EXTRA-ordinary differential equations
Attempts to an analysis of differential-algebraic systems.

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1 Introduction

For about 15 years differential-algebraic equations have been an intensively discussed field of applied mathematics.
DAE's arise in models that couple dynamical parts with constraints and invariants, respectively. The most popular fields of applications are the simulation of electric circuits, chemical reactions and vehicle system dynamics, but also optimal control problems. Further, there is a close relationship to singular perturbation problems.

Formally, most of the DAE's are of the form

$$A(x,t)x' + g(x,t) = 0 \tag{1.1}$$

with an everywhere singular leading coefficient matrix $A(x,t)$ of constant rank.
In 1971 *C. W. Gear* proposed to integrate DAE's numerically like regular differential equations by means of the so-called backward differentiation formulas (BDF). Up to now, this has been practice with partly great success, e.g. in electrical industry, where large systems have to be treated. It was only 10 years later, after some inexplicable and unexpectable failures, that a more detailed mathematical investigation of DAE's set in. This was initialized by a lecture of C.W. Gear in Oberwolfach in 1981 ([2]) and studies by L.R. Petzold ([3]) with the provocative title

DAE's are not ODE's,

which, however, only ment that, numerically, DAE's do not behave like (regular) ordinary differential equations (ODE's). Of course, equations of the form (1.1) *are* ODE's, too, but not regular ones.

Let's have a look at two very simple (maybe too simple) examples.

Example 1: On $x_2 > 0$, consider the system

$$\left.\begin{array}{rl} x_1' &= \alpha x_1 \\ x_2' &= x_3/x_2 \\ x_1^2 &+ x_2^2 = 1 \end{array}\right\} \tag{1.2}$$

Obviously, the flow is restricted to remain on the (obvious) constraint manifold given by the derivative-free third equation. Differentiating this equation leads to a second, "hidden" constraint given by

$$\alpha x_1^2 + x_3 = 0.$$

Clearly, initial values should meet both constraints.

Example 2: Given the special linear system in constant coefficient Kronecker normal form

$$\left.\begin{array}{rcl} x_1' \; +Wx_1 & = & q_1(t) \\ x_3' \quad +x_2 & = & q_2(t) \\ x_3 & = & q_3(t) \end{array}\right\} . \tag{1.3}$$

Again we have two constraints, the obvious one $x_3 = q_3(t)$ and the hidden one $x_2 = q_2(t) - q_3'(t)$. For solvability, say for $x_1, x_3 \in C^1$, $x_2 \in C$, the right-hand side q has to be continuous as usually, but q_3 has to be C^1, additionally.
Consequently, a linear map L representing the related IVP's on a compact interval may be stated to be injective. However, in its natural setting ($q \in C$) L is not a Fredholm map ([4]). Since L^{-1} is unbounded, the IVP is ill-posed in the sense of Tikhonov. And this is just what makes the numerical treatment so difficult.

In both examples the DAE's have index 2. Roughly speaking, the index μ is the maximal number of steps of the nested constraint manifolds, and $\mu - 1$ is the number of the inherent differentiations.
The higher the index, the more complex is the problem.

From the point of view of application, DAE's with low index (1-3) and low smoothness of the describing functions are of special interest. To an increasing extent the models are generated automatically by complex algorithms, where the dimensions are often very large and the equations are not given explicitly. Looking at the long equations that frequently cover several pages if they are explicitly available seems to be of little profit. It is strenuous and challenging as well to reveal those pieces of information and model structures that are valuable for the mathematical characterization.
A reliable numerical treatment of the models generated is more and more combined with more exact information on the structure, basing on a thorough DAE-analysis in the given coordinates, which represent physical quantities like voltage etc. in most cases.
Models of practical interest contain not very smooth functions that are far from being analytic or meromorphic. Sometimes, difficulties occur already with the second derivative. And there is hardly a possibility to get the hidden constraints explicitly. From this point of view the two examples above are already too trivial and misleading.

2 Briefly on reduction methods and transformation into formally integrable systems

If one succombs to the encitement of nice simple examples, one will think of applying the formal theory of differential systems here ([5]).
By means of prolongations and projections into lower-dimensional jet spaces the original system is transposed into a formally integrable system. Then, for sufficiently regular systems

$$f(x', x, t) = 0, \tag{2.1}$$

a formal index is defined to be the number of prolongation and projection steps needed to transform (2.1) into a formally integrable system. Finally, the formal index is found out to be finite. The resulting formally integrable system is an overdetermined system constituting a regular ODE as well as constraints.

The approach described above is closely related to the notion of the differentiation index (e.g. [6]). This notion is based on the so-called derivative array or compound function obtained from (2.1) by k-times formal differentiations

$$\left.\begin{array}{l} f(x^1, x, t) = 0 \\ f'_{x'}(x^1, x, t)x^2 + f'_x(x^1, x, t)x^1 + f'_t(x^1, x, t) = 0 \\ \cdots \\ f'_{x'}(x^1, x, t)x^{k+1} + \cdots = 0 \end{array}\right\} \tag{2.2}$$

System (2.2) is treated as a nonlinear equation in separate variables $x^i \in I\!R^m,\ i = 1, \ldots, k+1$.

The differentiation index μ is the smallest number such that (2.2) with $k = \mu$ can be solved for

$$x^1 = \mathcal{S}(x, t).$$

The resulting regular ODE

$$x' = \mathcal{S}(x, t) \tag{2.3}$$

is called the ODE underlying (2.1). Unfortunately, the ODE (2.3) is not equivalent to (2.1). The system (2.2) contains more information, which was not taken into account here, but should not get lost, namely the equations for the constraints. E. Griepentrog ([7]) has thoroughly worked out how to consider the constraints. Finally, the vectorfield of (2.3) is restricted to a constraint manifold given by a certain equation $r(x, t) = 0$. Hence, one does not have the underlying ODE in the $I\!R^m$, but instead

$$x' = \mathcal{S}(x, t)\,, \quad r(x, t) = 0. \tag{2.4}$$

(2.1) and (2.2) are equivalent under correspondingly strong regularity conditions .

