Curves in $\mathbb{P}^3$ with good restriction of the tangent bundle

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Introduction

The stability of the tangent bundle $\Theta_{\mathbb{P}^3}$ of the projective space does not yield the stability of its restriction to an arbitrary space curve. On the one hand we expect that for a general curve X in $\mathbb{P}^3$ the restriction of $\Theta_{\mathbb{P}^3}$ to the curve X is stable. On the other hand we may ask what the instability of the restriction implies for the geometry of the curve and its embedding?
It is evident to consider this problem in the context of the Hilbert scheme $\mathrm{Hilb}(d, g)$ which parameterizes all space curves in $\mathbb{P}^3$ of degree d and arithmetic genus g. To measure the instability we use a slight generalization of the well known Harder-Narasimhan polygon. We obtain an upper semicontinuous map $\mathrm{HNP} : \mathrm{Hilb}(d, g) \to \{\text{polygons}\}$ with finite image by assigning each curve $[X]$ the Harder-Narasimhan polygon of $\Theta_{\mathbb{P}^3}|_X$. We ask what is the image of the map HNP?
The first subquestion naturally arising is to decide if the stable polygon lies in the image. Under the assumption $g \leq \frac{4}{3}d - 4$, we show that this is indeed true. Furthermore, we derive good upper bounds for the map HNP. These bounds allow us to describe the image of HNP, for curves of degree less or equal to six. Moreover, we are able to decide which special properties the curves in different strata have. Such special properties are low values of i-gonalities of a curve, secant lines of high order and being contained in a smooth quadric surface. To illustrate this we regard in 2.4 the Hilbert scheme $\mathrm{Hilb}(6, 3)$ which has two strata in the open set parametrizing smooth curves.
In the first section we repeat the well known definition of (semi)stability. However, we do not require the schemes to be reduced. Then we give with theorem 1.1 a variant of a theorem of S. Shatz (see [8]). We end the first section with the definition of a Shatz stratification of a Hilbert schemes defined by a vector bundle. Furthermore, we give two examples of Shatz stratifications which provide us with invariants of rank-2 vector bundles on projective space $\mathbb{P}^n$.
The second section is dedicated to the study of the Shatz stratification of the Hilbert scheme of space curves defined by the tangent bundle $\Theta_{\mathbb{P}^3}$ of the projective space

$\mathbb{P}^3$.
This work is part of the authors thesis [5] which was supervised by H. Kurke. The starting point was the work of R. Hernández [6] where embeddings of a fixed curve were studied. The case of rational curves has been studied by L. Ramella in [9]. There exists a preprint [2] of Ghione, Iarrobino and Sacchiero where they study the pullback of tangent bundle of projective space to a rational curve mapped to the $\mathbb{P}^n$. Partially this article may be considered as a continuation of [4]. The definition of stability given there applies only to reduced curves while we work with the natural definition. Theorem 2.6 contains the results of [4].
In fall 1995 I had the possibility to discuss this work with A. Hirschowitz in Nice. His idea to tackle the problem via stick figures seems to be very promising. The "easy" part is to construct stick figures with stable restricted tangent bundle, while the smoothability is up to now beyond the authors possibilities, even so, the problem can be reduced to the calculation of the rank of a certain linear map.
The author wishes to thank his thesis supervisor H. Kurke for many fruitful discussions.

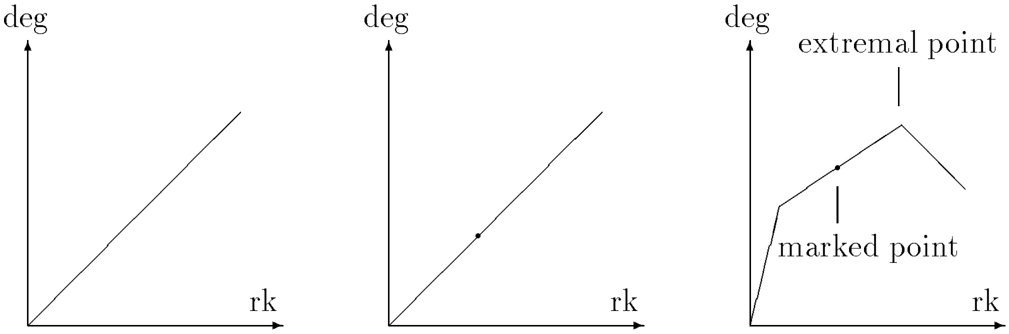
1 The Shatz stratification

1.1 Coherent sheaves on polarized curves

Let $(X, \mathcal{O}_X(1))$ be a one dimensional projective scheme equipped with an ample line bundle. We do not require that X is pure dimensional or reduced. If the Hilbert polynomial of the polarized scheme X is given by $\chi_X(n) = \chi(\mathcal{O}_X(n)) = dn+1-g$, we say that the pair $(X, \mathcal{O}_X(1))$ is of degree d and arithmetic genus g. The polarization of X allows the definition of the rank and the degree of a coherent X sheaf E. If the Hilbert polynomial of E is given by $\chi_E(n) = \chi(E(n)) = a_1 n + a_0$, then we define:

$$\mathrm{rk}(E) = \frac{a_1}{d} \qquad \deg(E) = a_0 - \chi(\mathcal{O}_X)\mathrm{rk}(E)\,.$$

These definitions coincide we the usual ones, if X is a smooth integral scheme. If the X-sheaf E has a positive rank, we define its slope $\mu(E)$ to be the quotient $\frac{\deg(E)}{\mathrm{rk}(E)}$.
A coherent X sheaf E is called (semi)stable, if for all proper subsheaves F of E the inequality $\deg(F)\mathrm{rk}(E)(\leq)\deg(E)\mathrm{rk}(F)$ holds. If both ranks are positive the last inequality is equivalent to the more convenient one $\mu(F)(\leq)\mu(E)$.
We will now define the *Harder-Narasimhan polygon* $\mathrm{HNP}(E)$, for any coherent X-sheaf. Our definition will differ from the usual one, because we add *marked points*. The Harder-Narasimhan polygon of the coherent sheaf E is the convex hull of the set $\{(\mathrm{rk}(F), \deg(F)) \,|\, F \subseteq E\}$. The points which determine the polygon are called *extremal points*. Those points which are not extremal but lying on the polygon are the marked points. We consider them as a part of the Harder-Narasimhan polygon.



