# Global Weak Solutions and Uniqueness for a Moving Boundary Problem for a Coupled System of Quasilinear Diffusion-Reaction Equations arising as a Model of Chemical Corrosion of Concrete Surfaces
(Part 3)

by

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## Abstract

We show existence and uniqueness for global weak solutions of a moving boundary problem for a coupled system of three quasi-linear diffusion-reaction equations. The model is briefly described. The proofs are based on Schauder's and Banach's fixed point theorems, the one-dimensional setting and they make use of relatively general and realistic assumptions on the production terms providing bounds on the weak solutions of the problem. The paper extends previously known results with constant coefficients to a quasi-linear setting.

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## 1. Introduction

### 1.1 Preliminaries

In this note we show existence and uniqueness for global weak solutions of a moving boundary problem for a coupled system of quasi-linear diffusion-reaction equations.

The model describes the advancement of the corrosion front in the concrete walls of sewer pipes, where, due to the reaction with sulfate generated from hydrogen sulfide arising from the sewage, calcium carbonate parts of the concrete wall are transformed into gypsum.

This transformation leads to density changes of the wall, subsequently increased stresses in the porous matrix and several other destabilizing effects. Here we concentrate on the non-mechanical chemical phase-change and neglect the density change.

Later on, the present model will be incorporated into a mechanical one. This is the main motivation for considering weak solutions in this paper.

The general feature of the model is: Hydrogen sulfide enters from the inside the part of the pipe, which is already corroded (i.e. the gypsum part), diffuses through the water-filled and the air-filled parts of the gypsum, reacts in the water-filled pores to sulfate and moves to the corrosion front, which is the (idealized) interface between corroded part (porous gypsum) and uncorroded part of the wall (also: cf. rem. (1.3.1 and 2)).

We imagine the cross section of the pipe as a circular ring with the outer radius R and initial (i.e. before corrosion) thickness d, draw a horizontal line through the center of the (cross section of the) pipe, position on this line an x-coordinate axis with its origin coinciding with the position of the inner boundary of the pipe wall before the onset of corrosion and pointing to the right. Thus the x-axis is at approximately the same level as the average sewage surface. Corrosion is considered in the direction of the $x$-axis, i.e., to the right. The position of the corrosion front is denoted by $s(t)$ with $t \geq 0$ denoting time. For more on this and on related models cf. [BoDJR], [De], [BoDR], [BoR].

The main result is theorem 2 (cf. section 3).

Before formulating the problem we make some general remarks on diffusion-reaction equations for porous media in the following section.

### 1.2 Some generalities

The underlying model is the diffusion-reaction equation specified for a porous medium, which can be written in a variety of ways: To be more specific, consider the diffusive-reactive flow in a porous domain $\Omega$ having porosity $\kappa = \kappa(x,t)$, tortuosity $\delta = \delta(x,t)$, source- and sink rates $f_i$, diffusion coefficient $E$, and define a concentration $u$ such that

$$\left.\begin{array}{l}\forall \text{ measurable (and physically reasonable) subdomains } \Omega' \subseteq \Omega: \\ \int_{\Omega'} \kappa u dx = \text{ (mass- or molar) content of the diffusive species in } \Omega\end{array}\right\} \tag{1}$$

Formulating FICK's (mass- or molar) flux relation as

$$j = -\delta E \nabla(\kappa u) \ , \tag{2}$$

or, alternatively, as

$$j = -\delta \kappa E \nabla u \ , \tag{3}$$

conservation of mass yields either

$$\frac{\partial(\kappa u)}{\partial t} - div(\delta E \nabla(\kappa u)) = f_1 \tag{4}$$

or

$$\frac{\partial(\kappa u)}{\partial t} - div(\delta \kappa E \nabla u) = f_2 \tag{5}$$

and the reaction rate(-s) $f_i$ are, in either case, of the form

$$f_i = f_i(\kappa, u, v) \tag{6}$$

with the dead variable $v$ standing for other reactants. We are dealing with a simple reaction, for which the corresponding $f$'s will be a product of powers of the porosity and of the participating concentrations.

In this note we will employ (2), (4). Substitution $w := \kappa u$ yields

$$\frac{\partial w}{\partial t} - div(\delta E \nabla w) = f. \tag{7}$$

Note that the porosity $\kappa$ (formally) disappears and $w = \kappa u$ (not $u$) is the physically relevant density of the dissolute mass.

At this point we would like to point out that the literature dealing with concise derivations of diffusion-reaction equations for porous media dealing with space dependent porosities and tortuosities seems to be very rare. The case of constant $\delta$ and $\kappa$ is dealt with in an abundance of papers. Homogenization results such as in [HoJ] deal mainly with constant effective diffusion coefficients or a diffusion law like (3) is *assumed.* Averaging techniques as in [BeBa], [BeC] e.g., do not yield more insight. On the other hand, the situation is similar to the one for *flow* in porous media and the question how Darcy's law (for non-constant coefficients) seems, in general, not quite clear (cf. the discussion in [Sch], e.g.), although the equivalent of (3) seems to be preferred.

Moreover, the difference between (2) and (3) is not merely formal – at least, if one has relations like $\kappa = \kappa(u(x,t))$ in mind.

### 1.3 The model

The resulting model is the following coupled system of diffusion-reaction equations with a moving boundary

$$\frac{\partial v_i}{\partial t} - (\overline{A}_i v_{ix})_x = \overline{f}_i \quad 0 < x < s(t),\ t > 0,\ i = 1, 2, 3, \tag{1}$$
$$v_i(x, 0) = v_{0i}(x) \quad 0 < x < s_0\ , \tag{2}$$
$$v_i(0, t) = \lambda_i \quad t > 0\ , \tag{3}$$
$$-\overline{A}_i v_{ix} = \overline{g}_i \text{ at } x = s(t),\ t > 0\ , \tag{4}$$
$$s'(t) = \overline{L}_2 v_3^m(s(t), t) \quad t > 0,\ m = \text{const.} \geq 1\ , \tag{5}$$
$$s(0) = s_0 > 0\ , \tag{6}$$

where

$$\overline{A}_i = \overline{A}_i(x, s(t), v(x, t)),\ v = (v_1, v_2, v_3),\ v = v(x, t), \tag{7}$$
$$\overline{B}_i = \overline{B}_i(x, s, v(x, t)),\ \overline{\kappa}_i = \overline{\kappa}_i(x, s, v(x, t)), \tag{8}$$
$$\overline{f}_i = \overline{f}_i(x, s, v(x, t)),\ \overline{f}_i := \begin{cases} \overline{K}, (v_2 - \overline{B}_1 v_1) & i = 1, \\ \overline{K}_2(B_2 v_1 - v_2) - \overline{K}_3 v_2 & i = 2, \\ \overline{K}_3 v_2 & i = 3, \end{cases} \tag{9}$$
$$\overline{L}_2 = \overline{L}_2(s(t), v(x, t))\ , \tag{10}$$
$$\overline{A}_{ir} = \overline{A}_{ir}(s(t), v(x, t))\ , \tag{11}$$
$$\begin{aligned} \overline{g}_i &= \overline{g}_i(s(t), \overline{s}(t), v(x, t), v^*(t)),\ \overline{g}_i(s, s, v, v^*) \\ &:= \begin{cases} \overline{A}_{ir}(s, v) \left\{ \frac{v_i - v^*}{\overline{s} - d} \right\} & i = 1, 2 \\ \overline{L}_2(s, v) v_3^{m+1} + & i = 3 \end{cases} \end{aligned} \tag{12}$$

