Criteria for the trivial solution of differential algebraic equations with small nonlinearities to be asymptotically stable

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Abstract

Differential algebraic equations consisting of a constant coefficient linear part and a small nonlinearity are considered. Conditions that enable linearizations to work well are discussed. In particular, for index-2 differential algebraic equations there results a kind of Perron-Theorem that sounds as clear as its classical model except for the expensive proofs.

1 Introduction

This paper deals with the question whether the zero-solution of the equation

$$Ax'(t) + Bx(t) + h(x'(t), x(t), t) = 0 \tag{1.1}$$

is asymptotically stable in the sense of Lyapunov. Equation (1.1) consists of a linear part characterized by the constant matrix-coefficients $A, B \in L(\mathbb{R}^m)$ and a small nonlinearity described by the function $h : \mathcal{D} \times [0, \infty) \to \mathbb{R}^m$, $\mathcal{D} \subseteq \mathbb{R}^m \times \mathbb{R}^m$ open, $0 \in \mathcal{D}$,

$$h(0, 0, t) = 0, \quad t \in [0, \infty).$$

The zero-function solves (1.1) trivially, i.e. the origin represents a stationary solution of (1.1).

The leading coefficient matrix A is not necessarily nonsingular, but if A is so, equation (1.1) represents a regular ordinary differential equation (ODE). For singular matrices A, there are differential-algebraic equations (DAEs) under consideration. The matrix pencil $\{A, B\}$ is assumed to be regular, i.e. the polynomial $p(\lambda) := \det(\lambda A + B)$ does not vanish identically. By $\sigma\{A, B\}$ and $\mathrm{ind}\{A, B\}$ we denote the finite spectrum and the Kronecker index of the pencil $\{A, B\}$, respectively. Recall that $\sigma\{A, B\}$ is the set of the roots of $p(\lambda)$.
The given function h is continuous together with its partial Jacobians $h'_{x'}$, h'_x. Moreover, h is small in the following sense. To each $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $|x| \le \delta(\varepsilon)$, $|y| \le \delta(\varepsilon)$, $t \in [0, \infty)$ yield

$$|h'_{x'}(y, x, t)| \le \varepsilon, \quad |h'_x(y, x, t)| \le \varepsilon. \tag{1.2}$$

Clearly, (1.1) covers the well-understood case of regular explicit ODEs

$$x'(t) = Bx(t) + g(x(t), t) \tag{1.3}$$

by $A = -I$, $h(y, x, t) \equiv g(x, t)$. The pencil $\{-I, B\}$ is always regular, further $\mathrm{ind}\{-I, B\} = 0$, $\sigma\{-I, B\} = \sigma(B)$. In this case Perron's Theorem (e.g. [1], [2]) applies immediately. Hence, if $\sigma\{A, B\}$ belongs to $\mathbb{C}^-$, then the trivial solution is asymptotically stable in the sense of Lyapunov.

Does this assertion hold true also in more general cases? If so, to what extend does it hold? Answers should be of great interest, since they constitute the background of further stability considerations via linearization and tracing back linear parts to the constant coefficient case.
Although the classical stability results formed by Poincaré, Perron and Lyapunov (e.g. [1], [2]) date back more than hundred years, the respective theory for DAEs is rather in its infancy.
For so-called transferable DAEs (1.1), in [3] the stability question is reduced to that for an inherent regular ODE relative to a certain invariant subspace. Unfortunately, this inherent state equation is not attainable in practice. On the other hand, criteria by linearization are expected to enable also practical determinations. For autonomous low index DAEs, stability via linearization is considered e.g. in [4], [5], [6]. Unlike regular ODEs nonautonomous DAEs involve nontrivial new difficulties in comparison with autonomous ones. A Lyapunov stability criterion for nonautonomous index-1 DAEs (1.1) is proved in [7]. However, even for autonomous DAEs (1.1) with $\mathrm{ind}\{A, B\} > 1$, this index may become an irrelevant detail of (1.1), that is, linearization does not work in those cases (e.g. [4] and Section 2 below).
If the matrix A is nonsingular, then, applying the Implicit Function Theorem, we can transform (1.1) into

$$x'(t) = -A^{-1}Bx(t) + g(x(t), t). \tag{1.4}$$

The pencil $\{A, B\}$ is regular, $\mathrm{ind}\{A, B\} = 0$, $\sigma\{A, B\} = \sigma(-A^{-1}B)$. Again by standard arguments, $\sigma\{A, B\} \subset \mathbb{C}^-$ yields the asymptotical stability of the trivial solution. Now, let us turn to the more interesting case of A being singular.
To make sure that $Ax'(t)$ in (1.1) may be considered as a somewhat leading term in general, we assume the inclusion

$$N := \ker A \subseteq \ker h'_{x'}(y, x, t), \quad (y, x, t) \in \mathrm{D} \times [0, \infty) \tag{1.5}$$

to be satisfied. Note that (1.5) holds for trivial reasons if $h(y, x, t)$ does not depend at all on its first argument. Due to condition (1.5) only those components of $x'(t)$ occur in the nonlinear part of (1.1) that are already involved in the leading term $Ax'(t)$.

Next, denote by $P \in L(\mathbb{R}^m)$ any projector matrix along N that is $P^2 = P$, $\ker P = N$. Then $Q := I - P$ projects onto the nullspace N, hence $A = A(P + Q) = AP$.
It is easily checked that (1.5) implies the identity

$$h(y, x, t) \equiv h(Py, x, t) \tag{1.6}$$

and vice versa. This suggests to reformulate equation (1.1) more precisely as

$$A(Px)'(t) + Bx(t) + h((Px)'(t), x(t), t) = 0. \tag{1.7}$$

In the following we indicate equation (1.1) as a shorter notation of (1.7). Naturally, now we should ask for solutions of (1.1) that belong to the class

$$C_N^1 := \{x(\cdot) \in C : \; Px(\cdot) \in C^1\}.$$

Only those components of the unknown function are expected to be from C^1 whose derivatives are really involved in (1.1). For the other components continuity will do.
At this place it should be mentioned that both the class C_N^1 and the formulation (1.7) are invariant of the special choice of the projector P. For nonsingular A, we have trivially $N = \ker A = \{0\}$, $P = I$, thus $C_N^1 = C^1$. However, if A is singular, $\operatorname{im} P \subset \mathbb{R}^m$ becomes a proper subspace, and C_N^1 is a larger class than C^1 in fact.

Example: Consider the two-dimensional system

$$x_1'(t) + x_1(t) + \alpha(t)x_1(t)^2 \;=\; 0, \tag{1.8}$$

$$x_2(t) + \beta(t)x_1(t)^2 + \gamma(t)x_2(t)^2 \;=\; 0, \tag{1.9}$$

with continuous, uniformly bounded on $[0, \infty)$ scalar functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$. Obviously, all the above assumptions on h are satisfied. In particular, (1.5) holds due to $h'_{x'} = 0$. Further, we have

$$A = \operatorname{diag}(1, 0), \quad B = I, \quad \det(\lambda A + B) = \lambda + 1, \quad \operatorname{ind}\{A, B\} = 1,$$

and $P = \operatorname{diag}(1, 0)$ is a possible choice. The respective class C_N^1 consists of all continuous functions $x(\cdot) = (x_1(\cdot), x_2(\cdot))^T$, the first component of which is continuously differentiable.
System (1.8), (1.9) shows once more that, looking for C^1 solutions instead of those from C_N^1, would necessitate more smoothness of the function h. However, in view of applications, we try for lower smoothness conditions if possible.
It is evident that the regular ODE (1.8) for $x_1(\cdot)$ can be treated again by standard arguments. Its zero-solution is stable. The constraint equation (1.9) determines the second component in dependence of the first one, and $x_1(t) \to 0$ $(t \to \infty)$ yields $x_2(t) \to 0$ $(t \to \infty)$.
Obviously, to cover all neighbouring solutions of the trivial one in the complete system (1.8), (1.9) we should vary only the initial data of the first component. Observe that $\sigma\{A, B\} = \{-1\} \subset \mathbb{C}^-$ and that the trivial solution is asymptotically stable in this modified sense. □