HNP of a stable, a semistable, and a non semistable sheaf

Let E and E' be two coherent sheaves. We write $\mathrm{HNP}(E) \leq \mathrm{HNP}(E')$, if all extremal and marked points of E lie below the polygon $\mathrm{HNP}(E')$ or are extremal or marked points of $\mathrm{HNP}(E')$.
By standard arguments we have a unique filtration $0 \subset E_1 \subset E_2 \ldots \subset E_l = E$ of the coherent sheaf E, such that:

- The quotient E_{k+1}/E_k is semistable, for any $k = 0, \ldots, l-1$;
- The inequality $\mu(E_{k+1}/E_k) < \mu(E_k/E_{k-1})$ holds, for $k = 1, \ldots, l-1$.

This filtration is called the *Harder-Narasimhan filtration* of E. The points $\{(0,0), (\mathrm{rk}(E_1), \deg(E_1)), \ldots, (\mathrm{rk}(E), \deg(E))\}$ are the extremal points of $\mathrm{HNP}(E)$.

1.2 The Shatz stratification

Let $X \to S$ be a projective morphism of relative dimension one. If $\mathcal{O}_X(1)$ is an ample X-line bundle and $\mathcal{E}$ a S-flat X-modul we can define the the map

$$\begin{array}{rrcl} \mathrm{HNP}_{\mathcal{E}}: & S & \to & \{\text{polygons}\} \\ & s & \mapsto & \mathrm{HNP}(\mathcal{E} \otimes \mathcal{O}_{X_s})\,. \end{array}$$

Here X_s inherits the polarization of X.

Theorem 1.1 *Assume S to be a noetherian scheme, then the map $\mathrm{HNP}_{\mathcal{E}}$ has finite image and is upper semicontinuous, i.e. the sets $\bigcup_{P' \geq P} \mathrm{HNP}^{-1}(P')$ are closed.*

Proof: We easily reduce to the case where S is a connected scheme. Then the Hilbert polynomial of $\mathcal{E}_s = \mathcal{E} \otimes \mathcal{O}_{X_s}$ does not depend on the point $s \in S$. Let this Hilbert polynomial be given by $\chi_{\mathcal{E}_s}(n) = an + b$.
Step 1: We have to bound the Hilbert polynomial $\chi_F(n) = a'n + b'$ of all subsheaves F of the sheaves $\mathcal{E}_s$. Obviously we have $a' \leq a$ and $b' = \chi(F) \leq h^0(F) \leq h^0(\mathcal{E}_s)$. Since S is a noetherian scheme there exists an upper bound for the dimension $h^0(\mathcal{E}_s)$. Hence the set A of all points $(\mathrm{rk}(F), \deg(F))$ which lie not below the line through $(0,0)$ and $(\mathrm{rk}(\mathcal{E}), \deg(\mathcal{E}))$ is finite. This shows the finiteness of the map $\mathrm{HNP}_{\mathcal{E}}$.

Step 2: Let $A = \{(r_1, d_1), (r_2, d_2), \ldots, (r_n, d_n)\}$. We will show that the subsets S_i of S defined by

$$S_i = \{s \in S \mid \text{there exists a } F \subset \mathcal{E}_s \text{ with } \mathrm{rk}(F) = r_i \text{ and } \deg(F) = d_i\}$$

are closed, for $i = 1, \ldots, n$. Let the Hilbert polynomial of such an F be given by $\chi_F(n) = a_i n + b_i$. The Quot scheme Q_i of quotients G of $\mathcal{E}$ with Hilbert polynomial $\chi_G(n) = (a - a_i)n + (b - b_i)$ is a projective scheme over S. Hence its image S_i is a closed subset of S.
Step 3: Let P be a convex polygon whose extremal and marked points are in the set A. Let N be the set of integers from 1 to n, for which the point (r_i, d_i) is a marked or extremal point of the polygon P. By M we denote the set of all indicies of points (r_i, d_i) which are not lying below the polygon P. Then by definition of the Harder-Narasimhan polygon we have:

$$\mathrm{HNP}^{-1}(P) = \left(\bigcup_{i \in N} S_i\right) \setminus \left(\bigcup_{i \in M \setminus N} S_i\right).$$

This proofs our assertion. $\square$

1.3 Shatz stratification defined by a vector bundle

Let E be a vector bundle on a polarized scheme $(Y, \mathcal{O}_Y(1))$. Let Hilb be a Hilbert scheme of curves in Y with universal curve $\mathcal{C}$.

$$\mathrm{Hilb} \overset{p}{\leftarrow} \mathcal{C} \overset{\pi}{\rightarrow} Y$$

Then $\mathcal{E} = \pi^* E$ is a Hilb-flat $\mathcal{O}_{\mathcal{C}}$-modul. Hence by theorem 1.1 we obtain a stratification of the Hilbert scheme by a vector bundle E.

