Iteration methods with incomplete corrections and solution of nonlinear equations

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Basing on the theory of dynamic systems with inclomplete corrections [1-2], various methods of Gauss-Seidel-type for the numerical solution of nonlinear equations are formulated and analyzed. For special classes of equations some of these methods have already been discussed (e.g. [3-5]).
The results of the present paper provide statements on the asymptotic stability of equilibria for new classes of nonlinear systems.

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1 The method of successive approximation

1.1 We consider the approximate solution of the equation

$$x = F(x), \quad x = \{\xi_1, \dots, \xi_N\} \in I\!R^N, \tag{1}$$

where $F(x) = \{f_1(x), \dots, f_N(x)\}$ is defined on $I\!R^N$ with the scalar functions $f_i(x) = f_i(\xi_1, \dots, \xi_N)$. Let the space $I\!R^N$ be half-ordered (cf. [6-8]) with respect to the cone $K_+ = \{x = \{\xi_1, \dots, \xi_N\} : \xi_i \geq 0, i = 1, \dots, N\}$ of nonnegative vectors, i. e., we define $x \leq y$ for $x,\, y \in I\!R^N$ if $y - x \in K_+$ holds true. Let a norm $\|.\|$ be given on $I\!R^N$. For $x = \{\xi_1, \dots, \xi_N\} \in I\!R^N$ let $|x|$ denote the vector $\{|\xi_1|, \dots, |\xi_N|\} \in I\!R^N$. For the function F we assume the condition

$$|F(x) - F(y)| \leq A|x - y|, \quad x,\, y \in I\!R^N \tag{2}$$

to be fulfilled with an $N \times N$ matrix $A = (a_{ij})$ with nonnegative entries and a spectral radius of

$$r(A) < 1. \tag{3}$$

In terms of coordinates, (1) is of the form

$$\xi_i = f_i\left(\xi_1, \dots, \xi_N\right), \quad i = 1, \dots, N\,, \tag{4}$$

and (2) corresponds to the Lipschitz-condition

$$\left|f_i\left(\xi_1, \dots, \xi_N\right) - f_i\left(\varsigma_1, \dots, \varsigma_N\right)\right| \leq a_{i1}|\xi_1 - \varsigma_1| + \dots + a_{iN}|\xi_N - \varsigma_N| \tag{5}$$

for the scalar functions f_i .

1.2 For the classical Picard iteration method

$$x_n = F(x_{n-1}), \quad n = 1, 2, \dots, \tag{6}$$

written in form of coordinates as

$$\xi_i^n = f_i\left(\xi_i^{n-1}, \dots, \xi_N^{n-1}\right), \quad i = 1, \dots, N;\ n = 1, 2, \dots \tag{7}$$

with $x_n = \{\xi_1^n, \dots, \xi_N^n\}$, the following statements of convergence are well-known (e.g. [9-11]).

Theorem 1 *:*

(i) Equation (1) has exactly one solution $x_* \in I\!R^N$. For an arbitrary initial guess $x_0 \in I\!R^N$ the sequence (6) converges to x_* in $I\!R^N$.

(ii) The estimation

$$\|x_n - x_*\| \le C(\varepsilon)(r(A)+\varepsilon)^n \|x_0 - x_*\|, \quad n = 1, 2, \ldots \tag{8}$$

holds for each arbitrary $\varepsilon > 0$ with a constant $C(\varepsilon) < \infty$ depending on ε and the chosen norm $\|.\|$.

(iii) If there are no principal vectors for those eigenvalues λ of the matrix A for which $|\lambda| = r(A)$, then we have the stronger estimation

$$\|x_n - x_*\| \le C(r(A))^n \|x_0 - x_*\|, \quad n = 1, 2, \ldots \tag{9}$$

with a constant $C < \infty$ depending on the norm $\|.\|$ only.

The inequality (9) holds if e.g. the vectors Ay or $A^k y$, with k fixed, have positive components for each nontrivial $y \in K_+$. In particular, (9) holds true for focussing or acute matrices A ([8]). Under these conditions $r(A)$ is a simple eigenvalue of A, and $|\lambda| < r(A)$ holds for all the other eigenvalues λ of A.

For large N, the calculation of the spectral radius $r(A)$ is rather extensive, however, there are efficient simple algorithms that provide appropriate estimations for $r(A)$ (e.g. [7-8]). The inequalities (8) and (9) for the simple Picard-iteration (6) are sharp and cannot be improved. Hence, we are striving for iteration methods that converge faster.

