_t}\ \|u\|^{1-\theta}_{p,B_t}\right]$$

whenever $0<t<dist(x_\circ,\partial\Omega)$ with an suitable $\theta\,(0<\theta\le 1/(2m))$ (cf. lemma 2.1), we may argue as before to obtain from (3.14),

$$(3.16)\qquad \begin{cases} \|D^m u\|_{p,B_\sigma} \le\ c\left(\dfrac{R}{R-\sigma}\right)^{\gamma} R^{-m}\ \|u\|_{p,B_R} \\ \forall 0<\sigma<R<dist(x_\circ,\partial\Omega),\ ^{8} \end{cases}$$

where the constant c depends only on n,m,N,p and $c_\circ/\nu_\circ$. This concludes the proof of the theorem. ■

Remark that the constant γ in (3.16) may be estimated by $\chi^{\overline{k}}m^2<\gamma\le\chi^{\overline{k}+1}m^2$.

COROLLARY 3.1. *Assume* (1.2). *Let* $u\in W^{m,p}(\Omega;\mathbb{R}^N)$ *be a weak solution of system* (1.1). *Then for each multi-index* $\beta=(\beta_1,\dots,\beta_n)$ *the function* $D^\beta u$ *belongs to the space* $W^{m,p}_{loc}(\Omega;\mathbb{R}^N)$ *and is a weak solution refering to* (1.1). *Furthermore, for any integer* $h\ge 1$ *and for all concentric balls* $B_\sigma\subset\subset B_R\subset\Omega$, *we have*

$$(3.17)\qquad \|D^{hm}u\|_{p,B_\sigma} \le\ c\left(\frac{R}{R-\sigma}\right)^{h\gamma} R^{h\gamma}\ \|u\|_{p,B_R},$$

where $c=const.>0$ *depending only on* n,m,N,p,h *and* $c_\circ/\nu_\circ$. (*Here* γ *denotes the constant which appears in* (3.1)).

PROOF. - Let $B_\sigma\subset\subset B_R\subset\Omega$ and $h\in\mathbb{N}\ge 1$ be arbitrarily fixed. Define $\sigma_j=\sigma+j(R-\sigma)/h$ [9] $(j=1,\dots,h)$.

Since $u\in C^\infty(\overline{\Omega'};\mathbb{R}^N)\ \ \forall\Omega'\subset\subset\Omega$, using integration by parts an elementary calculation shows that for any multi-index $\beta=(\beta_1,\dots,\beta_n)$ the function $D^\beta u$ satisfies the integral identity (1.3). Consequently, $D^\beta u\in W^{m,p}_{loc}(\Omega;\mathbb{R}^N)$ is a weak solution of the system (1.1).

[8] Notice that by the absolute continuity of the LEBESGUE-integral one may obtain (3.16) also for each $0<\sigma<R\le dist(x_\circ,\partial\Omega)$.

[9] Remark that $\sigma\le\sigma_j<\sigma_{j+1}\le R\ \ (j=1,\dots,k-1)$.

Now, applying (3.1) to the function $D^\beta u$ ($|\beta|= (h-j)m; 1 \le j \le h$) replacing σ by σ_{j-1} and R by σ_j yields

$$(3.18) \qquad \|D^m(D^\beta u)\|_{p,B^{(j-1)}} \le \ c \left(\frac{\sigma_j}{\sigma_j - \sigma_{j-1}} \right)^\gamma \sigma_j^{-m} \ \|D^\beta u\|_{p,B^{(j)}},$$

where c and γ are the same constants as in (3.1) (cf. theorem 3.1) ($B^{(j)} = B_{\sigma_j}; j = 0, \ldots, h$).

Then summation over $|\beta|= m(h-j)$ in (3.18) implies,

$$(3.19) \qquad \|D^{m(h-j+1)}u\|_{p,B^{(j-1)}} \le \ c \left(\frac{hR}{R-\sigma} \right)^\gamma R^{-m} \ \|D^{m(h-j)}u\|_{p,B^{(j)}}$$

($j = 1, \ldots, h$).