Example 1: Jumping lines of vector bundles (cf. [1])
Let E be a semistable vector bundle on $\mathbb{P}^n$ with $rk(E) = 2$ and $c_1(E) = 0$. Let $\mathrm{Hilb}(1,0)$ be the Grassmannian of all lines in $\mathbb{P}^n$. By the Grauert-Mülich theorem the restriction of E to the general line l of $\mathrm{Hilb}(1,0)$ is isomorphic to $\mathcal{O}_l \oplus \mathcal{O}_l$, hence semistable. The complement of this stratum is a divisor in the linear system $|c_2(E)H|$, where H is the pullback of a hyperplane via the Plücker embedding. This divisor is called the divisor of jumping lines of the vector bundle E.

Example 2: Jumping conics of vector bundles
Let E be a stable vector bundle on $\mathbb{P}^n$ with $\mathrm{rk}(E) = 2$ and $c_1(E) = -1$. The Hilbert scheme $\mathrm{Hilb}(2,0)$ is the scheme of all conics in $\mathbb{P}^n$, i.e. the Hilbert scheme of all closed subschemes X of $\mathbb{P}^n$ with Hilbert polynomial $\chi_X(n) = 2n + 1$. By the Grauert-Mülich theorem we again get the semistability of the restriction of E to the general conic.

If $n = 2$, the scheme $\mathrm{Hilb}(2,0)$ is isomorphic to the $\mathbb{P}^5$ and the jumping conics are parametrized by a divisor of degree $c_2(E) - 1$. This slightly generalizes a result of Hulek [7] where the considered conics are double lines.
If $n \geq 3$, we consider the Grassmannian $\mathrm{Grass}(2, \mathbb{P}^n)$ of planes in $\mathbb{P}^n$ together with the universal plane $\mathcal{P}$.

$$\mathrm{Grass}(2, \mathbb{P}^n) \xleftarrow{p} \mathcal{P} \xrightarrow{q} \mathbb{P}^n$$

If G is the dual of the vector bundle $p_* q^* \mathcal{O}_{\mathbb{P}^n}(2)$, then $\mathrm{Hilb}(2,0)$ is isomorphic to $\mathbb{P}(G)$. Hence $\mathrm{Hilb}(2,0)$ is a $\mathbb{P}^5$-bundle over $\mathrm{Grass}(2, \mathbb{P}^n)$:

$$\mathrm{Hilb}(2,0) \xrightarrow{\pi} \mathrm{Grass}(2, \mathbb{P}^n)\,.$$

Let $\mathcal{O}_{\mathrm{Grass}(2,\mathbb{P}^n)}(1)$ be the ample generator of the Picard group of $\mathrm{Grass}(2, \mathbb{P}^n)$, then the Picard group of $\mathrm{Hilb}(2,0)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ with the generators $\mathcal{O}_G(1)$ and $\pi^* \mathcal{O}_{\mathrm{Grass}(2,\mathbb{P}^n)}(1)$. By straightforward calculations we see that the jumping conics are the zero set of a global section of the line bundle $\mathcal{O}_G(c_2(E) - 1) \otimes \pi^* \mathcal{O}_{\mathrm{Grass}(2,\mathbb{P}^n)}(2c_2(E) - 1)$. Hence we get the divisor of jumping conics.

2 Restricting the tangent bundle to space curves

In this section we study a Shatz stratification of the Hilbert scheme $\mathrm{Hilb}(d, g)$ of space curves in $\mathbb{P}^3$ of degree d and arithmetic genus g. We first fix some notations. The projective space $\mathbb{P}^3$ is $\mathbb{P}(V)$ where V is a vector space of dimension four. Let E be the vector bundle $\Theta_{\mathbb{P}^3}(-1)$. For a space curve X in $\mathbb{P}^3$ we denote the restriction $E \otimes \mathcal{O}_X$ by E_X. From the Euler sequence restricted to X

$$0 \to \mathcal{O}_X(-1) \to V^\vee \otimes \mathcal{O}_X \to E_X \to 0$$

we see that E_X is a spanned vector bundle of degree $d = \deg(X)$.

2.1 Some subsheaves of E_X

We list here some typical subsheaves of E_X which reflect the geometry of the embedding. In many examples these subsheaves appear in the Harder-Narasimhan filtration of E_X.
The subsheaf defined by a point
If P is a geometric point of $\mathbb{P}(V)$, then P is given by a three dimensional subspace

W' of V. We define a quotient $F_P^\vee$ of $E_X^\vee$ via the diagram

$$\begin{array}{ccccccccc} & & & & 0 & \to & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & W'_X & \to & L & \to 0 & \\ & & & & \downarrow & & \downarrow & & \\ 0 \to & E_X^\vee & \to & & V_X & \to & \mathcal{O}_X(1) & \to 0 & \\ & \downarrow & & & \downarrow & & \downarrow & & \\ 0 \to & F_P^\vee & \to & & W_X & \to & \tau & \to 0\,. & \end{array}$$

This defines a subsheaf F_P of E_X of rank one and degree length(τ). Tensorizing the exact sequence $W' \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}^3_{\mathbb{P}}(1) \to k(P) \otimes \mathcal{O}^3_{\mathbb{P}}(1) \to 0$ with the structure sheaf of X we obtain $\tau \cong \mathcal{O}_{X \cap P}(1)$. So we have:

$$\deg(F_P) = \begin{cases} 0 & \text{if } P \notin X; \\ 1 & \text{if } P \in X. \end{cases}$$

The subsheaf defined by a line If $l \subset \mathbb{P}^3$ is a line not contained in X, then analogously we get a subsheaf F_l of rank two and degree length$(l \cap X)$.