**Some remarks.**

1. (9) models the exchange between water- and air-filled parts of the corroded part (gypsum, e.g.) and the reaction of hydrogen sulfide to sulfate ([De], [BoDJR],[BoDR]). For the sake of expositional simplicity, the position of the inner boundary of the corrosion product remains fixed at its initial position (= 0). The corresponding modifications taking the lower density (and therefore the larger volume) of the corrosion product into account has been dealt with in [BoR]. $s(0) > 0$ means that at the beginning of the process we are considering, there is already some corrosion product. The case $s(0) = 0$ is more of mathematical than of practical interest, since the very beginning of the actual corrosion is likely to require a completely different model. One-dimensional mbp's for problems which are related to ours and which are dealing with a *single* equation have been addressed in [AnR], e.g. Mathematically, the case $s(0) = 0$ leads to some sort of a degeneracy.

2. Although the chemical reaction transforming the sulfur ions (as part of hydrogen sulfide) to sulfate is a simple first order reaction, for which engineers sometimes use $m \approx 0.5$. For the mathematics of the problem this is essential since it destroys local Lipschitz-continuity of the term on the right hand side of (1.3.5).

3. Local solutions for very general one-dimensional quasilinear *single* parabolic equations are obtained in [FaP]. [FrRZ] consider a mbp for three weakly coupled semilinear parabolic equations in a context which is vaguely related to ours. [Am] is likely to yield a wealth of (time-wise) local solutions for our problem, although one would have to do similar work as in this note and in [BoR] to obtain estimates and *global* weak solutions. At the practical level, [Ri] is conceptually related to this paper. He considers a *single* semi-linear parabolic mbp for diffusion through an ash layer. The methods in [Ri] and in [FrRZ] yield classical solutions and they are not applicable to quasilinear problems. The problem of polymer swelling, which is formally related to the expansion of the corrosion product in this paper, is considered in several articles ([FaMP],[AnR],[CoRT], e.g.).

## 2. Notations. Technicalities

$L^p(G)$, $W^{s,p}(G)$, $H^s(G)$, $C^{0,\alpha}(G)$ and $C^{0,\alpha-}(G) := \bigcap_{0<\beta<\alpha} C^{0,\beta}(G)$ are the usual Lebesgue-, Sobolev- and Hölder spaces with the norms $|\cdot|_p$, $\|\cdot\|_{s,p}$, $\|\cdot\|_s$ and $|\cdot|_{0,\alpha}$, resp. $V := \{v = (v_1, v_2, v_3) : v_i \in H^1(0,1) \text{ and } v_i(0) = 0\}$ and $H := L^2(0,1)^3$ are normed by $\|v\| := \left(\sum_{i=1}^{3} |v_{iy}|^2_{L^2(0,1)}\right)^{1/2}$ and $|v| := \left(\sum_{i=1}^{3} |v_i|^2_{L^2(0,1)}\right)^{1/2}$, resp.
Usually we won't distinguish between the norm in $H$ and in $L^2(0,1)$. For appropriate functions $w = w(y, \ldots)$ and $v(t) = v(t, \ldots)$ and we write $w_y := \frac{\partial w}{\partial y}$ and $v' := \frac{\partial v}{\partial t}$, resp. Let $S = (0,T)$ be a (time-)interval, $X$ a normed space. $[M \to N]$ stands for the set of all maps from the set $M$ into the set $N$. $L^p(S;X)$, $C^{k,\mu}(S;X)$ and $W^{s,p}(S;X)$ are the usual spaces of $B$-measurable functions $\in [S \to X]$, of $k$ times $F$-differentiable functions $\in [\overline{S} \to X]$ having $\mu$-Hölder continuous derivatives and of Sobolev-functions $\in [S \to X]$ (with corresponding integrability assumptions on all (up to the $s$-th) derivative (if $s$ is an integer) and defined by interpolation, if $s$ is not an integer), resp., cf. [KuJF], [Ze], [Zi], [GGZ]).

For functions $w = w(x,t)$ we set $w(t) := w(\cdot, t)$, $w'(t) := \frac{\partial w}{\partial t}(\cdot, t)$, $w_x := \frac{\partial w}{\partial x}$.

References within the same section are made without prefix, references to other sections contain the number of that section. (3.4) means reference (4) in section 3, e.g. Usually, c stands for a nonnegative constant.

The following lemma collects some repeatedly used arguments. In some way, the first part with $\Theta = \frac{1}{2}$ is one of several essential gaps which the proofs go through.

**Lemma 1.** *Let $\varepsilon > 0$, $\Theta \in \left[\frac{1}{2}, 1\right)$.*

(i) *There are constants $\hat{c} = \hat{c}(\Theta)$, $c_\varepsilon > 0$ such that*

$$|v|_\infty \le \hat{c}|v|^{1-\Theta}\|v\|^{\Theta} \le \hat{c}(\varepsilon\|v\| + c_\varepsilon|v|) \quad \forall v \in V \quad .$$

(ii) *Let $s \in W^{1,1}(S), v, \varphi \in V, s'(t) \ge 0$ a.e. Then there is a constant c:*

$$\begin{aligned} s'(t)(yv_y, \varphi) &= s'(t)\{v(1)\varphi(1) - (yv, \varphi_y) - (v, \varphi)\} , \\ s'(t)(yv_y, v) &= \frac{1}{2}s'(t)\{v(1)^2 - |v|^2\} \le \frac{1}{2}|s'(t)|\{c_1|v|^{2(1-\Theta)}\|v\|^{2\Theta} - |v|^2\} . \end{aligned}$$

(iii) *Let $\hat{c}, \varepsilon, c_\varepsilon, \Theta$ be as above, $v \in V$, $s > 0$. Then*

$$\begin{aligned} \frac{1}{s}|v(1)|^2 \le \frac{1}{s}|v|_\infty^2 &\le (\hat{c})^2\{s^{2\Theta-1}|v|^{2(1-\Theta)}\}(s^{-1}\|v\|)^{2\Theta} \\ &\le \frac{\varepsilon}{s^2}\|v\|^2 + c_\varepsilon(\hat{c})^{\frac{2}{1-\Theta}}|v|^2 \ . \end{aligned}$$