The example discussed above demonstrates an important peculiarity of DAEs. One has to deal with constraints like (1.9), but also with so-called hidden ones (cf. §2 below). Naturally, the initial values $x_0 := x(t_0)$ of solutions satisfy all relevant constraints, i.e., x_0 is consistent at t_0. However, how to state initial value problems? Formulations like

$x(t_0) = x^0$, $x^0 \in \mathbb{R}^m$ is consistent at t_0, are nice but unfit for practical use. In general, one has no idea on how the constraints look like. On the other hand, simply stating

$$x(t_0) = x^0 \in \mathbb{R}^m$$

would yield unsolvable problems. In the following, we try to pick up and fix the free integration constants involved by means of a certain projector matrix $\Pi \in L(\mathbb{R}^m)$ that can be computed practically in terms of A, B. We state

$$\Pi x(t_0) = \Pi x^0, \qquad x^0 \in \mathbb{R}^m, \tag{1.10}$$

as initial condition.
Note that, in case of nonsingular A, we obtain again $\Pi = I$, of course. In example (1.8), (1.9) the choice of $\Pi = P = \mathrm{diag}(1, 0)$ is convenient. Π depends on the pencil $\{A, B\}$ in general, and on its index in particular.

Definition: The zero-solution of (1.1) is stable in the sense of Lyapunov if there is a certain projector $\Pi \in L(\mathbb{R}^m)$ and, for each $t_0 \geq 0$

(i) a value $\tau > 0$ can be found such that the initial value problem (1.1), (1.10) with $|\Pi x^0| \leq \tau$ has a C^1_N-solution $x(\cdot, x^0, t_0)$ defined at least on $[t_0, \infty)$, and further

(ii) a value $\varrho(\eta) > 0$ to each $0 < \eta \leq \tau$ can be found so that $|\Pi x^0| \leq \varrho(\eta)$ yields $|x(t, x^0, t_0)| \leq \eta$ for $t \geq t_0$.

The trivial solution of (1.1) is asymptotically stable in the sense of Lyapunov if it is stable and, for all sufficiently small $|\Pi x^0|$, it holds that

$$x(t, x^0, t_0) \longrightarrow 0 \qquad (t \to \infty).$$

No doubt, this is a straightforward generalization of the classical notion for regular ODEs which is recovered by $\Pi = I$. As mentioned before, a respective stability result for (1.1) with an index-1 pencil $\{A, B\}$ is given in [7]. It says that $\sigma\{A, B\} \subset \mathbb{C}^-$ implies the trivial solution to be asymptotically stable, whereby $\Pi = P$ is chosen. In particular, this assertion applies to the special system (1.7), (1.8) and confirms the stability behaviour we discussed before.

It should be mentioned that, in the above Lyapunov stability notion, the projector matrix $\Pi \in L(\mathbb{R}^m)$ can be replaced by any matrix $C \in L(\mathbb{R}^m)$ with the only property $\ker C = \ker \Pi$. This fact can be realized easily by using the relations $C = C\Pi$, $\Pi = \Pi C^+ C$, where $C^+ \in L(\mathbb{R}^m)$ indicates the Moore-Penrose inverse of C. Hence, in particular, Lyapunov stability does not depend on the special choice of the projector Π, the only relevant characteristic feature is its nullspace, but that is fully determined by the DAE itself.

One might expect that, in general, $\sigma\{A, B\} \subset \mathbb{C}^-$ yields the trivial solution to be asymptotically stable. In Section 2, this tentative, somewhat coarse conjecture, is discussed by means of examples. After that, we derive the main result of the present paper (Theorem 3.3, Section 3), a stability criterion for the index 2 case. Section 4 contains the detailed proofs.

2 A tentative conjecture and counterexamples

The good experience with regular ODEs and index-1 DAEs of the form (1.1), which corresponds to matrix pencils $\{A, B\}$ of index zero or one, gives rise to the *tentative conjecture* that the origin is an asymptotically stable stationary point if $\sigma\{A, B\} \subset \mathbb{C}^-$. We know this to become true for $\operatorname{ind}\{A, B\} \leq 1$. Thereby, we have $\ker \Pi = \ker A$.

If the nonlinearity in (1.1) disappears, i.e., $h(y, x, t) \equiv 0$, the above conjecture also holds true. The projector Π projects along the infinite eigenspace of the matrix pencil $\{A, B\}$. As shown in [8], if $\operatorname{ind}\{A, B\} = k$, the projector Π can be constructed by a special matrix chain as $\Pi = P_0 P_1 \cdots P_{k-1}$, where $A_0 := A$, $B_0 := B$, $A_{i+1} := A_i + B_i(I - P_i)$, $P_i \in L(\mathbb{R}^m)$ projects along $\ker A_i$, $B_{i+1} := B_i P_i$, $i \geq 1$. Then, equation

$$Ax'(t) + Bx(t) = 0$$

can be reduced to

$$\begin{gathered} P_0 P_1 \cdots P_{k-1} x'(t) + P_0 P_1 \cdots P_{k-1} A_k^{-1} B x(t) = 0, \\ (I - P_0 \cdots P_{k-1}) x(t) = 0, \end{gathered}$$

while $\sigma\{A, B\}$ consist of exactly those eigenvalues of $P_0 P_1 \cdots P_{k-1} A_k^{-1} B$ whose associated eigenspaces belong to $\operatorname{im} P_0 \cdots P_{k-1}$.
Unfortunately, our conjecture is wrong if there are nonlinearities in (1.1), even in the case of $\operatorname{ind}\{A, B\} = 2$.

Example 1 Given the autonomous system

$$\left.\begin{aligned} x_1' - x_2 &= 0, \\ x_1 - x_2^3 &= 0, \\ x_3' - \alpha x_3 &= 0, \\ x_4 - x_2 + x_3 &= 0, \end{aligned}\right\} \tag{2.1}$$

which can be rewritten in compact form (1.1) by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}, \quad h(y, x, t) = \begin{bmatrix} 0 \\ -x_2^3 \\ 0 \\ 0 \end{bmatrix}.$$

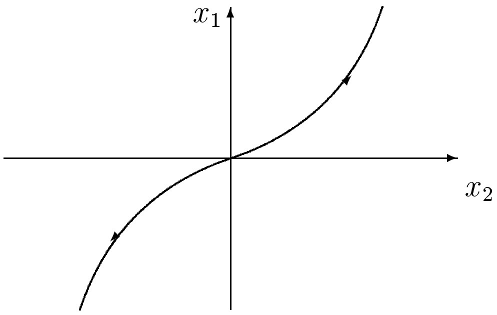
$\alpha \in \mathbb{R}$ is a parameter.
This special function h satisfies all conditions we agreed upon in Section 1. Choosing $P = \operatorname{diag}\{1, 0, 1, 0\}$ we consider

$$A_1 := A + B(I - P) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

Since A_1 is singular, we know that $\operatorname{ind}\{A,B\} > 1$. Further, we realize that

$$\{z \in I\!R^4 : z \in \ker A_1, BPz \in \operatorname{im} A_1\} = \{0\}.$$

Consequently (cf. [9]), the matrix pencil $\{A,B\}$ has index 2. Furthermore, $p(\lambda) = \det(\lambda A + B) = \lambda - \alpha$, thus $\sigma\{A,B\} = \{\alpha\}$. For $\alpha < 0$, our conjecture promises asymptotical stability for the origin. However, taking a look at the flow-picture in the (x_1,x_2)-plan we can realize immediately that the solutions move away from the origin. Hence, our conjecture is wrong.



□

As we shall see below, the problem with Example 1 is that linearization does not work in this case. The DAE (2.1) does not represent an index-2 DAE although we have $\operatorname{ind}\{A,B\} = 2$. System (2.1) is rather a singular index-1 DAE having a singularity at $x_2 = 0$. In Section 3 below we shall formulate convenient *structural conditions* that enable linearization and exclude this kind of singularities.
Our next example makes clear that even if linearization works and $\operatorname{ind}\{A,B\} = 2$, additional smoothness and boundedness conditions for h have to be satisfied.