Curves on smooth quadrics
Let Q be a smooth quadric in $\mathbb{P}^3$. Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Let X be a curve contained in Q of bidegree (a, b). The twisted tangent bundle $\Theta_Q(-1)$ is isomorphic to $\mathcal{O}_Q(1, -1) \oplus \mathcal{O}_Q(-1, 1)$. Hence we obtain two subline bundles L_1 and L_2 of E_X of degree $b - a$ and $a - b$. Since E_X is of degree $a + b$ we conclude that E_X can not be stable if $a \leq 2b$. The lines of bidegree $(1, 0)$ (resp. $(0, 1)$) are b-secant lines of X (resp. a-secants).

2.2 Two technical results

Let X be an irreducible curve in $\mathbb{P}^3 = \mathbb{P}(V)$ not contained in a hyperplane. Any subbundle G of E_X defines a morphism ϕ to the Grassmannian of subspaces of dimension rk(G) of $\mathbb{P}^3$. Here we will study these maps for subbundles of rank one and two. The results we obtain can be used to bound the degrees of these subbundles.

Let L be a subbundle of E_X of rank one. From the Euler sequence we obtain surjections $V^\vee \otimes \mathcal{O}_X \stackrel{\pi}{\to} E_X/L$ and $\Lambda^2 V^\vee \otimes \mathcal{O}_X \stackrel{\Lambda^2\pi}{\to} \det(E_X/L)$. These surjections define a morphism $\phi_\pi : X \to \text{Grass}(1, \mathbb{P}^3) \subset \mathbb{P}(\Lambda^2 V^\vee)$. By p we denote the dimension of the kernel of the homomorphism $H^0(\Lambda^2\pi) : \Lambda^2 V^\vee \to H^0(\det(E_X/L))$.

Lemma 2.1 *Suppose that in the image of the morphism ϕ_π there are two skew lines, then the inequality $p \leq 3$ holds. Moreover, p=3 implies that X lies on a smooth quadric Q and the line bundle L is induced by one of the rulings of Q.*

Proof: The number p gives the maximal codimension of a linear subspace Y of $\mathbb{P}^5 = \mathbb{P}(\Lambda^2 V^\vee)$ which contains the image of ϕ_π. Furthermore, we know that $\mathrm{im}(\phi_\pi)$ is contained in the Grassmannian $\mathrm{Grass}(1, \mathbb{P}^3)$. By construction of ϕ_π we have the incidence relation $x \in \phi_\pi(x)$, for any point x of X.
$p = 5$ would imply that the image is only one line l, and X is contained in l. This is impossible.
If we assume $p = 4$, then the image of ϕ_π is contained in a linear subspace Y of dimension one in $\mathrm{Grass}(1, \mathbb{P}^3)$. However, any two lines parameterized by such a subspace Y intersect. We conclude $p \leq 3$.
The rest of the proof is devoted to the case $p = 3$. Geometrically spoken we shall show that the intersection of the Grassmannian $\mathrm{Grass}(1, \mathbb{P}^3)$ with a linear subspace W of codimension three in $\mathbb{P}^5$ is a smooth conic defined by one of the rulings of a smooth quadric Q in $\mathbb{P}^3$, if $W \cap \mathrm{Grass}(1, \mathbb{P}^3)$ is irreducible and contains two skew lines.
Let $\{x^i \,|\, i = 0, \ldots, 3\}$ be a base of the vector space V. Let $\{e_i \,|\, i = 0, \ldots, 3\}$ be a the dual base for $V^\vee$ and $\{e_{ij} \,|\, 0 \leq i < j \leq 3\}$ the resulting base of $\Lambda^2 V^\vee$. Since there are two skew lines in the image of ϕ_π, we may assume the two lines defined by the direct sum decomposition $V = < x^0, x^1 > \oplus < x^2, x^3 >$ are in the image of ϕ_π. These two lines l_1 and l_2 correspond to the points $(1:0:0:0:0:0)$ and $(0:0:0:0:0:1)$ of $\mathbb{P}(\Lambda^2 V^\vee)$. The image of ϕ_π is contained in three hyperplanes given by the linear forms

$$h_i = a_{i0}e_{01} + a_{i1}e_{02} + a_{i2}e_{03} + a_{i3}e_{12} + a_{i4}e_{13} + a_{i5}e_{23},$$

for $1 \leq i \leq 3$. We define the matrix $\mathbf{A}$ by $\mathbf{A} = (a_{ij})_{(i=1,2,3\; j=0,\ldots,5)}$. Since $l_i \in \mathrm{im}(\phi_\pi)$ we have $a_{i0} = 0 = a_{i5}$. By Z we denote the set of all points lying on a line parameterized by $\mathrm{im}(\phi_\pi)$. In order to prove the lemma we show that Z is a smooth quadric. Since Z is dominated by a $\mathbb{P}^1$-bundle with base $\mathrm{im}(\phi_\pi)$ we see that Z is irreducible.
Let $P = (A : B : C : D) \in \mathbb{P}(V)$ be a given point. We obtain a linear map $\alpha_P : V^\vee \xrightarrow{\mathrm{B}} \Lambda^2 V^\vee \xrightarrow{\mathrm{A}} k^3$ defined by $\alpha_P(w) = (h_1(w \wedge P), h_2(w \wedge P), h_3(w \wedge P))$ where we write $w \wedge P$ for $w \wedge (Ae_0 + Be_1 + Ce_2 + De_3)$. There exists a line l containing P which lies in all three hyperplanes defined by h_i, if the kernel of α_P is at least of dimension two. In other terms: P is a point of Z iff the rank of the matrix product $\mathbf{A} \circ \mathbf{B}$ is at most two. The matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{pmatrix} -B & A & 0 & 0 \\ -C & 0 & A & 0 \\ -D & 0 & 0 & A \\ 0 & -C & B & 0 \\ 0 & -D & 0 & B \\ 0 & 0 & -D & C \end{pmatrix}.$$

