ON THE EPI-COMPACTNESS OF EQUI-LOWER SEMI-CONTINUOUS FUNCTIONS.

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Abstract. In this paper we show that equi-lsc. functions from a topological vector space X to the extended reals are epi-compact without assuming the local compactness or the second countablity of the underlying space X. We also show that weakly equi-lower semicontinuous functions from a Banach space X to the extended reals are Mosco-compact. Finally, we apply these results to prove the Mosoc-compactness of families of integral functionals that arise in optimization problems.

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• Introduction

Given a Hausdorff topological space (X, τ), the epigraphical convergence of lower semicontinuous functions from X to the extended reals plays an important role in optimization theory particularly in terms of finding valid approximations to solutions of difficult optimization problems. Different notions of convergence of closed subsets of $X \times I\!R$ lead to different types of epigraphical convergence. The classical notion of Painlevé-Kuratowski convergence of closed sets leads to the Painlevé-Kuratowski epi-convergence of lower semicontinuous functions, which we shall simply refer to as epi-convergence. If (X, τ) is second countable, then every sequence of lower semicontinuous functions from X to the extended reals has epi-converging subsequence (cf. [1]). In [5], Dolecki, Salinetti and Wets showed that for a collection of equi-lower semicontinuous functions, epi- convergence coincide with pointwise convergence. This fact was used in the same paper to obtain results of the Arzelá-Ascoli type about the relative compactness of equi-lower semicontinuous families of functions in the case of a locally compact space (X, τ). In the first part of this paper, we slightly modify the approach taken in [5] in order to obtain compactness results for families of equi-lsc. functions without assuming the local compactness or the second countability of the underlying space. This approach will further lead to the compactness of weakly equi- lower semicontinuous functions with respect to Mosco convergence which is a type of convergence that uses both the weak and the strong topologies of the underlying space X.

In addition to their theoretical importance, compactness results of the type mentioned above are used to prove the epigraphical convergence of nets of functionals that belong to certain classes. In the last part of this paper and following the approach of Attouch [1] and Dal Maso [4], we apply the new results to prove Mosco convergence of certain families of integral functionals that arise in optimization problems.

1. Preliminaries

We start by recalling the basic definitions and the main results concerning epi-convergence. Let (X, τ) be a topological space. Allowing for some notational abuse, we will use τ also to denote the product topology on $X \times I\!R$ (the product of τ and the natural topology on $I\!R$). Let f be a function from X to the extended reals $\overline{I\!R}$. The *effective domain* of f is

$$\operatorname{dom} f = \{x \in X \ | f(x) < +\infty\},$$

and its *epigraph* is

$$\operatorname{epi} f = \{(x, \beta) \in X \times I\!R | \beta \geq f(x)\}.$$

Let N be a general index space . We associate with N a filter $\mathcal{N}^\infty(D)$ and a grill $\mathcal{N}^\#(D)$. For a filtered collection of functions $\{ f_\nu \ \ , \nu \in H \in \mathcal{H} \}$, we define

$$\operatorname{li}_\tau f_\nu(x) = \sup_{G \in \mathcal{G}_\tau(x)} \sup_{H \in \mathcal{H}} \inf_{\nu \in H} \inf_{y \in G} f_\nu(y), \tag{1.1}$$

$$\operatorname{ls}_\tau f_\nu(x) = \sup_{G \in \mathcal{G}_\tau(x)} \inf_{H \in \mathcal{H}} \sup_{\nu \in H} \inf_{y \in G} f_\nu(y), \tag{1.2}$$

where $\mathcal{G}_\tau(x)$ is the family of τ-open neighborhoods at x. The family f_ν is said to e_τ-converge to f, if for all x we have

$$\operatorname{li}_\tau f_\nu(x) = \operatorname{ls}_\tau f_\nu(x),$$

and we use $\lim_\tau f_\nu$ to denote the epigraphical limit of such a family.

A function is called *τ-lower semicontinuous* (τ-lsc.), if $\forall x \in X$ and $\forall \varepsilon > 0, \exists\, G \in \mathcal{G}_\tau(x)$ such that

$$\inf_{y \in G} f(y) \geq \min[f(x) - \varepsilon, \varepsilon^{-1}].$$

A filtered collection of functions $\{f_\nu, \nu \in H \in \mathcal{H}\}$ is *τ-equi-lower semicontinuous* (τ-equi-lsc.) at x, if $\exists\, \varepsilon_x \geq 0$ such that for all $\varepsilon \in (0, \varepsilon_x)$, $\exists H \in \mathcal{H}, \exists G \in \mathcal{G}_\tau(x)$
such that for all $\nu \in H$, we have

$$\inf_{y \in G} f_\nu(y) \geq \min[f_\nu(x) - \varepsilon, \varepsilon^{-1}].$$

The collection is said to be τ-equi-lsc., if it is τ-equi-lsc. at every x.