Now, iterating (3.19) starting with $j = 1$ gives

$$\begin{aligned}
\|D^{mh}u\|_{p,B_\sigma} \ &\le \ c \left(\frac{hR}{R-\sigma} \right)^\gamma R^{-m} \ \|D^{m(h-1)}u\|_{p,B^{(1)}} \le \\
&\le \ c^2 \left(\frac{hR}{R-\sigma} \right)^{2\gamma} R^{-2m} \ \|D^{m(h-2)}u\|_{p,B^{(2)}} \le \ \cdots \\
&\cdots \ \le \ c^h \left(\frac{hR}{R-\sigma} \right)^{h\gamma} R^{-hm} \ \|u\|_{p,B_R},
\end{aligned}$$

what concludes the proof of assertion (3.17). ■

4 Proof of Theorem 1.1

Let $l, k \in I\!N$ ($0 \le k \le l+1 \le m$) and $B_R \subset \Omega$ be arbitrarily fixed. Since in the particular case $R/2 < \sigma < R$ the assertion (1.5) is trivially fulfilled, without loss of generality we may assume that $0 < \sigma \le R/2$.

For almost all $y \in B_1(0)$ we set

$$(4.1) \qquad w(y) = u(x_\circ + Ry) - P_l^{x_\circ,R}(x_\circ + Ry),$$

where $P_l^{x_\circ,R}$ denotes the "*mean value polynomial*" of u of the order l for the ball B_R. Using the transformation formula of the LEBESGUE - integral an elementary calculation shows that $w \in \mathrm{W}^{m,p}_{loc}(B_1(0); I\!R^N)$ is a weak solution of the system (1.1) in the unit ball $B_1(0)$.

Next, let s denote the smallest integer satisfying $s > n/p$. Setting $h = [(s+l+1)/m]+1$ [10] by (2.2) (cf. lemma 2.2) we get for all $\tau \in (0, 1/2]$,

$$
(4.2)\qquad \begin{aligned}
\|D^{l+1}w\|^p_{p,B_\tau} &\le \omega_n \tau^n \sup_{B_\tau} \|D^{l+1}w\|^p \le \\
&\le c\tau^n \, \|w\|^p_{hm,p,B_{1/2}} = c\tau^n \sum_{j=0}^{hm} \|D^j w\|^p_{p,B_{1/2}} .
\end{aligned}
$$

On the other hand, by (2.1) we have

$$
(4.3)\qquad \|D^j w\|_{p,B_{1/2}} \le c \left[\|w\|_{p,B_{1/2}} + \|D^{hm}w\|_{p,B_{1/2}} \right] .
$$

Combining (4.3) and (3.17) with $\sigma = 1/2$ and $R = 1$ yields,

$$
(4.4)\qquad \|D^j w\|^p_{p,B_{1/2}} \le c \, \|w\|^p_{p,B_1} .
$$

$(j = 0, \ldots, hm)$.

Now, inserting estimate (4.4) into (4.2) gives

$$
(4.5)\qquad \|D^{l+1}w\|^p_{p,B_\tau} \le c\tau^n \, \|w\|^p_{p,B_1} \qquad \forall\, 0 < \tau \le 1/2.
$$

Finally, using the transformation formula of the LEBESGUE-integral (setting $\tau = \sigma/R$) with help of POINCARÉ's inequality (2.3) from (4.5) we get

$$
(4.6)\qquad \begin{aligned}
\int_{B_\sigma} \|D^{l+1}u\|^p \, dx &\le c \left(\frac{\sigma}{R}\right)^n R^{-p(l+1)} \int_{B_R} \|u - P^{x_\circ,R}\|^p \, dx \le \\
&\le c \left(\frac{\sigma}{R}\right)^n R^{-p(l+1-k)} \int_{B_R} \|D^k(u - P^{x_\circ,R})\|^p \, dx
\end{aligned}
$$

for all $0 < \sigma < R \le dist(x_\circ, \partial\Omega)$.

Now, assertion (1.5) is obtained after having applied once more the POINCARÉ - inequality (2.3) combined with (4.6). ■

[10] Here $[a]$ denotes the greatest integer $\le a$.

5 Appendix

PROOF OF LEMMA 2.2 - Let $B_R(x_\circ) \subset I\!R^n$ be arbitrarily chosen. Let $v \subset \mathrm{W}^{k,q}(B_R)$ ($k \in I\!N \geq 1; 1 \leq q < \infty$).