Example 2 Given the DAE

$$\left.\begin{aligned} x_2' + x_1 &= 0,\\ x_2 + q(t)x_3^2 &= 0,\\ x_3' - \alpha x_3 &= 0, \end{aligned}\right\} \tag{2.2}$$

which can be described in terms of (1.1) as

$$A = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -\alpha \end{pmatrix}, \quad h(y,x,t) = \begin{pmatrix} 0\\ q(t)x_3^2\\ 0 \end{pmatrix}.$$

In (2.2), $\alpha < 0$ indicates a parameter and $q(t)$ is a continuous, uniformly bounded, scalar function. Again, all conditions for h given in Section 1 are fulfilled. Moreover, we derive that $\operatorname{ind}\{A,B\} = 2$, $\sigma\{A,B\} = \{\alpha\}$.
On the other hand, the last two equations of (2.2) yield

$$\begin{aligned} x_3(t) &= e^{\alpha t}x_3(0),\\ x_2(t) &= -q(t)e^{2\alpha}x_3(0)^2. \end{aligned}$$

Considering the first equation from (2.2), i.e. $x_1 = -x_2'$, we learn that continuity of the function $q(\cdot)$ is not adequate for this problem. To obtain just a continuous solution component x_1 we have to demand that $q(\cdot)$ is C^1. Then we derive

$$x_1(t) = (q'(t) + 2\alpha q(t))e^{2\alpha t}x_3(0)^2. \tag{2.3}$$

An appropriate projector to state the initial condition (1.10) is $\Pi = \mathrm{diag}\{0,0,1\}$. Obviously, $\Pi = P = \mathrm{diag}\{0,1,1\}$ would not do in this case.
As far as the asymptotical behaviour of $t \to \infty$ is concerned, there are no problems with the solution components $x_2(\cdot)$, $x_3(\cdot)$. However, to make sure that $\sigma\{A,B\} \subseteq \mathbb{C}^-$ yields also $x_1(t) \to 0$ $(t \to \infty)$, we have to demand the uniform boundedness of $q'(t)$ additionally. In terms of the function h, our additional assumptions mean that h has continuous partial derivatives h_t', h_{tx}'', too. Further, the relation

$$h_t'(0,0,t) = 0, \qquad t \in [0,\infty) \tag{2.4}$$

as well as the inequality

$$|h_{tx}''(0,x,t)| \leq c|x| \quad \text{for small} \ \ |x|$$

with a certain constant c are given. Considering this additional regularity and smallness of the function h, now $\sigma\{A,B\} = \{\alpha\} \subset \mathbb{C}^-$ implies the zero-solution to be asymptotically stable. This simple fact will be confirmed once more by Theorem 3.3 below. □

We know the above conjecture to be somewhat coarse. To improve it, one has to add

– structural conditions that guarantee linearization to work, but also

– more regularity and smallness conditions for the nonlinearity h.

3 A positive result for the case $\mathrm{ind}\{A,B\} = 2$

In this part we study equation (1.1) with an index-2 matrix pencil $\{A,B\}$. For that case, we shall verify our improved conjecture (cf. Section 2) and give precise formulations of all additional assumptions needed, respectively.
For more clarity, we recall the standard assumptions used in Section 1 and indicate them as (A).

Assumption (A):

(i) $h : \mathcal{D} \times [0,\infty) \to \mathbb{R}^m$, $\mathcal{D} \subseteq \mathbb{R}^m \times \mathbb{R}^m$ open,

$$0 \in \mathcal{D}, \quad h(0,0,t) = 0 \quad \text{for} \quad t \in [0,\infty),$$

h is continuous together with its partial Jacobians $h_{x'}'$, h_x'.

(ii) To each small $\varepsilon > 0$, a $\delta(\varepsilon) > 0$ can be found such that $|x| \le \delta(\varepsilon)$, $|x'| \le \delta(\varepsilon)$ yield

$$h'_{x'}(x',x,t)| \le \varepsilon, \quad |h'_x(x',x,t)| \le \varepsilon$$

uniformly for all $t \in [0,\infty)$.

(iii) $\{A,B\}$ is a regular matrix pencil.

(iv) $N := \ker A \subseteq \ker h'_{x'}(x',x,t)$ for all $(x',x,t) \in \mathcal{D} \times [0,\infty)$.

Further, let us formulate certain additional smoothness and smallness conditions as suggested by Example 2 in §2.

Assumption (B):

(i) The function $\tilde{h}$ defined by $\tilde{h}(x,t) := (I - AA^+)h(0,x,t)$ has also continuous partial derivatives $\tilde{h}'_t$, $\tilde{h}''_{tx}$, $\tilde{h}''_{xx}$.

(ii) $\tilde{h}'_t(0,t) = 0$ for all $t \in [0,\infty)$.

(iii) A constant κ can be found so that, with $\delta(\varepsilon)$ from (A), $|x| \le \delta(\varepsilon)$ yields

$$|\tilde{h}''_{xt}(x,t)| \le \kappa\varepsilon \quad \text{for all} \quad t \in [0,\infty).$$

(iv) $\tilde{h}''_{xx}(x,t)$ is uniformly bounded by a constant $\bar{\kappa} \ge 0$.

For the special DAE (2.2) it holds that $AA^+ = \mathrm{diag}\{1,0,1\}$, $\tilde{h}(x,t) = (0, q(t)x_3^2, 0)^T$. If the function $q(\cdot)$ and its derivative $q'(t)$ are uniformly bounded as discussed in Example 2 (§2), then this special $\tilde{h}$ fulfils (B).

The following matrices and subspaces will be used below (cf. [9]):

$N := \ker A, \quad S := \{z \in \mathbb{R}^m : Bz \in \mathrm{im}\, A\},$

$Q \in L(\mathbb{R}^m), \quad Q^2 = Q, \quad \mathrm{im}\, Q = N, \quad P := I - Q,$

$A_1 := A + BQ,$

$N_1 := \ker A_1, \quad S_1 := \{z \in \mathbb{R}^m : BPz \in \mathrm{im}\, A_1\},$

$Q_1 \in L(\mathbb{R}^m), \quad Q_1^2 = Q_1, \quad \mathrm{im}\, Q_1 = N_1, \quad P_1 := I - Q_1,$

$A_2 := A_1 + BPQ_1,$

$V \in L(\mathbb{R}^m), \quad V^2 = V, \quad \mathrm{im}\, V = N \cap S, \quad U := I - V.$

It is well-known that the pencil $\{A,B\}$ has index 2 if and only if A_2 is nonsingular but A, A_1 are singular (e.g. [9]). Moreover, if $\{A,B\}$ has index 2, we can use the decomposition $\mathbb{R}^m = N_1 \oplus S_1$. Hence, a convenient choice of the projector Q_1 is given by this decomposition. In the following we agree to have, more precisely, $\mathrm{im}\, Q_1 = N_1$, $\ker Q_1 = S_1$, and consequently (e.g. [3], [9]),

$$Q_1 = Q_1 A_2^{-1} BP = Q_1 A_2^{-1} B, \quad Q_1 Q = 0. \tag{3.1}$$

The relations (3.1) lead to

$$(PP_1)^2 = PP_1, \quad (PQ_1)^2 = PQ_1,$$
$$A_2^{-1}A = P_1P, \quad A_2^{-1}B = A_2^{-1}BPP_1 + Q_1 + Q.$$

Scaling equation (1.1) by A_2^{-1} yields

$$\begin{aligned}(PP_1x)'(t) - QQ_1(PQ_1x)'(t) + A_2^{-1}BPP_1x(t) + Q_1x(t) + Qx(t)\\ +A_2^{-1}h((PP_1x)'(t) + (PQ_1x)'(t),\ x(t), t) = 0.\end{aligned} \tag{3.2}$$

