Differentiable Selections of Set-Valued Mappings and Asymptotic Behaviour of Random Sets in Infinite Dimensions *

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Abstract

We consider set-valued mappings acting between two linear normed spaces and having convex closed images. Our aim is to construct selections with directional differentiability properties up to the second order, using certain tangential approximations of the mapping. The constructions preserve measurability and lead to a directionally differentiable Castaing representation of measurable multifunctions admitting the required tangential approximation. A generalized set-valued central limit theorem for random sets in infinite dimensional spaces is presented. The results yield asymptotic distributions of measurable selections forming the Castaing representation of the multifunction.

Keywords: First- and second-order differentiable set-valued mapping, selections, Wijsman topology, Castaing representation, central limit theorems

1 Introduction

The problem of finding selections of set-valued mappings (multifunctions) with some regularity properties is an indispensable topic of set-valued analysis.
Let $F : X \rightrightarrows Y$ be a multifunction acting between two linear normed spaces X and Y. A mapping $f : X \to Y$ is a *selection* of F if $f(x) \in F(x)$ for all $x \in X$. By the axiom of choice, such a mapping exists whenever F has non-empty images. One is usually interested in finding selections with additional properties.
This paper is devoted to the question how differentiability properties of a multifunction are inherited by its selections.
One of the most celebrated results is the Michael's selection theorem ([25]) on the existence of continuous selections of lower semicontinuous mappings with closed convex

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images. There is extensive literature elaborating sufficient conditions for the existence of Lipschitz-continuous selections [3, 4, 15, 18]. Fundamental results on the existence of measurable selections are contained in [12, 13, 23]. We refer to [4] for an overview of the basic facts how selections inherit measurability, Lipschitz-continuity etc., as well as a presentation of some special selections and their properties.

This paper aims at generalizing some of the results in [15] for mappings valued in infinite dimensions as well as at presenting generalized set-valued central limit theorems. These results lead to nontrivial consequences about the asymptotic behaviour of measurable selections. We present two extensions of the constructions suggested in [15]. One of them uses certain tangential approximation of the graph and results in finding a continuous selection that is Hadamard directionally differentiable at given points. The second construction defines a continuous selection that is twice directionally differentiable at certain points at which a second-order approximation of the multifunction exists. The properties of the multifunctions required here are usually regarded as a kind of differentiability property.

Several concepts of differentiability of set-valued mappings have been introduced in the literature. They typically represent some cone-approximation of the multifunction. Furthermore, corresponding second-order tangent sets have been developed. Some geometrical concepts are presented in [4], which correspond to looking at certain tangent cone to the graph of the multifunction. A stronger differentiability notion related to interesting phenomena in optimization, called proto-differentiability, has been introduced by Rockafellar [28]. We work with another attractive concept of differentiability called semi-differentiability. This concept goes back to Penot [27] and corresponds to a concept of tangential approximation due to Shapiro, investigated in various papers [32, 33, 34].

The notion of semi-differentiability is slightly modified in this paper by using a metric convergence on the hyperspace of nonempty closed subsets of Y, which is better suited to the infinite-dimensional case.

Semi-differentiability plays an important role in the so-called delta-approach, which provides information about the asymptotic behaviour of stochastic processes. Generally speaking, delta-theorems establish central limit formulas for random elements in Banach spaces. There is a set-valued version of a delta-theorem (see [22, 35]) dealing with random sets in $\mathbb{R}^n$. We shall prove such a theorem for random sets in Polish spaces. It will be shown that mappings with semi-differentiability properties admit Castaing representation such that, given a central limit formula for the arguments, all selections satisfy the same central limit formula. The asymptotic behaviour of random sets and, respectively, measurable selections can be characterized up to the second order provided second-order differentiability of the underlying multifunction.

The existence of differentiable selections is treated in [15, 18, 20]. Gautier and Morchadi consider the so-called Steiner selection and its remarkable properties in [20]. Conditions for differentiability properties of Steiner selections are identified also in [15, 18]. The Steiner center of convex sets is well-defined in finite dimensional spaces only. It is shown in [36] that some natural attempts to generalize the definition for sets in infinite dimensional spaces have unsatisfactoried results. For example, if a set moves in a continuous way, its center does not have to change continuously. This is the reason why we try to obtain selections by using another approach. We use the existence of a continuous partition of unity in the space X, which is a general tool often used while constructing selections. In [18], Dommisch exploits the existence of a smooth partition of unity in $\mathbb{R}^n$ to prove the existence of a smooth selection under some additional conditions.

The paper is organized as follows. In section 2, we formulate the necessary notions, recall preliminary facts and show some basic properties of the derivative.
Section 3 contains constructions of a Hadamard directionally differentiable selection of a mapping that is semi-differentiable at some points and of a twice directionally differentiable selection of a mapping that admits second order approximation at some points. In section 4, measurability properties, Castaing representation theorems, delta-theorems and convergence in distribution for random sets and their selections are discussed.

2 Differentiability of multifunctions

Throughout the whole paper we will consider set-valued mappings $F : X \rightrightarrows Y$ defined on a linear normed space X and having nonempty closed convex values in a linear normed space Y. We denote the graph of F by $\operatorname{graph} F$.
Let $\mathcal{F}(Y)$ denote the hyperspace of nonempty closed sets in Y, and let $\mathcal{F}_c(Y)$ stand for the corresponding hyperspace of nonempty closed convex sets. We endow these spaces with the Wijsman topology determined by the metric $d(x, y) = \|x - y\|$ and denoted by τ, i.e., τ is the weak topology determined by the family $\{d(y, \cdot) : y \in Y\}$. We recall some basic facts concerning the topological space $(\mathcal{F}(Y), \tau)$ and refer to [6, 7] for further details. A subbase of τ consists of all sets of the form:

$$\{A \in \mathcal{F}(Y) : d(y, A) < \varepsilon\} \quad \text{and} \quad \{A \in \mathcal{F}(Y) : d(y, A) > \varepsilon\} \quad (y \in Y,\ \varepsilon > 0).$$

A sequence of closed sets $\{A_n\}$, $A_n \subseteq Y$ converges to some $A \subseteq Y$ in $(\mathcal{F}(Y), \tau)$ (written $A_n \xrightarrow{W} A$) if and only if the sequence of the distance functions converges pointwise, i.e.,

$$A_n \xrightarrow{W} A \quad \text{if and only if} \quad d(y, A_n) \to d(y, A) \text{ for all } y \in Y,$$

The space $(\mathcal{F}(Y), \tau)$ is metrizable if and only if Y is separable. For $\{y_i\}$ being a countable dense subset of the separable metric space (Y, d) the following metric is compatible with the Wijsman topology:

$$\rho_d(A, B) = \sum_{i=1}^{\infty} 2^{-i} \min\{1, |d(y_i, A) - d(y_i, B)|\}.$$

Moreover, if (Y, d) is a complete separable metric space, then $(\mathcal{F}(Y), \tau)$ is a Polish space, i.e., it is separable and there is a complete metric compatible with the topology (cf. [8]).
The following notions of differentiability of set-valued mappings will be used.