For a filtered collection $\{C_\nu, \nu \in H \in \mathcal{N}^\infty(D)\}$ of subsets of X, we define

$$\operatorname{Li}_\tau C_\nu = \bigcap_{H \in \mathcal{N}^\#(D)} \operatorname{cl}(\bigcup_{\nu \in H} C_\nu),$$

$$\operatorname{Ls}_\tau C_\nu = \bigcap_{H \in \mathcal{H}} \operatorname{cl}(\bigcup_{\nu \in H} C_\nu),$$

where the closure is taken with respect to the topology τ. The above equalities define the Painlevé-Kuratowski convergence of τ-closed subsets of X. We say C_ν PK-converge to C, if

$$\mathrm{Li}_\tau\, C_\nu = \mathrm{Ls}_\tau\, C_\nu.$$

The relationship between Painlevé-Kuratowski convergence of τ-closed subsets of X and e_τ-convergence of lsc. functions is expressed by the following two theorems :

Theorem 1.1. [8]. *For a filtered collection of functions* $\{ f_\nu, \nu \in N \}$, *we have*

$$\mathrm{epi}\,\mathrm{li}_\tau\, f_\nu = \mathrm{Ls}_\tau\, \mathrm{epi}\, f_\nu,$$

$$\mathrm{epi}\,\mathrm{ls}_\tau\, f_\nu = \mathrm{Li}_\tau\, \mathrm{epi}\, f_\nu.$$

One of the consequences of the above theorem is that $\mathrm{li}_\tau\, f_\nu$ and $\mathrm{ls}_\tau\, f_\nu$ are τ-lsc. This is due to the intersection formulas for $\mathrm{Ls}_\tau\,\mathrm{epi}\, f_\nu$ and $\mathrm{Li}_\tau\,\mathrm{epi}\, f_\nu$ and the fact that a function is τ-lsc, if and only if $\mathrm{epi}\, f$ is τ-closed in $X \times I\!R$. For every subset C of X, we can define an indicator function δ_C :

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C; \\ +\infty & \text{otherwise.} \end{cases}$$

It is clear that δ_C is τ-lsc., if and only if C is τ-closed. We also have

Theorem 1.2. [1] *For a filtered collection C_τ of subsets of X, the following holds:*

$$\mathrm{li}_\tau\, \delta_{C_\nu} = \delta_{\mathrm{Ls}_\tau\, C_\nu},$$

$$\mathrm{ls}_\tau\, \delta_{C_\nu} = \delta_{\mathrm{Li}_\tau\, C_\nu},$$

where δ_C is the indicator function of C.

In order for a notion of convergence to define a topology, four properties must be satisfied by converging filtered families :

(i) If f_ν is a filtered family that converges to f, then every filtered subfamily of f_ν converges to f.

(ii) If f_ν is a filtered family and $f_\nu \equiv f$, then f_ν converges to f.

(iii) A filtered family f_ν converges to f, if every filtered subfamily of f_ν has a filtered subfamily converging to f.

(iv) If a filtered family f_ν converges to f, then for each ν and every filtered family f_ν^μ converging to f_ν, one can extract from f_ν^μ a filtered subfamily converging to f.

For an arbitrary space X, e_τ-convergence satisfies the first three conditions and it satisfies the last condition, if and only if (X, τ) is locally compact(cf. [6], page 29).

In general, there is no relationship between pointwise convergence and e_τ-convergence. However, for a family of τ-equi-lsc functions, we have the following relationship

Theorem 1.3. [5] *Suppose $\{f_\nu, \nu \in H \in \mathcal{H}\}$ is a filtered collection of functions defined on the space (X, τ) with values in the extended reals. Then,*

(i) If the collection is τ-equi-lsc., then e_τ-convergence and pointwise convergence are equivalent.

(ii) If the collection e_τ and pointwise converges to $f > -\infty$, then the collection is τ-equi-lsc.

In [5], the above theorem was used to prove the following compactness result:

Thorem 1.4. [5, Theorem 4.6.] *Suppose that (X, τ) is locally compact, then any τ-equi-lsc. family contains a filtered subfamily that e_τ and pointwise converges to a τ-lsc. function. If the collection is bounded, then there is a subfamily that e_τ and pointwise converges to a bounded function in $\mathcal{E}_\tau(X)$.*

Furthermore, recall that f is an *upper semi-continuous* (usc.), if $-f$ is lsc. For a filtered collection of functions, we define

$$\mathrm{Li}_\tau \,\mathrm{hypo}\ f_\nu = -(\mathrm{li}_\tau - f_\nu),$$

and

$$\mathrm{Ls}_\tau \,\mathrm{hypo}\ f_\nu = -(\mathrm{ls}_\tau - f_\nu).$$

We say that a collection $\{f_\nu\}$ converges in the $-e_\tau$ topology if

$$\mathrm{Ls}_\tau \,\mathrm{hypo}\ f_\nu = \mathrm{Li}_\tau \,\mathrm{hypo}\ f_\nu.$$

Clearly, every property in $(\mathcal{E}_\tau(X), e_\tau)$ has its counterpart in $(-\mathcal{E}_\tau(X), -e_\tau)$, where $-\mathcal{E}_\tau(X)$ is the space of τ-upper semicontinuous(usc.) functions on X.

Let $\overline{C(X)} = \mathcal{E}_\tau(X) \cap -\mathcal{E}_\tau(X)$ be the space of continuous extended real-valued functions and let $\overline{e_\tau}$ be the joint topology of e_τ and $-e_\tau$ (i.e. a filtered collection of functions $\overline{e_\tau}$ converges, if and only if it e_τ and $-e_\tau$-converges.). As a direct result of Theorem 1.4, we get

Theorem 1.5. [5, Proposition 4.7.] *Let (X, τ) be a locally compact Hausdorff space. Let $\mathcal{F}$ be a collection of functions in $\overline{C(X)}$. Then, $\mathcal{F}$ contains a $\overline{e_\tau}$-convergent subfamily, if and only if it is τ-equi-continuous.*

Our first goals in this paper is to prove results similar to Theorems 1.4 and 1.5 without the local compactness of (X, τ). The key to any compactness results of this type is the relationship between e_τ-convergence and the so called Fell topology on $\mathcal{E}_\tau(X)$ which we now briefly review.