(iv)
$$\begin{aligned} \frac{s'}{s}|v(1)|^2 \le \frac{s'}{s}|v|_\infty^2 &\le (\hat{c})^2 s' s^{2\Theta-1}|v|^{2(1-\Theta)}(s^{-1}\|v\|)^{2\Theta} \\ &\le \frac{\varepsilon}{s^2}\|v\|^2 + c_\varepsilon(\hat{c})^{\frac{2}{1-\Theta}} s^{\frac{2\Theta-1}{1-\Theta}} (s')^{\frac{1}{1-\Theta}}|v|^2 \ . \end{aligned}$$

**Proof.** For the first part of (i) cf. [Ag]. All the other parts are based on this and on obvious applications of Young's inequality and on integration by parts.□

## 3. Transformation onto a fixed domain. Results

(1.3.1-12) can be reformulated on a fixed domain by the transformation

$$\left.\begin{aligned} &(x,t) \in [0, s(t)] \times \overline{S} \mapsto (y,t) \in [0,1] \times [0,T], \\ &y := x/s(t),\ u(y,t) := v(x,t) - \lambda(t),\ u = (u_1, u_2, u_3),\ \lambda = (\lambda_1, \lambda_2, \lambda_3). \end{aligned}\right\} \quad (1)$$

Using the transformed coefficients

$$
\begin{aligned}
A_i &= A_i(y, s(t), u(y,t) + \lambda(t)) := \overline{A}_i(x, s(t), v(x,t)), && \text{(2.a)}\\
f_i &= f_i(y, s(t), u(y,t) + \lambda(t)) := \overline{f}_i(x, v(x,t)), && \text{(2.b)}\\
g_i &= g_i(y, s(t), u(1,t) + \lambda(t), v^*(t)) := \overline{g}_i(x, s(t), v(x,t), v^*(t)) && \\
&\qquad (i = 1, 2,\ y = 1), && \text{(2.c)}\\
g_3 &= g_3(y, s(t), u(1,t) + \lambda(t), v^*(t)) := \overline{g}_3(u_3 + \lambda_3)\ (y = 1), && \text{(2.d)}\\
L_2 &= L_2(s(t), u(1,t) + \lambda(t)) := \overline{L}_2(s(t), v(s(t),t)), && \text{(2.e)}\\
A_{ir} &= A_{ir}(s(t), u(1,t) + \lambda(t)) := \overline{A}_{ir}(s(t), v(s(t),t)), && \text{(2.f)}\\
h_i &= h_i(s, s', y, \eta) := \frac{s'}{s} y\eta, && \text{(2.g)}
\end{aligned}
$$

(1.3.1-6) transforms to

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} - \frac{1}{s^2(t)}(A_i u_{iy})_y &= f_i - \lambda_i' + h_i(s, s', y, u_{iy}), && (3)\\
u_i(y,0) &=: u_{0i}(y) := v_{0i}(y s_0) - \lambda_i(0), \quad y \in (0,1), && (4)\\
u_i(0,t) &= 0 \quad t > 0, && (5)\\
-\frac{1}{s(t)} A_i \cdot \frac{\partial u_i}{\partial u}\bigg|_{y=1} &= g_i(1, s(t), u(1,t) + \lambda(t), v^*(t)) && (6)\\
s'(t) &= L_2(u_3(1,t) + \lambda_3(t))^m \quad t > 0, && (7)\\
s(0) &= s_0. && (8)
\end{aligned}
$$

We will use the following assumptions on the transformed coefficients

$$
\left.\begin{array}{l}
A_i : [0,1] \times [s_0, \infty) \times (\mathbb{R}^+)^3 \to \mathbb{R} \quad , y \mapsto A_i(y,s,u)\\
\text{is measurable for all } (s,u) \in [s_0,\infty) \times (\mathbb{R}^+)^3,\\
(s,u) \mapsto A_i(y,s,u) \text{ is continuous for all } y \in [0,1],\\
\text{there are bounds } A_{ij} > 0 : A_{i0} \le A_i(y,s,u) \le A_{i1}\\
\text{for a.a. } y \text{ and for all } s, u, i = 1,2,3,
\end{array}\right\} \qquad (9)
$$

$$
\left.\begin{array}{l}
B_i, K_i : [0,1] \times [s_0, \infty) \times (\mathbb{R}^+)^3 \to \mathbb{R}\\
\text{are non-negative, bounded, measurable and}\\
(s,u) \mapsto K_i(y,s,u),\ B_i(y,s,u) \text{ are continuous, } i = 1,2,3.
\end{array}\right\} \qquad (10)
$$

$$
\left.\begin{array}{l}
L_2, A_{ir} : [s_0,\infty) \times (\mathbb{R}^+)^3 \to \mathbb{R}^+ \text{ are continuous, bounded}\\
\text{and non-negative, } i = 1,2.
\end{array}\right\} \qquad (11)
$$

For an "auxiliary problem" we will need the following coefficients

$$
\left.\begin{array}{l}
\hat{A}_i := \hat{A}_i(y,t), \quad \hat{A}_i \in L^\infty(G \times S) \text{ such that there are constants}\\
A_{ij} > 0 \text{ with } A_{i0} \le \hat{A}_i(y,t) \le A_{i1} \text{ a.e. } (A_{ij} \text{ - cf. } (9)),
\end{array}\right\} \qquad (12)
$$

$$\hat{K}_i := \hat{K}_i(y,t) \geq 0 \ \text{ a.e.}, \ \hat{K}_i \in L^\infty(G \times S), \quad i = 1,2, \tag{13}$$
$$\hat{B}_i := \hat{B}_i(y,t) \geq 0 \ \text{ a.e.}, \ \hat{B}_i \in L^\infty(G \times S), \quad i = 1,2, \tag{14}$$
$$\hat{A}_{ir} := \hat{A}_{ir}(t) \geq 0 \ \text{ a.e.}, \ \hat{A}_{ir} \in L^\infty(S), \quad i = 1,2, \tag{15}$$
$$\hat{f}_i := \hat{f}_i \text{ defined with } \hat{K}_i, \ \hat{B}_i \text{ in analogy to } f_i, \tag{16}$$
$$\hat{L}_2 := \hat{L}_2(y,t), \quad \hat{L}_2 \in L^\infty(G \times S). \tag{17}$$