Definition 2.1 *A mapping* $F : X \rightrightarrows Y$ *is called directionally differentiable at a point* $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ *in direction* $h \in X$*, if the limit*

$$F'(\bar{x}, \bar{y}; h) = \lim_{t \downarrow 0} \frac{F(\bar{x} + th) - \bar{y}}{t}$$

exists in $(\mathcal{F}(Y), \tau)$*.*

The original definition of semi-differentiability (cf.[27]) is modified in the following way:

Definition 2.2 *A mapping* $F : X \rightrightarrows Y$ *is called semi-differentiable at a point* $(x, y) \in \operatorname{graph} F$*, if the limit*

$$DF(x, y)(h_0) = \lim_{t \downarrow 0, h \to h_0} \frac{F(x + th) - y}{t}$$

exists in $(\mathcal{F}(Y), \tau)$ *for all* $h_0 \in X$*.*

The convergence $h \to h_0$ is understood as the norm-convergence in X. We recall that, originally, these two notions are associated with the Kuratowski-Painlevé convergence of the corresponding differential quotients in the literature. For Y being infinite dimensional, there is no topologization of this convergence in general (cf.[6]). This means that, in general, there is no topology on $\mathcal{F}(Y)$ such that the convergence of elements of $\mathcal{F}(Y)$ in the sense of Kuratowski-Painlevé will force convergence with respect to the topology. We require some stronger convergence in the definitions above. The Wijsman topology is a natural counterpart of the Kuratowski-Painlevé convergence for infinite dimensional spaces. Moreover, it plays a crucial role for measurability properties and convergence in distribution of random sets as we shall see later.
The various notions of first order differentiability are extended to higher order properties e.g. in [4, 3, 31]. The approach used to introduce a higher order derivative is to define sets that are higher order "tangent" to the graph of the multifunction. This way of introducing second order derivatives typically leads to a set-valued counterpart of second order directional derivative in the sense of Ben-Tal and Zowe. Let us illustrate this by discussing the notions considered in [3]. A set $A \subseteq I\!R^n$ is called pseudo-differentiable in [3] if

$$\liminf_{t \downarrow 0} \frac{A - x}{t} = \limsup_{t \downarrow 0} \frac{A - x}{t}.$$

The limit set builds a first-order derivative. The set A is second-order pseudo-differentiable with respect to y taken from the first-order derivative if

$$\liminf_{t \downarrow 0} \frac{A - x - ty}{t^2/2} = \limsup_{t \downarrow 0} \frac{A - x - ty}{t^2/2}.$$

Having in mind that the Kuratowski-Painlevé convergence of closed sets in $I\!R^n$ is characterized by the convergence of the distance function, it follows that A is pseudo-differentiable at x if the distance function $d(\cdot, A)$ is directionally differentiable, and A

is twice pseudo-differentiable if $d(\cdot, A)$ admits a second order directional derivative in the sense of Ben-Tal and Zowe, i.e., setting $d_A(\cdot) = d(\cdot, A)$

$$d''_A(x; h, v) = \lim_{t \downarrow 0} \frac{d_A(x + th + t^2 v/2) - d_A(x) - t d'_A(x; h)}{t^2/2}.$$

The second order semi-differentiability is considered there in a similar way, i.e., the multifunction $F : I\!R^m \rightrightarrows I\!R^n$ is called twice semi-differentiable at (x, y) in direction (h, z) if the limit

$$\lim_{t \downarrow 0, v \to v_0} \frac{F(x + th + t^2 v/2) - y - tz}{t^2/2}$$

exists for all $v_0 \in I\!R^m$. Here $z \in DF(x, y)(h)$. If F is single-valued, it is exactly second order Hadamard-directional differentiability in the sense of Ben-Tal and Zowe (cf.[10]). We prefer to work with a set-valued counterpart of the classical Hadamard second-order directional derivative. It will turn out that this is not only useful for constructing second-order directionally differentiable selections, but that this is the proper setting for the application we have in mind.

Definition 2.3 *The mapping $F : X \rightrightarrows Y$ is called twice directionally differentiable at a point $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ in direction $h \in X$ with respect to $z \in Y$ if the limit*

$$F''(\bar{x}, \bar{y})(h, z) = \lim_{t \downarrow 0} \frac{F(\bar{x} + th) - \bar{y} - tz}{t^2/2}$$

exists in $(\mathcal{F}(Y), \tau)$.

Definition 2.4 *The mapping $F : X \rightrightarrows Y$ is called twice semi-differentiable at a point $(x, y) \in \operatorname{graph} F$ if the limit*

$$DF(x, y)(h_0, z_0) = \lim_{t \downarrow 0, h \to h_0, z \to z_0} \frac{F(x + th) - y - tz}{t^2/2},$$

where $z \in DF(x, y)(h)$ exists in $(\mathcal{F}(Y), \tau)$ for all $h_0 \in X$ and $z_0 \in DF(x, y)(h_0)$.

The convergences $h \to h_0$ and $z \to z_0$ are taken with respect to the corresponding norms. Of course, one should ask the question whether, for all $z_0 \in DF(x, y)(h_0)$, there are elements $z \in DF(x, y)(h)$ that converge in norm to z. As a matter of fact, there exist such elements by virtue of Proposition 2.6 bellow.
Let us agree upon calling a mapping (twice) semi-differentiable at a certain point if the corresponding limit exists and is, in addition, nonempty. This requierment is implicitly included in the definitions above since only nonempty subsets are included in $\mathcal{F}(Y)$ and we consider the convergence there. Originally, the existence of the limit with respect to the Kuratowski-Painlevé convergence does not guarantee nonempty derivatives. The following straightforward observation justifies the Definitions 2.2 and 2.4.

Proposition 2.5 *Let $f : X \to Y$ be considered as a multifunction F by setting $F(x) := \{f(x)\}$. F is (twice) semi-differentiable if and only if f is (twice) Hadamard-directionally differentiable.*

Note that the first-order semi-derivatives are closed cones and the second-order derivatives are closed sets. Moreover, both derivatives are convex, if F has convex images. We will show that the first-order semi-derivatives are τ-continuous multifunctions of the directions and, thus, represent a continuous approximation of the multifunction. Correspondingly, the second-order derivatives are τ-continuous with respect to the pair (h, z).
We call a multifunction $F : X \rightrightarrows Y$ *τ-continuous* at a point $\bar{x} \in X$ if

$$\tau - \lim_{x \to \bar{x}} F(x) = F(\bar{x}).$$

Proposition 2.6 *Let $F : X \rightrightarrows Y$ be semi-differentiable (resp. twice semi-differentiable) at $(\bar{x}, \bar{y})$. Then, the first-order (resp. second-order) semi-derivative is a τ-continuous multifunction in the sense that*

$$DF(\bar{x}, \bar{y})(\bar{h}) = \tau - \lim_{h \to \bar{h}} DF(\bar{x}, \bar{y})(h) \quad \textit{and} \quad D^2F(\bar{x}, \bar{y})(\bar{h}, \bar{z}) = \tau - \lim_{h \to \bar{h}, z \to \bar{z}} D^2F(\bar{x}, \bar{y})(h, z),$$

where $z \in DF(\bar{x}, \bar{y})(h)$ and $\bar{z} \in DF(\bar{x}, \bar{y})(\bar{h})$.

Proof: Let a sequence $\{h_n\}$ be given. We consider $d(y, DF(\bar{x}, \bar{y})(h_n))$ for arbitrary but fixed $y \in Y$. Given an $\varepsilon > 0$, then there exists a positive number t_n (that can be taken smaller than t_{n-1}) for any n such that

$$|d(y, DF(\bar{x}, \bar{y})(h_n)) - d(y, t_n^{-1}(F(\bar{x} + t_n h_n) - \bar{y}))| \leq \varepsilon$$

by the definition of a semi-derivative and τ-convergence. For the same reason the following inequality holds for n large enough:

$$|d(y, DF(\bar{x}, \bar{y})(\bar{h}) - d(y, t_n^{-1}(F(\bar{x} + t_n h_n) - \bar{y}))| \leq \varepsilon.$$

Consequently, $|d(y, DF(\bar{x}, \bar{y})(h) - d(y, DF(\bar{x}, \bar{y})(h_n)| \leq 2\varepsilon$ for n large enough. Letting $\varepsilon \to 0$ we obtain the convergence of the distance functions and the continuity of the first-order semi-derivative.
Let now an element $\bar{z} \in DF(\bar{x}, \bar{y})(\bar{h})$ be chosen. In view of the continuity of $DF(\bar{x}, \bar{y})(\cdot)$, it is an easy observation that there exists a sequence $z_n \in DF(\bar{x}, \bar{y})(h_n)$ that converges to $\bar{z}$: $0 = d(z, DF(\bar{x}, \bar{y})(\bar{h}) = \lim_{n \to \infty} d(z, DF(\bar{x}, \bar{y})(h_n)$ which imply that there exist $z_n \in DF(\bar{x}, \bar{y})(h_n)$ such that $0 = \lim_{n \to \infty} \|z_n - z\|$. Pick up such a sequence. Similarly as above, we construct a sequence $t_n \downarrow 0$ and obtain, in the same way, an estimation

$$|d(y, DF(\bar{x}, \bar{y})(\bar{h}, \bar{z}) - d(y, DF(\bar{x}, \bar{y})(h_n, z_n))| \leq 2\varepsilon$$

via the second-order differential quotients. We conclude the desired assertion since y was arbitrary. $\square$