We say a set B in $X \times I\!R$ recedes vertically in the negative direction, if

$$(x, \alpha) \in B \Rightarrow (x, \beta) \in B \text{ for all } \beta \leq \alpha.$$

We say that a set A in $X \times I\!R$ recedes vertically in the positive direction, if

$$(x, \alpha) \in A \Rightarrow (x, \beta) \in A \quad \text{for all } \beta \geq \alpha.$$

We remark that if A_ν recede vertically in the positive (negative) direction, then $\bigcup_{\nu \in N} A_\nu$ also recedes vertically the positive (negative) direction. We also remark that if A recedes vertically in one direction, then A^c , the complement of A, recedes in the opposite direction. Now let $\mathcal{G}_-$ be the collection of all τ-open sets in $X \times I\!R$ that recedes in the negative direction and let $\mathcal{K}$ be the collection of τ-compact sets in $X \times I\!R$.

For any set Q in $X \times I\!R$, we define

$$\mathcal{F}^Q = \{A \in X \times I\!R | A \cap Q = \emptyset\},$$

$$\mathcal{F}_Q = \{A \in X \times I\!R | A \cap Q \neq \emptyset\}.$$

Let $\mathcal{E}_\tau(X)$ be the space of all τ-closed subsets of $X \times I\!R$ that recede in the positive direction. Clearly, $\mathcal{E}_\tau(X)$ corresponds to the collection of epigraphs of τ-lsc functions defined on X.

Now we define the F_τ topology (the Fell topology) on $\mathcal{E}_\tau(X)$: it is the topology that has the following subbase:

$$\{\mathcal{F}_G | G \in \mathcal{G}_-\},$$

$$\{\mathcal{F}^K | K \in \mathcal{K}\}.$$

This is a hit and miss topology where you "miss" the τ-compact sets and you "hit" the τ-open sets that recede in the negative direction. Theorems 1.6 to 1.8 record some of the most important (and well known) properties of the topology F_τ. We include the proofs to emphasize the fact that no local compactness of the underlying space is needed.

Theorem 1.6. *Let (X, τ) be a Hausdorff topological space. Then, the space $(\mathcal{E}_\tau(X), F_\tau)$ is compact.*

Proof. We use Alexander's characterization of compactness. Let I, J be arbitrary index sets. Suppose

$$(\bigcap_{i\in I} \mathcal{F}_{k_i}) \cap (\bigcap_{j\in J} \mathcal{F}^{G_j}) = \emptyset, \tag{1.3}$$

where $K_i \in \mathcal{K}$ and $G_j \in \mathcal{G}_-$.

Clearly, $\mathcal{F}_{K_i}$ and $\mathcal{F}^{G_j}$ are F_τ closed for all i in I and all j in J.

Let $G = \bigcup_{j\in J} G_j$. Then, G is τ-open and it recedes in the negative direction and hence $G \in \mathcal{G}_-$.

Now (1.3) holds, if and only if $\bigcap_{i\in I}(\mathcal{F}_{K_i} \cap \mathcal{F}^G) = \emptyset$ which in turn holds, if and only if $\exists\, i_0 \in I$ such that $K_{i_o} \subset G$. For if we assume $K_{i_o} \subset G$, then $\mathcal{F}_{K_{i_o}} \cap \mathcal{F}^G = \emptyset$ and

$$\bigcap_{i\in I}(\mathcal{F}_{K_i} \cap \mathcal{F}^G) = \emptyset.$$

If we assume that K_i is not contained in G for any $i \in I$, then G^c is τ-closed and it recedes in the vertical positive direction. Thus, it is in $\mathcal{E}_\tau(X)$ and $G^c \in F_{K_i}$ for all i and $G^c \in F^G$. Hence, $\bigcap_{i\in I}(\mathcal{F}_{K_i} \cap \mathcal{F}^G) \neq \emptyset$.

Now since K_{i_o} is τ-compact, $\exists j_1, ..., j_q \in J$ such that

$$K_{i_o} \subset G_{j_1} \cap ... \cap G_{j_q}.$$

Thus,

$$\mathcal{F}_{K_{i_o}} \bigcap (\bigcap_{i=1}^{q} \mathcal{F}^{G_{j_i}}) = \emptyset,$$

and consequently, the space is compact. ◻

Remarks: The function $f_\infty \equiv +\infty$ is an element in $\mathcal{E}_\tau(X)$, since $\operatorname{epi} f_\infty = \emptyset$ is closed in $X \times I\!R$. To say that a collection f_ν epi-converges to f_∞ means that this collection will eventually miss every compact set in $X \times I\!R$. Moreover, if we consider the space $\mathcal{E}^o_\tau(X) = \mathcal{E}_\tau(X) \setminus \{f_\infty\}$ with the relative topology that it inherits from F_τ, then this space is not compact even if (X, τ) is locally compact. Hence, adding f_∞ and the open sets associated with it, is a one point compactification of $(\mathcal{E}^o_\tau(X), F_\tau)$. In general, the space $(\mathcal{E}_\tau(X), F_\tau)$ is not metric and therefore it is possible that it is not sequentially compact despite the fact that it is compact.