Under the assumption that $A \neq 0$, a base of the kernel of $\mathbf{B}^t$ is given by:

$$\ker(\mathbf{B}^t) = < (0, D, -C, 0, 0, A),\ (D, 0, -B, 0, A, 0),\ (C, -B, 0, A, 0, 0) > .$$

Hence the condition $P \in Z$ is equivalent to

$$f(A,B,C,D) = \det \begin{pmatrix} 0 & D & -C & 0 & 0 & A \\ D & 0 & -B & 0 & A & 0 \\ C & -B & 0 & A & 0 & 0 \\ & & & \mathbf{A} & & \end{pmatrix} = 0\,.$$

f is a homogeneous form of degree three. Since $a_{i5} = 0$ the cubic form f is a product of a A with a quadric form $q(A,B,C,D)$. However, Z is not contained in a plane, so Z is contained in the quadric Q defined by q. This quadric Q must be irreducible. Q contains two skew lines l_1 and l_2. Hence Q is a smooth quadric. $\square$

Let F be a subbundle of E_X of rank two. We obtain a surjection $V^\vee \otimes \mathcal{O}_X \stackrel{\nu}{\to} E_X/F$ which defines a morphism $X \stackrel{\phi_\nu}{\to} \mathbb{P}(V^\vee)$. By q we denote the dimension of the kernel of $V^\vee \stackrel{H^0(\nu)}{\to} H^0(E_X/F)$. We have the following

Lemma 2.2 *The inequality $q \leq 2$ holds, and equality implies that F is induced by a $\deg(F)$-secant l of X.*

Proof: The image of ϕ_ν is contained in a linear subspace of $\mathbb{P}(V^\vee)$ of codimension q. Since X is not contained in a plane we have $q \leq 2$. However $q = 2$ implies that F is the subsheaf of E_X generated by the image of ν. This finishes the proof because of section 2.1. $\square$

2.3 Upper bounds for $\mathrm{HNP}(E_X)$

Let X be a smooth projective curve. For an integer $i \geq 2$, we define the $i-$*gonality* $m_i(X)$ of X to be the minimal degree of a line bundle L with $h^0(L) = i$. The classical gonality of X is the number $m_2(X)$. By Clifford's theorem we know that $m_i(X) \geq \min\{2i-2, 2g-2\}$. We need these i-gonalities to bound the degree of subbundles of E_X.

Theorem 2.3 *Let X be a smooth space curve in $\mathbb{P}^3$ not contained in a plane. Let L and F be subbundles of E_X of degree one and two.*

1. *The degree of the line bundle L satisfies $\deg(L) \leq \max\{1, d - m_3(X)\}$;*

2. *If X is not contained in a smooth quadric, then we have the stronger inequality $\deg(L) \leq \max\{1, d - m_4\}$;*

3. *If X is a curve of bidegree (a,b) in smooth quadric Q and $\deg(L) \neq \pm(a-b)$, then the inequality $\deg(L) \leq \max\{1, d - m_4\}$ holds;*

4. *For the degree of the rank two subbundle F the inequality $\deg(F) \leq d - m_2$ holds;*

5. *If X has no $\deg(F)$-secant, then we have the bound* $\deg(F) \leq d - m_3$.

Proof: Suppose that L is a line bundle contained in E_X. We consider the surjection $V^\vee \otimes \mathcal{O}_X \xrightarrow{\pi} E_X/L$. If $\deg(L) > 1$ there exist skew lines in the image of ϕ_π (see sections 2.1 and 2.2). So by lemma 2.1 we have $h^0(\det(E_X/L)) \geq 3$. Since $\deg(\det(E_X/L)) = d - \deg(L)$ and the very definition of $m_3(X)$ we conclude $\deg(L) \leq d - m_3(X)$.
The rest of the proof works analogously. □

This theorem gives good bounds for space curves of low degree. As a corollary we obtain the following table, where we list all possible Harder-Narasimhan polygons for smooth space curves X of genus g with degree d. By $P_{d_1,d_2,d}$ we mean the polygon with marked and extremal points $(0,0)$, $(1,d_1)$, $(2,d_2)$ and $(3,d)$. We write $P_{*,d_2,d}$ if there exists no marked or extremal point with rank one. Hence in this notation the Harder-Narasimhan polygon of a stable bundle is $P_{*,*,d}$.

(d,g)	$d_1 \leq$	$d_2 \leq$	polygon	remarks
$(3,0)$	1	2	$P_{1,2,3}$	$E_X \cong \mathcal{O}_X(P)^{\oplus 3}$, for a geometric point $P \in X$.
$(4,0)$	2	3	$P_{2,3,4}$	$E_X \cong \mathcal{O}_X(2P) \oplus \mathcal{O}_X(P)^{\oplus 2}$
$(5,0)$	3	4	$P_{3,4,5}$	$E_X \cong \mathcal{O}_X(3P) \oplus \mathcal{O}_X(P)^{\oplus 2}$, if the curve
				X lies on a smooth quadric,
			$P_{2,4,5}$	$E_X \cong \mathcal{O}_X(P) \oplus \mathcal{O}_X(2P)^{\oplus 2}$ otherwise.
$(4,1)$	1	2	$P_{*,*,4}$	E_X is stable.
$(5,1)$	1	3	$P_{*,*,5}$	E_X is stable.
$(6,1)$	2	4	$P_{2,4,6}$	E_X is not stable but semistable.
$(5,2)$	1	3	$P_{*,*,5}$	E_X is stable.
$(6,3)$	2	4	$P_{2,4,6}$	E_X is not stable but semistable;
				$\Leftrightarrow$ X is hyperelliptic;
				$\Leftrightarrow$ X lies on a smooth quadric;
				$\Leftrightarrow$ X has a 4-secant.
			$P_{*,*,6}$	E_X is stable.