We decompose

$$x = Px + Qx = PP_1x + PQ_1x + Qx =: u + v + w,$$

and decouple (3.2) into the system

$$u'(t) + PP_1A_2^{-1}Bu(t) + PP_1A_2^{-1}h(u'(t) + v'(t), u(t) + v(t) + w(t), t) \ = \ 0, \tag{3.3}$$
$$v(t) + PQ_1A_2^{-1}h(u'(t) + v'(t), u(t) + v(t) + w(t), t) \ = \ 0, \tag{3.4}$$
$$\begin{aligned}-QQ_1v'(t) + w(t) + PQ_1A_2^{-1}Bu(t)\\ +QP_1A_2^{-1}h(u'(t) + v'(t), u(t) + v(t) + w(t), t) \ = \ 0.\end{aligned} \tag{3.5}$$

Because of $\operatorname{im} QQ_1 = N \cap S$ the latter equation (3.5) splits up further into

$$Uw(t) + UQA_2^{-1}Bu(t) + UQA_2^{-1}h(u'(t) + v'(t), u(t) + v(t) + w(t), t) \ = \ 0, \tag{3.6}$$
$$\begin{aligned}-QQ_1v'(t) + Vw(t) + VQP_1A_2^{-1}Bu(t)\\ +VQP_1A_2^{-1}h(u'(t) + v'(t), u(t) + v(t) + w(t), t) \ = \ 0.\end{aligned} \tag{3.7}$$

If the nonlinearity h disappears, equation (3.3) simplifies to a regular explicit linear ODE for $u(\cdot)$ that has the invariant subspace $\operatorname{im} PP_1$. (3.4) realizes $v(t) = 0$, hence it results that $x(t) = u(t) + w(t) = (I - PQ_1A_2^{-1}B)u(t) = (I - PQ_1A_2^{-1}BPP_1)PP_1u(t)$. Note that $\Pi_{\text{can}} := (I - PQ_1A_2^{-1}BPP_1)PP_1$ is also a projector. It holds that $\ker \Pi_{\text{can}} = \ker PP_1 = N \oplus N_1$. Further, $\operatorname{im} \Pi_{\text{can}}$ represents the finite eigenspace of the matrix pencil (cf. [8]), while the vectors Uw correspond to that part of the infinite eigenspace that has simple structure. The vectors Vw and v form the respective part for Jordan blocks of order 2.

In [10] we find the relation

$$\operatorname{im} A = \ker((PQ_1 + UQ)A_2^{-1}), \tag{3.8}$$

which will become very helpful to realize the appropriate structural conditions below.

Lemma 3.1 *Given* (A), $\operatorname{ind}\{A, B\} = 2$. *Additionally, let*

$$\operatorname{im} h'_{x'}(x', x, t) \subseteq \operatorname{im} A \quad \textit{for} \ \ (x', x, t) \in \mathcal{D} \times [0, \infty). \tag{3.9}$$

Then the identity

$$(PQ_1 + UQ)A_2^{-1}h(y, x, t) \equiv (PQ_1 + UQ)A_2^{-1}(I - AA^+)h(0, x, t) \tag{3.10}$$

is valid.

Proof: Due to (3.9) we have

$$(PQ_1 + UQ)A_2^{-1}(h(x,y,t) - h(0,x,t)) = \int_0^1 (PQ_1 + UQ)A_2^{-1}h'_{x'}(sy,x,t)y\mathrm{d}s = 0.$$

On the other hand, (3.8) implies

$$(PQ_1 + UQ)A_2^{-1} = (PQ_1 + UQ)A_2^{-1}(I - AA^+).$$

$\square$

Note that condition (3.9) further specifies the possible structure of (1.1). By Lemma 3.1, the equations (3.4) and (3.6) are much simpler now, namely

$$\begin{aligned} v(t) + PQ_1A_2^{-1}\tilde{h}(u(t) + v(t) + w(t), t) &= 0, \\ Uw(t) + UQA_2^{-1}Bu(t) + UQA_2^{-1}\tilde{h}(u(t) + v(t) + w(t), t) &= 0. \end{aligned}$$

We put them together compactly to

$$y(t) + UQA_2^{-1}BPP_1z(t) + (PQ_1 + UQ)A_2^{-1}\tilde{h}(y(t) + (PP_1 + VQ)z(t), t) = 0, \quad (3.11)$$

where

$$y := v + Uw, \qquad z := u + Vw. \quad (3.12)$$

Clearly, if $x(\cdot)$ satisfies the original DAE (1.1), then (3.11) is satisfied by $y(\cdot) = PQ_1x(\cdot) + UQx(\cdot)$ and $z(\cdot) = PP_1x(\cdot) + VQx(\cdot)$.
Equation (3.11) suggests to realize y as a function of z and t by applying the Implicit Function Theorem.

Lemma 3.2 *Let* (A), (B) *as well as* (3.9) *be given,* $\mathrm{ind}\{A, B\} = 2$. *Then, for sufficiently small* $\varepsilon > 0$ *and the corresponding* $\delta(\varepsilon) > 0$ *from* (A), *there is a uniquely determined function* $f : \mathcal{D}(\varepsilon) \times [0, \infty) \to \mathbb{R}^m$,
$\mathcal{D}(\varepsilon) := \left\{ z \in \mathbb{R}^m : (1 + 2|UQA_2^{-1}BP|)|(PP_1 + VQ)z| \le \frac{1}{2}\delta(\varepsilon) \right\}$
satisfying

(i) $f(z,t) + (PQ_1 + UQ)A_2^{-1}\tilde{h}(f(z,t) + (PP_1 + VQ)z, t) + UQA_2^{-1}BPP_1z = 0$, $z \in \mathcal{D}(\varepsilon)$, $t \in [0, \infty)$.

(ii) f *is continuous and has continuous partial derivatives* f'_z, f''_{zz}, f'_t, f''_{zt}.

(iii) *It holds that*

$$\begin{gathered} f(0,t) = 0, \quad f'_t(0,t) = 0, \quad f'_z(0,t) = -UQA_2^{-1}BPP_1, \\ (PQ_1 + UQ)f(z,t) = f(z,t) = f((PP_1 + VQ)z, t), \quad z \in \mathcal{D}(\varepsilon),\ t \in [0, \infty). \end{gathered}$$

For the proof we refer to Section 4 below.

Lemma 3.2 enables us to rewrite (3.11) locally equivalently as

$$y(t) = f(u(t) + Vw(t), t)$$

and, in more detail, as

$$\begin{aligned} v(t) &= Py(t) = Pf(u(t) + Vw(t), t), \\ Vw(t) &= Qy(t) = Qf(u(t) + Vw(t), t). \end{aligned} \tag{3.13}$$

Considering once more equation (3.5) we know that $v'(t)$ is needed. Our concept of C^1_N-solvability means in terms of the decomposition used that $u(\cdot) = PP_1x(\cdot)$, $v(\cdot) = PQ_1x(\cdot)$ are from C^1 and $w(\cdot) = Qx(\cdot)$ is just continuous. The idea is now to further specify the structure of (1.1) by supposing that

$$Pf(z,t) = Pf(PP_1z, t), \quad z \in \mathcal{D}(\varepsilon),\ t \in [0, \infty) \tag{3.14}$$

or, equivalently, $Pf'_z(z,t)VQ \equiv 0$. In other words, $f'_z(z,t)$ is forced to map $N \oplus N_1$ into N. By this we meet the natural smoothness of the solution, and we are allowed then to differentiate equation (3.13) with respect to t. We derive

$$v'(t) = Pf'_z(u(t), t)u'(t) + Pf'_t(u(t), t). \tag{3.15}$$

Now, expressions for $v(t)$, $v'(t)$, $Uw(t)$ in terms of $u(t)$, $u'(t)$, $Vw(t)$ are available. Inserting them into the equations (3.3) and (3.7), there results a system

$$\begin{aligned} u'(t) &= -PP_1A_2^{-1}Bu(t) + \varphi(u'(t), u(t), Vw(t), t), \\ Vw(t) &= \psi(u'(t), u(t), Vw(t), t) \end{aligned}$$

that could be transformed locally into a system that reads

$$u'(t) = -PP_1A_2^{-1}Bu(t) + g(u(t), t), \tag{3.16}$$

$$Vw(t) = k(u(t), t). \tag{3.17}$$

Together with

$$v(t) + Uw(t) = f(u(t) + Vw(t), t) \tag{3.18}$$

provided by Lemma 3.2, we arrive at a local decoupling of (1.1). If $x(\cdot, x^0, t_0)$ solves the initial value problem for (1.1) and the initial condition

$$PP_1x(t_0) = PP_1x^0, \quad x^0 \in \mathbb{R}^m, \tag{3.19}$$

with sufficiently small $|PP_1x^0|$, then $u(\cdot) = PP_1x(\cdot, x^0, t_0)$ satisfies the regular ODE (3.16), but also

$$u(t_0) = PP_1x^0.$$

Moreover, the components $v(\cdot) = PQ_1x(\cdot, x^0, t_0)$ and $w(\cdot) = Qx(\cdot, x^0, t_0)$ satisfy (3.17), (3.18). The resulting idea is to use such decouplings to construct all neighbouring solutions of the zero-solution.