Theorem 1.7. *Let X be a Hausdorff space. Then, the points in $(\mathcal{E}_\tau(X), F_\tau)$ are closed, i.e. $(\mathcal{E}_\tau(X), F_\tau)$ is T_1.*

Proof. One has :

$$\forall A \in \mathcal{E}_\tau(X), \quad A = (\bigcap_{y \in A} \mathcal{F}_{\{y\}}) \bigcap \mathcal{F}^{A^c},$$

and A is the intersection of F_τ-closed sets and hence it is F_τ-closed. □

The F_τ topology on $\mathcal{E}_\tau(X)$ can be also described in the following manner: it is the topology generated by the following subbase:

$$\{\mathcal{F}^{K,a} | K \in \mathcal{K}, a \in I\!R\} \text{ and } \{\mathcal{F}_{G,a} | G \in \mathcal{G}, a \in I\!R\},$$

where $\mathcal{K}$ is the collection of τ-compact sets in X, $\mathcal{G}$ is the collection of τ-open sets in X, and

$$\mathcal{F}^{K,a} = \{E \in \mathcal{E}_\tau(X) | E \cap (K \times]-\infty, a]) = \emptyset\},$$

$$\mathcal{F}_{G,a} = \{E \in \mathcal{E}_\tau(X) | E \cap (G \times]-\infty, a[) \neq \emptyset\}.$$

It is a routine check to see that the above definition is equivalent to our original definition of F_τ. Now using the above definition we can prove the following :

Theorem 1.8. *Any bounded collection of τ-lsc. functions in $\mathcal{E}_\tau(X)$ is F_τ-closed.*

Proof. Consider the set

$$B^a(D) = \{\, f \mid f \text{ is } \tau\text{-lsc. on } X \text{ and } f \leq a \text{ on } D\,\} = \bigcap_{x \in D} \{\, f \text{ is } \tau\text{-lsc., } f(x) \leq a\,\}.$$

The above set is the intersection of F_τ-closed sets and hence it is F_τ-closed.

Similarly, we can show that $\forall a \in \bar{I\!R}$, $\forall\ G$ open in X, the set

$$B_a(G) = \{f | f \text{ is lsc. and } f \geq a \text{ on } G\}$$

is closed. $\square$

2. The compactness of τ-lsc. functions

We start by proving a theorem that will allow us to connect e_τ-convergence and the F_τ topology for families of τ-equi-lsc functions without local compactness and thus the notion of e_τ-convergence for τ-equi-lsc functions becomes " topological".

Proposition 2.1. *Let $\{f_\nu\}$ be a filtered τ-equi-lower semicontinuous family in $\mathcal{E}_\tau(X)$. Then,*

$$F_\tau\text{-convergence of } \{f_\nu\} \Rightarrow \text{ pointwise convergence of } \{f_\nu\}.$$

Proof. Recall that for a filtered family of functions, we can define the pointwise convergence as follows: $\forall x \in X$, let

$$\begin{aligned} \operatorname{li} f_\nu(x) &= \sup_{H \in \mathcal{H}} \inf_{\nu \in H} f_\nu(x), \\ \operatorname{ls} f_\nu(x) &= \inf_{H \in \mathcal{H}} \sup_{\nu \in H} f_\nu(x). \end{aligned}$$

f_ν converges pointwise at x, if $\operatorname{li} f_\nu(x) = \operatorname{ls} f_\nu(x)$. Now $\forall x \in X$, pick $\alpha \leq f(x)$. The point (x, α) is τ-compact in $X \times I\!R$. Due to the F_τ-convergence of f_ν, there exists $H \in \mathcal{H}$ such that $\forall \nu \in H$, $\operatorname{epi} f_\nu \cap (x, \alpha) = \emptyset$ and thus

$$f_\nu(x) \geq \alpha, \forall \nu \in H.$$

Hence,

$$\inf_{\nu \in H} f_\nu(x) \geq \alpha \text{ and } \operatorname{li}_\tau f_\nu(x) \geq \alpha.$$

Taking the sup over all $\alpha \leq f(x)$, we get

$$\operatorname{li} f_\nu(x) \geq f(x).$$

Furthermore, F_τ-convergence implies that for any set O such that O is open in $X \times I\!R$ and $O \cap \operatorname{epi} f \neq \emptyset$, there exists $H_O \in \mathcal{H}$ such that

$$O \cap \operatorname{epi} f_\nu \neq \emptyset, \quad \forall \nu \in H_O.$$

Suppose now $y \in \operatorname{epi} f$, then for any set O that is open in $X \times I\!R$ and contains y, we have $O \cap \operatorname{epi} f \neq \emptyset$ which implies

$$\exists H_O \in \mathcal{H} \text{ such that } \forall \nu \in H_O, \quad O \cap \operatorname{epi} f_\nu \neq \emptyset.$$

Every member of $\mathcal{H}$ meets every member of $\mathcal{N}^{\#}(D)$. Therefore, for every $H \in \mathcal{N}^{\#}(D)$, we have

$$(\bigcup_{\nu \in H} \operatorname{epi} f_\nu) \bigcap O \neq \emptyset,$$

and since O was an arbitrary open set containing y, we get

$$y \in \operatorname{cl}(\bigcup_{\nu \in H} \operatorname{epi} f_\nu)$$

and

$$y \in \bigcap_{H \in \mathcal{N}^{\#}(D)} \operatorname{cl}(\bigcup_{\nu \in H} \operatorname{epi} f_\nu).$$

Therefore, $y \in \operatorname{Li}_\tau \operatorname{epi} f_\nu$ and since y was an arbitrary point in $\operatorname{epi} f$, we have $\operatorname{ls}_\tau f_\nu(y) \leq f(x)$, $\forall x \in X$. Hence,

$$\sup_{G \in \mathcal{G}(x)} \inf_{H \in \mathcal{H}} \sup_{\nu \in H} \inf_{y \in G} f_\nu(y) \leq f(x),$$

and $\forall G$, $\exists H_1 \in \mathcal{H}$ such that

$$\sup_{\nu \in H_1} \inf_{y \in G} f_\nu(y) \leq f(x). \tag{2.1}$$

The fact that $\{f_\nu\}$ are τ-equi-lsc. implies that $\exists \varepsilon_x$ such that $\forall \varepsilon \in (0, \varepsilon_x)$, $\exists H_2 \in \mathcal{H}$ such that

$$\inf_{y \in G} f_\nu(y) \geq \ f_\nu(x) - \varepsilon, \qquad \forall \nu \in H_2,$$

and hence

$$\sup_{\nu \in H_2} \inf_{y \in G} f_\nu(y) \geq \sup_{\nu \in H_2} f_\nu(x) - \varepsilon. \tag{2.2}$$

