Duality Construction of Moduli Spaces

Georg Hein

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Introduction

In §1 of Faltings' article [5] a "GIT-free" construction is given for the moduli spaces of vector bundles on curves using generalized theta functions. Incidentally, this construction is implicitly described in Le Potier's article [12]. The aim of this paper is to generalize the *duality construction* to projective surfaces.

For a rank two vector bundle E on the projective plane $\mathbb{P}^2$, the divisor D_E of its jumping lines is a certain generalization of the Chow divisor of a projective scheme. We give a generalization of this divisor for coherent sheaves on surfaces. Using this duality we construct the moduli space of coherent sheaves on a surface that does not use Mumford's geometric invariant theory. Furthermore, we obtain a finite morphism from this moduli space to a linear system, which generalizes the divisors of jumping lines.

Applying this construction to curves, we get exactly Faltings' construction. The moduli space we construct here can also be obtained by using GIT. This construction is carried out in §8.2 of the book [9] of Huybrechts and Lehn. Le Potier obtained this moduli space in [11] for surfaces with "many lines" (see §4 for an exact definition). However, it is the modest hope of the author that the construction presented here provides new insight into the geometry of moduli spaces.

First we outline this concept, which generalizes the famous *strange duality* to moduli of coherent sheaves on surfaces. To do so we define duality between schemes in part 1, giving three examples of "natural dualities". In section 2 the duality construction is given. In order to avoid a too technic presentation of the construction itself, we defer the proofs to the following section. The last section is dedicated to the Barth morphism.

In order to simplify the discussion we restrict ourselves to moduli spaces of sheaves of rank two with trivial determinant. The interested reader will be able to extend this to arbitrary rank and determinant.

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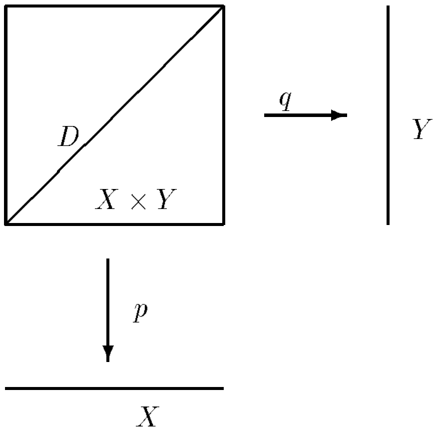
1 Duality of schemes

1.1 Definitions

Let $(X, \mathcal{O}_X(D_X))$ and $(Y, \mathcal{O}_Y(D_Y))$ be two schemes with line bundles. A duality between these two pairs is given by a nontrivial section $s \in H^0(\mathcal{O}_X(D_X)) \otimes H^0(\mathcal{O}_Y(D_Y))$. We will identify s with its vanishing divisor $D = V(s) \subset X \times Y$.



We obtain a rational morphism

$$\begin{array}{rcccc} s_X : & X & - - - \to & |D_Y| \\ & x & \mapsto & q(D \cap p^{-1}(x)) \, . \end{array}$$

The base locus of this morphism s_X consists of all points x of X such that the vertical component $x \times Y$ is contained in D. This motivates the following

Definition: The duality D between (X, D_X) and (Y, D_Y) is called
generated, if s_X is a morphism;
generated ample, if s_X is a finite morphism;
very ample, if s_X is an embedding.

1.2 Examples

The first example demonstrates that the above definitions are something with which we are familiar.

The duality of a linear system Let X be a given scheme with an effective divisor D_X, and let $Y \subset |D_X|$ be a linear system. We take D to be the incident divisor, i.e. $D = \{(x, H) \,|\, x \in H\}$. Then the notions for D given in the above definition correspond to those for the linear system.