Reduction methods (e.g. [6],[7]) realize the process of transition to (2.4) successively, like the transformation steps to a formally integrable system, where, usually, not all equations are differentiated, but only the derivative-free ones that have been filtered out by projection.
In this connection let us regard the simple linear DAE with constant coefficients

$$Ax' + Bx = q. \tag{2.5}$$

Prolongation provides

$$\left.\begin{array}{rl} Ax^1 + Bx &= q \\ Ax^2 + Bx^1 &= q' \end{array}\right\},$$

and after the projection step this yields the system

$$\left.\begin{array}{rl} Ax^1 + Bx &= q \\ (I - AA^+)Bx^1 &= (I - AA^+)q' \end{array}\right\}.$$

On the other hand, in a reduction step, the derivative free part of (2.5) is filtered out first, i.e.,

$$(I - AA^+)Bx = (I - AA^+)q$$

and the result is added to (2.5) in differentiated form then. In both cases the DAE (2.5) is transformed into the system

$$\left.\begin{array}{rcl} Ax' + AA^+Bx &=& AA^+q \\ (I - AA^+)Bx' &=& (I - AA^+)q' \\ (I - AA^+)Bx &=& (I - AA^+)q \end{array}\right\}.$$

Supposed the matrix pencil $\{A, B\}$ is regular and it has index 1, the matrix $A + (I - AA^+)B$ becomes nonsingular, hence we arrive at a special form of (2.4), namely

$$\left.\begin{array}{rl} x' = & (A + (I - AA^+)B)^{-1}(-AA^+Bx + AA^+q(t) + (I - AA^+)q'(t)) \\ & (I - AA^+)(Bx - q(t)) = 0 \end{array}\right\}.$$

On the basis of reduction steps P.J. Rabier and W.C. Rheinboldt (e.g. [8]) have studied DAE's and found, among other things, nice solvability statements.

All these approaches have in common that they require a high regularity of f and that recursively, hence,so to speak hidden conditions of rank constancy have to be agreed upon again and again. Moreover, the vectorfield in (2.4) is relevant for the DAE on the corresponding constraint manifold only. Outside of this invariant manifold this vectorfield is absolutely irrelevant, which becomes obvious e.g. in an asymptotic stability behaviour, which has nothing in common with that of (2.1).

Even if one can realize this reduction procedure (which is very doubtful for serious applications), one will be confronted with the overdetermined system (2.4) in the given coordinates. Transition to the coordinates of the constraint manifold might

be successful in practice in trivial cases only.
For quite a long time numerical analysts have been confronted with the problem of integrating regular ODE's numerically in such a way that invariants are taken into account and kept. C.W. Gear ([9]) has shown that this problem actually requires the numerical solution of a DAE with index 2.

Altogether, the reduction to (2.4) involves inadequately high demands to regularity. Example (1.3) makes very clear that C^1 does not represent the appropriate class for the solution and that a larger class has to be considered. For semi-explicit systems

$$\left.\begin{array}{rcl} x_1' + g_1(x_1, x_2, t) &=& 0 \\ g_2(x_1, x_2, t) &=& 0 \end{array}\right\}, \tag{2.6}$$

with index 1, i.e., with an everywhere nonsingular partial Jacobian $g_{22}'(x_1, x_2, t)$, the above fact becomes obvious. In this case (2.4) is of the form

$$\left.\begin{array}{l} x_1' + g_1(x_1, x_2, t) = 0 \\ x_2' + g_{22}'(x_1, x_2, t)^{-1}\{-g_{21}'(x_1, x_2, t)g_1(x_1, x_2, t) + g_t'(x_1, x_2, t)\} = 0 \\ g_2(x_1, x_2, t) = 0 \end{array}\right\}. \tag{2.7}$$

A consistent initial value (x_0, t_0) has to fulfil the condition $g_2(x_{0,1}, x_{0,2}, t_0) = 0$. On the other hand, in a neighbourhood of (x_0, t_0) the second equation of (2.6) can be solved directly by means of the theorem on implicit functions for

$$x_2 = h(x_1, t).$$

Now it becomes more obvious what the DAE is really made of, namely

$$\left.\begin{array}{rcl} x_1' + g_1(x_1, h(x_1, t), t) &=& 0 \\ x_2 &=& h(x_1, t) \end{array}\right\}. \tag{2.8}$$

No daubt, (2.8) is simpler than (2.7). We want to go on from that analytical local decoupling in the following. It seems to be possible to develop such an analysis for DAEs with low index.
Several investigations (e.g. [10], [11]) have been available for the simpler case of DAE's with index 1 since the middle of the eighties. Further, let us mention F. Takens ([12]), who investigated small gradient systems (index 1)

$$x_1' = f(x_1, x_2), \quad g(x_1, x_2) = 0$$

in connection with the approximating singularly perturbed system

$$x_1' = f(x_1, x_2), \quad \varepsilon x_2' = g(x_1, x_2)$$

already in 1976. In particular, singularities of the vector field were characterized in case of $g : I\!R \times I\!R \to I\!R$.