We call the quadruple $(s,u)$, $u = (u_1, u_2, u_3)$, a *weak solution of* (2)-(8) if

$$s \in W^{1,\infty}(S), \tag{18}$$
$$u \in L^2(S;V) \cap H^1(S;V^*) \cap L^\infty(S;(([0,1])^3) \cap [\overline{S} \to L^\infty(0,1)^3] \tag{19}$$

and $\forall \varphi = (\varphi_1, \varphi_2, \varphi_3) \in V$, a.a. $t \in S$:

$$\left.\begin{aligned}
&(u'(t),\varphi) + \tfrac{1}{s^2(t)}\sum_{i=1}^{3}(A_i u_{iy}, \varphi_{iy}) + \tfrac{1}{s(t)}\sum_{i=1}^{3} g_i(1, s(t), u(1,t) \\
&\qquad +\lambda(t), u(1,t) + \lambda(t))\varphi_i(1) \\
&= \sum_{i=1}^{3}(f_i(\cdot, s(t), u(t) + \lambda(t), u(t) + \lambda(t)), \varphi_i) \\
&\qquad + \sum_{i=1}^{3}(\lambda_i'(t), \varphi_i) + \sum_{i=1}^{3}\tfrac{s'(t)}{s(t)}(y u_{iy}, \varphi_i),
\end{aligned}\right\} \tag{20}$$

$$u(0) = u_0, \tag{21}$$
$$s'(t) = L_2(1, s(t), u(1,t) + \lambda(t))(u_3(1,t) + \lambda_3(t))^m \ \text{ for a.a. } t \in S, \tag{22}$$
$$s(0) = s_0. \tag{23}$$

In complete analogy to (18)-(23) we define a weak solution for the system (3)-(8) with $A_i, B_i, K_i, L_2, A_{ir}$ (in the definition of $g_i$) replaced by coefficients $\hat{A}_i = \hat{A}_i(t,y), \ldots, \hat{A}_{ir} = \hat{A}_{ir}(t,y)$, $\hat{L}_2 = \hat{L}_2(y,t)$ (cf. (12)-(17)). The corresponding substitutes for (18)-(23) will be marked as $(\widehat{18}) - (\widehat{23})$. $\qquad (\widehat{18}) - (\widehat{23})$

**Theorem 1.** *Let the coefficients satisfy (12)-(17) and let*

$$\lambda_i \in W^{1,2}(0,T), \ \lambda_i(t) \geq 0 \ \forall t \in [0,T], \ i = 1,2,3, \tag{24}$$
$$u_{0i} \in L^\infty(0,1), \ u_{0i}(y) + \lambda_i(0) \geq 0 \ \textit{for a.a. } y \in [0,1], \ i = 1,2,3, \tag{25}$$
$$0 < s_0 < d, \tag{26}$$
$$v_i^* \in L^2(0,T), \ i = 1,2, \tag{27}$$
$$k_i \geq \max\{u_{0i}(y) + \lambda_i(t), \lambda_i(t), v_i^*(t), y \in [0,1], t \in [0,T]\}, \ i = 1,2, \tag{28}$$
$$k_3 \geq \max\{u_{03}(y) + \lambda_3(t), \lambda_3(t), y \in [0,1], t \in [0,T]\} \tag{29}$$

*and assume*

$$\frac{\hat{K}_2(y,t)\hat{B}_2(y,t)}{\hat{K}_2(y,t)+\hat{K}_3(y,t)} \leq \hat{B}_1(y,t) \text{ for a.a. } y \in (0,1),\ t \in S, \tag{30}$$

$$\inf_{y,t} \hat{B}_1(y,t) \geq \frac{k_2}{k_1} \text{ and } \frac{\hat{K}_2(y,t)\hat{B}_2(y,t)}{\hat{K}_2(y,t)+\hat{K}_3(y,t)} \leq \frac{k_2}{k_1}$$
$$\text{for a.a. } y \in (0,1),\ t \in S, \tag{31}$$

*and*

$$T|\hat{L}_2|_\infty k_3^m < d - s_0. \tag{32}$$

*Then*

*(i) Problem* $(\widehat{18}) - (\widehat{23})$ *admits a weak solution* $(s,u)$.

*(ii) One has*

$$\left.\begin{array}{l} 0 \leq u_i(y,t)+\lambda_i(t) \leq k_i \text{ for a.a. } t \in [0,T] \text{ and} \\ \text{for all } y \in [0,1],\ s(t)-s_0 \in [0,b_1],\ b_1 := T|\tilde{L}_2|_\infty k_3^m, \end{array}\right\} \tag{33}$$

*and there is a constant* $c^*$ *depending at most on the constants* $k_i$ *in (28), (29), on the* $L^\infty$*-norms of* $u_0$ *and* $\lambda$ *and on the* $L^\infty$*-bounds for* $\hat{A}_i, \ldots, \hat{L}_2$ *in (12)-(17) such that*

$$\left.\begin{array}{lcl} \|u\|_{L^2(S;V)\cap H^1(S;V^*)} & \leq & c^*, \\ \|s\|_{W^{1,\infty}(S)} & \leq & c^*. \end{array}\right\} \tag{34}$$

For the main theorem we need a variant of (30), (31):

$$\left.\begin{array}{l} \frac{K_2(y,s,u)(B_2(y,s,u)}{K_2(y,s,u)+K_3(y,s,u)} \leq B_2(y,s,u) \text{ for a.a. } y \in [0,1], \\ \text{a.a. } (s,u) \in [s_0,b_1] \times \mathbb{R}^3_+, \end{array}\right\} \tag{35}$$

$$\left.\begin{array}{l} B_1(y,s,u) \geq \frac{k_2}{k_1} \text{and } \frac{k_2}{k_1} \geq \frac{K_2(y,s,u)(B_2(y,s,u)}{K_2(y,s,u)+K_3(y,s,u)} \\ \text{for a.a. } y \in [0,1],\ (s,u) \in [s_0,b] \times \mathbb{R}^3_+, \end{array}\right\} \tag{36}$$

and a variant of (32):

$$T|L_2|_\infty k_3^m < d - s_0. \tag{37}$$

**Remarks.**

1. (37) will be needed to guarantee that $s(t) < d$. This condition is *not needed,* if the fluxes for $u_1$ and $u_2$ are assumed to vanish at $y = 1$. The latter assumption can be made for the practical models we have in mind (cf. [BoR], [BoDJR], [BoDR]), which (partly) justifies the title *global solutions.*
2. (35) and (36) seem to be essential to guarantee the existence of an appropriate invariant region for the concentrations (also: cf. rem. 2.1 in [BoR]). In practice one has $B_1 = B_2$, implying (35).

**Theorem 2.** *Let the coefficients satisfy (9)-(11), (35)-(37) and (24)-29). Then there is a solution* $(s,u)$ *of problem (18)-(23) satisfying (33).*

**Remark 3.** The boundedness assumptions on most coefficients can be considerably relaxed.

**Theorem 3.** *Let the assumptions of theorem 2 be fulfilled and assume all the coefficients to be locally Lipschitz-continuous. Moreover, assume that there is at least one solution* $(s,u)$ *such of (3.18)-(3.23) such that*

$$u \in L^4(0,T;V). \tag{38}$$

*Finally, (38) can be obtained if* $\lambda_j \in W^{1,4}(S)$.

**Remark 4.** There is a whole variety of other regularity assumptions also leading to uniqueness. Furthermore, even under the assumptions of theorem 1, the solutions of theorem 1 are more regular than stated. Theorem 3 can be extended to a well-posedness statement, such, that the solutions depend locally Lipschitz-continuous on the data and on most coefficient-functions. We do not go into detail.