Since $H_1 \cap H_2 \in \mathcal{H}$, combining (2.1),(2.2) gives us:

$$\exists H \in \mathcal{H} \quad \text{such that } \sup_{\nu \in H} f_\nu(x) - \varepsilon \leq f(x).$$

and

$$\inf_{H \in \mathcal{H}} \sup_{\nu \in H} f_\nu(x) - \varepsilon \leq f(x).$$

Since the above is true for all $\varepsilon < \varepsilon_x$, we get

$$\operatorname{ls} f_\nu(x) \leq f(x),$$

and the proof is complete. $\square$

Remark : The above proposition relates pointwise convergence to F_τ-convergence whereas Theorem 1.3. relates pointwise convergence to e_τ- convergence.

Lemma 2.2. *Let $\{f_\nu\}$ be filtered collection of functions in $\mathcal{E}_\tau(X)$. Suppose $\operatorname{epi} f \subset \operatorname{Li}_\tau \operatorname{epi} f_\nu$. Then, for any set G such that G is open in $X \times I\!R$ and $\operatorname{epi} f \cap G \neq \emptyset$, $\exists H_G \in \mathcal{H}$ such that $\operatorname{epi} f_\nu \cap G \neq \emptyset$.*

Proof. By our assumption, $\operatorname{epi} f \cap G \neq \emptyset$ implies that

$$\operatorname{Li}_\tau \operatorname{epi} f_\nu \cap G \neq \emptyset.$$

Hence, for all $H \in \mathcal{N}^\#(D)$,

$$(\bigcup_{\nu \in H} \operatorname{epi} f_\nu) \cap G \neq \emptyset,$$

which means that there is $H_G \in \mathcal{H}$ such that for all $\nu \in H_G$, $\operatorname{epi} f_\nu \cap G \neq \emptyset$, (recall that $\mathcal{H}$ consists of all the subsets of N that meet every set in $\mathcal{N}^\#(D)$). $\square$

Lemma 2.3. *Let $\{f_\nu\}$ be filtered collection of functions in $\mathcal{E}_\tau(X)$. Suppose that $\operatorname{Ls}_\tau \operatorname{epi} f_\nu \subset \operatorname{epi} f$. Then, for any compact set K, $K \cap \operatorname{epi} f = \emptyset$ implies that $\exists H_K \in \mathcal{H}$ such that $\operatorname{epi} f_\nu \cap K = \emptyset$.*

Proof. Assume that the claim of the lemma is not true. Then, there is a compact set K such that $K \cap \operatorname{epi} f = \emptyset$ and for every $H \in \mathcal{H}$ we can find $\nu \in H$ such that $\operatorname{epi} f_\nu \cap K \neq \emptyset$. Then, there exists $H' \in \mathcal{N}^\#(D)$ such that $\operatorname{epi} f_\nu \cap K \neq \emptyset$ for every ν in H'. Since K is compact, $\{\operatorname{epi} f_\nu \cap K, \nu \in H'\}$ admits at least one cluster point y in K. Hence, for every H in $\mathcal{H}$, we have

$$y \in \operatorname{cl}(\bigcup_{\nu \in H} \operatorname{epi} f_\nu) \cap K.$$

Thus,

$$y \in (\operatorname{Ls}_\tau \operatorname{epi} f_\nu) \cap K,$$

which contradicts our assumption that $\mathrm{Ls}_\tau \operatorname{epi} f_\nu \subset \operatorname{epi} f$. $\square$

Theorem 2.4. *Let $\{f_\nu\}$ be a filtered τ-equi-lsc. family in $\mathcal{E}_\tau(X)$. Then,*

$$e_\tau \text{ -convergence } \Longleftrightarrow F_\tau \text{ -convergence } .$$

Proof. Lemmas 2.2 and 2.3. prove that

$$e_\tau \text{ -convergence } \Longrightarrow F_\tau \text{ -convergence } .$$

To prove the other direction of the implication we use Theorem 2.1 to get pointwise convergence. Pointwise convergence and the fact that $\{f_\nu\}$ are τ equi-lsc. give us e_τ-convergence via Theorem 1.3. $\square$

Here we need to emphasize that the above theorem does not mean that e_τ-convergence and convergence in the F_τ topology are compatible for any family in $\mathcal{E}_\tau(X)$. Such claim is impossible due to lack of local compactness of X as we remarked in the beginning of the first section.

Finally, we obtain the desired generalization of Theorem 1.4. which asserts that families of τ-equi-lsc. functions are in some sense e_τ-compact despite the fact that e_τ-convergence is not topological due to lack of local compactness of (X, τ).

Theorem 2.5. *Any τ-equi-lsc. collection of τ-lsc functions contains a filtered subfamily that e_τ and pointwise converges in $\mathcal{E}_\tau(X)$. If the collection is bounded, then the subfamily converges to a bounded function in $\mathcal{E}_\tau(X)$.*

Proof. Theorems 1.6, 2.4 and 1.8. $\square$

We also have a the following generalization of Theorem 1.5.