Proposition 2.7 *Let $F : X \rightrightarrows Y$ be semi-differentiable (resp. twice semi-differentiable) at $(\bar{x}, \bar{y})$. Then, the first-order (resp. second-order) semi-derivative is a lower semicontinuous multifunction, i.e., whenever $V \cap DF(\bar{x}, \bar{y})(\bar{h}) \neq \emptyset$ for some open set $V \subseteq Y$, there is a neighbourhood $U \subseteq X$ of $\bar{h}$, such that $V \cap DF(\bar{x}, \bar{y})(h) \neq \emptyset$ for all $h \in U$.*

Proof: We show the statement for the first-order semi-derivative; the property for the second-order derivative follows in the same way. Assume the opposite. This means that there is an open set $V \subseteq Y$ such that $V \cap DF(\bar{x}, \bar{y})(\bar{h}) \neq \emptyset$ and for any neighbourhood $U_\epsilon \subseteq X$ of $\bar{h}$, there is a point $h_\epsilon \in U_\epsilon$ such that $V \cap DF(\bar{x}, \bar{y})(h_\epsilon) = \emptyset$. Let $z \in V \cap DF(\bar{x}, \bar{y})(\bar{h})$. There exists a $\delta > 0$ such that $\tilde{z} \in V$ for all $\|\tilde{z} - z\| \leq \delta$. Taking $\epsilon = 1/n$ we obtain a sequence $h_n \to \bar{h}$. At the same time, it holds $d(z, DF(\bar{x}, \bar{y})(h_n)) \geq \delta$ and $d(z, DF(\bar{x}, \bar{y})(\bar{h})) = 0$. This is a contradiction to the τ-continuity proved in the previous proposition. □

The different differentiability concepts for finite dimensions are compared in [5, 28]. Continuous tangential approximations of set-valued mappings, also in infinite dimensions, are considered in [32, 33], where differentiability properties of the metric projection are investigated. In [5], it is shown for mappings in finite-dimensional spaces that such tangential approximations, if they exist, coincide with the semi-derivatives, where at that place the semi-derivative is understood with respect to the Kuratowski-Painlevé convergence of the differential quotients. Furthermore, in [5, 28], it is shown that proto-differentiability, resp. tangential differentiability for pseudo-Lipschitzian multifunctions implies semi-differentiability.
It is not difficult to establish the latter relation also in our setting.
Recall that a set-valued mapping $F : X \rightrightarrows Y$ is said to be *pseudo-Lipschitzian* at a point $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ if there exist neighbourhoods $U \subseteq X$ of $\bar{x}$ and $V \subseteq Y$ of $\bar{y}$, and a positive constant L such that

$$d(F(x_1) \cap V, F(x_2) \cap V) \leq L\|x_1 - x_2\|, \tag{1}$$

whenever $x_1,\, x_2 \in U$. Here $d(A, B)$ stands for the Hausdorff distance, i.e.,

$$d(A, B) = \max\{\sup_{y \in B} d(y, A), \sup_{y \in A} d(y, B)\}.$$

Proposition 2.8 *If $F : X \rightrightarrows Y$ is pseudo-Lipschitzian at a point $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ and tangentially differentiable in every direction at that point, then it is semi-differentiable at $(\bar{x}, \bar{y})$ and the semi-derivative coincide with the directional derivative.*

Proof: Let sequences $\{h_n\}$, $h_n \in X$ and $\{t_n\}$ be given such that $h_n \to h_0$ and $t_n \downarrow 0$. We denote by U and V the neighbourhoods of $\bar{x}$ and $\bar{y}$, respectively, such that the inequality (1) holds with some constant L. Let $\varepsilon > 0$ be given and $y \in Y$ be an arbitrary point. The following sequence of inequalities holds for n large enough:

$$\begin{aligned}
&|d(y, \frac{F(\bar{x} + t_n h_n) - \bar{y}}{t_n}) - d(y, F'(\bar{x}, \bar{y})(h_0))| \\
&\leq |d(y, \frac{F(\bar{x} + t_n h_0) - \bar{y}}{t_n}) - d(y, F'(\bar{x}, \bar{y})(h_0))| + \\
&|d(y, \frac{F(\bar{x} + t_n h_0) - \bar{y}}{t_n}) - d(y, \frac{F(\bar{x} + t_n h_n) - \bar{y}}{t_n})| \\
&\leq \varepsilon + t_n^{-1}[d(\bar{y} + t_n y, F(\bar{x} + t_n h_n)) - d(\bar{y} + t_n y, F(\bar{x} + t_n h_0))]
\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon + t_n^{-1}|d(\bar{y}+t_n y, F(\bar{x}+t_n h_n)\cap V) - d(\bar{y}+t_n y, F(\bar{x}+t_n h_0)\cap V)| \\
&\leq \varepsilon + t_n^{-1} d(F(\bar{x}+t_n h_n)\cap V, F(\bar{x}+t_n h_0)\cap V) \\
&\leq \varepsilon + L\|h_n - h_0\| \leq (1+L)\varepsilon
\end{aligned}
$$

Consequently, $d(y, t_n^{-1}[F(\bar{x}+t_n h_n) - \bar{y}])$ converges to $d(y, F'(\bar{x},\bar{y};h_0))$, which proves the assertion. $\square$

3 Existence of directionally differentiable selections

The main idea of the constructions presented in this section is to exploit the approximation of the multifunction and the existence of a continuous partition of unity, subordinated to a given open covering of a linear normed space.
We recall that, given a linear normed space X and an open covering $X = \cup_{\alpha\in\Lambda} V_\alpha$ of it, a continuous partition of unity subordinated to $\{V_\alpha\}_\Lambda$ is a union $\{g_\alpha\}_\Lambda$ of continuous functions g_α such that:

- $0 \leq g_\alpha(x) \leq 1$ for all $x \in X$,
- $\cup_{\alpha\in\Lambda}(\operatorname{supp} g_\alpha) = X$,
- for any $\alpha \in \Lambda$ there exists a finite number of $\beta \in \Lambda$ such that $\operatorname{supp} g_\alpha \cap \operatorname{supp} g_\beta \neq \emptyset$,
- for any $\alpha \in \Lambda$ one has $\operatorname{supp} g_\alpha \subseteq V_\alpha$.

Of course, the existence of such a partition is known for a larger class of spaces, but this formulation corresponds to our setting.
Let us denote the metric projection of $y \in Y$ onto a closed convex set A by $\Pr(y, A)$. First we need some results concerning the continuity properties of the metric projection, because this will play a key role for the constructions presented in the remainder of this section.
At this place, another topology on the hyperspace $\mathcal{F}(Y)$ becomes useful. Let m denotes the Mosco-topology on $\mathcal{F}(Y)$.
A sequence $\{A_n\}$ of weakly closed subsets of Y converges to a set $A \subseteq Y$ in $(\mathcal{F}(Y), m)$ if both of the following conditions are met:

(i) for each $y \in A$ there are points $y_n \in A_n$ such that $\{y_n\}$ converges in norm to y;

(ii) whenever points $y_k \in A_{n_k}$, $k = 1, 2, \ldots$, are given, then the weak convergence of $\{y_n\}$ to $y \in Y$ implies $y \in A$.

This convergence is introduced by Mosco ([26]) as a modification of the sequential characterization of Kuratowski-Painlevé convergence (valid in a first countable space) in order to take into consideration the weak topology.
Recall that Y has the Kadec-Klee property if weak convergence and convergence of the norms imply norm convergence. The following relation between the Mosco- and the Wijsman-topology is known:

Lemma 3.1 *([6]) Let Y be a normed linear space. Then $\tau = m$ on $\mathcal{F}_c(Y)$ if and only if Y is reflexive and the dual space Y^* has the weak* Kadec-Klee property, i.e., weak* convergence in Y^* and convergence of the norms imply norm convergence in Y^*.*

A reflexive Banach space with a strongly convex norm and the Kadec-Klee property is called an *E-space*. We will use the following result:

Lemma 3.2 *([19]) Let Y and its dual Y^* be E-spaces and $A,\ A_n \in \mathcal{F}_c(Y)$, $n = 1, 2, \ldots$. Then the following three statements are equivalent:*

(i) $m - \lim_{n\to\infty} A_n = A$

(ii) $\lim\limits_{n\to\infty} d(y, A_n) = d(y, A)$ *for all* $y \in Y$

(iii) $\lim\limits_{n\to\infty} \mathrm{Pr}(y, A_n) = \mathrm{Pr}(y, A)$ *with respect to the norm-convergence in Y.*

The metric projection onto a convex set is unique provided that the norm is strongly convex. The lemma shows that the Mosco-topology is the weakest one that guarantees strong continuity of the metric projection onto moving convex sets.
Examples of E-spaces are Euclidean spaces, Hilbert spaces, uniformly convex Banach spaces (see e.g.[19]). Each reflexive space is potentially an E-space, since it can be renormed with an equivalent locally uniformly convex norm by a theorem of Troyanski (cf.[17], Theorem 1, p.164).