Strange duality Let C be a smooth projective curve of genus g over the complex numbers. We fix two positive integers m and n and a theta characteristic A, i.e. $A \in \mathrm{Pic}^{g-1}(C)$ and $A^{\otimes 2} \cong \omega_C$. We consider the following moduli schemes.

$$\begin{array}{rcl} X & = & \mathrm{U}_C(n, n(g-1)) \\ & = & \left\{ E \;\middle|\; \begin{array}{l} E \text{ semistable } C\text{-vector bundles with} \\ \mathrm{rk}(E) = n \text{ and } \deg(E) = n(g-1) \end{array} \right\} \\ D_X & = & \{E \in X \,|h^0(E) = h^1(E) > 0\} \\ Y & = & \mathrm{SU}_C(m) \\ & = & \left\{ F \;\middle|\; \begin{array}{l} F \text{ semistable } C\text{-bundles with} \\ \mathrm{rk}(E) = m \text{ and } \det(E) \cong \mathcal{O}_C \end{array} \right\} \\ D_Y & = & \{F \in Y \,|h^0(F \otimes A) = h^1(F \otimes A) > 0\} \\ D & = & \{(E, F) \,|\, h^0(E \otimes F) = h^1(E \otimes F) > 0\} \end{array}$$

The line bundle $\mathcal{O}_{X \times Y}(D)$ is isomorphic to $p^*\mathcal{O}_X(D_X)^{\otimes m} \otimes q^*\mathcal{O}_Y(D_Y)^{\otimes n}$. By the duality we obtain a linear map from $H^0(X, \mathcal{O}_X(D_X)^{\otimes m})^\vee \xrightarrow{s} H^0(Y, q^*\mathcal{O}_Y(D_Y)^{\otimes n})$. According to the Verlinde formula, both spaces have the same dimension. The natural conjecture that s is an isomorphism is called the *Strange Duality Conjecture*. For more details on this topic, see Beauville's survey article [2].

Duality between moduli spaces on polarized surfaces The next example is our main example, the notions of which will be used for the remainder of the article. We describe the moduli spaces here only set theoretically. Their construction uses the duality, thus giving a rough idea of the construction which is the object of the following section. The concept of semistability used here is the Mumford (slope) semistability.
Let $(S, \mathcal{O}_S(1) = \mathcal{O}_S(H))$ be a projective polarized surface. Fix a class $c_2 \in H^4(S, \mathbb{Z})$. We will consider a duality between the following two coarse moduli spaces:

$$\begin{array}{rcl} X & = & M_S(2, 0, c_2) \\ & = & \left\{ E \;\middle|\; \begin{array}{l} E \text{ semistable torsion free sheaf on } S, \text{ with} \\ \mathrm{rk}(E) = 2 \quad \det(E) \cong \mathcal{O}_S \quad c_2(E) = c_2 \end{array} \right\} \end{array}$$

$$\begin{array}{rcl} Y & = & M_{|H|}(2, \omega_{|H|}) \\ & = & \left\{ F \;\middle|\; \begin{array}{l} F \text{ semistable torsion sheaf on } S, \text{ with} \\ Z = \mathrm{supp}(F) \in |H| \quad \mathrm{rk}_Z(F) = 2 \quad \det_Z(F) \cong \omega_Z \end{array} \right\} \end{array}$$

The duality will be given by the $X \times Y$ divisor

$$D = \{(E, F) | H^*(E \otimes F) \neq 0\}\,.$$

2 The duality construction

Using the notation introduced in the last example, we give a construction of the coarse moduli scheme X using the duality morphism s. X will be obtained together with a polarization and the *Barth morphism*, which will be finite by construction. The steps for this construction are listed below, and proofs for all pertinent theorems are provided in the next section.

Boundedness of X There exists a projective scheme Q and a torsion free sheaf $\mathcal{E}$ on $Q \times S$ flat over Q which (over)parameterizes the moduli problem. More precisely denote by p, q the two projections

$$Q \xleftarrow{p} Q \times S \xrightarrow{q} S\,.$$

For any sheaf E of X let Q_E be the subscheme

$$Q_E = \{q \in Q \mid \mathcal{E}_q \cong E\}$$

of Q. All Q_E are required to be connected and nonempty. Since this is the same starting point as in the GIT construction, it is obvious that Q can be taken to be a suitable Quot scheme (see [6]).