## 4. Proofs of Theorems 1 and 2

### Proof of Theorem 1.

A similar problem with constant coefficients has been dealt with in [BoR]. The arguments there carry over to the situation of theorem 1. □

### Proof of Theorem 2.

We employ Schauder's fixed point theorem. In order to define an appropriate fixed point-operator, fix $p \in [1,\infty)$, set

$$b_1 := T \cdot |L_2|_\infty k_3^m \ (k_3 \text{ - cf. } (3.29)) \tag{1}$$

and note that the set

$$\begin{aligned} &M := \{(s,u) \in C([0,T]) \times L^p(S;C([0,1])^3) : s(0) = s_0,\ s(t) - s_0 \in [0,b_1] \\ &\text{for all } t \in [0,T],\ |u_i(\cdot,t) + \lambda_i(t)|_\infty \le k_i \text{ a.a. } t \in S\} \ (k_i \text{ - cf. } (3.28),(3.29)) \end{aligned}$$

is convex and closed as well as bounded in $Y := C([0,T]) \times L^p(S;C([0,1])^3)$. We define the (fixed point) operator $Q$ with

$$\operatorname{dom}(Q) := M \ \text{ and } \ Q : (\overline{s},\overline{u}) \mapsto (s,u), \tag{2}$$

where $(s,u)$ is the (weak) solution of $(3.\widehat{22}-\widehat{28})$ with $(\overline{s},\overline{u})\in M$ and

$$\begin{aligned}
\hat{A}_i &:= A_i(y,\overline{s}(t),\overline{u}(y,t)), \quad \hat{B}_i := B_i(y,\overline{s}(t),\overline{u}(y,t)),\\
\hat{K}_i &:= K_i(y,\overline{s}(t),\overline{u}(y,t)), \quad \hat{A}_{ir} := A_{ir}(1,\overline{s}(t),\overline{u}(1,t)),\\
\hat{E}_{3j} &:= E_{3j}(y,\overline{s}(t),\overline{u}(1,t)), \quad \hat{L}_2 := L_2(\overline{s}(t),\overline{u}(1,t)).
\end{aligned}$$

By theorem 1, $Q$ is well-defined,

$$Q(M)\subseteq L^2(S;V)\cap H^1(S;V^*)\cap L^\infty(S;C^{\frac{1}{2}}([0,1]))\cap[\overline{S}\to L^\infty(0,1)^3]\cap M \quad (3)$$

and there are constants $c$, independent of $(\overline{s},\overline{u})$, such that

$$\left.\begin{array}{l}
|s|_{W^{1,\infty}(S)}\le c,\\
\|u\|_{L^2(S;V)\cap H^1(S;V^*)\cap L^\infty(S;C([0,1])^3)}\le c\\
\text{for all } (s,u)\in Q(M).
\end{array}\right\} \quad (4)$$

We have

**Lemma 1.** *(i) $Q(M)$ is relatively compact in $Y$.*
*(ii) Let $(u_n)\subset L^2(S;V)\cap H^1(S;V^*)\cap L^\infty(S;C([0,1]^3)=:X$ be bounded, $u\in X$, and $u_n\to u$ in $L^2(S;H)$. Then $u_n\to u$ in $L^r(S;C([0,1]))$ for all $r\in[1,\infty)$.*

**Proof of Lemma 1 (i).** By Aubin's lemma (cf. [Ze]) there is a subsequence (we drop the subsequence index) and a limit $u$ such that $u_n\to u$ in $L^2(S;H)$. This and $L^2(S;V)\cap H^1(S;V^*)\hookrightarrow C(\overline{S};H)$ imply $u_n\to u$ in $L^r(S;H)$ for all $r\in[1,\infty)$. Let $\eta\in[2,4)$, $b:=4/\eta$, $\frac{1}{a}+\frac{1}{b}=1$. By lemma 2.1(i) and by Hölder's inequality:

$$\|u_n-u_m\|_{L^\eta(S;C(\overline{G})^3)}\le c|u_n-u_m|^{1/2}_{L^{a\eta/2}(S;H)}\|u_n-u_m\|^{1/2}_{L^2(S;V)} \text{ for all } n,m.$$

Therefore: $u_n\to u$ in $L^\eta(S;C(\overline{G})^3)$ for all $\eta\in[2,4)$. This and the boundedness of $(u_n)$ in $L^\infty(S;C(\overline{G})^3)$ imply (i).

(ii) follows similarly. □

It remains to show

**Lemma 2.** *The (fixed point) operator $Q$ is continuous.*

**Proof.** Let $(\overline{s}_n, \overline{u}_n)$, $(\overline{s}, \overline{u}) \in M$, $\overline{s}_n \to \overline{s}$ in $C(\overline{G})$, $\overline{u}_n \to u$ in $L^p(S; C(\overline{G}))$, (5i) set $(s, u) := Q(\overline{s}, \overline{u})$, $(s_n, u_n) := Q(\overline{s}_n, \overline{u}_n)$, $\overline{w}_n := \overline{u}_n - u$, $w_n := u_n - u$. By the estimates in (4) there is a constant $d^*$ (independent of $n$) such that

$$\left.\begin{array}{l} |s_n|_{W^{1,\infty}(S)} + |s|_{W^{1,\infty}(S)} \leq d^*, \\ \|u_n\|_{L^2(S;V)\cap H^1(S;V^*)\cap L^\infty(S;C([0,1])^3)} \leq d^*, \\ \|u\|_{L^2(S;V)\cap H^1(S;V^*)\cap L^\infty(S;C([0,1])^3)} \leq d^*. \end{array}\right\} \tag{5ii}$$

We claim

$$s_n \to s \text{ in } C(\overline{S}) \quad \text{and} \quad u_n \to u \text{ in } L^2(S; H). \tag{6}$$

To this end consider the equations in $(3.\widehat{18}) - (3.\widehat{23})$ for $(s_n, u_n)$, subtract the ones for $(s, u)$, split some of the expressions, use $\varphi := w_n := u_n - u$ as test function and integrate with respect of time from 0 to $t \in (0, T]$. This results in

$$\begin{aligned} |w_n(t)|^2 + A_{1n}(t) + G_{1n}(t) = A_{2n}(t) + A_{3n}(t) + F_{1n}(t) + G_{2n}(t) \\ + H_{1n}(t) + R_n(t) =: B_n(t) + H_{2n}(t) + H_{3n}(t) \end{aligned} \tag{7i}$$