Lemma 3.3 *Let Y and its dual Y^* be E-spaces. Furthermore, let the multifunction $F : X \rightrightarrows Y$ be τ-continuous at $\bar{x}$ and have nonempty closed convex images. Suppose that the single-valued mapping $y : X \to Y$ is continuous at $\bar{x}$ with respect to the norms in X and Y. Then the metric projection $f(x) = \mathrm{Pr}(y(x), F(x))$ is continuous at $\bar{x}$ with respect to the norms.*

Proof: The statement follows immediately from the cited Lemma 3.2 and the fact that Mosco and Wijsman topologies agree on $\mathcal{F}_c(Y)$. $\square$

Theorem 3.4 *Let both Y and Y^* be E-spaces. Suppose $F : X \rightrightarrows Y$ to be a τ-continuous multifunction with non-empty closed convex images. We assume that for all $x \in X$ there is a $y \in F(x)$ such that F is semi-differentiable at (x, y).*
Given such a point $(\bar{x}, \bar{y})$, there exists a continuous selection f of F with $f(\bar{x}) = \bar{y}$ which is Hadamard-directionally differentiable at $\bar{x}$ with a derivative

$$f'(\bar{x}; h) \in DF(\bar{x}, \bar{y}; h),\ \ \forall h \in X.$$

Proof: Let $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ be the chosen point. We define a neighbourhood $U_{\bar{x}}$ of $\bar{x}$ by fixing some $\varepsilon > 0$ and setting $U_{\bar{x}} = \bar{x} + \varepsilon B$, where B is the open unit ball in X. By Proposition 2.6, $DF(\bar{x}, \bar{y})(\cdot)$ is a continuous set-valued mapping. Hence, there exists a continuous selection $\bar{z}(h) \in DF(\bar{x},\bar{y})(h)$. Let us consider the restriction of $\bar{z}(\cdot)$ to the unit sphere and define $z(\cdot)$ by setting:

$$z(h) = \begin{cases} \bar{z}(h) & \text{if } \|h\| = 1, \\ \|h\|\bar{z}(h/\|h\|) & \text{if } 0 < \|h\| < 1, \\ 0 & \text{if } h = 0. \end{cases}$$

Consequently, $z(\cdot)$ is a continuous positive homogeneous mapping defined on the closed unit ball of X. It is obvious that $z(h) \in DF(\bar{x},\bar{y})(h)$.
Take a point $x \in U_{\bar{x}}$. It can be represented as

$$x = \bar{x} + \|x - \bar{x}\| \frac{x - \bar{x}}{\|x - \bar{x}\|} = \bar{x} + \lambda h$$

by setting

$$\lambda = \|x - \bar{x}\| \text{ and } h = \frac{x - \bar{x}}{\|x - \bar{x}\|}$$

Consider $F(x),\ x \in U_{\bar{x}}$ and define

$$f_{\bar{x}}(x) = \Pr(\bar{y} + z(x - \bar{x}), F(x)).$$

Using Lemma 3.3, we conclude that $f_{\bar{x}}$ is continuous at any point $x \in U_{\bar{x}}$. Further, we consider all points $x \in X$ such that $\|x - \bar{x}\| \geq \varepsilon$. There exists a $y \in F(x)$ such that F is semi-differentiable at (x, y) and we can construct a neighbourhood U_x and a continuous mapping $f_x : U_x \to Y$ in the same way. We set $U_x^c = X \setminus U_{\bar{x}}$ and define neighbourhoods

$$\begin{aligned} V_{\bar{x}} &= U_{\bar{x}}, \\ V_x &= \frac{\varepsilon}{2} U_x \text{ for all } x \in U_x^c. \end{aligned}$$

It is clear that

$$\Big(\bigcup_{x \in U_x^c} V_x\Big) \cup V_{\bar{x}} = X.$$

Let $\{g_\alpha\}_\Lambda$ be the partition of the unity subordinated to this covering. This means that any g_α corresponds to a neighbourhood V_x. We denote this neighbourhood by V_α and the mapping f_x by f_α. Now set:

$$f(x) = \sum_{\alpha \in \Lambda} g_\alpha(x) f_\alpha(x).$$

$f(\cdot)$ is well-defined by $\operatorname{supp} g_\alpha \subseteq V_\alpha$, where f_α is defined. Note that f is continuous as a sum of continuous functions. Given a point x, $\sum_{\alpha \in \Lambda} g_\alpha(x) f_\alpha(x)$ is a finite sum by the local finiteness of the partition of unity. It is a convex combination of points $f_\alpha(x) \in F(x)$ and so $f(x) \in F(x)$. Consequently, f is a continuous selection of F.
Since the point $\bar{x}$ and some part of $V_{\bar{x}}$ are not overlapped by other open sets V_x, there exists a function $g_{\bar{\alpha}}$ such that $g_{\bar{\alpha}}(x) = 1$ for all x in some smaller neighbourhood $\bar{V} \subseteq V_{\bar{x}}$.

Consequently, $f(x) = f_{\bar{x}}(x)$ on $\bar{V}$. Therefore, $f(\bar{x}) = \bar{y}$ and we are left with the task to show that $f_{\bar{x}}$ is Hadamard-directionally differentiable at $\bar{x}$ with $f'(\bar{x}; h) \in DF(\bar{x}, \bar{y})(h)$ for all $h \in X$. We have to prove

$$\lim_{t \downarrow 0, h \to h_0} \frac{f_{\bar{x}}(\bar{x} + th) - f_{\bar{x}}(\bar{x})}{t} = z \quad \text{for some } z \in DF(\bar{x}, \bar{y})(h_0)$$

By construction $f_{\bar{x}}(\bar{x} + th) = \Pr(\bar{y} + z(th), F(\bar{x} + th))$. Consequently,

$$\begin{aligned} & \| \frac{f_{\bar{x}}(\bar{x} + th) - f_{\bar{x}}(\bar{x})}{t} - z(h_0) \| \\ = \;& t^{-1} \| \Pr(\bar{y} + z(th), F(\bar{x} + th)) - \bar{y} - tz(h_0) \| \\ = \;& t^{-1} \| \Pr(\bar{y} + tz(h), F(\bar{x} + th)) - \bar{y} - tz(h) + tz(h) - tz(h_0) \| \\ \leq \;& t^{-1} d(\bar{y} + tz(h), F(\bar{x} + th)) + \| z(h) - z(h_0) \| \end{aligned}$$

The first term converges to 0 by the definition of a semi-derivative and the second term by the construction of $z(\cdot)$. Setting $f'(\bar{x}; h_0) = z(h_0)$ we obtain the desired property. This completes the proof. □

Unfortunately, the constructed selection is differentiable only at a given point. An assumption about the semi-differentiability of the multifunction at each point of its graph would not lead to the directional differentiability of the selection constructed in this manner. The only "globalization" is the following:

Corollary 3.5 *Under the assumptions of the previous theorem suppose that $x_i \in X$, $i \in I$, where I is some index-set, are such that*

(i) there are $\varepsilon_i > 0$ with $B(x_i, \varepsilon_i) \cap B(x_j, \varepsilon_j)$ $\forall i \neq j$,

(ii) for all $i \in I$, there is a $y_i \in F(x_i)$: F is semi-differentiable at (x_i, y_i).

Then there is a continuous selection f of F such that for all $i \in I$, $f(x_i) = y_i$ and f is Hadamard-directionally differentiable at x_i with $f'(x_i; h) \in DF(x_i, y_i)(h)$, for all $h \in X$.