Elements of Y give sections in a Q-line bundle $\mathcal{L}$ (see §3.1)

We will show that there exists a Q-line bundle $\mathcal{L}$ and a global section $s_F \in H^0(Q, \mathcal{L})$, for any $F \in Y$. The vanishing locus of s_F is given by

$$V(s_F) = \{q \in Q \mid H^*(S, \mathcal{E}_q \otimes F) \neq 0\}\,.$$

Base points correspond to unstable objects (see §3.2)

The base locus $B(\mathcal{L})$ with respect to the sections given by Y is the scheme theoretic intersection of all $V(s_F)$ for all $F \in Y$. Since Q is noetherian we can write

$$B(\mathcal{L}) = \bigcap_{i=0}^{N} V(s_{F_i})\,.$$

It will be shown that $B(\mathcal{L})$ consists exactly of those points $q \in Q$ for which the sheaf $\mathcal{E}_q$ is not semistable.

Properness of X (see §3.3)

We have to show that semistable limits of semistable families exist.

The line bundle $\mathcal{L}$ is X-positive (see §3.4)

It will be shown that the degree of $\mathcal{L}$ on a curve C parameterizing semistable objects is zero only if the curve parameterizes Jordan-Hölder equivalent sheaves.

The duality construction The rational morphism $s = (s_{F_0} : \ldots : s_{F_N})$ from Q to $\mathbb{P}^N$ leads to a morphism $\bar{Q} \xrightarrow{\varphi} \mathbb{P}^N$ after a blow up of Q. We consider the following diagram

$$\begin{tikzcd} \bar{Q} \arrow[r, "\varphi_0"] \arrow[d, "\pi"] \arrow[dr, "\varphi"] & X \arrow[d, "\varphi_1"] \\ Q \arrow[r, dotted, "s"] & \mathbb{P}^N \end{tikzcd}\,.$$

Here $\varphi = \varphi_1 \circ \varphi_0$ is the Stein factorization of φ. Hence the Barth morphism φ_1 is finite. By the above, any point of X corresponds to exactly one Jordan-Hölder equivalence class of semistable bundles.

3 Details and proofs

3.1 The Q-line bundle $\mathcal{L}$ and its invariant sections

$\mathcal{L}$ is defined to be the determinant bundle $\mathcal{L} = det(p_!(\mathcal{E} \otimes q^*F))^{-1}$. The definition does not depend on the choice of $F \in Y$, because these elements coincide in the Grothendieck group $K_0(S)$.

Framed elements of Y give sections in $\Gamma(\mathcal{L})$
Let F be a semistable element in Y with support $Z \in |H|$. Then there exists a short exact sequence

$$0 \to F \to \mathcal{O}_Z(M)^{\oplus 3} \xrightarrow{\alpha} \omega_Z^\vee(3M) \to 0\,.$$

We remark that $M >> 0$ can be chosen for all $F \in Y$. Define s_F to be the section $\det(Rp_*(\mathcal{E} \otimes q^*\alpha))$. It is clear from the construction that the vanishing divisor $V(s_F)$ is supported on those $z \in Q$, for which $H^*(\mathcal{E}_z \otimes q^*F)$ is not zero.
Remark: By abuse of notation we simply write s_F and do not explicitly refer to the framing.

Global sections in $\mathcal{L}^{\otimes k}$
Let $\tilde{F}$ be a rank two vector bundle on a curve $\tilde{Z}$ from the linear system $|kH|$. We require the determinant of $\tilde{F}$ to be isomorphic to $\omega_{\tilde{Z}}$. Using adjunction to express $\omega_{\tilde{F}}$ and ω_F, the following computation in the Grothendieck group $K_0(S)$ shows that $[\tilde{F}] = k[F]$ for any $F \in Y$:

$$\begin{aligned}
[\tilde{F}] &= ([\mathcal{O}_S] - [\mathcal{O}_S(-kH)])([\mathcal{O}_S] + [K_S(kH)]) \\
&= ([\mathcal{O}_S] - [\mathcal{O}_S(-H)])\left(\sum_{i=0}^{k-1}[\mathcal{O}_S(-iH)])([\mathcal{O}_S] + [K_S(kH)]\right) \\
&= ([\mathcal{O}_S] - [\mathcal{O}_S(-H)])k([\mathcal{O}_S] + [K_S(H)]) \;=\; k[F]\;.
\end{aligned}$$

Hence $\tilde{F}$ (together with a framing) defines a global section in $\mathcal{L}^{\otimes k}$.