$$s_n(t) - s(t) = L_{1n}(t) + L_{2n}(t), \tag{7ii}$$

where

$$\begin{aligned} A_{1n}(t) &:= \int_0^t \frac{1}{\overline{s}_n^2(\tau)} \sum_{i=1}^3 (A_i(\cdot, \overline{s}_n(\tau), \overline{u}_n(\tau)) w_{niy}(\tau), w_{niy}(\tau)) d\tau \\ &\geq d_1 \int_0^t \|w_n(\tau)\|^2 d\tau \quad (d_1 := (s_0 + b_1)^{-2} \sum_{i=1}^3 A_{i0}), \\ G_{1n}(t) &:= \sum_{i=1}^3 \int_0^t \frac{1}{\overline{s}_n(\tau)} g_i(1, \overline{s}_n(\tau), \overline{u}_n(1, \tau), w_n(1, \tau), 0) w_n(1, \tau) d\tau \geq 0, \\ A_{2n}(t) &:= \int_0^t \frac{1}{\overline{s}_n(\tau)^2} \sum_{i=1}^3 ([A_i(\cdot, \overline{s}_n, \overline{u}_n) - A_i(\cdot, \overline{s}_n, \overline{u})] u_{iy}, w_{niy}) d\tau \\ &\leq s_0^{-2} \sum_{i=1}^3 |[A_i(\cdot, \overline{s}_n, \overline{u}_n) - A_i(\cdot, \overline{s}, \overline{u})] u_{iy}|_{L^2(S;H)} \cdot \|w_n\|_{L^2(S;V)} \\ &\leq 2d^* s_0^{-2} \cdot \sum_{i=1}^3 |[A_i(\cdot, \overline{s}_n, \overline{u}_n) - A_i(\cdot, \overline{s}, \overline{u})] u_{iy}|_{L^2(S;H)} =: \hat{A}_{2n}(t) \\ &\qquad (d^* \text{ - cf. } (5)), \\ A_{3n}(t) &:= \int_0^t \left( \frac{1}{\overline{s}_n(\tau)^2} - \frac{1}{\overline{s}(\tau)^2} \right) \sum_{i=1}^3 (A_i(\cdot, \overline{s}, \overline{u}) u_{iy}, w_{ny}) d\tau \end{aligned}$$

$$
\begin{aligned}
&\le |\overline{s}_n - \overline{s}|_{C([0,T])} d_2 =: \hat{A}_{3n}(t) \quad (d_2 := 4 s_0^{-4}(s_0 + b_1) \sum_{i=1}^{3} A_{i1} d^* \\
&\qquad \text{(cf. (5))}, \\
F_{1n}(t) &:= \int_0^t a_f(\overline{u}_n + \lambda, w_n, w_n) d\tau \le d_3 \int_0^t |w_n(\tau)|^2 d\tau, \\
G_{2n}(t) &:= \int_0^t \sum_{j=1}^{2} [E_{3j}(\cdot, \overline{s}_n(\tau), \overline{u}_n(1,\tau)) - E_{3j}(\cdot, \overline{s}(\tau), \overline{u}(1,\tau))] \\
&\qquad (u_3(1,\tau) + \lambda_3(\tau))^{m-1+j} \times w_{3n}(1,\tau) d\tau =: \hat{G}_{2n}(t). \\
H_{1n}(t) &:= \sum_{i=1}^{3} \int_0^t (s_n'(\tau) - s'(\tau)) \cdot \overline{s}_n^{-1}(\tau)(y u_{niy}, w_{ni}) d\tau, \\
H_{2n}(t) &:= \sum_{i=1}^{3} \int_0^t s_n'(\tau) \left( \frac{1}{\overline{s}_n} - \frac{1}{\overline{s}} \right) (y u_{niy}, w_{ni}) d\tau, \\
H_{3n}(t) &:= \sum_{i=1}^{3} \int_0^t \frac{s'(\tau)}{\overline{s}(\tau)} (y w_{niy}, w_{ni}) d\tau, \\
L_{1n}(t) &:= \int_0^t L_2(\overline{s}_n, \overline{u}_n(1,\tau))[(u_{n3}(1,\tau) + \lambda_3(\tau))^m] \\
&\qquad - (u_3(1,\tau) + \lambda_3(\tau))^m] w_{3n}(1,\tau) d\tau, \\
|L_{1n}(t)| &\le d_4 \int_0^t |w_{n3}(1,\tau)|^2 d\tau \le d_4 c \int_0^t |w_{n3}(\tau)| \cdot \|w_{n3}(\tau)\| d\tau \\
&\le \varepsilon \int_0^t \|w_n(t)\|^2 d\tau + (d_4 c)^2 c_\varepsilon \int_0^t |w_n(\tau)|^2 d\tau,
\end{aligned}
$$

where $d_4 := \sup_n \operatorname{vraimax}_\tau L_2(\overline{s}_n(\tau), \overline{u}_n(\tau)) \cdot m \cdot (2(d^*)^{m-1})$ and $c$ stems from lemma 2.1(i). $\varepsilon$ and $c_\varepsilon$ are related via Young's inequality,

$$
\begin{aligned}
L_{2n}(t) &:= \int_0^t [L_2(\overline{s}_n(\tau), \overline{u}_n(1,\tau)) \\
&\qquad - L_2(\overline{s}(\tau), \overline{u}(1,\tau))](u_3(1,\tau) + \lambda_3(\tau))^m w_{n3}(1,\tau) d\tau, \\
|L_{2n}(t)| &\le d_5 \int_0^t [L_2(\overline{s}_n(\tau), \overline{u}_n(1\tau,)) - L_2(\overline{s}(\tau), \overline{u}(1,\tau))] d\tau =: \hat{L}_{2n}(t),
\end{aligned}
$$

where $d_5 := \sup_{n,\tau} |u_3(1,\tau) + \lambda_3(\tau)|^m |w_{n3}(1,\tau)| < \infty$ (because of (5ii)).

The remaining expressions in the equation for $w_n$ are collectively denoted by $R_n$ (see (7i)).

Plugging the relations $(3.\widehat{22})$ for $s_n'$ and $s'$, resp., into the expressions $H_{1n}(t)$ $, \ldots,$ $H_{3n}(t)$ and using similar estimates as for $L_{2n}(t)$, one obtains

$$\begin{aligned}|H_{1n}(t)| \leq d_6 \int_0^t |L_2(\overline{s}_n(\tau), \overline{u}_n(\tau)) - L_2(\overline{s}(\tau), \overline{u}(\tau))|^2 d\tau \\ + \varepsilon \int_0^t \|w_n(\tau)\|^2 d\tau + c_\varepsilon d_7^2 \int_0^t \|u_n(\tau)\|^2 |w_n(\tau)|^2 d\tau,\end{aligned}$$

where $d_6 := \sup\limits_{n,i,\tau} |w_{ni}(\tau)|_\infty d^*(d^* + |\lambda|_\infty)^m s_0^{-1} < \infty$ (cf. (5ii)) and $d_7 := \sup\limits_{\tau} |L_2(\overline{s}(\tau), \overline{u}(1,\tau))| \cdot m \cdot \sup\limits_{n,\tau}\{|u_{n3}(1,\tau) + \lambda_3(\tau)|^m + |u_3(1,\tau) + \lambda_3(\tau)|^m\} < \infty$ (cf. (5ii)). Furthermore:

$$|H_{2n}(t)| \leq d_8 |\overline{s}_n - \overline{s}|_\infty,$$

where $d_8 := \sup\limits_{n,\tau} |L_2(\overline{s}_n(\tau), \overline{u}_n(1,\tau))| \cdot \sup\limits_{n}\{|s_n|_\infty + |s|_\infty\} \int_0^T \|u_n(\tau)\| \cdot |w_n(\tau)| d\tau < \infty$ (cf. (5ii)), and

$$\begin{aligned}|H_{3n}(t)| &\leq d_9 \int_0^t \|w_n(\tau)\| \cdot |w_n(\tau)| d\tau \\ &\leq \varepsilon \int_0^t \|w_n(\tau)\|^2 d\tau + c_\varepsilon d_9^2 \int_0^t |w_n(\tau)|^2 d\tau,\end{aligned}$$

where $d_9 := \sup\limits_{\tau} |L_2(\overline{s}(\tau), \overline{u}(1,\tau))| \cdot s_0^{-1} < \infty$.