An immediate consequence of this corollary and Proposition 2.8 is the following statement:

Corollary 3.6 *Let Y and Y^* be E-spaces. Suppose $F : X \rightrightarrows Y$ to be a pseudo-Lipschitzian multifunction with non-empty closed convex images. We assume that for all $x \in X$ there is a $y \in F(x)$ such that F is directionally differentiable at (x, y) in any direction.*
Let the points $x_i \in X$, $i \in I$, where I is some index-set, are chosen such that

(i) there are $\varepsilon_i > 0$ with $B(x_i, \varepsilon_i) \cap B(x_j, \varepsilon_j)$ $\forall i \neq j$,

(ii) for all $i \in I$, let $y_i \in F(x_i)$: F is directionally differentiable at (x_i, y_i).

Then there is a continuous selection f of F such that for all $i \in I$, $f(x_i) = y_i$ and f is Hadamard-directionally differentiable at x_i with $f'(x_i;h) \in F'(x_i,y_i)(h)$ for all $h \in X$.

In [18], Dommisch establishes the following result on this topic. The existence of a smooth partition of unity in $I\!R^n$ is exploited there to prove the existence of a smooth selection in the following situations: F acts from a closed subset of $I\!R^m$ in $I\!R^n$ and has convex images and open preimages, i.e., $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X for all $y \in I\!R^n$. The proof of this assertion can be carried over also to the case X is a weakly countably determined Banach space and Y is a linear normed space.
A Banach space X is called *weakly countably determined* if there exists a countable collection $\{K_n\}$ of weak* compact subsets of X^{**} such that for every $x \in X$ and $u \in X^{**} \setminus X$ there exists an index n such that $x \in K_n$ and $u \notin K_n$.

Theorem 3.7 *Let X be a weakly countably determined Banach space and Y be a linear normed space. Assume that F has nonempty closed convex images and open preimages, i.e., $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X for all $y \in F(X)$.*
Then F admits Gâteaux differentiable selection.

Proof: The set $\bigcup_{y \in F(X)} F^{-1}(y)$ forms an open covering of X. By virtue of Corollary 3.3, Chap.VIII in [16] the countably determined Banach space X admits a Gâteaux differentiable partition of unity subordinated to this covering, i.e., the functions g are Gâteaux differentiable. Each function g_α corresponds to some set $F^{-1}(y)$ and we note this by writing g_y instead of g_α. We define

$$f(x) = \sum_{y \in F(x)} g_y(x) y.$$

Given a point x, $\sum_{y \in F(x)} g_y(x)y$ is a finite sum by the local finiteness of the partition of unity. It is a finite convex combination of the points $y \in F(x)$ such that $x \in \operatorname{supp} g_y$ and, thus, $f(x) \in F(x)$. We show that f is Gâteaux differentiable. Let $\bar{x}$ be a given point, and $\{x_n\}$ be a sequence converging in norm to $\bar{x}$. By the local finiteness of the partition, there exists a neighbourhood of $\bar{x}$ that has a nonempty untersection with finitely many $\operatorname{supp} g_y$. Therefore, in the definition of $f(x_n)$ the same terms $g_y(x_n)y$ vanish for all x_n with n large enough. Consequently, $f(x)$ is Gâteaux differentiable at $\bar{x}$ with derivative $f'(x) = \sum_{y \in F(x)} g'_y(x)y$. □

Note that the assertion remains true in the situation that $\operatorname{dom} F$ is a proper closed subset of X. In this case, we simply have to include $X \setminus \operatorname{dom} F$ to the open covering of X.
The selection constructed in the latter theorem has stronger differentiability properties globally. However, the assumptions of the theorem are not satisfied when F is an upper semicontinuous mapping for instance. It is known (cf.[6]) that the preimages $F^{-1}(y)$ are closed provided F is upper semicontinuous.
We use the same idea to construct a selection that is twice Hadamard-directionally differentiable. The essential difference to the construction of Theorem 3.4 is that a second-order approximation is assumed at one particular point only.

Theorem 3.8 *Let Y and Y^* be E-spaces, and $F : X \rightrightarrows Y$ be a τ-continuous multifunction with non-empty closed convex images. We assume that for all $x \in X$ there is a $y \in F(x)$ such that F is semi-differentiable at (x, y). Furthermore, let F be twice semi-differentiable at a given point $(\bar{x}, \bar{y})$.*
Then there exists a continuous selection f of F with $f(\bar{x}) = \bar{y}$, which is twice Hadamard-directionally differentiable at $\bar{x}$ with derivatives

$$f'(\bar{x}; h) \in DF(\bar{x}, \bar{y})(h), \quad \text{and } f''(\bar{x}; h) \in D^2F(\bar{x}, \bar{y})(h, f'(\bar{x}; h)) \text{ for all } h \in X.$$

Proof: Let $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ be the chosen point. In the same way as in the proof of Theorem 3.4, we define an ε-neighbourhood $U_{\bar{x}}$ of $\bar{x}$ and the mapping $z : h \mapsto z(h) \in DF(\bar{x}, \bar{y})(h)$. By virtue of Proposition 2.7, $D^2F(\bar{x}, \bar{y})(h, \bar{z}(h))$ is an lower semicontinuous mapping for h on the unit ball. Moreover, it has closed and convex images and, consequently, we may apply Michael's selection theorem. We choose a continuous selection $\bar{v}$ of $D^2F(\bar{x}, \bar{y})(h, \bar{z}(h))$ for h on the unit sphere. Define $v : \bar{B} \to Y$ with $\bar{B}$ being the closed unit ball in X by setting

$$v(h) = \begin{cases} \bar{v}(h) & \text{if } \|h\| = 1, \\ \|h\|^2/2\bar{v}(h/\|h\|) & \text{if } h: \ 0 < \|h\| < 1, \\ 0 & \text{if } h = 0. \end{cases}$$

Let us observe that $v \in D^2F(\bar{x}, \bar{y})(h, z)$ if and only if $\lambda^2/2v \in D^2F(\bar{x}, \bar{y})(h, \lambda z)$ for $\lambda > 0$. Therefore, it holds that $v(h) \in D^2F(\bar{x}, \bar{y})(h, z(h))$ for $\|h\| \leq 1$. Take a point $x \in U_{\bar{x}}$ and represent it as

$$x = \bar{x} + \|x - \bar{x}\| \frac{x - \bar{x}}{\|x - \bar{x}\|} = \bar{x} + \lambda h$$

where

$$\lambda = \|x - \bar{x}\| \text{ and } h = \frac{x - \bar{x}}{\|x - \bar{x}\|}$$

Choose $y \in F(x)$ that are metric projections of $\bar{y} + z(x - \bar{x}) + v(x - \bar{x})$ onto $F(x)$ and define

$$f_{\bar{x}}(x) = \Pr(\bar{y} + z(x - \bar{x}) + v(x - \bar{x}), F(x)).$$

Using Lemma 3.3, we conclude again that $f_{\bar{x}}$ is continuous at any point $x \in U_{\bar{x}}$. We proceed further as in the previous proof. The only difference that has to be worked out is to show that $f_{\bar{x}}$ is twice Hadamard-directionally differentiable at $\bar{x}$ with a directional derivative belonging to the corresponding semi-derivative of the multifunction. By construction $f_{\bar{x}}(\bar{x} + th) = \Pr(\bar{y} + z(th) + v(th), F(\bar{x} + th)$. Without loss of generality we may consider only directions h and sequences $\{h_n\}$ on the unit sphere, i.e., $\|h\| = 1$ and $\|h_n\| = 1$. The following chain of relations holds true:

$$\begin{aligned} & \left\| \frac{f_{\bar{x}}(\bar{x} + t_n h_n) - f_{\bar{x}}(\bar{x})}{t_n} - z(h_0) \right\| \\ = \ & t_n^{-1} \| \Pr(\bar{y} + t_n z(h_n) + t_n^2/2 v(h_n), F(\bar{x} + t_n h_n)) - \bar{y} - t_n z(h_0) \| \end{aligned}$$

$$
\begin{aligned}
&= t_n^{-1}\| \Pr(\bar{y} + t_n z(h_n) + t_n^2/2v(h_n), F(\bar{x} + t_n h_n)) - \\
&\quad \bar{y} - t_n z(h_n) - t_n^2/2v(h_n) + t_n z(h_n) + t_n^2/2v(h_n) - t_n z(h_0)\| \\
&\le t_n^{-1} d(\bar{y} + t_n z(h_n) + t_n^2/2v(h_n), F(\bar{x} + th)) + \|z(h) + t_n/2v(h) - z(h_0)\|
\end{aligned}
$$