A DAE analysis we are aiming at should characterize the DAE (2.1) in terms of the original model, that is, in actual information on f. What we need, e.g. for

creating appropriate numerical methods, are criteria that *guarantee* the regularity of the inherent ODE, its stability etc., and which may be checked in practice.
From this point of view, assertions like " if (2.1) represents a regular ODE on a smooth manifold, then ..." are nice, but *l'art pour l'art*. Now we show that a direct analysis basing on local decoupling like (2.8) provides nice and nontrivial extensions of the standard ODE theory for lower index DAEs, which are important for applications, too. Hence, *l'art pour la practique.*

3 Linear continuous coefficient DAE's

In this section we try to decouple the given DAE

$$A(t)x'(t) + B(t)x(t) = q(t), \quad t \in J, \tag{3.1}$$

into its characteristic parts by means of certain projections. Not surprisingly, certain subspaces play an important role in this approach.
Since we have in mind applications to nonlinear DAEs (2.1), we set great store by having contiunous coefficients A, B, but not smoother ones in general.Then we may expect to handle linearizations of (2.1) along solutions, i.e., equations (3.1) with $A(t) = f'_{x'}(x'(t), x(t), t)$, $B(t) := f'_x(x'(t), x(t), t)$.
More precisely, for a given interval $J \subseteq I\!R$, the coefficients A, B are assumed to belong to $C(J, L(I\!R^m))$, but the leading nullspace

$$N(t) := \ker A(t), \quad t \in J, \tag{3.2}$$

is supposed to depend smoothly on t, i.e., to be spanned by a C^1-basis, or, equivalently, there is a projector function $Q \in C^1(J, L(I\!R^m))$ such that $Q(t)^2 = = Q(t)$, $imQ(t) = N(t)$, $t \in J$.

In addition to Q we introduce $P := I - Q$. Taking into account the trivial relations

$$A(t)Q(t) \equiv 0, \quad A(t)P(t) \equiv A(t)$$

and further

$$A(t)x'(t) = A(t)\{(Px)'(t) - P'(t)x(t)\} \tag{3.3}$$

we know that (3.1) involves the derivative of $(Px)(t) = P(t)x(t)$, but that of the nullspace component $(Qx)(t)$ is not involved at all.
Naturally, we should look for solutions of (3.1) from the function space C^1_N,

$$C^1_N(J, I\!R^m) := \{w \in C(J, I\!R^m) : Pw \in C^1(J, I\!R^m)\}.$$

Note that C^1_N does not depend on the choice of the projector function since $P = P\bar{P}$, $\bar{P} = \bar{P}P$ are true for each two projectors P and $\bar{P}$ along N.
Moreover, due to

$$A\{(Px)' - P'x\} = A\{(\bar{P}x)' - \bar{P}'x\}$$

we may agree to use the expression Ax' as an abbreviation of $A\{(Px)' - P'x\}$ with any C^1 projector P.

Next, considering the homogeneous system

$$A(t)x'(t) + B(t)x(t) = 0 \tag{3.4}$$

we find the subspace

$$S(t) := \{z \in \mathbb{R}^m : B(t)z \in imA(t)\}$$

to be relevant because it contains all solutions. However, $S(t)$ is filled by solutions of (3.4) only in case of index-1 DAEs.

Definition:
The DAE (3.1) is index-1 tractable on J if and only if

$$N(t) \cap S(t) = \{0\}, \quad t \in J. \tag{3.5}$$

The index-1 condition (3.5) is well-known to be equivalent with the full-rank of the matrix (cf. Appendix A).

$$G_1(t) := A(t) + B(t)Q(t), \quad t \in J. \tag{3.6}$$

Hence, with $G_1^{-1}A = G_1^{-1}G_1P = P$, $G_1^{-1}B = G_1^{-1}BQ + G_1^{-1}BP = Q + G_1^{-1}BP$, equation (3.1) scaled by G_1^{-1} reads

$$Px' + Qx + G_1^{-1}BPx = G_1^{-1}q.$$

Multiplying by P and Q, respectively, we arrive at the decoupled version

$$\left.\begin{array}{rcl} Px' + PG_1^{-1}BPx &=& PG_1^{-1}q \\ Qx + QG_1^{-1}BPx &=& QG_1^{-1}q \end{array}\right\},$$

and, more precisely, at

$$\left.\begin{array}{c} (Px)' - P'(I - QG_1^{-1}B)Px + PG_1^{-1}BPx = PG_1^{-1}q + P'QG_1^{-1}q \\ Qx = -QG_1^{-1}BPx + QG_1^{-1}q \end{array}\right\}. \tag{3.7}$$

It should be mentioned that

$$Q_{\text{can}} := QG_1^{-1}B, \quad P_{\text{can}} := I - Q_{\text{can}}$$

represent again projectors. Q_{can} projects onto N along S and is said to be the canonical projector for the index 1 case. Note that Q_{can} is continuous.
System (3.7) shows how to state an initial condition, namely

$$P(t_0)x(t_0) = P(t_0)x^0, \quad x^0 \in \mathbb{R}^m \tag{3.8}$$

i.e., the initial condition should fix the free integration constants of the inherent in (3.7) regular ODE for the component $u := Px$,

$$u' - P'P_{\text{can}}u + PG_1^{-1}Bu = PG_1^{-1}q + P'QG_1^{-1}q. \tag{3.9}$$

The subspace $imP(t)$ is easily checked to be invariant for the regular ODE (3.9), that is, $u(t_0) \in imP(t_0)$ implies $Q(t)u(t) \equiv 0$.