Let us recall once more the structural conditions used for (1.1) and denote them by (C).

Assumption (C):

(i) $\operatorname{im} h'_{x'}(x', x, t) \subseteq \operatorname{im} A$, $(x', x, t) \in \mathcal{D} \times [0, \infty)$.

(ii) $f'_z(z, t)$ maps $N \oplus N_1$ into N for $z \in \mathcal{D}(\varepsilon)$, $t \in [0, \infty)$.

Now, we are ready to formulate our main result that sounds as clear as its classical model.

Theorem 3.3 *Given the Assumptions* (A), (B), (C) *and* $\operatorname{ind}\{A, B\} = 2$, $\sigma\{A, B\} \subset \mathbb{C}^-$. *Then the trivial solution of* (1.1) *is asymptotically stable in the sense of Lyapunov with* $\Pi := PP_1$.

The proof will be carried out in Section 4.

Remarks:

1) Theorem 3.3 generalizes the results for autonomous DAEs in [2]. This is not as trivial as one might think when coming from the regular ODE case.

2) The nullspace $\ker(PP_1) = N \oplus N_1$ is nothing else but the infinite eigenspace of the pencil $\{A, B\}$. Instead of PP_1 for stating the initial conditions we can use any matrix C that has the kernel $N \oplus N_1$.

3) Concerning condition (C), it is somewhat difficult to check its second part in practice. The function $y = f(z, t)$ is only implicitly given by the equation

$$y + UQA_2^{-1}BPP_1z + (PQ_1 + UQ)A_2^{-1}\tilde{h}(y + (PP_1 + VQ)z, t) = 0. \quad (3.20)$$

We close this section by providing sufficient criteria for (C)(ii) to be valid, which are given in terms of the original data of (1.1). For different index-2 DAEs those criteria are proposed in [4] and [10], respectively.

Lemma 3.4 *Let* (A), (B) *as well as* (C)(i) *be given,* $\operatorname{ind}\{A, B\} = 2$. *Then each of the following* 4 *conditions implies condition* (C)(ii) *to be satisfied:*

(i) $\tilde{h}(x, t) = \tilde{h}(Px, t)$ *for* $(0, x, t) \in \mathcal{D} \times [0, \infty)$.

(ii) $\tilde{h}(x, t) - \tilde{h}((PP_1 + PQ_1 + UQ)x, t) \in \operatorname{im} A$ *for* $(0, x, t) \in \mathcal{D} \times [0, \infty)$.

(iii) $\tilde{h}(x, t) - \tilde{h}(Px, t) \in \operatorname{im} A_1$, $(0, x, t) \in \mathcal{D} \times [0, \infty)$.

(iv) $\{z \in \mathbb{R}^m : Bz + h'_x(y, x, t)z \in \operatorname{im} A\} \cap N = \{z \in \mathbb{R}^m : Bz \in \operatorname{im} A\} \cap N$ *for* $(y, x, t) \in \mathcal{D} \times [0, \infty)$.

The proof is given in Section 4.

Although criterion (iv) related to certain subspaces looks somewhat strange, it seems to be very useful in practice, e.g. in circuit simulation ([11]).
Systems in Hessenberg form are often of special interest, that is

$$x_1' + B_{11}x_1 + B_{12}x_2 + g_1(x_1, x_2, t) \;=\; 0, \tag{3.21}$$
$$B_{21}x_1 + g_2(x_1, t) \;=\; 0. \tag{3.22}$$

System (3.21), (3.22) is a Hessenberg form equation of size 2 if $B_{21}B_{12}$ is assumed to be nonsingular. With $A = \mathrm{diag}(I, 0)$, $AA^+ = \mathrm{diag}(I, 0)$ we find

$$\tilde{h}(x,t) = \begin{pmatrix} 0 \\ g_2(x_1, t) \end{pmatrix},$$

further $N = \left\{ \binom{z_1}{z_2} \in I\!R^m : z_1 = 0 \right\}$ and thus $\tilde{h}(x,t) = \tilde{h}(Px, t)$.
Applying Lemma 3.4(i) we conclude the following assertion.

Corollary 3.5 *Condition* (C) *is valid in case of Hessenberg form equations* (1.1) *of size* 2.

4 Proofs

Proof of Lemma 3.2:

Let the conditions (A) and (B) be satisfied, further $\mathrm{ind}\{A, B\} = 2$. The structural condition (C)(i) (which is the same as relation (3.9)) is also assumed to be valid. Since A_2^{-1} is a constant nonsingular matrix, the smallness conditions (A)(ii), (B)(iii) apply also to $A_2^{-1}h$, $A_2^{-1}\tilde{h}$. Hence, there is a $\delta(\varepsilon) > 0$ to each small $\varepsilon > 0$ such that $|x| \le \delta(\varepsilon)$, $|y| \le \delta(\varepsilon)$, $t \in [0, \infty)$ yield

$$\begin{aligned} |A_2^{-1}h'_{x'}(y,x,t)| &\le \varepsilon, \\ |A_2^{-1}h'_x(y,x,t)| &\le \varepsilon, \\ |A_2^{-1}\tilde{h}''_{xt}(y,x,t)| &\le \kappa\varepsilon. \end{aligned} \tag{4.1}$$

Because of

$$\begin{aligned} A_2^{-1}h(y,x,t) &= A_2^{-1}\{h(y,x,t) - h(0,0,t)\} \\ &= \int_0^1 \{A_2^{-1}h'_{x'}(sy, sx, t)y + A_2^{-1}h'_x(sy, sx, t)x\}\mathrm{d}s \end{aligned}$$

and

$$A_2^{-1}\tilde{h}'_t(x,t) = A_2^{-1}\{\tilde{h}'_t(x,t) - \tilde{h}'_t(0,t)\} = \int_0^1 A_2^{-1}\tilde{h}''_{xt}(sx,t)x\mathrm{d}s,$$

from (4.1) it follows immediately that

$$|A_2^{-1}h(y,x,t)| \leq \varepsilon(|y|+|x|) \tag{4.2}$$

and

$$|A_2^{-1}\tilde{h}'_t(x,t)| \leq \kappa\varepsilon|x| \tag{4.3}$$

hold true for $|x| \leq \delta(\varepsilon)$, $|y| \leq \delta(\varepsilon)$, $t \in [0,\infty)$. Consider the function (cf. (3.11))

$$\begin{aligned} F(y,z,t) &:= -(PQ_1+UQ)A_2^{-1}\tilde{h}(y+(PP_1+VQ)z,t) - UQA_2^{-1}BPP_1z \\ &= -(PQ_1+UQ)A_2^{-1}h(0,y+(PP_1+VQ)z,t) - UQA_2^{-1}BP(PP_1+VQ)z \end{aligned}$$

mapping from $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ into $\mathbb{R}^m$. Denote $\kappa_1 := |PQ_1+UQ|$ and choose ε small enough to realize $2\kappa_1\varepsilon < 1$. Then, F is well-defined on $B(\varepsilon) \times \mathcal{D}_0(\varepsilon) \times [0,\infty)$, where

$$\begin{aligned} B(\varepsilon) &:= \Big\{y \in \mathbb{R}^m : |y| \leq \frac{1}{2}\delta(\varepsilon)\Big\}, \\ \mathcal{D}_0(\varepsilon) &:= \Big\{z \in \mathbb{R}^m : |PP_1z+VQz| \leq \frac{1}{2}\delta(\varepsilon)\Big\}. \end{aligned}$$