2.4 The Hilbert scheme $Hilb(6,3)$ and its stratification

We want to study the last part of the above table more detailed. In the case of the Hilbert scheme Hilb(6, 3) we can illustrate how the Harder-Narasimhan polygon reflects the geometry of the curve and its embedding.

Lemma 2.4 *Let $X \subset \mathbb{P}^3$ be a smooth space curve of genus $g = 3$ and degree $d = 6$, then the restriction E_X of the vector bundle E to X is semistable. Moreover, the following conditions are equivalent:*

1. *E_X is not stable;*
2. *E_X contains a line bundle of degree 2;*
3. *E_X contains a rank-2 vector bundle of degree 4;*
4. *X lies on a smooth quadric;*
5. *X is contained in a quadric.*
6. *X has 4-secant;*
7. *X is hyperelliptic.*

Proof: First we remark that the map $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{O}_X(1))$ is an isomorphism. By 2.3 we see that the vector bundle E_X is semistable.
2. $\Rightarrow$ 4. See lemma 2.1.
4. $\Rightarrow$ 5. trivial.
5. $\Rightarrow$ 2. If X is lying on a quadric the map $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_X(2))$ is not injective. Since both cohomology groups have the same dimension the map is not surjective. Hence the map

$$H^0(\mathcal{O}_X(1)) \otimes H^0(\mathcal{O}_X(1)) \to H^0(\mathcal{O}_X(2))$$

is not surjective. By Serre duality we conclude that the dual homomorphism

$$\mathrm{Ext}^1(\mathcal{O}_X(1), \omega_X(-1)) \to \mathrm{Hom}[H^0(\mathcal{O}_X(1)), H^1(\omega_X(-1))]$$

is not injective. So we find a nontrivial extension

$$0 \to \omega_X(-1) \to F \to \mathcal{O}_X(1) \to 0\,,$$

which is exact with respect to the functor H^0. Hence we get the following diagram:

$$\begin{array}{ccccccccc} & & & & V \otimes \mathcal{O}_X & = & V \otimes \mathcal{O}_X & & \\ & & & & \downarrow \psi & & \downarrow & & \\ 0 \to & & \omega_X(-1) & \to & F & \to & \mathcal{O}_X(1) & & \to 0\,. \end{array}$$

The ker-coker sequence of the above diagram is

$$0 \to \ker \psi \to E_X^\vee \xrightarrow{\partial} \omega_X(-1) \to \mathrm{coker}\psi \to 0\,.$$

Since F is a nontrivial extension the morphism ∂ can not be zero. Hence we get a morphism $\partial^\vee : \omega_X^\vee(1) \to E_X$. The degree of $\omega_X^\vee(1)$ is 2.

4. $\Rightarrow$ **6.** If X is on a smooth quadric Q, then it must be of bidegree $(2,4)$. Hence any line of type $(1,0)$ on Q is a 4-secant line of X.
6. $\Rightarrow$ **7.** If l is a 4-secant line of X, then the projection $\mathbb{P}^3 \setminus l \to \mathbb{P}^1$ induces a degree-2 morphism from X to $\mathbb{P}^1$.
7. $\Rightarrow$ **2.+3.** Let D be a hyperelliptic divisor on X, i.e. $\deg(D) = 2 = h^0(\mathcal{O}_X(D))$. Twisting the Euler sequence

$$0 \to E_X^\vee \to V^\vee \otimes \mathcal{O}_X \mathcal{O}_X(1)$$

by $\mathcal{O}_D$ and taking cohomology we get $\dim(\mathrm{Hom}(E_X, \mathcal{O}_D)) \geq 2$. We take two linear independent homomorphisms α and β of $\mathrm{Hom}(E_X, \mathcal{O}_X(D))$. Obviously we have $\ker(\alpha)$ is a rank two subbundle of degree 4 while $\ker(\alpha) \cap \ker(\beta)$ is a invertible subbundle of rank 2 because E_X is semistable.
3. $\Rightarrow$ **6.** This follows by theorem 2.3. □

If we denote by $\mathrm{Hilb}^0(6,3)$ the open subset of the Hilbert scheme $\mathrm{Hilb}(6,3)$ parameterizing smooth space curves, then we have seen that the Shatz stratification of $\mathrm{Hilb}^0(6,3)$ consists of exactly two strata: The open stratum $HNP_E^{-1}(P_{*,*,6})$ and the closed stratum $D_1 = HNP_E^{-1}(P_{2,4,6})$. Using lemma 2.4 we can determine the structure of D_1 equipped with the reduced subscheme structure:

Lemma 2.5 *The stratum D_1 is a irreducible smooth divisor in* $\mathrm{Hilb}^0(6,3)$.