Now the solutions of the IVP (3.1), (3.8) are represented by

$$\begin{aligned} x &= Px + Qx = u + Qx \\ &= P_{\text{can}} u + QG_1^{-1} q, \end{aligned}$$

where $u \in C^1$ solves the inherent regular ODE (3.9), but also the initial condition $u(t_0) = P(t_0)x^0$. Obviously, the consistent initial value is

$$x_0 := x(t_0) = P_{\text{can}}(t_0)x^0 + Q(t_0)G_1(t_0)q(t_0).$$

We have $P(t_0)x_0 = P(t_0)x^0$, but not $x_0 = x^0$, in general.
For solutions of the homogeneous system (3.4) we find the expression

$$x = P_{\text{can}} u = P_{\text{can}} \mathcal{U} P(t_0) x^0 =: Xx^0,$$

with the fundamental solution matrix $\mathcal{U}$ of (3.9), $\mathcal{U}(t_0) = I$. For $x^0 \in S(t_0)$, it holds that

$$x(t_0) = P_{\text{can}}(t_0)P(t_0)x^0 = P_{\text{can}}(t_0)x^0 = x^0,$$

that is, $S(t_0)$ is the set of consistent initial values for the homogeneous system. Let us summarize what we know in the following theorem.

Theorem 3.1 *Given an index-1 DAE* (3.1).

(i) For each $t_0 \in J$, $x^0 \in \mathbb{R}^m$, $q \in C(J, \mathbb{R}^m)$, the IVP (3.1)(3.8) *is uniquely solvable on the given interval J.*

(ii) Exactly one solution of the homogeneous equation (3.4) *passes through each $x_0 \in S(t_0)$, at t_0.*

The matrix function used above

$$X(t) := P_{\text{can}}(t)\mathcal{U}(t)P(t_0) \tag{3.10}$$

is said to be the fundamental solution matrix of the DAE. It is uniquely determined by the IVP

$$AX' + BX = 0, \quad P(t_0)(X(t_0) - I) = 0.$$

The problem with that fundamental solution matrix lies in its singularity. It is easily verified that $\ker X(t) \equiv N(t_0)$ holds. Hence, instead of an inverse we are confronted with generalized inverses, say X^- defined by the relations $X^- X X^- = X^-, XX^-X = X, XX^- = P_{\text{can}}, X^-X = P_{\text{can}}(t_0)$. In particular, Green functions are developed in this way (e.g. [13]).

If the index-1 condition (3.5) does not hold, the situation is much more complicated. Iterrupting (3.5) at isolated points may cause bifurcations etc. Up to now, there has been no comprehensive systematic analysis of such singularities.
On the other hand, if (3.5) fails on the whole interval J, we may be confronted with a higher index DAE. The best understood higher index DAEs are those having index 2.
To give a precise definition we introduce additional matrix functions and subspaces, namely

$$A_1 := G_1(I - PP'Q),$$

which has the same rank as G_1,

$$N_1(t) := \ker A_1(t),$$

$$S_1(t) := \{z \in I\!R^m : B(t)P(t)z \in imA_1(t)\}, \quad t \in J.$$

The subspace $N_1(t)$ has the same dimension as the intersection $N(t) \cap S(t)$.

Definition:
The DAE (3.1) is index-2 tractable on J if and only if the intersection $N(t) \cap S(t)$ has constant dimension on J and further

$$N_1(t) \cap S_1(t) = \{0\}, \quad t \in J. \tag{3.11}$$

Relation (3.11) allows to use the further projector $Q_1(t)$ onto $N_1(t)$ along $S_1(t)$, and $P_1(t) := I - Q_1(t)$.
Due to the basic linear algebra properties (Appendix A), the matrix function

$$G_2 := A_1 + BPQ_1 = A + (B - AP')Q + BPQ_1$$

remains nonsingular. Moreover, the relations

$$Q_1 = Q_1 G_2^{-1} BP, \quad Q_1 Q = 0$$

become true. In the consequence, the decomposition $x = PP_1x + PQ_1x + Qx$ makes sense.
Taking into account the further identities

$$G_2^{-1}A = P_1 P, \quad G_2^{-1}B = G_2^{-1}BPP_1 + Q_1 + Q + P_1PP'Q$$

we decouple the DAE (3.1) into its essential parts in a similar way as we did in the index-1 case. Then, the following system results (cf. [14]):

$$\left.\begin{array}{rcl} PP_1x' + PP_1P'Qx + PP_1G_2^{-1}BPP_1x & = & PP_1G_2^{-1}q \\ -QQ_1PQ_1x' + QQ_1Q'x + Qx + QP_1G_2^{-1}BPP_1x & = & QP_1G_2^{-1}q \\ PQ_1x & = & PQ_1G_2^{-1}q \end{array}\right\}. \tag{3.12}$$

A priori, $Q_1(t)$ depends continuously on t because $A_1(t)$ has constant rank.
Suppose additionally that Q_1 belongs to C_N^1 so that PQ_1 and $PP_1 = P - PQ_1$ are

from C^1.
Then, with the denotations $u := PP_1x$, $v := PQ_1x$, $w := Qx$, system (3.12) transforms into

$$\left.\begin{array}{rcl} u' - (PP_1)'(u+v) + PP_1G_2^{-1}Bu &=& PP_1G_2^{-1}q \\ -QQ_1v' + QQ_1(PQ_1)'(u+v) + w + QP_1G_2^{-1}Bu &=& QP_1G_2^{-1}q \\ v &=& PQ_1G_2^{-1}q \end{array}\right\}. \tag{3.13}$$

Now it is evident that initial conditions should be directed to the component $u = PP_1x$, say in the form

$$(PP_1)(t_0)x(t_0) = (PP_1)(t_0)x^0, \quad x^0 \in I\!R^m. \tag{3.14}$$

By similar arguments as for the index 1 case the next assertion may be proved.