More precisely, for all $y, \bar{y} \in B(\varepsilon)$, $z \in \mathcal{D}_0(\varepsilon)$, $t \in [0,\infty)$, we have

$$\begin{aligned} &F(0,0,t) = 0, \\ &|F(y,z,t) - F(\bar{y},z,t)| \leq \kappa_1\varepsilon|y-\bar{y}|, \\ &|F(y,z,t)| \leq \kappa_1\varepsilon|y+(PP_1+VQ)z| + |UQA_2^{-1}BP|\,|PP_1z+VQz| \\ &\quad \leq \frac{1}{2}|y| + \frac{1}{2}(1+2|UQA_2^{-1}BP|)|PP_1z+VQz|. \end{aligned}$$

Form the set $\mathcal{D}(\varepsilon) \subseteq \mathcal{D}_0(\varepsilon)$,

$$\mathcal{D}(\varepsilon) := \Big\{z \in \mathbb{R}^m : (1+2|UQA_2^{-1}BP|)|PP_1z+VQz| \leq \frac{1}{2}\delta(\varepsilon)\Big\},$$

such that for each fixed $z \in \mathcal{D}(\varepsilon)$, $t \in [0,\infty)$, $F(\cdot,z,t)$ maps the closed ball $B(\varepsilon)$ into itself. Since $F(\cdot,z,t)$ is contractive with $\varepsilon\kappa_1 < \frac{1}{2}$ due to Banach's Fixed Point Theorem, there is a uniquely determined function

$$f : \mathcal{D}(\varepsilon) \times [0,\infty) \longrightarrow B(\varepsilon) \subseteq \mathbb{R}^m$$

such that, for all $z \in \mathcal{D}(\varepsilon)$, $t \in [0,\infty)$,

$$\begin{aligned} &f(z,t) \equiv F(f(z,t),z,t), \\ &f(0,t) = 0, \quad |f(z,t)| \leq \frac{1}{2}\delta(\varepsilon), \\ &(PQ_1+UQ)f(z,t) = f(z,t) = f((PP_1+VQ)z,t) \end{aligned} \tag{4.4}$$

hold true.
By the Implicit Function Theorem, the smoothness of F is passed on to the function

f. Since F is continuous together with its partial derivatives F'_y, F'_z, F'_t, F''_{yy}, F''_{yz}, F''_{yt}, F''_{zz}, F''_{zt} (cf. (A), (B)), the implicitly given function f is also continuous and possesses continuous partial derivatives f'_z, f'_t, f''_{zz}, f''_{zt}. Further with

$$F'_y(y,z,t) = -(PQ_1 + UQ)A_2^{-1}\tilde{h}'_x(y + (PP_1 + VQ)z, t)$$

we have

$$\begin{aligned} &F'_y(0,0,t) = 0, \\ &|F'_y(y,z,t)| \leq \kappa_1 \varepsilon, \end{aligned}$$

hence the matrix $I - F'_y(y,x,t)$ remains nonsingular uniformly for $y \in B(\varepsilon)$, $z \in \mathcal{D}(\varepsilon)$, $t \in [0,\infty)$. The relations

$$f'_z(0,t) = -UQA_2^{-1}BPP_1, \quad f'_t(0,t) = 0, \quad t \in [0,\infty)$$

are obtained immediately by differentiating (4.4) and considering

$$F'_z(0,0,t) = -UQA_2^{-1}BPP_1, \qquad F'_t(0,0,t) = 0.$$

□

Next we derive some further properties of the function f to be used in the proof of Theorem 3.3 below.

Corollary 4.1 *With $\kappa_3 := |P|\,|PP_1 + VQ + UQA_2^{-1}BPP_1|$, for all $z \in \mathcal{D}(\varepsilon)$, $t \in [0,\infty)$, it holds that*

$$|f'_t(z,t)| \leq \frac{\varepsilon\kappa_1}{1-\varepsilon\kappa_1}\,\kappa\delta(\varepsilon), \tag{4.5}$$

$$|Pf'_z(z,t)| \leq \frac{\varepsilon\kappa_1}{1-\varepsilon\kappa_1}\,\kappa_3, \tag{4.6}$$

$$f''_{zt}(0,t) = 0. \tag{4.7}$$

Moreover, there are constants κ_4, κ_5 such that $|f''_{zz}(z,t)| \leq \kappa_4$ and

$$|f''_{zt}(z,t)| \leq \varepsilon\kappa_5 \quad \textit{for} \quad z \in \mathcal{D}(\varepsilon),\ t \in [0,\infty). \tag{4.8}$$

Proof: First of all we have

$$|(I - F'_y(y,z,t))^{-1}| \leq \frac{1}{1-\varepsilon\kappa_1}$$

for all $y \in B(\varepsilon)$, $z \in \mathcal{D}(\varepsilon)$, $t \in [0,\infty)$. From $f'_t = (I - F'_y)^{-1}F'_t$ and (4.3) we conclude

$$|f'_t(z,t)| \leq \frac{\varepsilon\kappa_1}{1-\varepsilon\kappa_1}\,\kappa|f(z,t) + (PP_1 + VQ)z| \leq \frac{\varepsilon\kappa_1}{1-\varepsilon\kappa_1}\,\kappa\delta(\varepsilon).$$

From $f'_z = (I - F'_y)^{-1} F'_z = -(I - F'_y)^{-1}(PQ_1 + UQ)A_2^{-1}\tilde{h}'_x(PP_1 + VQ) - (I - F'_y)^{-1}UQA_2^{-1}BPP_1$ we obtain

$$\begin{aligned}
&|f'_z(z,t) + UQA_2^{-1}BPP_1| \leq \\
&\quad \leq \frac{1}{1-\varepsilon\kappa_1}|(PQ_1 + UQ)A_2^{-1}\tilde{h}'_x(f(z,t) + (PP_1 + VQ)z, t)\{PP_1 + VQ - UQA_2^{-1}BPP_1\}| \\
&\quad \leq \frac{\varepsilon\kappa_1}{1-\varepsilon\kappa_1}|PP_1 + VQ - UQA_2^{-1}BPP_1|,
\end{aligned}$$

hence,

$$\begin{aligned}
|Pf'_z(z,t)| &= |P(f'_z(z,t) + UQA_2^{-1}BPP_1)| \leq \\
&\leq |P|\frac{1}{1-\varepsilon\kappa_1}\varepsilon\kappa_1|PP_1 + VQ + UQA_2^{-1}BPP_1| \leq \frac{\varepsilon\kappa_1}{1-\varepsilon\kappa_1}\,\kappa_3.
\end{aligned}$$

Since $f'_z = (PQ_1 + UQ)f'_z = f'_z(PP_1 + VQ)$ and $(PP_1 + VQ)(PQ_1 + UQ) = 0$, we may express the second derivative f''_{zz} simply as

$$f''_{zz} = (I - F'_y)^{-1}F''_{zz}.$$

Due to (B)(iv), F''_{zz} is uniformly bounded, therefore f''_{zz} is so, too.
Finally, using the above arguments once more we find the expression

$$\begin{aligned}
f''_{zt} &= (I - F'_y)^{-1}\{F''_{yt}f'_z + F''_{zt}\} \\
&= -(I - F'_y)^{-1}(PQ_1 + UQ)A_2^{-1}\tilde{h}''_{xt}\{PP_1 + VQ - f'_z\}.
\end{aligned}$$