3.2 Semistability

Assume that H is big enough, which means that $\mathcal{O}_S(H)$ is globally generated, and the following two conditions hold:
(i) $H^2 > 4c_2$, and
(ii) The positive generator a of the $\mathbb{Z}$ ideal $\{D.H \mid D \in \mathrm{NS}(S)\}$ satisfies $a > c_2$.
Under these assumptions we have the following

Theorem/Definition 3.1 *For a torsion free rank two S-sheaf E with $\det(E) \cong \mathcal{O}_S$ and $c_2(E) = c_2$ the following four conditions are equivalent:*

1. *There exists an $F \in Y$ such that $H^*(S, E \otimes F) = 0$;*
2. *For all rank one subsheaves $M \subset E$, the inequality $c_1(M).H \leq 0$ holds;*
3. *The restriction of E to a general divisor $Z \in |H|$ is semistable, i.e. all Z-line bundles contained in E_Z have nonpositive degree;*

4. For $Z \in |H|$ general, there exists a Z-line bundle A such that $H^*(Z, L \otimes E|_Z) = 0$.

If one of these conditions is satisfied we call E a semistable S-sheaf.

Proof: **(1) $\Rightarrow$ (2)** Suppose there exists an $M \subset E$ such that $c_1(M).H > 0$. Then M restricted to $Z = \mathrm{supp}(F)$ is of positive degree. Therefore the Euler characteristic of $M \otimes F$ is positive. Since the sheaf is one-dimensional there are global sections. Hence there are global sections in $H^0(E \otimes F)$, which contradicts the assumption of (1).
(2) $\Rightarrow$ (3) This is a restriction theorem that follows from Bogomolov's inequality (see [3]). For a complete proof of this implication see [9] theorem 7.3.5.
(3) $\Rightarrow$ (4) This result goes back to Raynaud ([14]). For a shorter proof see [7].
(4) $\Rightarrow$ (1) Denote the genus of Z by g. It follows by Riemann-Roch that $A \in \mathrm{Pic}^{g-1}(Z)$. The condition $H^*(E|_Z \otimes A) = 0$ is satisfied on a nonempty open subset of the Jacobian $\mathrm{Pic}^{g-1}(Z)$. Hence the condition $H^*(E|_Z \otimes \omega_Z \otimes A^{-1}) = 0$ defines an open and nonempty subset of the Jacobian. Consequently we can choose F to be a direct sum $A \oplus (\omega_Z \otimes A^{-1})$. $\square$

3.3 Properness

Although the properness of the moduli functor X is well known, a new proof is given here which is shorter than Langton's original proof in [10]. The main idea to get "more and more" semistable extensions of a generic semistable family by elementary transformations comes from Langton's proof. However, using the invariant functions, we can control the maximal number of elementary transformations required. Therefore the proof fits into the concept of the duality construction.

Theorem 3.2 *([10]) Let R be a discrete valuation ring with* $\mathrm{Spec}(R) = \{0, \eta\}$. *Let $\mathcal{E}_\eta$ be a semistable torsion free sheaf on $\eta \times S$. Then there exists an extension $\mathcal{E}_R$ of $\mathcal{E}_\eta$ which is semistable in the special fiber as well.*

Proof: We consider the following morphisms:

$$\mathrm{Spec}(R) \xleftarrow{p} \mathrm{Spec}(R) \times S \xrightarrow{q} S \quad .$$

Since the Quot scheme is projective, there exist torsion free extensions of $\mathcal{E}_\eta$. For an extension $\mathcal{E}$, we define its badness[1] $b(\mathcal{E}, F)$ with respect to an $F \in Y$ as

$$b(\mathcal{E}, F) = \begin{cases} \infty & \text{if } \mathrm{supp}\, R^1 p_*(\mathcal{E} \otimes q^* F) = \mathrm{Spec}(R)\,, \\ \mathrm{length}(R^1 p_*(\mathcal{E} \otimes q^* F)) & \text{otherwise.} \end{cases}$$

The absolute badness $b(\mathcal{E})$ of $\mathcal{E}$ is defined to be the minimum of all these numbers:

$$b(\mathcal{E}) = \min_{F \in Y} \{ b(\mathcal{E}, F) \}\,.$$

[1]We use the word badness because b measures how far $\mathcal{E}$ is from being a semistable extension. So badness zero implies semistability.