Theorem 3.2 *Given an index-2 DAE* (3.1) *with* $Q_1 \in C^1_N$.

(i) For each $t_0 \in J$, $x^0 \in I\!R^m$, $q \in C(J, I\!R^m)$, $PQ_1G_2^{-1}q \in C^1(J, I\!R^m)$ *the IVP* (3.1), (3.14) *is uniquely solvable on the given interval.*

(ii) At t_0 *exactly one solution of the homogeneous DAE passes through each* $x_0 \in S^{[1]}(t_0)$, *where*

$$\begin{array}{rcl} S^{[1]}(t) &:=& im\Pi_{\mathrm{can}}(t), \\ \Pi_{\mathrm{can}} &:=& (I - QQ_1(PQ_1)' - QP_1G_2^{-1}B)PP_1. \end{array}$$

It should be mentioned that $S^{[1]}(t) \subset S(t)$ is a proper subspace. Π_{can} is a projector function, $\Pi_{\mathrm{can}}(t)$ projects along $N(t) \oplus N_1(t)$. Both, the subspace $S^{[1]}(t)$ as well as the projector function Π_{can} may be shown to be independent of the choice of the projectors P, Q we started with.
The fundamental solution matrix X given by

$$AX' + BX = 0, \ (PP_1)(t_0)(X(t_0) - I) = 0$$

has the representation

$$X = \Pi_{\mathrm{can}}\mathcal{U}P(t_0)P_1(t_0), \ \ker X(t) = N(t_0) \oplus N_1(t_0),$$

where $\mathcal{U}$ denotes the fundamental matrix of the inherent regular ODE

$$u' - (PP_1)'u + PP_1G_2^{-1}Bu = PP_1G_2^{-1}q + (PP_1)'PQ_1G_2^{-1}q.$$

Apart from the greater technical expense, Theorem 3.2 sounds as simple as Theorem 3.1, and index 2 DAEs behave quite similar as index 1 DAEs from this point of view. On the other hand, there is an essential difference relative to the linear map $\mathfrak{A}$ given by

$$\mathfrak{A}x := Ax' + Bx, \quad x \in C^1_N(J, I\!R^m). \tag{3.15}$$

Theorem 3.3

(i) If the index-1 condition holds true, then $\mathfrak{A}$ is surjective.

(ii) If the index-2 condition holds true and $Q_1 \in C^1_N$, then

$$im\ \mathfrak{A} = \{q \in C(J, I\!R^m) : PQ_1G_2^{-1}q \in C^1(J, I\!R^m)\}$$

is a proper subset of $C(J, I\!R^m)$.

Relating on a compact interval J, $\mathfrak{A}$ is a Fredholm map in the index-1 case, but $\mathfrak{A}$ is no more Fredholm for index-2 problems. Due to the inherent differentiation of the component $PQ_1G_2^{-1}q$, the index-2 IVP (3.1), (3.14) is essentially ill-posed in the sense of Tikhonov while the index-1 IVP remains well-posed.
Fortunately, for index-2 problems the ill-posedness is somewhat harmless, and we are able to manage the numerical problems well in many cases.

In view of the asymptotical stability and further questions, transformations of the unknown functions $x(t) = F(t)\bar{x}(t)$ and scalings of the DAE (3.1) by $E(t)$ are of certain interest. The coefficients of the resulting DAE for $\bar{x}(\cdot)$ are

$$\bar{A} := EAF, \quad \bar{B} := EBF + EAF'. \tag{3.16}$$

Theorem 3.4 *Let $F \in C^1_N(J, L(I\!R^m))$, $E \in C(J, L(I\!R^m))$ be nonsingular.*

(i) Then $\bar{A}$ has a smooth nullspace $\bar{N}$.

(ii) It holds that $\bar{N}(t) \cap \bar{S}(t) = F(t)^{-1}(N(t) \cap S(t))$, i.e., the index-1 property is invariant.

(iii) $Q_1 \in C^1_N$ implies $\bar{Q}_1 \in C^1_{\bar{N}}$.

(iv) The index-2 property is invariant.

It should be mentioned that index-μ-tractability generalizes the notion of global index μ introduced in [22] in terms of a possible reduction of the DAE to Kronecker normal form by making a linear smooth transformation of the variable and scaling the DAE.
A linear DAE (3.1) is said to be in Kronecker normal form if it has the special coefficients

$$A(t) = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}, \quad B(t) = \begin{pmatrix} W(t) & 0 \\ 0 & I \end{pmatrix},$$

and J represents a constant nilpotent block, say with index μ, i.e., $J^\mu = 0$, $J^{\mu-1} \neq 0$. The nilpotency index μ is said to be the global index of the given DAE.
If a linear DAE has global index μ, then it is index-μ-tractable . On the other hand, using transformations of lower smoothness $F \in C^1_N$, $E \in C$, each index-1-tractable DAE can be reduced to its Kronecker normal form ([15]). The corresponding assertion concerning index-2-tractable DAEs is under preparation.

Interesting particular results may be proved for DAEs with periodic coefficients (e.g. [15]).

Definition:
Two DAEs with T-periodic coefficients are called (periodically) equivalent if there are a T-periodic transform $F \in C^1_N$ and a periodic scaling $E \in C$ that connect the coefficients by (3.16).
As in the regular ODE case we call $X(T)$ the monodromy matrix. It plays the expected role, in fact.

By considerable effort it is possible to generalize the well-known results of Lyapunov for DAEs having index 1 or index 2 with $Q_1 \in C^1_N$.

Theorem 3.5 *(" Lyapunov for DAEs "):*

(i) If linear T-periodic DAEs are equivalent, then their monodromy matrices are similar.