Proof: We proceed by steps.
Step 1: D_1 is irreducible
Let Q be the fiber product $\mathbb{P}^1 \times \mathbb{P}^1$ and D a divisor of bidegree $(2,4)$ on the surface Q. By U_0 we denote the open subset of $|D|$ corresponding to smooth curves. By W we denote the vector space $H^0(\mathcal{O}_Q(1,1))$. An open subset U_1 of $\mathbb{P}([W^{\oplus 4}]^\vee)$ parameterizes smooth embeddings of Q as a quadric hypersurface in $\mathbb{P}^3$. Hence we get a morphism $U_0 \times U_1 \to \mathrm{Hilb}^0(6,3)$. The image of these morphism is D_1 by lemma 2.4. This proves the irreducibility of D_1.
Step 2: D_1 is a divisor
We consider the morphisms:

$$\mathrm{Hilb}^0(6,3) \xleftarrow{p} \mathrm{Hilb}^0(6,3) \times \mathbb{P}^3 \xrightarrow{q} \mathbb{P}^3 .$$

By $\mathcal{J}$ we denote the ideal sheaf of the universal subscheme $\mathcal{C} \subset \mathrm{Hilb}^0(6,3) \times \mathbb{P}^3$. We obtain the following long exact sequence:

$$0 \to p_*(\mathcal{J} \otimes q^*\mathcal{O}_{\mathbb{P}^3}(2)) \to p_*(\mathcal{O}_{\mathrm{Hilb}^0(6,3)\times\mathbb{P}^3} \otimes q^*\mathcal{O}_{\mathbb{P}^3}(2)) \xrightarrow{\gamma}$$

$$\xrightarrow{\gamma} p_*(\mathcal{O}_{\mathcal{C}} \otimes q^*\mathcal{O}_{\mathbb{P}^3}(2)) \to R^1p_*(\mathcal{J} \otimes q^*\mathcal{O}_{\mathbb{P}^3}(2)) \to 0 .$$

By lemma 2.4 the support of $\mathrm{coker}(\gamma)$ is D_1. Since γ is a morphism of two rank-10 vector bundles it defines a section

$$s_\gamma = \Lambda^{10}\gamma \in H^0(\det[p_*(\mathcal{O}_{\mathrm{Hilb}^0(6,3)\times\mathbb{P}^3} \otimes q^*\mathcal{O}_{\mathbb{P}^3}(2))]^{-1} \otimes \det[p_*(\mathcal{O}_{\mathcal{C}} \otimes q^*\mathcal{O}_{\mathbb{P}^3}(2))]) .$$

D_1 is the degeneration set of s_γ, consequently D_1 is a divisor.
Step 3: The divisor D_1 is smooth
Since $\mathrm{Hilb}^0(6,3)$ is smooth of dimension 24, it is enough to compute the dimension of the tangent space of $[X] \in D_1$. By lemma 2.4 we know that X is contained in a smooth quadric Q. Moreover, this quadric is unique. Hence the tangent space of $[X]$ in D_1 is isomorphic to flat families of closed immersions $\tilde{X} \to \tilde{Q} \to \mathbb{P}^3 \times \mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$ over $\mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$ with special fiber $X \to Q \to \mathbb{P}^3$.
The deformations of Q form a vector space of dimension 9, because the linear system $|Q|$ is of dimension 9.
Now we consider a given a deformation $\tilde{Q} \to \mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$ of the quadric Q. Since Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ we see that $\tilde{Q}$ is isomorphic to $Q \times \mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$. Hence the deformations of X in $\tilde{Q}$ are isomorphic to deformations of X in Q. The Q-linear system $|X|$ is of dimension 14. This finishes the proof. □

2.5 Curves with good restricted tangent bundle

Now we define the condition *having a good restricted tangent bundle* for space curves which we will study in this part of the paper.
If the tangent bundle E_X is semistable and the cohomology group $H^1(E_X(1)) = H^1(\Theta_{\mathbb{P}^3} \otimes \mathcal{O}_X)$ vanishes, then we call X a curve with good restricted tangent bundle.
Let X be a space curve of degree d and genus g. Suppose that X is connected and not contained in a plane[1]. From the Euler sequence we directly conclude $h^0(E_X(1)) \geq 15$. Hence for a curve with good restricted tangent bundle we must have

$$h^0(E_X(1)) = \chi(E_X(1)) = 4\chi(\mathcal{O}_X(1)) - \chi(\mathcal{O}_X) = 4d - 3(g-1) \geq 15\,.$$

Hence the condition $g \leq \frac{4}{3}d - 4$ is necessary for curves with good restricted tangent bundle. The following result shows that this condition is also sufficient.

Theorem 2.6 *Let (d,g) be a pair of positive integers satisfying the inequality $g \leq \frac{4}{3}d - 4$, then there exist a smooth space curve X of degree d and genus g with good restricted tangent bundle. Moreover, for $g \geq 2$ and $d \neq 6$, there exists such a curve X with stable restricted tangent bundle E_X.*

Proof: We prove this theorem inductively. Having a curve X with good restricted tangent bundle we "produce" a new one $Z = X \cup Y$ where Y is a twisted cubic curve. Then we show that Z has good restricted tangent bundle. Moreover, Z may be deformed to a smooth curve Z' which has good restricted tangent bundle.
We say that $(*)$ holds for a pair of integers (d,g), if there exists a smooth space curve X of degree d and genus g with good restricted tangent bundle E_X. If, furthermore, E_X is stable we say that $(**)$ holds for the pair (d,g).
Step 1: The twisted cubic curve
The twisted cubic curve X will play a central role in our proof. We know from

[1] If X is contained in a plane H the rank-2 subbundle of E_X defined by H destabilizes E_X.

part 2.3 that E_X is semistable. If P is an arbitrary point of X, then the vector bundle $E_X(1)$ is isomorphic to $\mathcal{O}_X(4P)^{\oplus 3}$. The degree of the normal bundle N_X is 10. We now show that N_X is isomorphic to $\mathcal{O}_X(5P)^{\oplus 2}$. We suppose that N_X would not be isomorphic to this direct sum. Since N_X is a quotient of $E(1)$ we conclude $N_X \cong \mathcal{O}_X(6P) \oplus \mathcal{O}_X(4P)$. From the unique surjection $N_X \to \mathcal{O}_X(4P)$ we obtain the commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 \to & \Theta_X & \to & F & \to & \mathcal{O}_X(6P) & \to 0 \\ & \| & & \downarrow & & \downarrow & \\ 0 \to & \Theta_X & \to & E_X(1) & \to & N_X & \to 0 \end{array}$$