Theorem 2.6. *Let (X, τ) be a Hausdorff space. Let $\mathcal{F}$ be a collection of functions in $\overline{C(X)}$. Then, $\mathcal{F}$ contains a $\overline{e_\tau}$-convergent subfamily, if and only if it is τ-equi-continuous.*

Proof. If $\mathcal{F}$ is an τ-equi-lsc. collection, then $\mathcal{F}$ must contain a subcollection that converges in both e_τ and -e_τ topologies by Theorem 2.5. Moreover, if $\mathcal{F}$ is not τ-equi-lsc., then for some collection $\{f_\nu, \;\; \nu \in H \in \mathcal{N}^\infty(D)\}$ in $\mathcal{F}$ and some x we have

$$\lim_\tau f_\nu(x) < f_\nu(x) = -(\mathrm{ls} - f_\nu)(x) \leq -(\mathrm{ls}_\tau - f_\nu)(x).$$

Hence, there is no subcollection of $\{f_\nu\}$ that hypo-converges to $\lim_\tau f_\nu$ which contradicts our assumption about $\mathcal{F}$. Similarly, we obtain a contradiction if we assume that $\mathcal{F}$ is not τ-equi-usc. $\square$

It was noted in [5] that since $\overline{e_\tau}$-convergence implies uniform convergence on compact sets of X, the preceding theorem gives us the standard Arzelá-Ascoli Theorem.

3. The case of the Mosco Topology

Let X be a reflexive Banach space. For a filtered family, the Mosco convergence in $\mathcal{E}_w(X)$, the space of weak-lsc. functions on X, is defined as follows : f_ν Mosco converges to f, if $\forall x \in X$

$$\operatorname{ls} f_\nu(x) \leq f(x) \leq \operatorname{li}_w f_\nu(x),$$

where $\operatorname{ls} f_\nu$ is $\operatorname{ls}_\tau f_\nu$ with τ being the norm topology of X and $\operatorname{li}_w f_\nu$ is $\operatorname{li}_\tau f_\nu$ with τ being the weak topology of X.

In [1], Beer showed that for sequences of functions, the Mosco convergence is compatible with a "hit" and "miss" topology F_m. This topology is given by the following subbase:

$$\{\mathcal{F}_G | G \in \mathcal{G}_-\} \text{ and } \{\mathcal{F}^{K_w} | k_w \in K_w\},$$

where everything is as defined for the F_τ topology except that K_w is now the collection of weakly compact sets in X. We now show that the Mosco-convergence is compatible with F_m-convergence for filtered families of w-equi-lsc. functions (not just sequences !). We then extend the results of the first part of this paper to the Mosco case by considering the space $(\mathcal{E}_w(X), F_w)$, where F_w is the hit and miss topology defined in the first section with τ being the weak topology on X.

Corollary 3.1. *The space $(\mathcal{E}_w(X), F_w)$ is compact.*

Proof. Theorem 1.6. with τ being the weak topology on X. $\square$

Corollary 3.2. *If f_ν is an w-equi-lsc. family in $\mathcal{E}_w(X)$, then F_w- convergence of f_ν implies the pointwise convergence of f_ν.*

Proof. Theorem 2.1. with τ being the weak topology on X. $\square$

Theorem 3.3. *If f_ν is a w-equi-lsc. family in $\mathcal{E}_w(X)$, then*

$$F_w \text{ -convergence } \iff \text{ Mosco convergence } \iff F_m \text{ -convergence}$$

Proof. In general we have:

$$\text{Mosco convergence } \Longrightarrow F_m \text{ -convergence } \Longrightarrow F_w \text{ -convergence } .$$

The first implication is a result of Lemmas 2.3 and 2.4 with the appropriate topologies. The second implication is a result of the fact that the subbase of the the F_m topology contains more elements than the subbase of the F_w topology. Now assume $\{f_\nu\}$ converges in the F_w topology and that $\{f_\nu\}$ are w-equi-lsc. Then, by Corollary 3.2., we get pointwise convergence of f_ν. Moreover, it was shown in [5] that w-equi-lower semicontinuity and pointwise convergence imply Mosco convergence. $\square$

Theorem 3.4. *Any w-equi-lsc collection in $\mathcal{E}_w(X)$ contains a subfamily that Mosco and pointwise converges to a w-lsc. function. Moreover, if the family is bounded then it contains a subfamily that converges Mosco and pointwise to a bounded w-lsc function.*

Proof. The space $(\mathcal{E}_w(X), F_w)$ is compact by Theorem 1.6. Combining Theorem 2.5 (with τ being the weak topology) and Theorem 3.3 will complete the proof. $\square$

As a direct corollary of the above theorem we obtain

Corollary 3.5. *Let C_ν be a filtered collection of subsets of X. Assume that for every x in X there exists $H \in \mathcal{H}$ such that $\forall \nu \in H$, we either have*
(i) there exists an open set W such that $W \subset C_\nu$,
or
(ii) there exists an open set W contained in the complement of C_ν.
Then, C_ν must contain a filtered subfamily that Mosco converges to a weakly closed (possibly empty) subset of X.

To show the necessity of the w-equi-lsc condition for Theorem 3.4, we have the following example:

Example 3.6. [3] Consider the net (sequence) $A_n = \text{con}\{\theta, e_1 + e_{n+1}\}$ in l_2, where con denotes the convex hull of a set, θ is the origin of l_2 and e_n are the usual base elements of l_2. Then, δ_{A_n} converges to $\{\theta\}$ in the F_s topology(F_s is F_τ with τ being the norm topology on X) and converges to $\text{con}\{\theta, e_1\}$ in the F_w topology. Hence, A_n cannot have any Mosco convergent subnet.