(ii) If the monodromy matrices of linear T-periodic DAEs are similar, then these DAEs are equivalent .

(iii) A linear T-periodic DAE is equivalent to a $2T$-periodic real (T-periodic complex) DAE in constant coefficient Kronecker normal form.

Surprisingly, apart from the technical amount in the proof, this assertion sounds as simple and transparent as the original reduction theorem of Lyapunov, which was proved more than 100 years ago.
Note that also the representation theorem of Floquet holds true for DAEs.

4 Nonlinear DAEs

Considering nonlinear equations

$$f(x'(t), x(t), t) = 0 \tag{4.1}$$

we may try to form an analogous chain of subspaces, projectors and matrices by using the partial Jacobians $f'_y(y, x, t), f'_x(y, x, t)$ pointwise instead of $A(t)$, $B(t)$ in Section 3 (cf. e.g. [14], [16]). Roughly speaking, we are aiming at the following situation:

" *The DAE* (4.1) *has index μ if the linearized DAE has it, and vice versa.*"

However, except for the index 1 case, which is well-understood , there remain a lot of open questions on how to take into account the different kinds of rotating subspaces appropriately.

From the analytical point of view, the behaviour of the leading nullspace of (4.1), that is, $\ker f'_y(y,x,t)$, is of great importance - even in the index 1 case. In [17], Ch. Lubich pointed out that, in case the leading nullspace varies with (y,x), the solutions involve certain derivatives, similarly as the solutions of the linear index-2 DAEs (cf. Section 3). On the other hand, an index 1 DAE (4.1) whose leading nullspace is invariant of y and x does not show this unpleasant feature.

To measure the sensibility of solutions with respect to inherent differentiations the so-called perturbation index (e.g. [18]) has been introduced. It provides essential information on the difficulty of the problem from the numerical point of view.
One might conjecture that just rotating subspaces cause the perturbation index to be higher than the (formal) differentiation index.

For more transparency, only quasilinear DAEs of the form

$$A(t)x'(t) + g(x(t),t) = 0 \tag{4.2}$$

are considered in the following. $A(t)$, $g(t)$, $g'_x(x,t)$ are supposed to be continuous on $J_0 \subset I\!R$ and $\mathcal{D}_0 \times J_0 \subseteq I\!R^m \times I\!R$, respectively. The nullspace $N(t) := \ker A(t)$ is assumed to be smooth (cf. Section 3). $Q(t)$ denotes again a C^1 projector function onto $N(t)$, $P(t) := I - Q(t)$.

In Appendix B it is shown how the results obtained for (4.2) immediately apply to the general DAE (4.1) provided that the leading nullspace is invariant of y and x.
In order to treat the problems (4.1), whose leading nullspace $\ker f'_y(y,x,t)$ varies with (y,x), we propose to change to the trivially enlarged system having a constant leading nullspace (Appendix C), but emphasize once more that the differentiation index increases by that transformation.

Clearly, all solutions of (4.2) should belong to the class C^1_N introduced in Section 3.

Definition:
The DAE (4.2) is index-1 tractable on $J \times \mathcal{D} \subseteq J_0 \times \mathcal{D}_0$ if and only if

$$N(t) \cap S(x,t) = \{0\}, \quad t \in J,\, x \in \mathcal{D} \tag{4.3}$$

holds, where

$$S(x,t) := \{z \in I\!R^m : g'_x(x,t)z \in imA(t)\}.$$

Further, introduce the set

$$\mathcal{M}_{(1)}(t) := \{w \in \mathcal{D}_0 : g(w,t) \in imA(t)\},\ t \in J_0,$$

containing all solutions of (4.1).
Now, the subspace $S(x,t)$ manifests its geometrical meaning

$$S(x,t) = T_x\mathcal{M}_{(1)}(t) \quad \text{for} \quad x \in \mathcal{M}_{(1)}(t).$$

Restricted to $t \in J, x \in \mathcal{M}_{(1)}(t)$, the index-1 condition (4.3) says that the leading nullspace $N(t)$ and the tangent space $T_x\mathcal{M}_{(1)}(t)$ have to intersect transversally. However, condition (4.3) applies also to elements outside of $\mathcal{M}_{(1)}(t)$.

Due to the basic linear algebra properties (cf. Appendix A), the index-1 condition is satisfied if and only if the matrix function

$$G_1(x,t) := A(t) + g'_x(x,t)Q(t) \tag{4.4}$$

remains nonsingular. Hence, we have got a nice criterion for checking index 1 and for detecting singularities, respectively. Using the decoupling technique described above and applying the implicit function theorem we are able to prove the next local solvability assertion, which is a straightforward generalization of the classical regular ODE case.

Theorem 4.1 *Given an index-1 DAE* (4.1), $t_0 \in J$, $x_0 \in \mathcal{M}_{(1)}(t_0)$, $(x_0,t_0) \in \mathcal{D} \times J$.

(i) Then, there passes exactly one solution through (x_0,t_0).

(ii) For sufficiently small $\tau > 0$ *the IVPs*

$$\begin{aligned} &A(t)x'(t) + g(x(t),t),t) = 0, \quad P(t_0)(x(t_0) - x^0) = 0,\\ &|P(t_0)(x_0 - x^0)| \le \tau, \quad x^0 \in \mathbb{R}^m, \end{aligned} \tag{4.5}$$

are locally uniquely solvable.

(iii) The solution $x(\cdot, t_0, x^0)$ *of* (4.5) *depends continuously differentiably on* x^0.

It seems that the IVP solutions provided by Theorem 4.1 can be continued as long as they do not leave the index-1 domain. Recall that impasse points and bifurcations may occur at points where the index-1 condition is lost.