1°) In order to prove assumption (1.4) we proceed as follows: Let $c_\beta \in I\!R$ ($|\beta| \leq k$) denote the coefficients of the polynomial $P_k^{x_\circ,R}$, i.e.

$$(5.1) \qquad P_k^{x_\circ,R}(x) = \sum_{|\beta|\leq k} c_\beta (x - x_\circ)^\beta \quad \text{for } x \in I\!R^n.$$

It is readily seen that for each $|\alpha| \leq k$,

$$(5.2) \qquad D^\alpha P_k^{x_\circ,R}(x) = \sum_{\substack{|\beta|\leq k \\ \alpha\leq\beta}} c_\beta\, \alpha! \binom{\beta}{\alpha} (x - x_\circ)^\beta \quad \forall x \in I\!R^n,$$ [11]

where $\alpha! = \alpha_1! \cdots \alpha_n!$ and $\binom{\beta}{\alpha} = \frac{\beta!}{(\beta-\alpha)!\alpha!}$.

By an elementary calculation one can see that (1.4) is fulfilled if and only if the coefficients c_β ($|\beta| \leq k$) satisfying the following system of linear equations,

$$(5.3) \qquad \sum_{|\beta|\leq k} c_\beta a_{\alpha\beta} = b_\alpha \quad (|\alpha| \leq k), \quad \text{where}$$

$$b_\alpha = \int_{B_R} D^\alpha v dx\ ,$$

$$a_{\alpha\beta} = \begin{cases} \alpha! \binom{\beta}{\alpha} \displaystyle\int_{B_R} (x - x_\circ)^{\beta-\alpha} dx & \text{if} \quad \beta \leq \alpha \\ 0 \quad \text{otherwice} \end{cases}$$

($|\alpha|, |\beta| \leq k$).

It is immediately obvious that the matrix $\mathbf{A} = \{a_{\alpha\beta} \mid \ |\alpha|,\ |\beta| \leq k\}$ is of triangular form. Moreover, $a_{\alpha\alpha} = \alpha!\, meas(B_R) > 0$ ($|\alpha| \leq k$), what proves the regularity of $\mathbf{A}$. Hence, there exists exactely one vector $\mathbf{c} = \{c_\beta \mid \ |\beta| \leq k\}$ such that (5.3) is fulfilled.

2°) Let $l \in I\!N$ ($0 \leq l < k$) and let $\beta = (\beta_1, \ldots, \beta_n)$ be a multi-index with the length $|\beta| = l$. Since by (1.4) the mean value of the function $D^\beta(v - P_{k-1}^{x_\circ,R})$ taken over the ball B_R vanishes, we are in a position to apply the known POINCARÉ - inequality. Hence

[11] We say that $\alpha \leq \beta$ if and only if $\alpha_j \leq \beta_j$ $\forall j = 1, \ldots, n$.

$$(5.4) \qquad \int_{B_R} |D^\beta(v - P_{k-1}^{x_\circ,R})|^q \, dx \;\leq\; c_* R^q \sum_{|\alpha|=l+1} \int_{B_R} |D^\alpha(v - P_{k-1}^{x_\circ,R})|^q \, dx,$$

where the constant c_* depends only on n and q.

Summation over $|\beta|= l$ now provides, according to (5.4)

$$(5.5) \qquad |v - P_{k-1}^{x_\circ,R}|^q_{l,q,B_R} \leq \; cR^q \, |v - P_{k-1}^{x_\circ,R}|^q_{l+1,q,B_R} \, .$$

Finally, let $s, t \in I\!N$ $(0 \leq s < t \leq k)$ be arbitrarily chosen. Then we conclude the proof of the lemma by applying (5.5) iteratively starting with $l = s$ until $t - 1$, i.e.

$$|v - P_{k-1}^{x_\circ,R}|^q_{s,q,B_R} \leq \; cR^q \, |v - P_{k-1}^{x_\circ,R}|^q_{s+1,q,B_R} \leq \; \cdots \; \leq \; cR^{q(t-s)} \, |v - P_{k-1}^{x_\circ,R}|^q_{t,q,B_R},$$

where $c = c(n, q, s, t)$. ■

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