In particular, $\tilde{h}''_{xt}(0,t) = 0$ (cf. (4.1)) leads now to $f''_{zt}(0,t) = 0$. Moreover, we may estimate

$$\begin{aligned}
|f''_{zt}(z,t)| &\leq \frac{\kappa_1}{1-\varepsilon\kappa_1}\,\kappa\varepsilon(|PP_1 + VQ| + |f'_z(z,t)|) \\
&\leq \kappa_5 \cdot \varepsilon.
\end{aligned}$$

□

Let us stress once more that, if a C^1_N-function $x(\cdot)$ in the neighbourhood of the origin solves the DAE (1.1), then it satisfies also the identity

$$(PQ_1 + UQ)x(t) = f((PP_1 + VQ)x(t), t).$$

With the denotations $u := PP_1x$, $v := PQ_1x$, $w = Qx$ this reads

$$v(t) + Uw(t) = f(u(t) + Vw(t), t).$$

In particular, it holds that

$$v(t) = Pf(u(t) + Vw(t), t).$$

Now, the structural condition (C)(ii) casts the nullspace component out from the function Pf such that

$$v(t) = Pf(u(t), t) \tag{4.9}$$

results. Differentiating yields the expression

$$v'(t) = Pf_z'(u(t),t)u'(t) + Pf_t'(u(t),t). \tag{4.10}$$

Rewrite equation (1.1) scaled by A_2^{-1}, that is equation (3.2), as

$$\begin{aligned} &PP_1x'(t) + A_2^{-1}BPP_1x(t) + (I - PP_1)x(t) + QQ_1(PQ_1x)(t) \\ &\quad -QQ_1(PQ_1x)'(t) + A_2^{-1}h(PP_1x'(t) + (PQ_1(x)'(t), x(t), t) = 0. \end{aligned} \tag{4.11}$$

This formulation suggests to replace the terms PQ_1x, $(PQ_1x)'$ by means of (4.9) and (4.10), respectively. Then we arrive at the DAE

$$PP_1x'(t) + A_2^{-1}BPP_1x(t) + (I - PP_1)x(t) + H(x'(t), x(t), t) = 0, \tag{4.12}$$

where the nonlinearity H is introduced as the following map from $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ into $\mathbb{R}^m$:

$$\begin{aligned} H(x', x, t) \;:=\; & A_2^{-1}h(PP_1x' + Pf_z'(x,t)PP_1x' + Pf_t'(x,t), x, t) \\ & -QQ_1\{Pf_z'(x,t)PP_1x' + Pf_t'(x,t) - Pf(x,t)\}. \end{aligned}$$

Recall that $Pf(x,t) = Pf(PP_1x, t)$ due to (C)(ii). For $t \in [0, \infty)$, $|PP_1x'| \le \frac{1}{2}\delta(\varepsilon)$, $PP_1x \in \mathcal{D}(\varepsilon)$ the inequalities (4.5), (4.6) yield the estimation

$$\begin{aligned} &|PP_1x' + Pf_z'(x,t)PP_1x' + Pf_t'(x,t)| \le \\ &\qquad \le \frac{1}{2}\delta(\varepsilon) + \frac{\varepsilon\kappa_1}{1 - \varepsilon\kappa_1}(\kappa_3 + 2|P|\kappa)\,\frac{1}{2}\delta(\varepsilon) \le \delta(\varepsilon), \end{aligned}$$

supposed ε is chosen small enough to realize

$$\frac{\varepsilon\kappa_1}{1 - \varepsilon\kappa_1}\,(\kappa_3 + 2|P|\kappa) \le 1, \quad \varepsilon\kappa_1 < \frac{1}{2}. \tag{4.13}$$

By this, the function H is well-defined for $|PP_1x'| \le \frac{1}{2}\delta(\varepsilon)$, $|x| \le \delta(\varepsilon)$, $PP_1x \in \mathcal{D}(\varepsilon)$, $t \in [0, \infty)$.

Lemma 4.2 *Given a regular index-2 matrix pencil $\{A, B\}$, $\tilde{A} := PP_1$, $\tilde{B} := A_2^{-1}BPP_1 + (I - PP_1)$. Then, $\{\tilde{A}, \tilde{B}\}$ is a regular pencil of index 1 the finite spectrum of which coincides with that of $\{A, B\}$, i.e.,*

$$\sigma\{\tilde{A}, \tilde{B}\} = \sigma\{A, B\}.$$

Proof: Obviously, $\tilde{A}z = 0$, $\tilde{B}z \in \operatorname{im} \tilde{A}$ imply $z = 0$, i.e., $\{\tilde{A}, \tilde{B}\}$ is regular and $\operatorname{ind}\{\tilde{A}, \tilde{B}\} = 1$. Consider $(\lambda A + B)z = 0$, or equivalently,

$$\begin{aligned} (\lambda A_2^{-1}A + A_2^{-1}B)z = 0, \quad \text{i.e.,} \\ (\lambda(PP_1 - QQ_1) + A_2^{-1}BPP_1 + Q_1 + Q)z = 0. \end{aligned} \tag{4.14}$$

Multiplying by PQ_1 gives $PQ_1z = 0$, further $QQ_1z = QQ_1PQ_1z = 0$. Thus (4.14) simplifies to

$$\lambda PP_1z + A_2^{-1}BPP_1z + Qz = 0.$$

On the other hand, starting with

$$(\lambda\tilde{A} + \tilde{B})z = 0, \quad \text{i.e.,}$$
$$\lambda PP_1z + A_2^{-1}BPP_1z + PQ_1z + Qz = 0,$$

we also find that $PQ_1z = 0$ has to be true. □

At this place let us mention that the eigenvectors associated to finite eigenvalues have the property $z = PP_1z + Qz = (I - QP_1A_2^{-1}BP)PP_1z$, and $\operatorname{im}(I - QP_1A_2^{-1}BP)PP_1$ represents the finite eigenspace that has dimension $\dim \operatorname{im}PP_1 = m - \dim N - \dim N \cap S$.

Proof of Theorem 3.3:
As we have shown above, each small C_N^1-solution of (1.1) satisfies (4.12). On the contrary, if $x(\cdot)$ is a C_N^1-solution of (4.12), by multiplying by $(PQ_1 + UQ)$ we find the relation

$$(PQ_1 + UQ)x(t) + (PQ_1 + UQ)A_2^{-1}\tilde{h}(x(t),t) + UQA_2^{-1}BPP_1x(t) = 0$$

to be satisfied, hence

$$(PQ_1 + UQ)x(t) = f((PP_1 + UQ)x(t),t)$$

and due to the structural condition (C)(ii)

$$\begin{aligned} PQ_1x(t) &= Pf(PP_1x(t),t), \\ (PQ_1x)'(t) &= Pf'_z(PP_1x(t),t)(PP_1x)'(t) + Pf'_t(PP_1x(t),t). \end{aligned} \tag{4.15}$$

Then, we have

$$\begin{aligned} H(x'(t),x(t),t) &= A_2^{-1}h(PP_1x'(t) + PQ_1x'(t),x(t),t) - QQ_1\{PQ_1x'(t) - PQ_1x(t)\} \\ &= A_2^{-1}h(x'(t),x(t),t) - QQ_1x'(t) + QQ_1x(t). \end{aligned}$$

Inserting this expression into (4.12) we obtain (4.11), i.e., each small C_N^1 solution of (4.12) satisfies the original DAE (1.1).
The function H has a priori the property $N \oplus N_1 = \ker PP_1 \subseteq \ker H'_{x'}(x',x,t)$ such that the identity

$$H(x',x,t) \equiv H(PP_1x',x,t)$$

is valid. Naturally, the solutions of (4.12) belong to the class

$$C^1_{N\oplus N_1} := \{x \in C : PP_1x \in C^1\},$$

which is larger than $C_N^1 = \{x \in C : PP_1 x \in C^1, PQ_1 x \in C^1\}$. From this point of view, the solutions of (1.1) seem to be smoother than those of (4.12). However, the representation (4.15) shows clearly that due to the smoothness of f, which results from (A), (B), (C), each solutions of (4.12) has a continuously differentiable component $PQ_1 x(t)$, too, i.e. each solution of (4.12) has C_N^1-regularity. Consequently, (1.1) and (4.12) are equivalent.
Now we show that the respective stability result for the index-1 case ([7], Lemma 4.1) applies to equation (4.12).
By construction, it holds that

$$H(0,0,t) = 0, \qquad t \in [0,\infty).$$

H is continuous with continuous derivatives

$$H'_{x'} = A_2^{-1} h'_{x'} \{PP_1 + Pf'_z PP_1\} - QQ_1 Pf'_z PP_1 \tag{4.16}$$

and

$$\begin{aligned} H'_x \;=\; & A_2^{-1} h'_{x'} \{Pf''_{zz} PP_1 x' PP_1 + Pf''_{tz} PP_1\} + A_2^{-1} h'_x \\ & - QQ_1 \{Pf''_{zz} PP_1 x' PP_1 + Pf''_{tz} PP_1 - Pf'_z PP_1\}. \end{aligned} \tag{4.17}$$