If the index-1 condition (4.3) fails uniformly, then we may expect a higher index problem. This happens in example (1.2) above, which can be characterized to have index 2.
Note that for (1.2) it becomes characteristic that $S_1(x) \cap N_1(x) = \{0\}$ holds, where $S_1(x) := \{z \in \mathbb{R}^3 : x_1z_1 + x_2z_2 = 0\} = S(x)$, $N_1(x) := \{z \in \mathbb{R}^3 : z_1 = 0, z_2 + z_3/x_2 = 0\}$, but $N := \{z \in \mathbb{R}^3 : z_1 = 0, z_2 = 0\} \subset S(x)$, $\dim(N \cap S(x)) = 1$.

Next we turn to the linearizations along a given solution, say $x_*(\cdot) : J_* \to \mathbb{R}^m$ of (4.2). Denote $B(t) := g'_x(x_*(t),t)$ such that $A(t)$, $B(t)$ are the coefficients of the linearized along x_* DAE. Further, we will now use the projectors, subspaces etc. related to the linear DAE with the coefficients $A(t)$, $B(t)$, which we introduced in Section 3. In particular, let

$$\Pi_{(\mu)}(t) := \begin{cases} P(t) & \text{in case of } \mu = 1 \\ P(t)P_1(t) & \text{in case of } \mu = 2 \end{cases}$$

Moreover, we use the canonical projector function

$$\Pi_{\mathrm{can}(\mu)}(t) := \begin{cases} P_{\mathrm{can}} & \text{for } \mu = 1 \\ \Pi_{\mathrm{can}} & \text{for } \mu = 2 \end{cases}.$$

As we have pointed out above, the relation

$$im\Pi_{\mathrm{can}(1)}(t) = T_{x_*(t)}\mathcal{M}_{(1)}(t)$$

becomes true in the index-1 case. Such a relation is expected to be true also for $\mu = 2$, but then we would have to describe the corresponding complicated subset $\mathcal{M}_{(2)}(t) \subset \mathcal{M}_{(1)}(t)$ in some detail.
As we have seen for the linear DAEs, the canonical projector functions $\Pi_{\mathrm{can}(\mu)}$ are rather complex, whereas the projector functions $\Pi_{(\mu)}$ are fairly simple. Furthermore, the $\Pi_{(\mu)}$ belong to the class C^1, whereas the $\Pi_{\mathrm{can}(\mu)}$ belong to C^1_N exclusively.
In this sense, $\Pi_{(\mu)}$ can be regarded as a practicable substitute of $\Pi_{\mathrm{can}(\mu)}$. Essential preconditions for this are the following inherent properties

$$\begin{aligned} ker\Pi_{\mathrm{can}(\mu)}(t) &= ker\Pi_{(\mu)}(t), \\ P(t)\Pi_{\mathrm{can}(\mu)}(t) &= P(t)\Pi_{(\mu)}(t). \end{aligned}$$

Now, also $im\Pi_{(\mu)}(t)$ can be considered as a substitute of the tangent space $T_{x_*(t)}\mathcal{M}_{(\mu)}(t)$.

Taking into consideration the solvability statements for linear DAEs and the description of admissible right-hand sides by the Theorems 3.1 - 3.3, we introduce now the function spaces $C_{(\mu)}$, $\mu = 1, 2$, equipped with the norms $\|\cdot\|_{(\mu)}$ by

$$\begin{aligned} C_{(1)} &:= C, \quad \|\cdot\|_{(1)} := \|\cdot\|_\infty \\ C_{(2)} &:= \{q \in C : PQ_1G_2^{-1}q \in C^1\}, \|q\|_{(2)} := \|q\|_\infty + \|(PQ_1G_2^{-1}q)'\|_\infty \end{aligned}$$

Theorem 4.2 *Let $x_* \in C^1_N(J_*, \mathbb{R}^m)$ solve the DAE (4.2) on the compact interval $J_* \ni t_0$. Let the linearized at x_* DAE have index $\mu \in \{1, 2\}$.*

If $\mu = 2$, certain additional structural and smoothness conditions are supposed.

(i) Then, for sufficiently small $\tau > 0$, $\sigma > 0$ the IVPs

$$A(t)x'(t) + g(x(t), t) = q(t), \tag{4.6}$$

$$\Pi_{(\mu)}(t_0)(x(t_0) - x^0) = 0, \tag{4.7}$$

$$|\Pi_{(\mu)}(t_0)(x_*(t_0) - x^0)| \le \tau, \quad \|q\|_{(\mu)} \le \sigma$$

are locally uniquely solvable on J_*.

(ii) The solution $x(\cdot, t_0, x^0)$ of the IVP (4.2), (4.7) depends continuously differentiably on x^0, and $\frac{\partial x}{\partial x^0}$ satisfies the variational DAE.

(iii) The solution of (4.6), (4.7) depends continuously on $(x^0, q) \in I\!R^m \times C_{(\mu)}$.

This assertion is proved by applying the implicit function theorem after carefully preparing the properties of the maps involved (cf. [4],[19]).

The additional smoothness and structural conditions mentioned in Theorem 4.2 allow, on the one hand, the necessary inherent differentiation in the index 2 case. On the other hand, they guarantee that the index-2 property of the linearization can be extended to a neighbourhood by certain restrictions of the structure. For linear DAEs and DAEs in Hessenberg form of size 2 (e.g. [6]) these structural conditions are always fulfilled. For more general classes of DAEs, however, their description is technically rather expensive.