Since $\mathcal{E}_\eta$ is semistable the badness $b(\mathcal{E})$ has to be finite. We suppose that $\mathcal{E}$ is an extension with minimal possible badness. If the badness is zero, the special fiber $\mathcal{E}_0$ is semistable by 3.1. Hence we may assume that $b(\mathcal{E}) > 0$. Since $\mathcal{E}_0$ is not semistable, there is a surjection $\mathcal{E}_0 \to L \otimes \mathcal{J}_Z$ with $L.H < 0$, $\mathcal{J}_Z$ being the ideal sheaf of a codimension two subscheme of S. We choose an element $F \in Y$ subject to the following three open conditions:

(i) $H^0(L \otimes F) = 0$,
(ii) $b(\mathcal{E}, F) = b(\mathcal{E})$,
(iii) $\mathrm{supp}(F) \cap Z = \emptyset$.

Define the elementary transformation $\mathcal{E}'$ of $\mathcal{E}$ by the exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to L \otimes \mathcal{J}_Z \to 0\,.$$

Applying the functor $p_*(- \otimes q^*F)$ to that sequence, we obtain

$$\begin{array}{ccccccc} p_*(L \otimes q^*F) & \to R^1p_*(\mathcal{E}' \otimes q^*F) \to & R^1p_*(\mathcal{E} \otimes q^*F) \to & R^1p_*(L \otimes q^*F) & \to 0\,. \\ \| & & & \nparallel & \\ 0 & & & 0 & \end{array}$$

This contradicts the minimality assumption on the badness of $\mathcal{E}$ by the very definition of this number. □

3.4 X-positivity of the line bundle $\mathcal{L}$

We have to consider the equivalence classes of semistable sheaves parameterized by our moduli space X using the following equivalence relation.

Definition (trivially connected equivalence) Two semistable S-sheaves E and E' are called trivially connected if there exists a connected projective curve B and an B-flat family $\mathcal{E}$ on $B \times S$ such that
- the determinant line bundle $\mathcal{L}_B = \det(pr_{B!}(\mathcal{E} \otimes pr_S^* F))^{-1}$ on B is trivial and
- there are points b and b' in B with $E \cong \mathcal{E}_b$ and $E' \cong \mathcal{E}_{b'}$.

There is a second equivalence relation that reflects the geometry of the sheaves. We start with some preparations. If τ is a coherent S-sheaf of dimension zero, then we define its trivialisation $\mathrm{triv}(\tau)$ by

$$\mathrm{triv}(\tau) := \bigoplus_{P \in S} k(P)^{\oplus \mathrm{length}_P(\tau)}\,.$$

For a torsion free sheaf G, let $G^{\vee\vee}$ be its double dual and $\tau(G)$ be the cokernel of the injection $G \hookrightarrow G^{\vee\vee}$. Define by

$$\mathrm{triv}(G) = G^{\vee\vee} \oplus \mathrm{triv}(\tau(G))$$

the trivialisation[2] of G.

Define the graded object of a stable sheaf E to be E itself: $\mathrm{gr}_H(E) = E$.

[2]This definition is good enough for our purposes. However it should be replaced by $\mathrm{triv}(G) = G^{\vee\vee} \ominus \mathrm{triv}(\tau(G))$.

If E is a semistable but not stable sheaf, then there exists a short exact sequence $0 \to A' \to E \to A'' \to 0$ with A' a saturated subsheaf of E and $c_1(A').H = 0$. In this case we define the graded object $\mathrm{gr}_H(E)$ of E to be the direct sum $A' \oplus A''$.

Definition (Jordan-Hölder equivalence) Two X-sheaves E and E' on S are called Jordan-Hölder equivalent if and only if $\mathrm{triv}(\mathrm{gr}_H(E)) \cong \mathrm{triv}(\mathrm{gr}_H(E'))$.

This definition implies, in particular, that the equivalence class of a stable vector bundle consists of one element up to isomorphism. In the course of the next result, it will be shown that Jordan-Hölder equivalence coincides with trivially connected equivalence.