The first term converges to 0 by the definition of a second order semi-derivative, and the second term by the construction of $z(\cdot)$. This proves the first-order directional differentiability of the selection and we obtain $f'(\bar{x}; h_0) = z(h_0)$. The second-order property is established by the inequalities:

$$
\begin{aligned}
&\left\| \frac{f_{\bar{x}}(\bar{x} + t_n h_n) - f_{\bar{x}}(\bar{x}) - t_n f'_{\bar{x}}(\bar{x}; h_n)}{t_n^2/2} - v(h_0) \right\| \\
&= t_n^{-2}\| \Pr(\bar{y} + t_n z(h_n) + t_n^2/2v(h_n), F(\bar{x} + t_n h_n)) - \bar{y} - t_n z(h_n) - t_n^2/2v(h_0)\| \\
&= t_n^{-2} 2\| \Pr(\bar{y} + t_n z(h_n) + t_n^2/2v(h_n), F(\bar{x} + t_n h_n)) - \\
&\quad \bar{y} - t_n z(h_n) - t_n^2/2v(h_n) + t_n^2/2v(h_n) - t_n^2/2v(h_0)\| \\
&\le t_n^{-2} 2d(\bar{y} + t_n z(h_n) + t_n^2/2v(h_n), F(\bar{x} + th)) + \|v(h_n) - v(h_0)\|
\end{aligned}
$$

The first term converges to 0 by the definition of a second order semi-derivative, and the second term by construction. We have $f''(\bar{x}; h_0) = v(h_0)$, as required. $\square$

Remark 3.9 Renorming of the image space
As already mentioned, every reflexive space can be renormed with an equivalent locally uniformly convex norm (see Theorem 1, p.164, and Theorem 2, p.37 in [17]). The new norm, although compatible with the original one, generates a new Wijsman topology on $\mathcal{F}(Y)$. It has been proved in [9] that, given a linear normed space Y and a family π of equivalent norms, the family of distance functionals $\{d_p(y, \cdot) : y \in Y, p \in \pi\}$ determines a weak topology on $\mathcal{F}_c(Y)$ that coincides with the slice topology. We do not provide information about the latter topology because it will not be essentially for our paper and refer the interested reader to [6]. Assuming Y to be a Banach space, the slice topology on $\mathcal{F}_c(Y)$ agrees with a Wijsman topology that is determined by a norm such that the dual norm has the weak* Kadec-Klee property. If, in addition, Y is reflexive, then both topologies coincide with the Mosco topology. Therefore, we may modify the assumptions of the latter two theorems in the following way:

Corollary 3.10 *Let Y be a reflexive Banach space. Suppose $F : X \to \mathcal{F}_c(Y)$ to be m-continuous (equivalently: continuous with respect to the slice topology). We assume that for all $x \in X$ there is a $y \in F(x)$ such that F is (twice) semi-differentiable at (x, y). Given such a point $(\bar{x}, \bar{y})$, there exists a continuous selection f of F with $f(\bar{x}) = \bar{y}$, which is (twice) Hadamard-directionally differentiable at $\bar{x}$ with derivatives*

$$f'(\bar{x}; h) \in DF(\bar{x}, \bar{y})(h), \quad \textit{(resp. } f''(\bar{x}; h) \in D^2F(\bar{x}, \bar{y})(h, f'(\bar{x}; h)) \textit{) for all } h \in X$$

4 Measurability, Castaing representation and Delta-Theorems

We equip the linear normed space X with a σ-algebra $\mathcal{A}$. The Borel σ-algebra will be denoted by $\mathcal{B}(X)$. The following definition of measurability will be used (see also

[6, 4]):

Definition 4.1 *A mapping $f : (X, \mathcal{A}) \to (Y, \mathcal{B}(X))$ is measurable if for any open set $G \subseteq Y$ the preimage $f^{-1}(G) = \{x \in X \ : \ f(x) \in G\}$ belongs to $\mathcal{A}$.*
A set-valued mapping $F : (X, \mathcal{A}) \rightrightarrows (Y, \mathcal{B}(X))$ is measurable if for any open set $G \subseteq Y$ $F^{-1}(G) = \{x \in X \ : \ F(x) \cap G \neq \emptyset\} \in \mathcal{A}$.

Recall that $f : (X, \mathcal{A}) \to Y$ is called *a measurable selection* of F if f is measurable and $f(x) \in F(x)$ almost surely.
It is a fundamental result (cf. [23]) that a closed-valued measurable multifunction in a Polish image space admits a measurable selection. Furthermore, for a multifunction F with non-empty closed values in a Polish target space we can choose a countable family of measurable selections $\{f_n\}$ that pointwise fills up the values of the multifunction:

$$\text{for each } x \in X,\ F(x) = cl(\cup_{n=1}^{\infty} f_n(x)).$$

Such a countable family is called *a Castaing representation* for the multifunction. Moreover, the existence of such a representation characterizes measurability.
Our aim is to construct measurable selections resp. Castaing representations of a multifunction $F : X \rightrightarrows Y$ with additional directional differentiability properties. The motivation for this investigation is given by the so-called delta-approach. Delta-theorems are concerned with the asymptotic distribution of functions of random elements. The following theorem is known.

Theorem 4.2 *([35]) Suppose that X and Y are Banach spaces and let $f : (X, \mathcal{B}(X)) \to (Y, \mathcal{B}(Y))$ be measurable and Hadamard-directionally differentiable at some point $\bar{x} \in X$. Suppose that $t_n(x_n - \bar{x})$ are some random elements of X converging in distribution to some element z, written*

$$t_n^{-1}(x_n - \bar{x}) \xrightarrow{d} h,$$

while $t_n \downarrow 0$ and h is a random element in some separable subspace of X. Then

$$t_n(f(x_n) - f(\bar{x})) \xrightarrow{d} f'(\bar{x}; h).$$

Here $\xrightarrow{d}$ denotes convergence in distribution.

Recall that convergence in distribution of a sequence of random elements x_n, $x_n : (\Omega, \mathcal{A}, P) \to X$, means the weak* convergence of the measures $\mu_n = P \circ x_n^{-1}$ that these elements induce on the space X. A sequence of probability measures μ_n on a separable metric space X weakly* converges to μ if one of the following equivalent conditions is satisfied (cf.[11]):

(i) $\liminf_n P\{x_n \in G\} \geq P\{x \in G\}$ for all open sets G,

(ii) $\limsup_n P\{x_n \in C\} \leq P\{x \in C\}$ for all closed sets C,

(iii) $\lim_{n\to\infty} \int g(x)\mu_n(dx) = \int g(x)\mu(dx)$ for all bounded continuous functions $g : X \to I\!R$.

A closed-valued multifunction $F : (\Omega, \mathcal{A}) \rightrightarrows (Y, \mathcal{B}(Y))$ can be identified with a single valued mapping $\tilde{F} : (\Omega, \mathcal{A}) \to (\mathcal{F}(Y), \mathcal{B}(\mathcal{F}(Y)))$. As already mentioned, if (Y, d) is Polish, then $(\mathcal{F}(Y), \tau)$ is Polish, too. We consider the Borel sigma algebra on $\mathcal{F}(Y)$ generated by the Wijsman topology and ask whether the measurability of $\tilde{F}$ is related to the measurability of F. An answer to this question is due to Hess (see [21]): Assuming Y to be separable, F is measurable if and only if $\tilde{F}$ is so. Then, in a natural way, convergence in distribution of $F_n : (\Omega, \mathcal{A}) \rightrightarrows Y$ to some $F : (\Omega, \mathcal{A}) \rightrightarrows Y$ can be understood as the weak* convergence of the measures induced on $\mathcal{F}(Y)$ by $\tilde{F}_n$.
Convergence in distribution of set-valued mappings with images in $I\!R^n$ is considered in [29] and some investigations for the infinite-dimensional case are given in [30]. It has been shown in [29] that convergence in distribution of random sets $A_k \in I\!R^n$, $k = 1, 2, \ldots$, is equivalent to the convergence in distribution of the distance functions $d(\cdot, A_k)$ as random processes.
The first set-valued version of a delta-theorem was formulated by King [22] for random sets in $I\!R^n$. Another version is given in [35] by Shapiro. Corresponding results for random sets in infinite dimensions seem not to be known. There is also no characterization for the asymptotic behaviour of the ensemble of measurable selections. Let us emphasize that the distribution of a random set does not determine the ensemble of distributions of its measurable selections, as it has been elaborated in [1] (see also the references therein). In [1] the set of distributions of measurable selections of random sets has been studied from a different point of view. The primary object there is a given probability distribution on some compact subset of a complete separable metric space. The problem which distributions on the space are induced by selections of random sets with the the given probability distribution is investigated. In [2] "almost" Castaing representations are considered.
We first generalize the set-valued delta theorem for random sets in complete separable metric spaces and then formulate two results that establish a relation between the asymptotic distribution of random sets and their measurable selections. We are not able to characterize the behaviour of all measurable selections, but of countably many selections that form a Castaing representation of the set-valued mapping.