Choosing $\varepsilon$ sufficiently small (compared with $d_1$, from the estimate of $A_{1n}(t)$) and estimating the terms appearing at the right hand side of (7i) from above as on the preceding lines, one arrives at

$$|w_n(t)|^2 + \int_0^t \|w_n(\tau)\|^2 d\tau \leq P_n(t) + \int_0^t Q_n(\tau)|w_n(\tau)|^2 d\tau, \qquad (7\text{iii})$$

where

$$\begin{aligned}P_n(t) &:= \hat{A}_{2n}(t) + \hat{A}_{3n}(t) + \hat{G}_{2n}(t) + \hat{L}_{2n}(t) + d_8|\overline{s}_n - \overline{s}|_\infty, \\ Q_n(t) &:= d_3 + (d_4 c)^2 c_\varepsilon + d_8^2 c_\varepsilon \|u_n(t)\|_\varepsilon^2 + d_9^2 c_\varepsilon.\end{aligned}$$

Gronwall's inequality and a straightforward estimate yield

$$|w_n(t)|^2 \leq P_n(t) \cdot \exp\Big(\int_0^t Q_n(\tau)d\tau\Big) \leq P_n(T) d_{10} \quad \forall t, \qquad (7\text{iv})$$

where $d_{10} := \sup\limits_{n} \exp(\int_0^T Q_n(\tau)d\tau) < \infty$ (cf. (5ii)).

It remains to show

$$w_n \to 0 \ \text{ in } C(\overline{S}; H). \tag{8i}$$

(5i) implies for a sub-sequence

$$\left.\begin{array}{l} \overline{u}_{n_j}(t) \to \overline{u}(t) \ \text{ in } C([0,1])^3 \text{ f.a.a. } t \in [0,T] \text{ and} \\ \overline{s}_{n_j}(t) \to \overline{s}(t) \ \text{ in } \mathbb{R} \ \text{ for all } t \in [0,T]. \end{array}\right\} \tag{9i}$$

This, continuity of the coefficients involved and Lebesgue's theorem imply

$$P_{n_j}(T) \to 0 \ \text{ as } j \to \infty. \tag{9ii}$$

Therefore, by (7iii),

$$w_{n_j} \to 0 \ \text{ in } C(\overline{S}; H) \ \text{ as } j \to \infty.$$

The convergence of the *whole* sequence $(w_n)$ in $C(\overline{S}; H)$ follows by contradiction. This yields (8i). Moreover, (8i), (7ii) and (7iii) imply

$$w_n \to 0 \ \text{ in } L^2(S; V). \tag{8ii}$$

Set $\Delta L_2^n(\tau) := L_2(\overline{s}(\tau), \overline{u}(1,\tau)) - L_2(\overline{s}_n(\tau), \overline{u}_n(1,\tau))$.

$(3.\widehat{22})$ implies

$$\begin{aligned} |s_n(t) - s(t)| \le & \int_0^t |\Delta L_2^n(\tau)| \cdot |(u_{n3}(1,\tau) + \lambda_3(\tau))|^m d\tau \\ & + \int_0^t |L_2(\overline{s}(\tau), \overline{u}(1,\tau))| \cdot |(u_{n3}(1,\tau) + \lambda_3(\tau))^m \\ & - (u_3(1,\tau) + \lambda_3(\tau))^m | d\tau \\ \le & \, d_{11} \int_0^T |\Delta L_2^n(\tau)| d\tau + d_{12} \int_0^t |w_{n3}(1,\tau)| d\tau \\ \le & \, d_{11} \int_0^T |\Delta L_2^n(\tau)| d\tau + d_{12} c \int_0^t \|w_n(\tau)\| d\tau, \end{aligned}$$

where

$$d_{11} := \sup_{n,\tau} |(u_{n3}(1,\tau) + \lambda_3(\tau))^m| < \infty \ \text{ and}$$

$$d_{12} := \sup_{n,\tau} |L_2(\overline{s}(\tau), \overline{u}(1,\tau))| \cdot m \cdot 2(d^* + |\lambda|_\infty)^{m-1} < \infty.$$

By (8ii) and (9i):

$$s_{n_j} \to s \ \text{ in } C(\overline{S})$$

and, again by contradiction, the whole sequence $(s_n)$ converges. Now, (8.i) and lemma 1(ii) imply $u_n \to u$ in $L^p(S; C([0,1]))$. This and the convergence of $(s_n)$ implies: $Q$ is continuous. $\square$

**Remark.** We have not been able to employ Nemyzki-type arguments. Moreover, using the boundedness properties (5ii), some of the boundedness assumptions imposed on the coefficients could be relaxed.

## 5. Proof of Theorem 5

At first we show uniqueness. The main tools are: The $L^\infty$-estimates for the solutions, combined with the other estimates and a repeated use of Gronwall's inequality.

Assume there are two solutions $(s_j, u_j)$, $j = 1, 2$, satisfying (3.18)-(3.23), (3.33) and let $u_1$ satisfy (3.38). Set

$$s := s_2 - s, \quad \text{and} \quad w := u_2 - u_1 \quad .$$

Let $\varepsilon > 0$. (3.18), (3.19), (3.22), (3.23) and local Lipschitz-continuity of $L_2$ imply

$$\begin{aligned} |s(t)| &\leq c \int_0^t |w(1,\tau)| + |s(\tau)| d\tau \\ &\leq c \int_0^t |w(1,\tau)| d\tau + t^{1/2} \left[ \int_0^t |s(\tau)|^2 d\tau \right]^{1/2}, \end{aligned} \tag{1}$$

and

$$|s'(t)| \leq c[|w(1,t)| + |s(t)|]. \tag{2}$$

Gronwall's inequality and (1) yield

$$|s(t)| \leq c \int_0^t |w(1,\tau)| d\tau. \tag{3}$$

This and (2) imply

$$|s'(t)| \leq c[|w(1,t)| + \int_0^t |w(1,\tau)| d\tau]. \tag{4}$$

Subtract the equation (3.20) for $u_1$ from the one for $u_2$, use $\varphi := w$ as test function, split the nonlinear terms to arrive at