2 Main result

2.1 Let $M = (\mu_{ij})$ be an $N \times N$ matrix with the entries $0 \le \mu_{ij} \le 1, \quad i, j = 1, \ldots, N$. Later on, we will investigate iteration methods in which the approximation $x_n = \{\xi_1^n, \ldots, \xi_N^n\}$ is determined from the equations

$$\xi_i^n = f_i\left((\mu_{i1}\xi_1^n + (1-\mu_{i1})\xi_1^{n-1}, \ldots, \mu_{iN}\xi_N^n + (1-\mu_{iN})\xi_N^{n-1}\right), \tag{10}$$

where the initial guess $x_0 = \{\xi_1^0, \ldots, \xi_N^0\} \in I\!R^N$ can be chosen arbitrarily. We write the equations (10) more easily in the form

$$\xi_i^n = \varphi_i\left(\xi_1^n, \ldots, \xi_N^n; \xi_1^{n-1}, \ldots, \xi_N^{n-1}\right), \quad i = 1, \ldots, N, \tag{11}$$

by introducing the functions

$$\begin{aligned}\varphi_i&(\varsigma_1, \ldots, \varsigma_N; \eta_1, \ldots, \eta_N) \\ &= f_i(\mu_{i1}\varsigma_1 + (1-\mu_{i1})\eta_1, \ldots, \mu_{iN}\varsigma_N + (1-\mu_{iN})\eta_N) .\end{aligned} \tag{12}$$

In numerical methods (10) the transition from x_{n-1} to x_n requires to solve the N scalar equations (11) with respect to the N unknown components $\xi_1^n, \ldots, \xi_N^n$ of the vector x_n.
(11) provides again the Picard-iteration (7) in case M is the zero-matrix.

2.2 An $N \times N$ matrix $G = (g_{ij})$ is said to be positive (with respect to the cone K_+), which we write as $G \geq 0$, if all entries g_{ij} are nonnegative. In case $G_1 - G_2 \geq 0$ holds, we write $G_2 \leq G_1$.

We construct an auxiliary matrix $B = (b_{ij})$ for the matrices A and M by means of the definition $b_{ij} = a_{ij}\mu_{ij}$. Then $a_{ij} \geq 0$ and $0 \leq \mu_{ij} \leq 1$ imply

$$0 \leq B \leq A. \tag{13}$$

As known, (13) implies the relation $r(B) \leq r(A)$. Thus, $r(B) < 1$ holds naturally and the matrix $I - B$ is regular, where I denotes the unit matrix. Let us further introduce the matrix

$$A_0 = (I - B)^{-1}(A - B). \tag{14}$$

¿From the inequalities (3) and (13) we derive the estimation (cf. [8])

$$r(A_0) \leq r(A), \tag{15}$$

which will be important in the following theorem.

Theorem 2 *:*

(i) For arbitrary $y = \{\eta_1, \ldots, \eta_N\} \in I\!R^N$ the system

$$\varsigma_i = \varphi_i(\varsigma_i, \ldots, \varsigma_N; \eta_1, \ldots, \eta_N), \quad i = 1, \ldots, N \tag{16}$$

has exactly one solution $x(y) = \{\xi_1, \ldots, \xi_N\}$ each. Hence, (10) uniquely determines the sequence of the $x_n = \{\xi_1^n, \ldots, \xi_N^n\}$ for an arbitrarily given starting value $x_0 = \{\xi_1^0, \ldots, \xi_N^0\}$.

(ii) For each initial guess $x_0 \in I\!R^n$ the sequence of the x_n that is defined by (10) converges to the only solution x_* of the equation (1).

(iii) For the method (10) we have statements analogous to Theorem 1 (ii) and (iii) if $r(A)$ is replaced by the spectral radius $r(A_0)$ of the matrix (14), where $r(A_0) \leq r(A)$ is always true. If, additionally , A is irreducible and $M \neq 0$ holds, then the strong inequality

$$r(A_0) < r(A) \tag{17}$$

is satisfied.

Inequality (17) results from a conclusion by V. J. Stetsenko ([12]). A matrix A is called irreducible if $Ax \geq \alpha x,\ \alpha > 0,\ x = \{\xi_1, \ldots, \xi_N\} \in K_+,\ x \neq 0$ implies all entries ξ_i of the vector x to be positive.
The proof of Theorem 2 is given in the appendix.