Finally, we remark that if X has a separable dual, then the Mosco topology on the space of nonempty closed subsets of X is metrizable (cf [3]) and hence for such spaces Theorem 3.4 implies the sequential compactness of weakly equi-lsc. sequences.

4. Mosco compactness of weakly equi-lsc integral functionals

In this section we apply the results of section 3 to a particular family of integral functionals. We start by reviewing the basic definitions needed for this application. We use the same notation used by Dal Maso in [4].

Let Ω be an open subset of $\mathbb{R}^n$. Let $\mathcal{A}$ and $\mathcal{B}$ be respectively the collections of open subsets and Borel subsets of Ω. Let A and B be two subsets of Ω. We write $A \subset\subset B$, if $\text{cl}\, A$ is compact and contained in in B. We write $A << B$, if $\text{cl}\, A$ is compact and contained in $\text{int}\, B$. We say a collection $\mathcal{D}$ of subsets is *dense* in $\mathcal{A}$, if for any A and B in $\mathcal{A}$ such that $A \subset B$, there is a set $D \in \mathcal{D}$ such that $A << D << B$.

Now consider a functional $F : X \times \mathcal{A} \longrightarrow \overline{\mathbb{R}}$:

F is *lsc.*, if $\forall A \in \mathcal{A}$, $F(\cdot, A)$ is lsc.on X.

F is *increasing*, if $\forall x \in X$ and for all A and B in $\mathcal{A}$ such that $A \subset B$, we have $F(x, A) \leq F(x, B)$.

F is *inner regular*, if for all $x \in X$ and for all $A \in \mathcal{A}$,

$$F(x, A) = \sup_{B \in \mathcal{A},\, B << A} \{F(x, B)\}.$$

We also define F_-, the *inner regularization* of F,

$$F_-(x, A) = \sup_{B \in \mathcal{A},\, B << A} \{F(x, B)\}.$$

We say F is subadditive, if $\forall x \in X$ and $\forall A$ and $B \in \mathcal{A}$, we have $F(x, A \cup B) \leq F(x, A) + F(x, B)$. Similarly, we say F is superadditive, if $F(x, A \cup B) \geq F(x, A) + F(x, B)$ when

$A \cap B = \emptyset$. Finally, we say that F is a measure, if $\forall x \in X$, there is a Borel measure μ such that $F(x, A) = \mu(A)$ for all $A \in \mathcal{A}$. We note here that if $F(x, \emptyset) = 0, \forall x \in X$, then F is measure, if and only if it is inner regular, subadditive and superadditive on $\mathcal{A}$.

Theorem 4.1. *Let $F_n : X \times \mathcal{A} \longrightarrow \overline{I\!R}$ be a sequence of measures such that $\forall A \in \mathcal{A}$, $F_n(\cdot, A)$ are weakly lsc. on a separable reflexive space X . Suppose further that $\forall n$, $\forall x \in X$ and $\forall A \in \mathcal{A}$, $F_n(x, A) < G(x, A)$ for some measure $G : X \times \mathcal{A} \longrightarrow \overline{I\!R}$. Then, there exists a subsequence F_k such that F_k Mosco and pointwise converges to some weakly lower semicontinuous measure $F : X \times \mathcal{A} \longrightarrow \overline{I\!R}$.*

Proof. Let $\mathcal{D}$ be a countable dense collection of $\mathcal{A}$. Since Ω is an open subset of $I\!R^n$, such collection always exists (example 14.6 in [4]). For every $D \in D$, we can use Theorem 3.4. to find a subsequence that Mosco and pointwise converges to some weakly lsc. functional $F(\cdot, D)$. Using a diagnolization argument, we can then construct a subsequence F_k that pointwise and Mosco converges to F, for all $D \in \mathcal{D}$. Hence, by theorem 16.4. in [4], we get that $(\mathrm{ls}_s\, F_k)_- = (\mathrm{li}_w\, F_k)_- = F$, for all $A \in \mathcal{A}$. We now only need to show that $\mathrm{li}_s\, F_k(x, A) \le F(x, A)$, $\forall A \in \mathcal{A}$. Let $\varepsilon > 0$ and let K be a compact subset of A such that $G(A \backslash K)$ and let A' and A'' be subsets of $\mathcal{A}$ such that $K \subset A' \subset\subset A'' \subset\subset A$. Then, for all $x \in X$,

$$F_k(x, A) \le F_k(x, A'') + F_k(x, A \backslash K),$$

and hence

$$F_k(x, A) \le F_k(x, A'') + G(x, A \backslash K).$$

Taking limits of both sides, we get

$$\mathrm{ls}_s\, F_k(x, A) \le \mathrm{ls}_s\, F_k(x, A'') + G(x, A \backslash K),$$

and hence $\forall \varepsilon > 0$,

$$\mathrm{ls}_s\, F_k(x, A) \le F(x, A) + G(x, A \backslash K) \le F(x, A) + \varepsilon.$$

Thus, we have

$$(\mathrm{ls}_s\, F_k) = (\mathrm{li}_w\, F_k) = F.$$

Clearly, F is a weakly lsc, increasing and inner regular. The fact that it is also subadditive and superadditive on $\mathcal{A}$ follows immediately from the pointwise convergence of F_k to F and from the fact that every F_k is a measure. $\square$.

Remark: On one hand, Theorem 4.1 is weaker than similar results by Dal Maso (Theorems 16.9 and 18.6 in [1]) since these results do not require the integral functionals to be equi-lsc. On the other hand, Theorem 4.1 yields a Mosco-converging subsequence which is a stronger type of convergence than the one obtained in [1] . We also note that we required the space X to be reflexive and separable, only in order to make the Mosco topology on $X \times I\!R$ second countable (see the remark at the end of section 3).