Theorem 3.3 *Let B be a smooth projective connected curve and $\mathcal{E}_B$ be a family of sheaves on $B \times S$. Assume that the sheaf parameterized by the generic point of B is semistable. Then the B-line bundle $\mathcal{L}_B$ is trivial or ample. If $\mathcal{L}_B$ is trivial, then all S-sheaves parameterized by B are Jordan-Hölder equivalent.*

First we note that there are nontrivial sections in $\mathcal{L}_B$, because there are semistable objects parameterized by points of B. Therefore this line bundle is either trivial or ample. Consequently the proof reduces to showing that $\mathcal{O}_B \cong \mathcal{L}_B$ (by definition, the trivially connected equivalence) induces the Jordan-Hölder equivalence. To prove the above theorem some preparations will be needed. We retain the above notations.

Theorem 3.4 *([5] theorem I.4) Let C be smooth projective curve and $\mathcal{E}$ a vector bundle on $B \times C$ with $deg_C(\mathcal{E}_b) = 0$ for all $b \in B$. Denote the projections of $B \times C$ to the components by p and q. Suppose there exists a C-line bundle M such that $R^*p_*(\mathcal{E} \otimes q^*M) = 0$ holds. Then the C-objects parameterized by B are all S-equivalent.*

Proof: We proceed in steps.
Step 1: There exists a B-bundle G such that $\mathcal{E}_P \cong G$ for all points $P \in C$. Let P and Q be two arbitrary points of C. The set of all line bundles M in $\mathrm{Pic}^{g_C - 1}(C)$ with $R^*p_*(\mathcal{E} \otimes q^*M) = 0$ is open. Hence there is a line bundle $\tilde{M} \in \mathrm{Pic}^g(C)$ such that $\tilde{M}(-P)$ and $\tilde{M}(-Q)$ are in this open set. From the exact sequence $0 \to \tilde{M}(-P) \to \tilde{M} \to k(P) \to 0$ we obtain

$$p_*(\mathcal{E} \otimes q^*M) \cong p_*(\mathcal{E} \otimes q^*k(P)) \cong p_*(\mathcal{E}|_{B\times\{P\}}) \cong \mathcal{E}|_{B\times\{P\}}.$$

Analogously, $p_*(\mathcal{E} \otimes q^*M) \cong \mathcal{E}|_{B\times\{Q\}}$ which proves the assertion of the first step.
Step 2: Set $\mathcal{G} = p^*G$.
There are three distinct cases to be considered.
Case 1: G is stable
Since G is simple it follows that $N = q_*(\mathcal{G}^\vee \otimes \mathcal{E})$ is a C-line bundle. But $\mathcal{G} \otimes q^*N$ is isomorphic to $\mathcal{E}$, therefore all objects parameterized by B are isomorphic to $N \oplus N$.
Case 2: G is semistable but not stable

After a twist with a line bundle we may assume G to be of degree zero. By theorem 3.1 there exists a B-line bundle A such that $G \otimes B$ has no cohomology. This implies $R^*q_*(\mathcal{E} \otimes p^*A) = 0$, and, as in the first step, all C-objects parameterized by B are isomorphic.

Case 3: G is not semistable

Let $0 \subset A \subset G$ be the Harder-Narasimhan filtration of G, i.e. A is the subline bundle of G of maximal degree. We denote the quotient G/A by A'. By the uniqueness of A, $N = q_*(p^*A^\vee \otimes \mathcal{E})$ is a line bundle on C. We find that $p^*A \otimes q^*N$ is a subbundle of $\mathcal{E}$ with cokernel isomorphic to $p^*A' \otimes q^*N'$ for a C-line bundle N'. Using the short exact sequence

$$0 \to p^*A \otimes q^*(N \otimes M) \to \mathcal{E} \otimes q^*M \to p^*A' \otimes q^*(N' \otimes M) \to 0$$

to compute the degree of $R^*p_*(\mathcal{E} \otimes q^*M)$, we have

$$0 = \deg(A)\deg(N) + \deg(A')\deg(N').$$

Since $\mathcal{E}$ is a family of degree zero sheaves on C, it follows that $\deg(N') = -\deg(N)$. Hence we obtain the equality

$$0 = (\deg(A) - \deg(A'))\deg(N).$$

By assumption the first factor is strictly positive, thus $\deg(N) = \deg(N') = 0$. But this means that all objects parameterized by B are extensions of two line bundles of the same degree. □