The subbundle F of $E_X(1)$ contains the tangent bundle of X and determines a rank-2 subbundle $F(-1)$ of E_X. However such a subbundle is by theorem 2.3 given by a 2-secant line l of X. We consider the morphism $X \xrightarrow{\phi} \mathbb{P}^1$ induced by the projection of $\mathbb{P}^3$ from the line l. Since F contains the tangent sheaf of X the morphism $\phi^*\omega^1_{\mathbb{P}} \to \omega_X$ is trivial. Hence X is contained in a plane. This is impossible, so we see that $N_X \cong \mathcal{O}_X(5P)^{\oplus 2}$.

Step 2: (**) holds for (4,1), (5,1), (5,2), (6,3) and (6,4)
We first show the stability of the restriction of $\Theta_{\mathbb{P}^3}$ to the general curve. By the table given in part 2.3 and the study of Hilb$(6,3)$ it remains to show this for the pair $(6,4)$.
If X is a curve of bidegree (3,3) on a smooth quadric Q, we see directly that its degree and genus are 6 and 4. We easily see that the for 4-gonality $m_4(X) \geq 5$ holds. So we see by the theorem 2.3, that any line bundle $L \subset E_X$ is at most of degree one. Any line $l \subset \mathbb{P}^3$ not contained in Q is at most a 2-secant line of X. So we see that X-has no k-secant lines, for $k > 3$. Obviously the 3-gonality $m_3(X)$ satisfies by Cliffords theorem $m_3(X) \geq 4$. Hence again by theorem 2.3 we conclude that the maximal degree of a rank-2 subbundle of E_X is three. This shows the stability of E_X.
It remains to show the vanishing of the first cohomology group $H^1(E_X(1))$ in the above cases. We prove it only for the case (6,4) because the proof is the same in all cases. Suppose X is a curve of degree 6 and genus 4. If $h^1(E_X(1)) > 0$, then we obtain by Serre duality $h^0(E_X^\vee(-1) \otimes \omega_X) > 0$. However, $E_X^\vee(-1) \otimes \omega_X$ is a stable bundle of degree -6, which is a contradiction.

Step 3: (*) holds for the pairs (6,1) and (6,2)
By the table of part 2.3 it is enough to consider the case (6,2).
Let X be a curve of degree 6 and genus 2 in $\mathbb{P}^3$. We see that X can not be contained in a smooth quadric Q. The 4-gonality of X is given by $m_4(X) = 5$. Hence by theorem 2.3 the maximal degree of a line bundle contained in E_X is one. Also by theorem 2.3 we see that the maximal degree of a rank-2 subbundle of E_X is four. So E_X is semistable.[2]

[2]If X is a curve of degree 6 and genus 2, then X has in fact a 4-secant line. So E_X can not be stable. Indeed, such a curve has to be contained in s smooth cubic surface Y and by simple

Step 4: If X is a smooth curve in $\mathbb{P}^3$, then there exists a twisted cubic Y intersecting X quasitransversal in exactly k points, for any $k = 1, \ldots, 5$.
First we choose k points $P_1, \ldots, P_k$ on X which are in general position with respect to quadrics. Then we choose a smooth quadric Q containing these k points but no line connecting two of them. In the linear system of type $(1,2)$ on this quadric we choose a curve Y containing the k points. Y is irreducible and therefore smooth. But Y may intersect X in more than k points. Since Y is a twisted cubic curve we see that the twisted normal bundle $N_Y(-P_1 - \ldots - P_k)$ is globally generated and has no first cohomology. The global sections of $N_Y(-P_1 - \ldots - P_k)$ describe the tangent space of flat deformations of Y which pass through the points $P_1, \ldots, P_k$. Hence there exists a deformation of Y satisfying the above condition.
Step 5: If (*) (resp. (**)) holds for the pair (d, g) then it holds for the pair $(d+3, g+k-1)$, for $k = 1, \ldots, 5$.
Let X be a space curve of degree d and genus g such that the vector bundle $E_X(1)$ is semistable and the group $H^1(E_X(1))$ vanishes. Now we choose a twisted cubic Y intersecting X in exactly k points. Let Z be the union of the curves X and Y. From the exact sequence

$$0 \to \mathcal{O}_Z \to \mathcal{O}_X \oplus \mathcal{O}_Y \to \bigoplus_{i=1}^{k} k(P_i) \to 0$$

we see that Z has degree $d+3$ and arithmetic genus $g+k-1$. The semistability of E_X and E_Y implies the semistability of E_Z. Since $k \leq 5$ we obtain $H^1(E_Z(1) \otimes \mathcal{O}_Y(-P_1 - \ldots P_k)) = 0$. Hence $E_Z(1)$ has no first cohomology. Therefore (See [3] theorem 1.2.) Z can be deformed to a smooth curve Z'. Since semistability of the tangent bundle and vanishing of cohomology groups are open conditions we are done.
Step 6: (**) holds for (9,2)
We choose two elliptic space curves X and Y of degree 4 and 5 which intersect in exactly one point. The union $X \cup Y$ is of degree 9 and genus 2. Now we proceed like in the previous step. □

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numerical computations we find that exactly one of the 27 lines on Y is a 4-secant line of X.

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