2.3 According to Theorem 2 the speed of convergence of the iteration method (10) is characterized by the inequality

$$(18) \qquad \|x_n - x_*\| \leq C(q)q^n \|x_0 - x_*\|, \quad n = 1, 2, \ldots, x_0 \in I\!R^N,$$

where q can be an arbitrary number from the interval $(r(A_0), \infty)$.
Now, let Q be the set of all such numbers $q > 0$ for which the inequality (18) holds uniformly for all starting values $x_0 \in I\!R^N$. Obviously, Q is either the open interval (q_*, ∞) or the half-open interval $[q_*, \infty)$ with $q_* = \inf Q \leq r(A_0)$.
For a moment, we assume equation (1) to be of the special form

$$(19) \qquad x = Ax + d$$

with $d = \{d_1, \ldots, d_N\} \in I\!R^N$. Because of

$$|Ax - Ay| \leq A|x - y|, \quad x,\ y \in I\!R^N,$$

the condition (2) is fulfilled. In this case formula (10) simplifies to

$$(20) \qquad \begin{aligned} \xi_i^n &= a_{i1}\left(\mu_{i1}\xi_1^n + (1-\mu_{i1})\xi_1^{n-1}\right) + \cdots \\ &\quad + a_{iN}\left(\mu_{iN}\xi_N^n + (1-\mu_{iN})\xi_N^{n-1}\right) + d_i, \\ &\qquad i = 1, \ldots, N. \end{aligned}$$

Theorem 3 *: Let the sequence of approximations $x_n = \{\xi_1^n, \ldots, \xi_N^n\}$ for the solution x_* of the linear system (19) be defined by (20).*
Then the spectral radius $r(A_0)$ of the matrix (14) forms the infimum of the set Q, i.e. $q_* = r(A_0)$. Here $r(A_0) \in Q$ holds if and only if $r(A_0)$ is positive and if, moreover, there are no principal vectors to any eigenvalue λ of the matrix A_0 with $|\lambda| = r(A_0)$.

For shortness we will frequently apply the spectral condition already used in the assumption shortly as condition (a) in the following.
(a): There is no principal vector to any of the eigenvalues λ of the matrix A_0 with the property $|\lambda| = r(A_0)$.

According to Theorem 3 the estimation

$$(21) \qquad \|x_n - x_*\| \leq C(\varepsilon)(r(A_0) + \varepsilon)^n \|x_0 - x_*\|,\ n = 1,\ 2, \cdots$$

of the speed of convergence of the iterations (10) towards the solution x_* of the nonlinear equation (1), which results from the inequalities (2), (3), cannot be improved. In (21) ε is an arbitrarily small positive number, as usually. In case the condition (a) is given, the estimation (21) holds with $\varepsilon = 0$.
The proof of Theorem 3 is given in the appendix.

2.4 Let us now compare the speed of convergence of iteration methods (10) with different matrices M.
For $M^* = (\mu^*_{ij}),\ M^{**} = (\mu^{**}_{ij})$ with $\mu^*_{ij},\ \mu^{**}_{ij} \in [0,\,1]$ we construct the auxiliary matrices $B^* = (b^*_{ij}),\ B^{**} = (b^{**}_{ij})$ with $b^*_{ij} = a_{ij}\mu^*_{ij},\ b^{**}_{ij} = a_{ij}\mu^{**}_{ij}$, as well as $A^*_0 = (I - B^*)^{-1}(A - B^*), \quad A^{**}_0 = (I - B^{**})^{-1}(A - B^{**})$.
If $r(A^*_0) \geq r(A^{**}_0)$, then the speed of convergence of the iteration method

$$\xi^n_i = f_i(\mu^{**}_{i1}\xi^n_1 + \left(1 - \mu^{**}_{i1})\xi^{n-1}_1, \ldots, \mu^{**}_{iN}\xi^n_N + (1 - \mu^{**}_{iN})\xi^{n-1}_N\right), \qquad i = 1, \ldots, N,\ n = 1,\,2, \ldots\ , \tag{22}$$

which is guaranteed by the sharp estimation (21), is not less than the guaranteed speed of convergence of the method

$$\xi^n_i = f_i(\mu^{*}_{i1}\xi^n_1 + \left(1 - \mu^{*}_{i1})\xi^{n-1}_1, \ldots, \mu^{*}_{iN}\xi^n_N + (1 - \mu^{*}_{iN})\xi^{n-1}_N\right), \qquad i = 1, \ldots, N,\ n = 1,\,2, \ldots\ . \tag{23}$$

Theorem 4 *: Let $M^* < M^{**}$. Then*

$$r(A^*_0) \geq r(A^{**}_0), \tag{24}$$

and, consequently, the method (22) does not provide a convergence that is slower than that of the method (23).
Since equation (10) transformes into equation (7) for $M = 0$, the method (10) does not converge slower than the Picard-iteration (7).
For irreducible matrices A and $M \neq 0$ the strong inequality (17) becomes valid and (10) provides faster convergent sequences than (7) does.