Theorem 4.3 *Let X and Y be separable E-spaces and $F : X \rightrightarrows Y$ be a τ-continuous multifunction with non-empty closed convex images. Suppose that F is semi-differentiable at all points $(x, y) \in \operatorname{graph} F$.Let $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ be a given point. Then F admits a Castaing representation by continuous selections $\{f_n\}$ with the following property:*
For any point $(x, y) \in \operatorname{graph} F$ and for any $\varepsilon > 0$ there is a selection f_n such that $\|y - f_n(x)\| \le \varepsilon$ and f_n is Hadamard-directionally differentiable at some $\tilde{x} : \|x - \tilde{x}\| \le \varepsilon$ with $f_n'(\tilde{x}; h) \in DF(\tilde{x}, f_n(\tilde{x}))(h)$, $\forall h \in X$.
Moreover, all selections are Hadamard-directionally differentiable at $\bar{x}$ with $f_n'(\bar{x}; h) \in DF(\bar{x}, f_n(\bar{x}))(h)$ for all $h \in X$.
There is an index k such that $f_k(\bar{x}) = \bar{y}$.

Proof: By the separability of X, there exist countably many points x_n which span a dense subset of X. Consider $F(x_n)$ and choose $\{y_n^i\}$ to be a countable set of points that are dense in $F(x_n)$. This is possible again by separability arguments. Make sure that $(\bar{x}, \bar{y})$ belongs to the chosen couples of points and let, in particular, $(\bar{x}, y_j)$ be countable and dense set of points in $F(\bar{x})$. Fixing a point (x_n, y_n^i) we may construct a continuous selection f_{ni} through (x_n, y_n^i) which is Hadamard-directionally differentiable at x_n with directional derivatives belonging to the corresponding semi-derivative of F. We have to construct every selection f_{ni} directionally differentiable at the points (x_n, y_n^i) and $(\bar{x}, y_i)$ at the same time. The two points x_n and $\bar{x}$ can always be separated by some neighbourhoods and, consequently, we may apply Corollary 3.5 to construct f_{ni} with the required properties. The selection f_{ni} is measurable by its continuity. The family $\{f_{ni}\}$ is the Castaing representation of F we are looking for. □

Note that the semi-differentiability of F is actually not needed at all points of its graph. It is sufficient to have this property at a set of points that is countable and dense in graph F.
A multifunction satisfying the assumptions of the theorem admits a Castaing representation such that for every point of the graph there is a point close to it and a selection that is Hadamard-directionally differentiable at the latter point. In the setting of the delta-approach we are interested in obtaining a Castaing representation with differentiability properties at some particular point $x \in X$, and this has been established by the above result.
Now we are ready to prove a generalized delta-theorem for random sets in infinite-dimensional spaces.

Theorem 4.4 *Let X be a linear normed space and Y be a Polish space. Suppose that $F : X \rightrightarrows Y$ is a measurable multifunction that is semi-differentiable at the point $(\bar{x}, \bar{y})$. Let us assume that the random elements $x_n : (\Omega, \mathcal{A}) \to X$ satisfy a generalized central limit formula with limit $\bar{x}$, i.e.,*

$$t_n^{-1}(x_n - \bar{x}) \xrightarrow{d} h$$

while $t_n \downarrow 0$ and h is a random element in some separable subspace of X.
Given a point $\bar{y} \in F(\bar{x})$, F satisfies the central limit formula in the sense that

$$t_n^{-1}(F(x_n) - \bar{y}) \xrightarrow{d} DF(\bar{x}, \bar{y})(h)$$

as random sets in $\mathcal{F}(Y)$.
Moreover, if the additional assumptions of Theorem 4.3 are satisfied, then F admits a Castaing representation $\{f_k\}$ such that all f_k satisfy the generalized central limit formula

$$t_n^{-1}[f_k(x_n) - f_k(\bar{x})] \xrightarrow{d} f_k'(\bar{x}; h) \in DF(\bar{x}, f_k(\bar{x}))(h)$$

as random variables in Y.
There is an index k such that $f_k(\bar{x}) = \bar{y}$.

Proof: We have to show the convergence in distribution of the random sets. Let μ_n and μ be the corresponding probability measures induced on X by $h_n := t_n^{-1}(x_n - \bar{x})$ and by h, respectively. F is measurable and, therefore, the corresponding mapping $\tilde{F} : X \to \mathcal{F}(Y)$ is measurable. Let us consider the random elements $g_n(h_n) := t_n^{-1}(\tilde{F}(x_n) - \bar{y})$ and $g(h) = DF(\bar{x}, \bar{y})(h)$ in $\mathcal{F}(Y)$. The measures induced on the Polish space $\mathcal{F}(Y)$ by these elements are $\mu_n \circ g_n^{-1}$ and $\mu \circ g^{-1}$. According to Theorem 5.5 in [11] the central limit formula ($\mu_n \circ g_n^{-1} \to \mu \circ g^{-1}$) will be proved if $\lim_{n\to\infty} g_n(h_n) = g(h)$ holds for μ-almost all points h and h_n such that h_n approaches h. This is true by the definition of semi-differentiability. The existence of a Castaing representation with the desired asymptotic properties follows from Theorem 4.3 and Theorem 4.2. □

A relevant result is contained in the paper [24]. A multifunction acting between two infinite-dimensional spaces is considered there. The values $F(x)$ are supposed to be compact and $F(\bar{x}) = \bar{y}$ to be a singleton. The statement says that the measurable selections of F do not satisfy the central limit formula themselves, but there is a subsequence for which the formula holds.

Assuming second-order differentiability for the multifunction, we are able to obtain information on the asymptotic behaviour up to the second-order.

Theorem 4.5 *Let X be a linear normed space and Y be a Polish space. Suppose that $F : X \rightrightarrows Y$ is a measurable multifunction. Let us assume that the random elements $x_n : (\Omega, \mathcal{A}) \to X$ satisfy a generalized central limit formula with limit $\bar{x}$, i.e.,*

$$t_n^{-1}(x_n - \bar{x}) \xrightarrow{d} h$$

while $t_n \downarrow 0$ and h is a random element in some separable subspace of X.
Furthermore, let a point $\bar{y} \in F(\bar{x})$ and a random element $z \in DF(\bar{x}, \bar{y})(h)$ be given.

(i) Assume that F is twice semi-differentiable at $(\bar{x}, \bar{y})$ and $z(\cdot) \in DF(\bar{x}, \bar{y})(\cdot)$ with $z(h) = z$ is a continuous mapping with respect to the norms in X and Y. Then F satisfies the following second-order central limit formula:

$$t_n^{-2}(F(x_n) - \bar{y} - t_n z(h_n)) \xrightarrow{d} D^2F(\bar{x}, \bar{y})(h, z)$$

as random sets in $\mathcal{F}(Y)$, where $h_n = t_n^{-1}(x_n - \bar{x})$.