Compactness results are often used in the following manner : In order to find the Mosco limit a weakly equi-lower semicontinuous collection $\{F_\nu\}$, we only need to show that all converging subnets converge to the same limit (we already know that there is at least one converging subnet due to compactness). Therefore, we can assume that the Mosco limit of $\{F_\nu\}$ exists and we only need to identify it.

We now provide an example of a family of a weakly lsc. family of integrals .

Let P_n be a countable collection of probability measures on $\Omega \subset I\!R^n$ that are absolutely continuous with respect to some probability measure P_0 . Let U be a separable reflexive Banach space, and let Y be a Banach space. Let $D_n : U \longrightarrow Y$ be a collection of weakly equi-continuous operators. Let f_n be a collection of positive functions from $Y \times \Omega \longrightarrow \overline{I\!R}$ such that

(i) For all y in Y and for all n, $f_n(y, .)$ is a measurable function on Ω.

(ii) $f_n(\cdot, s)$ are equi-lsc. on Y uniformly in s: For all y in $\mathrm{dom}\, f$, for all $\varepsilon \geq 0$, there exists a weak neighborhood W of y such that

$$\forall y' \in W,\ \forall s \in S, \quad f_n(y', s) \geq f_n(y, s) - \varepsilon. \tag{4.1}$$

(iii) $\mathrm{dom}\, f = D \times S$ where D is weakly closed subset of Y.

Now consider the following functionals on $U \times \Omega$

$$I_n(u, A) = \int_A f_n(D_n(u), s) P_n(ds), \tag{4.2}$$

where $\mathrm{dom}\, I_n = D$. This type of integral functionals arises in problems in optimal control and two stage stochastic programs (cf. [7]). For example, U can be $L^2[0, T]$, the space of controls in a given control problem. $D_n : L^2[0, T] \longrightarrow C^1[0, T]$ can be integral operators representing perturbations of the solution of the dynamics of the problem. More specifically,

$$D_n(u) = x(t) = \int_o^T K_n(t, z) u(z) dt,$$

where K_n are the Kernels of these operators. The weak equi-lower continuity of D_n in this case is obtained with very mild conditions on K_n.
Now going back to (4.2), it is clear that for every n, I_n is a measure from $U \times \mathcal{A}$ to $\overline{\mathbb{R}}$. Also, for all $u \in D$, and $\forall \varepsilon > 0$, there is a weak neighbohood W of u such that of

$$\forall u' \in W,\ \forall s \in S, \quad f_n(D_n(u'), s) \geq f_n(D_n(u'), s) - \varepsilon. \tag{4.3}$$

Hence, for all $A \in \mathcal{A}$, there exists a weak neighborhood W' of u such that for all n, we have

$$\int_A f_n(u', s) P_n(ds) \geq \int_A f_n(u, s) P_n(ds) - \varepsilon, \tag{4.4}$$

and hence for all u' in W',

$$I_n(u', A) \geq I_n(u, A) - \varepsilon. \tag{4.5}$$

If u is not in D, then by assumption (ii), there exists a weak neighborhood W of u such that for all u' in W and all n, we have

$$I_n(u', A) = I_n(u, A) = +\infty. \tag{4.6}$$

Now (4.5) and (4.6) imply that for every $A \in \mathcal{A}$, $\{I_n\}$ are w-equi-lsc.

Proposition 4.2. *Let $I_n : X \times \mathcal{A} \longrightarrow \overline{\mathbb{R}}$ be the collection of integral functionals defined by (4.2). Suppose further that there a measure $G : X \times \mathcal{A} \longrightarrow \overline{\mathbb{R}}$ such that $I_n \leq G$. Then, there exists a Borel function $g : X \times \Omega \longrightarrow \overline{\mathbb{R}}$ and there exists and subsequence I_k such that $\forall A \in \mathcal{A}$,*

$$I_k(\cdot, A) \text{ Mosco converges to } I_0(\cdot, A),$$

where

$$I_0(x, A) = \int_A g(x, s) P_0(ds).$$

Proof. The proof follows imeadiately from Theorem 4.1. □

Remarks : The physical interpretation of the function g depends on the particular application. In homogenization problems, for exmple, it is the energy of the homogenized material. We also note that initially we did not have any convergence assumption on P_n and therefore the above proposition can be thought of as a compactness result for P_n (take for example $f_n \equiv 1$).

REFERENCES

1. H. Attouch, *Variational Convergence for Functions and Operators*, Pitman Advanced Publishing (1984).
2. G. Beer, *On the Mosco convergence of convex sets*, Bull. Australian Math. Soc. 38, 239-252, (1988).
3. G. Beer, *Topologies on closed and closed convex sets*, Kluwer Academic Publishers (1991).
4. G. Dal Maso, *An Introduction to* $\Gamma-$*Convergence*, Birkhauser (1992).
5. S. Dolecki, G. Salinetti and R. J.-B.Wets, *Convergence of functions : Equi-semicontinuity* , Trans. A.M.S. volume 276, Number 1, March 1983.
6. E. Klein and A. Thompson, *Theory of Correspondences*, Canadian Math. Soc. Series of Monographes and Advanced Texts, Wiley-Interscience Publication.
7. R. Lucchetti and R. J.-B. Wets, *Convergence of Minimum of Integral Functionals*, Statistics and Decision, 11, 69-84 (1993).
8. U. Mosco, *Convergence of Convex sets*, Adv. in Math. 3, 510-585, (1969).