It remains to show that the nonlinearity H is small in the sense of (A)(ii).
In the following, if we apply the inequalities (4.1), we choose $\delta(\varepsilon)$ small enough such that $\delta(\varepsilon) \le \varepsilon$ becomes true. For $\varepsilon > 0$ satisfying (4.13) and $|PP_1 x'| \le \frac{1}{2}\delta(\varepsilon)$, $|x| \le \delta(\varepsilon)$, $PP_1 x \in \mathcal{D}(\varepsilon)$, $t \in [0,\infty)$, we find

$$\begin{aligned} |H'_{x'}(x',x,t)| \;\le\; & \varepsilon\Big(|PP_1| + \kappa_3 \frac{\varepsilon\kappa_1}{1-\varepsilon\kappa_1}\Big) + |QQ_1| \frac{\varepsilon\kappa_1}{1-\varepsilon\kappa_1} \le c_1\varepsilon, \\ \le\; & \varepsilon(|PP_1| + \kappa_3 + 2\kappa_1|QQ_1|) = c_1\varepsilon, \\ |H'_x(x',x,t)| \;\le\; & \varepsilon + \varepsilon|P|\Big(\kappa_4 \cdot \frac{1}{2}\delta(\varepsilon) + \varepsilon\kappa_5\Big) + |QQ_1|\Big(\kappa_4 \frac{1}{2}\delta(\varepsilon) + \varepsilon\kappa_5\Big) + |QQ_1|\kappa_3 \frac{\varepsilon\kappa_1}{1-\varepsilon\kappa_1} \\ \le\; & \varepsilon\Big(1 + \varepsilon|P|\Big(\kappa_4 \cdot \frac{1}{2} + \kappa_5\Big) + |QQ_1|\Big(\kappa_4 \cdot \frac{1}{2} + \kappa_5\Big) + |QQ_1|\kappa_3 \cdot 2\Big) \\ \le\; & c_2\varepsilon. \end{aligned}$$

The condition $PP_1 x \in \mathcal{D}(\varepsilon)$ means $(1 + 2|UQP_1 A_2^{-1} BP|)|PP_1 x| \le \frac{1}{2}\delta(\varepsilon)$. If $PP_1 = 0$, it is satisfied trivially for all $x \in I\!R^m$. Denote $\delta^*(\varepsilon) := \frac{1}{2}\delta(\varepsilon)$ if $PP_1 = 0$, but otherwise

$$\delta^*(\varepsilon) := \min\{1, (1 + 2|UQP_1 A_2^{-1} BP)^{-1} |PP_1|^{-1}\} \cdot \frac{1}{2}\delta(\varepsilon).$$

Then, with $c := \max\{c_1, c_2\}$, the inequalities

$$|H'_{x'}(x',x,t)| \le c\,\varepsilon, \qquad |H'_x(x',x,t)| \le c\,\varepsilon$$

are fulfilled for all

$$|PP_1 x'| \le \delta^*(\varepsilon), \qquad |x| \le \delta^*(\varepsilon), \quad t \in [0,\infty).$$

Now, [8], Lemma 4.1 applies to (4.12). Since $\sigma\{\tilde{A},\tilde{B}\} \subseteq \mathbb{C}^-$ by Lemma 4.2 above, the trivial solution of (4.12) is asymptotically stable. Thereby, we can put $\Pi = PP_1$. In any case, Π has to have the nullspace $\ker \Pi = N \oplus N_1$, which represents the infinite eigenspace of the index-1 pencil $\{\tilde{A},\tilde{B}\}$ as well as the index-2 pencil $\{A, B\}$. $\square$

Proof of Lemma 3.4:

(i) Recall that for the projectors $V \in L(\mathbb{R}^m)$ onto $N \cap S = \operatorname{im} QQ_1$ we have $QV = V$, $VQ = QVQ$. The property $\tilde{h}(x,t) \equiv \tilde{h}(Px,t)$ simplifies equation (3.20) to

$$y = -(PQ_1 + UQ)A_2^{-1}\tilde{h}(Py + PP_1 z, t) - UQA_2^{-1}BPP_1 z,$$

that means, the function $y = f(z,t)$ implicitly given by (3.20) depends on $PP_1 z$ only, i.e.

$$f(z,t) \equiv f(PP_1 z, t),$$

thus $Pf(z,t) \equiv Pf(PP_1 z, t)$.

(ii) Since $\tilde{h}(x,t) - \tilde{h}((I - VQ)x, t)$ belongs to im A we may use the identity

$$(PQ_1 + UQ)A_2^{-1}\tilde{h}(x,t) \equiv (PQ_1 + UQ)A_2^{-1}\tilde{h}((PP_1 + PQ_1 + UQ)x, t).$$

Now, (3.20) reads

$$y = -(PQ_1 + UQ)A_2^{-1}\tilde{h}((PP_1 + PQ_1 + UQ)y + PP_1 z, t) - UQA_2^{-1}BPP_1 z.$$

Again it results that $f(z,t) \equiv f(PP_1 z, t)$ holds a priori.

(iii) $\tilde{h}(x,t) - \tilde{h}(Px,t) \in$ im A_1 yields $PQ_1 A_2^{-1}\tilde{h}(x,t) \equiv PQ_1 A_2^{-1}\tilde{h}(Px,t)$, hence $PQ_1 A_2^{-1}\tilde{h}'_x \equiv PQ_1 A_2^{-1}\tilde{h}'_x P$.

Multiplying the equation for f'_z

$$f'_z = -(PQ_1 + UQ)A_2^{-1}\tilde{h}'_x\{f'_z + (PP_1 + VQ)\}$$

by P we obtain

$$Pf'_z = -PQ_1 A_2^{-1}\tilde{h}'_x\{f'_z + (PP_1 + VQ)\} = -PQ_1 A_2^{-1}\tilde{h}'_x\{Pf'_z + PP_1\}.$$

Consequently, we have

$$Pf'_z(z,t) \equiv Pf'_z(z,t)PP_1.$$

(iv) Denote $w := VQx$ and compute

$$h(x', x, t) - h(x', (I - VQ)x, t) = \int_0^1 h'_x(x', x - (1-s)w, t)\mathrm{d}sw.$$

Because of $w \in N \cap S$, $N \cap S = N \cap \{z \in \mathbb{R}^m : Bz + h'_x(x', x - (1-s)w, t)z \in \operatorname{im} A\}$ we know that also

$$Bw + h'_x(x', x - (1-s)w, t)w \in \operatorname{im} A, \quad s \in [0,1],$$

is fulfilled. Therefore

$$h(x', x, t) - h(x', (I - VQ)x, t) + BVQx \in \operatorname{im} A,$$

hence

$$(PQ_1+UQ)A_2^{-1}h(x', x, t) = (PQ_1+UQ)A_2^{-1}h(x', (I-VQ)x, t) + (PQ_1+UQ)A_2^{-1}BVQx.$$

Because of $(PQ_1 + UQ)A_2^{-1}BVQ = 0$ we may conclude

$$(PQ_1 + UQ)A_2^{-1}\tilde{h}(x, t) = (PQ_1 + UQ)A_2^{-1}\tilde{h}((I - VQ)x, t)$$

and finally use the same arguments as we did for (ii). □

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