(ii) Suppose the assumptions of Theorem 4.3. If F is twice semi-differentiable at all points $(\bar{x}, y)$ with $y \in F(\bar{x})$, then F admits a Castaing representation $\{f_k\}$ such that all f_k satisfy the second-order central limit formula

$$t_n^{-2}[f_k(x_n) - f_k(\bar{x}) - t_n f_k'(\bar{x}; h_n)] \xrightarrow{d} f_k''(\bar{x}; h) \in D^2F(\bar{x}, f_k(\bar{x})(h, f_k'(\bar{x}; h))$$

as random variables on Y. Moreover, there is an index k such that $f_k(\bar{x}) = \bar{y}$.

Proof: The proof of the previous theorem carries over to this case, too, with the only difference that second-order differentiable selections are used to construct the Castaing representation.

Let μ_n and μ be the corresponding probability measures induced on X by $h_n = t_n^{-1}(x_n - \bar{x})$ and by h, respectively. We consider the random elements $g_n(h_n) = t_n^{-2}(\tilde{F}(x_n) - \bar{y} - t_n z(h_n))$ and $g(h) = DF^2(\bar{x}, \bar{y})(h, z(h))$ in $\mathcal{F}(Y)$. The measures induced on the Polish space $\mathcal{F}(Y)$ by these elements are $\mu_n \circ g_n^{-1}$ and $\mu \circ g^{-1}$. One has $\mu_n \circ g_n^{-1} \to \mu \circ g^{-1}$ if $\lim_{n\to\infty} g_n(h_n) = g(h)$ holds for μ-almost all points h and h_n such that $h_n \to h$. This is true by the given definition of second-order semi-differentiability. We can construct a Castaing representation of F as in Theorem 4.3 using selections that are first-order directionally differentiable at points (x_k, y_{kj}) and second-order differentiable at $(\bar{x}, y_j)$, where the points $(x_k, y_{kj}), k, j = 1, 2, \ldots$ are dense in the graph F and, in particular, $(\bar{x}, y_j)$ are dense in $F(\bar{x})$. The asymptotic properties of the selections follow by similar arguments as above. In order to see this, define $g_{nk}(h_n) = t_n^{-2}[f_k(x_n) - f_k(\bar{x}) - t_n f_k'(\bar{x}; h_n)]$ and $g_k(h) = f_k''(\bar{x}; h)$. We have $\lim_{n\to\infty} g_{nk}(h_n) = g_k(h)$ for all h by the second-order Hadamard-directional differentiability of f_k at $\bar{x}$. The statement holds by Theorem 5.5 in [11]. $\square$

Summarizing we can say that we do not have a result stating a central limit formula for all selections of the multifunction, but we are able to construct a Castaing representation with this property under relatively general assumptions. Having in mind the literature mentioned above, it is clear that statements of this kind should be possible for all measurable selections under very restrictive assumptions only.

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References

[1] Z. Artstein, Distributions of random sets and random selections, *Israel J. Math.* 46 (1984) 313-324

[2] Z. Artstein, On dense univalued representations of mulivalued maps, *Rendiconti der Circolo Mathematico di Palermo*, Serie II, Tomo XXXIII, (1983) 340-350

[3] J.-P. Aubin and A. Cellina, Differential Inclusions, Springer, Grundlehren der math. Wiss., 1984, Vol.264.

[4] J.-P. Aubin and H. Frankowska, Set-Valued Analysis, Birkhäuser, Boston 1990.

[5] A. Auslender and R. Cominetti, A comparative study of multifunction differentiability with applications in mathematical programming, *Mathematics of Operations Research*, 16 (1991), 240-258.

[6] G. Beer, Topologies on Closed and Closed Convex Sets, Kluwer, Mathematics and its Applications, Vol.268, 1993

[7] G. Beer, Wijsman convergence: a survey, *Set-Valued Analysis* 2 (1994) 77-94 .

[8] G. Beer, A Polish topology for the closed subsets of a Polish space, *Proc. Amer. Math. Soc.* 113(1991), 1123-1133.

[9] G. Beer, Wijsman convergence of convex sets under renorming, *Nonlinear Analysis, Theory, Methods& Applications*, 22 (1994) 2, 207-216.

[10] A. Ben-Tal and J. Zowe, A unified theory for first- and second-order conditions for extremum problems in topological vector spaces, *Mathematical Programming Study* 19 (1982) 39-76

[11] P. Billingsley, Convergence of probability measures, Wiley, New York, 1968.

[12] Ch. Castaing, Sur les multi-application measurables, *Inf. Rech. Op.* 1 (1967) 91-126

[13] Ch. Castaing and M. Valadier, Convex analysis and measurable multifunctions, Springer, Lecture Notes in Math., 1977, Vol.580.

[14] R. Cominetti, Metric regularity, tangent sets and second-order optimality conditions, *Applied Mathematics and Optimization* 21 (1990), 265-287.

[15] D. Dentcheva, Differentiable selections of set-valued mappings with application in stochastic programming, Preprint no. 96-28, Institute of Mathematics, Humboldt University of Berlin, 1996

[16] R. Deville, G. Godefroy and V. Zizler, Smoothness and renorming in Banach spaces, Longman Scientific & Technical, 1993.

[17] J. Diestel, Geometry of Banach spaces, Lecture Notes in Mathematics Vol.485, Springer-Verlag, Berlin, 1975.

[18] G. Dommisch, On the existence of Lipschitz-continuous and differentiable selections for multifunctions, *in* J.Guddat et al.(eds.): Parametric Optimization and related topics, Math. Res. Vol. 35, Akademie-Verlag Berlin, 1987, 60 - 73.

[19] A. Dontchev and T. Zollezzi, Well-posed optimization problems, Lecture Notes in Mathematics Vol.1543, Springer-Verlag, Berlin, 1993.

[20] S. Gautier and R. Morchadi, A selection of convex-compact-valued multifunctions with remarkable properties: the Steiner selection, *Numer. Funct. Anal. and Optimiz.* 13(5& 6) (1992) 513-522

[21] C. Hess, Quelques resultats sur la measurabilité des multifunctions à valeurs dans un espace métrique séparable, Séminaire d'Analyse Convexe de l'Université de Montpellier, Exposé 1, 1986.

[22] A.J. King, Generalized delta theorems for multivalued mappings and measurable selections, *Mathematics of Operations Research* 14 (1989) 4, 720-736

[23] K. Kuratowski and C. Ryll-Nardzewski, A general theorem on selectors, *Bull. Acad. Pol. Sc.* 13, 1965, 397-403.

[24] P. Lachout, On multifunction transforms of probability measures, *Annals of Operations Research*, 56 (1995) 241-250.

[25] E. Michael, Selected selection theorems, *Amer. Math. Montly* 63 (1956) 233-237.

[26] U. Mosco, Convergence of convex sets and solutions of variational inequalities, *Adv. in Math.* 3 (1969), 510-585.

[27] J.-P. Penot, Differentiability of relations and differential stability of perturbed optimization problems, *SIAM J. Control and Optimization* 22 (1984) 4, 529-551

[28] R. T. Rockafellar, Proto-differentiability of set-valued mappings and its application in optimization, *in:* H. Attouch et al. (eds.), Analyse Non Lineaire, Gauthier-Villars, Paris, 1989, 449-482.

[29] G. Salinetti and R.J.-B. Wets, On the convergence in distribution of measurable multifunctions (random sets), normal integrands, stochastic processes and stochastic infima, *Mathematics of Operations Research* 11 (1986) 3, 385-419.

[30] G.Salinetti and R.J.-B.Wets, Weak convergence of probability measures revisited, IIASA Working paper WP-87-30, Laxenburg, 1987.

[31] A. Seeger, Second-order directional derivatives in parametric optimization, *Mathematics of Operations Research* 13 (1988) 124-139.

[32] A. Shapiro, Existence and differentiability of metric projections in Hilbert spaces, *SIAM J. Optimization* 4 (1994) 1, 130-141.

[33] A. Shapiro, On differentiability of metric projections in $\mathbb{R}^n$: Boundary Case, *Proc.Amer. Math. Sic.* 99, 123-128.

[34] A. Shapiro, Directional differentiability of metric projections onto moving sets at boundary points, *Journal of Mathematical Analysis and Applications* 13 (1988) 392-403.

[35] A. Shapiro, Asymptotic analysis of stochastic programs, *Annals of Operations Research* 30 (1991), 169-186.

[36] R.A. Vitale, The Steiner point in infinite dimensions, *Israel Journal of Mathematics* 52 (1985) 245-250.