$$|w(t)|^2 + A(t) + B(t) = C(t) + D(t) + E(t) \quad \text{for a.a. } t \in S \tag{5}$$

where, with some constants $e_j > 0$ (for $j = 1, 2$), $e_j \geq 0$ otherwise,

$$
\begin{aligned}
A(t) &:= \int_0^t \frac{1}{s_1(\tau)^2})(A(\cdot, s_2(\tau), u_2(\tau) + \lambda(\tau))w(\tau), w(\tau))d\tau \\
&\geq 3e_1\|w\|^2_{L^2(S;V)} && (6) \\
B(t) &:= \int_0^t \sum_{i=1}^{2} A_{ir}(s_2, u_2 + \lambda_2)w_i^2(1,\tau) \cdot (s(\tau) - d)^{-1} d\tau \\
&\quad + \int_0^t L_2(s_2, u_2(1,\tau) + \lambda(\tau))[u_{23}(1,\tau) + \lambda_3(\tau))^m \\
&\quad - (u_{13}(1,\tau) + \lambda_3(\tau))^m] \times w_3(1,\tau)d\tau \\
&\geq 0, && (7) \\
C(t) &:= \int_0^t c_1(\tau) + c_2(\tau)d\tau \quad \text{with} \\
c_1(t) &:= \frac{1}{s_2^2(t)}([A(s_2(t), u_2(t)) - A(s_1, u_1)]u_{1y}(t), w_y(t)) \\
|c,(t)| &\leq e_3\{|s(t)| \cdot \|u_1(t)\| \cdot \|w(t)\| + |w(t)|_\infty \|u_1(t)\| \cdot \|w(t)\|\} \\
&\leq \varepsilon\|w(t)\|^2 + c_\varepsilon\|u_1(t)\|^2|s(t)|^2 + c_\varepsilon\|u_1(t)\|^4|w(t)|^2, \\
c_2(t) &:= \left(\frac{1}{s_2^2(t)} - \frac{1}{s_1^2(t)}\right)(A(s_1, u_1)u_{1y}, w_y), \\
|c_2(t)| &\leq e_4|s(t)| \cdot \|u_1(t)\| \cdot \|w(t)\| \\
&\leq \varepsilon\|w(t)\|^2 + c_\varepsilon\|u_1(t)\|^2|s(t)|^2, \\
|C(t)| &\leq \varepsilon\int_0^t \|w(\tau)\|^2 d\tau + c_\varepsilon\int_0^t \|u_1(\tau)\|^2|s(\tau)|^2 d\tau + c_\varepsilon\int_0^t \|u_1(\tau)\|^4|w(\tau)|^2 d\tau \\
&\overset{(3)}{\leq} \varepsilon\int_0^t \|w(\tau)\|^2 d\tau + c_\varepsilon\int_0^t |w(1,\tau)|d\tau + c_\varepsilon\int_0^t \|u_1(\tau)\|^4|w(\tau)|^2 d\tau \\
&\overset{(Lemma\ 1.1(i))}{\leq} \varepsilon\int_0^t \|w(\tau)\|^2 d\tau + c_\varepsilon\int_0^t (1 + \|u_1(\tau)\|^4)|w(\tau)|^2 d\tau && (8) \\
D(t) &:= \int_0^t d_1(\tau) + d_2(\tau)d\tau, \quad \text{where} \\
d_1(t) &:= \frac{s'(t)}{s_2(t)}\sum_{i=1}^{3}(yw_{iy}(t), u_{2i}(t)) \\
|d_1(t)| &\leq e_5\{|w(1,t)| + \int_0^t |w(1,\tau)|d\tau\}\|w(t)\| \cdot |u_2(t)| \\
&\leq c\{|w(t)|^{1/2}\|w(t)\|^{1/2} + \int_0^t |w(\tau)|^{1/2}\|w(\tau)\|^{1/2}d\tau\}\|w(t)\| \\
&\leq \varepsilon\|w(t)\|^2 + c_\varepsilon|w(t)|^2 + c_\varepsilon[\int_0^t |w(\tau)|^{1/2}\|w(\tau)\|^{1/2}d\tau],
\end{aligned}
$$

$$\int_0^t |d_1(\tau)|d\tau \le \varepsilon \int_0^t \|w(\tau)\|^2 d\tau + c_\varepsilon \int_0^t |w(\tau)|^2 d\tau + c_\varepsilon \int_0^t [\int_0^\tau |w|^{1/2}\|w\|^{1/2} d\xi]^2 d\tau$$
$$\le \varepsilon \int_0^t \|w(\tau)\|^2 d\tau + c_\varepsilon \int_0^t |w(\tau)|^2 d\tau, \tag{9}$$
$$d_2(t) := \frac{s'(t)}{s_2(t)} \sum_{i=1}^3 (y u_{2iy}, w_i),$$
$$\int_0^t |d_2(\tau)|d\tau \le e_6 \int_0^t \{|w(1,\tau)| + \int_0^\tau |w(1,\xi)|d\xi\} \|u_2(\tau)\| \cdot |w(\tau)| d\tau$$
$$\le \varepsilon \left[ \int_0^t \{|w(1,\tau)| + \int_0^\tau |w(1,\xi)|d\xi\}^2 d\tau \right] + c_\varepsilon \int_0^t \|u_2(\tau)\|^2 |w(\tau)|^2 d\tau$$
$$\le \varepsilon \int_0^t \|w(\tau)\|^2 d\tau + c_\varepsilon \int_0^t (1 + \|u_2(\tau)\|^2)|w(\tau)|^2 d\tau. \tag{10}$$

$E(t)$ collects all the remaining expressions.

Choose $\varepsilon > 0$ sufficiently small (compared with $e_1$ in (6)), set

$$h_1(t) := c_\varepsilon(1 + \|u_1(t)\|^4),$$

note that, due to the regularity assumption on $u_1, h_1 \in L^1(S)$ and put (5) and the estimates (6)-(9) together to arrive at

$$|w(t)|^2 + 2e_1 \int_0^t \|w(\tau)\|^2 d\tau \le |E(t)| + \int_0^t h_1(\tau)|w(\tau)|^2 d\tau. \tag{11}$$

The terms in $E(t)$ are easier to estimate than $C(t)$ and $D(t)$ and one obtains another function $h_2 \in L^1(S)$ such that for all $\varepsilon > 0$ there is a $c_\varepsilon > 0$ with

$$|E(t)| \le \varepsilon \int_0^t \|w(\tau)\|^2 d\tau + c_\varepsilon \int_0^t h_2(\tau)|w(\tau)|^2 d\tau. \tag{12}$$

(11), (12) and Gronwall's inequality imply uniqueness. The regularity statement follows by parabolic regularity and the observation that the leading coefficients are Hölder-continuous in $x$. □

## 6. References


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