It should be mentioned that, up to now, the *maximal index 2 structure* has not been clarified. C. Tischendorf ([20]) has obtained a structural criterion that is very useful in applications and generalizes the results in [19].
Note that Theorem 4.2 proves its value with respect to the numerical treatment of DAEs. There are analogous "discretized" versions (e.g. [20]) which show how to manage the computations well.
It should be mentioned once more that Theorem 4.2 (iii) means in fact that the DAE (4.2) has the perturbation index μ if it is index-μ tractable.
On the other hand, we are now stimulated to consider also solutions on infinite intervals, say $J_* = [t_0, \infty)$.

Definition:
Given a solution $x_* \in C^1_N([t_0, \infty), I\!R^m)$ of (4.2). Let (4.2) have index $\mu \in \{1, 2\}$.
x_* is said to be stable in Lyapunov's sense if there is a $\tau > 0$, and, moreover, there exists a $\delta(\varepsilon) > 0$ to each $\varepsilon > 0$ such that

(i) the IVPs

$$A(t)x'(t) + g(x(t), t) = 0, \quad \Pi_{(\mu)}(t_0)(x(t_0) - x^0) = 0, \quad |\Pi_{(\mu)}(t_0)(x_*(t_0) - x^0)| \leq \tau$$

are solvable on $[t_0, \infty)$.

(ii) $|\Pi_{(\mu)}(t_0)(x_*(t_0) - x^0)| \leq \delta(\varepsilon)$ implies $|x(t) - x_*(t)| \leq \varepsilon \quad$ for $t \geq t_0$.

x_* is called asymptotically stable if, additionally, $|\Pi_{(\mu)}(t_0)(x_*(t_0) - x^0)| \leq \tau_0$ implies $x(t) - x_*(t) \to 0 (t \to \infty)$ for sufficiently small $\tau_0 \leq \tau$.
Our definition reflects the geometrical meaning of Lyapunov stability properly. By means of that special statement of initial conditions, without evaluating the implicit state manifold, the neighbouring solutions on that manifold are catched properly to be compared with x_*.

Autonomous DAEs

$$Ax' + g(x) = 0 \tag{4.8}$$

seem to be essentially simpler than nonautonomous ones. In particular, in linear homogeneous constant coefficient DAEs all critical parts disappear, and the solution is smooth.
Let $x_* \in \mathcal{D}_0 \subseteq I\!R^m$ represent a stationary solution of (4.8), i.e., $g(x_*) = 0$. In this case, the linearized at x_* equation has constant coefficients A and $B := g'(x_*)$.
The well-known Lyapunov-Theorem of asymptotic stability of a stationary solution for DAEs sounds as simple as it does for regualr ODEs (cf. [21]).

Theorem 4.3 *Let $x_* \in \mathcal{D}_0, \quad g(x_*) = 0, g \in C^2(\mathcal{D}_0, I\!R^m)$. Let the matrix pencil $\{A, B\}$ be regular with index $\mu \in \{1, 2, 3\}$, $\sigma\{A, B\} \subset \mathbb{C}^-$.*
If $\mu > 1$, let some additional structural restrictions be satisfied.
Then, x_* is asymptotically stable.

Note that now all solutions have additional regularity, namely, they are C^1.

Next, let $x_* \in C^1_N$ be a T-periodic solution of (4.2) whose stability properties are to be considered. Then, supposed (4.2) is T-periodic, the coefficients $A(t), B(t)$ of the linearization are also T-periodic, and in the consequence all our subspaces, projectors etc. are so.

Now we can follow the way of Lyapunov to show stability via linearization, transformation of the linear part into constant coefficient form, and using a Lyapunov function for that simple case.
As an auxiliary problem we have to consider the stability of stationary solutions of DAEs

$$Ax'(t) + Bx(t) + h(x'(t), x(t), t) = 0 \tag{4.9}$$

with small nonlinearities. More precisely, assume $h(0, 0, t) \equiv 0$ and consider the trivial solution $x_* = 0$. Next, collect some assumptions to be satisfied for (4.9):
Let $\{A, B\}$ be regular with index $\mu \in \{1, 2\}$ and let $\ker A \subseteq \ker h'_y(y, x, t)$.
To each $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that

$$\left.\begin{array}{l} |h'_x(y,x,t)| \le \varepsilon \\ |h'_y(y,x,t)| \le \varepsilon \end{array}\right\} \quad \text{for} \quad |x| \le \delta(\varepsilon),\ |y| \le \delta(\varepsilon),\ t \in [t_0, \infty).$$

In case of $\mu = 2$, certain additional assumptions have to be satisfied. Then we obtain the result we are searching for and which sounds as nice as the original by Lyapunov for regular ODEs.

Lemma 4.4 *If $\sigma\{A, B\} \subset \mathbb{C}^-$, then the origin is an asymptotically stable point of* (4.9).

Finally, return to the T-periodic case of (4.2) with a T-periodic solution $x_*(\cdot)$.

Theorem 4.5 *Let g have an additional continuous derivative g''_{xx}, and let the linearized at x_* DAE be index 1.*
If the monodromy matrix $X(T)$ has all its eigenvalues in $\{z \in \mathbb{C} : |z| < 1\}$, then the periodic solution $x_*(\cdot)$ is asymptotically stable in the sense of Lyapunov.

The proof if given in [15]. A version for the index 2 case is hoped to be completed soon.

5 Conclusion

The modellation with DAEs and, hence, the numerical treatment as an essential part of modellation, is growing in importance. The direct numerical treatment of practically relevant lower index DAEs needs a further analysis (asymptotic behaviour, practicable index criteria, role of rotating subspaces, index changes, small parameters, singularities...). From this point of view, ODEs should be revisited and generalized to include DAEs. One can be optimistic in this respect.

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