Lemma 3.5 *The set of all line bundles L on S such that $L.H = 0$ and for which there exists a nontrivial homomorphism in $\mathrm{Hom}(E, L)$ for some $E \in X$ is bounded. Subsequently, these line bundles can be parameterized by a Noetherian scheme.*

Proof: It is enough to show that the set of Hilbert polynomials of these line bundles L is finite. For any such line bundle L, there is an exact sequence

$$0 \to L^{-1} \otimes \mathcal{J}_{Z_1} \to E \to L \otimes \mathcal{J}_{Z_2} \to 0.$$

Since E is semistable, $\mathcal{J}_{Z_2}$ is the ideal sheaf of some zero dimensional scheme. Using the above sequence, the second Chern class can be computed, and indeed, $c_2 = -L^2 + \mathrm{length}(Z_1) + \mathrm{length}(Z_2)$. Hence we conclude by the Hodge index theorem that L^2 is in the interval $[-c_2, 0]$ and that $(H.(K_S \pm L))^2 \leq H^2(K_S \pm L)^2$, which gives lower and upper bounds for $K_S.L$. The Hilbert polynomial of L with respect to H is determined completely by the numbers L^2, $L.K_S$ and $L.H$. □

Lemma 3.6 *There exists a positive number k such that, for all S-sheaves E and E' in the set X and all line bundles L on S with $H.L = 0$ and $\mathrm{Hom}(E, L) \neq 0$, the groups $\mathrm{Ext}^1(E, L(-kH))$ and $\mathrm{Ext}^1(E, E'(-kH)^{\vee\vee})$ vanish.*

Proof: By Serre duality $\mathrm{Ext}^1(E, L(-kH)) \cong H^1(E(K_S+kH)\otimes L^{-1})^\vee$. For any pair (E, L) there exists a number k such that the cohomology group vanishes. By lemma 3.5 the set of all these pairs is bounded. Hence there exists a global k.
The same argument shows the vanishing of $\mathrm{Ext}^1(E, E'(-kH)^{\vee\vee})$ for a given k.
□

Lemma 3.7 *There exists an integer k such that for all semistable X sheaves E on S the following holds:*
Let Z be a smooth curve in the linear system $|kH|$ such that $E|_Z$ is a vector bundle on Z. If $E|_Z \stackrel{\bar\alpha}{\to} \bar M$ is a surjection onto a Z-line bundle $\bar M$ of degree zero, then $\bar\alpha$ is the restriction of a morphism $E \stackrel{\alpha}{\to} M$ to Z, where M is a S-line bundle with $M.H = 0$.

Proof: This lemma follows from Bogomolov's inequality, as in the proof of (2) $\Rightarrow$ (3) of 3.1. For details see [4] theorem 2.3 or [9] theorem 7.3.5.

Proof of the positivity theorem 3.3
Since $\mathcal{L}_B$ is assumed to be the trivial line bundle, we may pass to a power $\mathcal{L}_B^{\otimes k}$. Choose k such that lemmas 3.6 and 3.7 apply. Now let F be a torsion sheaf supported on a smooth divisor $Z \in |kH|$ such that the global section of $\mathcal{L}_B^{\otimes k}$ defined by F is nontrivial. By construction it is clear that $\mathcal{E}$ restricted to Z is a vector bundle. Now by theorem 3.4 this yields the S-equivalence of the restriction $\mathcal{E}|_Z$, and from the proof given here, it follows that we are in one of the following two cases.
Case 1: All Z-vector bundles parameterized by B are stable and isomorphic. Let P and Q be two geometric points of the curve B. We consider the following long exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E}_P, \mathcal{E}_Q(-kH)^{\vee\vee}) \to \mathrm{Hom}(\mathcal{E}_P, \mathcal{E}_Q^{\vee\vee}) \stackrel{\alpha}{\to}$$

$$\stackrel{\alpha}{\to} \mathrm{Hom}(\mathcal{E}_P, \mathcal{E}_Q^{\vee\vee}|_Z) \to \mathrm{Ext}^1(\mathcal{E}_P, \mathcal{E}_Q(-kH)^{\vee\vee}) \to .$$