The proof of Theorem 4 is contained in the appendix.

3 Iteration methods of Gauß-Seidel-type

3.1 The classical Gauß-Seidel-method for the nonlinear system (4) consists of the equations

$$\left\{ \begin{array}{lcl} \xi^n_1 & = & f_1\left(\xi^n_1, \xi^{n-1}_2, \ldots, \xi^{n-1}_N\right), \\ \xi^n_2 & = & f_2\left(\xi^n_1, \xi^n_2, \xi^{n-1}_3, \ldots, \xi^{n-1}_N\right), \\ \ldots & & \\ \xi^n_N & = & f_N\left(\xi^n_1, \ldots, \xi^n_N\right), \end{array} \right. \tag{25}$$

for determining the iteration $x_n = \{\xi_1^n, \ldots, \xi_N^n\}$. The starting value $x_0 \in I\!R^N$ is chosen arbitrarily. The transition from x_{n-1} to x_n requires to solve N scalar equations with one scalar unknown each. By

$$M^* = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} \tag{26}$$

the ansatz (25) is contained in our general formula (10). The Gauß-Seidel-method (25) is investigated in detail e.g. in [3-5].

3.2 With the increasing development of computers, also block versions of the classical numerical iteration methods have met with interest. The block-wise version of the Gauß-Seidel-method (25) realizes the transition from $x_{n-1} = \left\{\xi_1^{n-1}, \ldots, \xi_N^{n-1}\right\}$ to $x_n = \{\xi_1^n, \ldots, \xi_N^n\}$ by means of the following system

$$\begin{aligned} \xi_1^n &= f_1\left(\xi_1^n, \ldots, \xi_{N_1}^n, \xi_{N_1+1}^{n-1}, \ldots, \xi_N^{n-1}\right), \\ &\cdots \\ \xi_{N_1}^n &= f_{N_1}\left(\xi_1^n, \ldots, \xi_{N_1}^n, \xi_{N_1+1}^{n-1}, \ldots, \xi_N^{n-1}\right), \\ \xi_{N_1+1}^n &= f_{N_1+1}\left(\xi_1^n, \ldots, \xi_{N_1+N_2}^n, \xi_{N_1+N_2+1}^{n-1}, \ldots, \xi_N^{n-1}\right), \\ &\cdots \\ \xi_{N_1+N_2}^n &= f_{N_1+N_2}\left(\xi_1^n, \ldots, \xi_{N_1+N_2}^n, \xi_{N_1+N_2+1}^{n-1}, \ldots, \xi_N^{n-1}\right), \\ &\vdots \\ \xi_{N_1+\cdots+N_{k-1}+1}^n &= f_{N_1+\cdots+N_{k-1}+1}\left(\xi_1^n, \ldots, \xi_N^n\right), \\ &\cdots \\ \xi_N^n &= f_N\left(\xi_1^n, \ldots, \xi_N^n\right), \end{aligned} \tag{27}$$

with $0 < N_1 < N_1 + N_2 < \cdots < N_1 + N_2 + \cdots + N_k = N$.

In order to compute x_n from x_{n-1} we now solve k nonlinear systems of $N_1, \ldots, N_k$ scalar equations (with the same number of unknowns). For $k = N$, (27) provides again the classical variant (25) .

Formula (10) contains the block version (27) if we use the matrix

$$M^{**} = \begin{pmatrix} J_{N_1N_1} & 0 & \cdots & 0 \\ J_{N_2N_1} & J_{N_2N_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J_{N_kN_1} & J_{N_kN_2} & \cdots & J_{N_kN_k} \end{pmatrix}, \tag{28}$$

where the rectangular blocks $J_{N_iN_j}$ consist of N_i rows and N_j columns, and all entries are one. The entries of the matrices 0 are zero. Since the inequality $M^* \leq M^{**}$ holds for the matrices determined in (26) and (28), the block method (27) does not converge slower than the classical method (25) due to Theorem 4. Usually, the block version (27) converges much faster.