Since E_P and E_Q are semistable the group $\mathrm{Hom}(\mathcal{E}_P, \mathcal{E}_Q(-kH)^{\vee\vee})$ vanishes. Hence by lemma 3.6 the morphism α is an isomorphism. The support of the cokernels of the nontrivial morphisms $\mathcal{E}_P$ to $\mathcal{E}_Q^{\vee\vee}$ can not change because they never meet the ample divisor Z. This shows the Jordan-Hölder equivalence of $\mathcal{E}_Q$ and $\mathcal{E}_P$.
Case 2: All Z vector bundles parameterized by B have a surjection to a Z-line bundle $\bar M$ of degree zero.
First we remark that $\bar M$ is the restriction of an S-line bundle M to Z by lemma 3.7. As in the first case, all sheaves parameterized by B are Jordan-Hölder equivalent. □

4 The Barth morphism

Our construction gives us the moduli space X together with a finite morphism $X \stackrel{\varphi_1}{\to} \mathbb{P}^N$, which we call the Barth morphism. In this section we show that this morphism for "surfaces with many lines" assigns a sheaf E its divisor of jumping curves. We say that a polarized surface $(S, \mathcal{O}_S(H))$ has many lines if the linear system $|H|$ is globally generated and the generic curve of this linear system is rational. By adjunction we have $H.(H+K_S) = -2$. Hence any given rank two sheaf E can be normalised such that $H.c_1(E) \in \{-3,-2,-1,0\}$ by twisting $\mathcal{O}_S(H+K_S)^{\otimes k}$. In the even case the assumptions made in 3.1 are not needed because of the following theorem.

Theorem 4.1 (Grauert-Mülich theorem) *If (S,H) is a surface with many lines and E a torsion free rank two sheaf on S with $c_1(E).H$ even, then E is semistable if and only if the restriction of E to the general curve l in the linear system $|H|$ is isomorphic to the direct sum of two isomorphic l-line bundles.*

Proof: See [13] II theorem, 2.1.4. or [8] Lemma 3.6.

This theorem provides us with a very explicit description of the dual moduli space Y as we will see in a moment. We have to distinguish the following two cases:

Case 1: $H.c_1(E) = -2$

By the Grauert-Mülich theorem, $H^*(E \otimes \mathcal{O}_l) = 0$ for E semistable and l general in $|H|$. Hence we may take $|H|$ to be the dual moduli space. Recalling our construction we work here with a square root of the line bundle $\mathcal{L}$.

Case 2: $H.c_1(E) = 0$

In this case we again identify Y with the complete linear system $|H|$. Note that for any $l \in |H|$ the dimension of the $\mathrm{Ext}^1_{\mathcal{O}_l}(\mathcal{O}_l, \mathcal{O}_l(K_l + H))$ is one. Hence there is a unique nontrivial extension ξ_l. By assigning to l this torsion sheaf ξ_l, $|H|$ is identified with the dual moduli space Y.

Any semistable sheaf D_E defines now a divisor D_E in the dual moduli space Y consisting of all curves $l \in |H|$ where $\otimes \mathcal{O}_l$ is not of the expected type. Therefore this divisor D_E is called the *divisor of jumping curves.* By straightforward calculations we find the

Proposition 4.2 *The degree d_E of the divisor D_E of jumping curves equals*

$$d_E = c_2(E) - 2 + \tfrac{(K_S - c_1(E)).c_1(E)}{2} \qquad \textit{if } H.c_1(E) = -2\,;$$

$$d_E = 2c_2(E) - 2 - c_1(E)^2 \qquad \textit{if } H.c_1(E) = 0\,.$$

If now we choose $N+1 = \binom{(H^2+1)+d_E}{d_E}$ curves $\{l_i\}_{i+0}^N$ in $|H|$ such that no divisor of degree d_E contains all the l_i, then the duality construction gives us a finite morphism $X \stackrel{\varphi_1}{\to} I\!P^N$. The name Barth morphism is used because in [1] Barth studied rank-2 vector bundles on the projective plane via their divisors of jumping lines. This morphism assigns every semistable rank two sheaf E its jumping divisor D_E. As a corollary of this construction we have the

Theorem 4.3 *The Barth morphism $X \stackrel{\varphi_1}{\to} I\!P^N$ is finite.*

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Georg Hein
Humboldt-Universität zu Berlin
Institut für Mathematik
Burgstr. 26
10099 Berlin (Germany)

hein@mathematik.hu-berlin.de