3.3 In both methods, (25) and (27), nonlinear systems of equations have to be solved to define the iteration x_n. This does not apply for the following rule

$$(29)\qquad \begin{array}{rcl} \xi_1^n &=& f_1\left(\xi_1^{n-1},\ldots,\xi_N^{n-1}\right), \\ \xi_2^n &=& f_2\left(\xi_1^n,\xi_2^{n-1},\ldots,\xi_N^{n-1}\right), \\ \cdots && \\ \xi_N^n &=& f_N\left(\xi_1^n,\ldots,\xi_{N-1}^n,\xi_N^{n-1}\right), \end{array}$$

which the matrix $M^{***} = (g_{ij})$ with $g_{ij} = 1$ for $i > j$ and $g_{ij} = 0$ for $i \leq j$ corresponds to in the general ansatz (10). Since, $M^{***} \leq M^* \leq M^{**}$ is obviously true, each of the methods (25) and (27) does not converge slower than (29).

Appendix

Proof of Theorem 2.:
First we show that the statement (i) holds true. For fixed $y = \{\eta_1,\ldots,\eta_N\} \in I\!R^N$ we write the system (16) in a compact way

$$(A.1)\qquad x = \Phi(x,\, y), \quad x = \{\xi_1,\ldots,\xi_N\} \in I\!R^N,$$

with $\Phi = \{\varphi_1,\ldots,\varphi_N\}$.
First of all, we establish an $I\!R^N$-norm in which the conditions for the principle of contracting mappings are fulfilled for the equation (A.1). Let $\|.\|_e$ denote the Euclidean norm $\|x\|_e = \sqrt{\xi_1^2+\cdots+\xi_N^2}$. Due to the property (3) there exists an $\varepsilon > 0$ fulfilling the inequality $r(A)+\varepsilon = q < 1$. Now we define the norm

$$(A.2)\qquad \|x\|_m = \sum_{k=0}^{m} q^{-k}\|A^k|x|\,\|_e.$$

Because of (5) the functions φ_i defined in (12) satisfy the Lipschitz-condition

$$|\varphi_i(x_1,y)-\varphi_i(x_2,y)| \leq a_{i1}\mu_{i1}|\xi_1-\varsigma_1|+\cdots+a_{iN}\mu_{iN}|\xi_N-\varsigma_N|,$$
$$\text{for } x_1 = \{\xi_1,\ldots,\xi_N\},\ x_2 = \{\varsigma_1,\ldots,\varsigma_N\} \in I\!R^N.$$

Since $0 \leq \mu_{ij} \leq 1$ holds true, we conclude the inequalities

$$|\varphi_i(x_1,y)-\varphi_i(x_2,y)| \;\leq\; a_{i1}|\xi_1-\varsigma_1|+\cdots+a_{iN}|\xi_N-\varsigma_N|,$$

$$i = 1,\ldots,N,$$

to hold, or, in a more compact form,

$$|\Phi(x_1, y) - \Phi(x_2, y)| \leq A|x_1 - x_2|. \tag{A.3}$$

On account of the positivity of the matrix A, the inequality (A.3) implies the inequalities

$$A^k|\Phi(x_1, y) - \Phi(x_2, y)| \leq A^{k+1}|x_1 - x_2|, \ k = 1, 2, \ldots \quad .$$

Moreover, $0 \leq z_1 \leq z_2$ immediately implies $\|z_1\|_e \leq \|z_2\|_e$ and, hence,

$$\|A^k|\Phi(x_1, y) - \Phi(x_2, y)|\ \|_e \leq \|A^{k+1}|x_1 - x_2|\ \|_e.$$

Thus, we have

$$\begin{aligned} \|\Phi(x_1, y) - \Phi(x_2, y)\|_m &= \sum_{k=0}^{m} q^{-k}\|A^k|\Phi(x_1, y) - \Phi(x_2, y)|\ \|_e \\ &\leq \sum_{k=0}^{m} q^{-k}\|A^{k+1}|x_1 - x_2|\ \|_e \end{aligned}$$

and, consequently,

$$\begin{aligned} \|\Phi(x_1, y) - \Phi(x_2, y)\|_m &\leq q\|x_1 - x_2\|_m - q\|\ |x_1 - x_2|\ \|_e \\ &\quad + q^{-m}\|A^{m+1}|x_1 - x_2|\ \|_e. \end{aligned}$$

As known,

$$r(A) = \lim_{n \to \infty} \sqrt[n]{\|A^n\|} \tag{A.4}$$

with $\|A^n\| = \max\{\|A^n x\| : \|x\| = 1\}$ holds true for matrix-norms $\|.\|$ induced by arbitrary $I\!R^N$-norms.
For sufficiently large m, (A.4) guarantees the relation

$$\begin{aligned} \|A^{m+1}|x_1 - x_2|\ \|_e \leq \|A^{m+1}\|_e\|\ |x_1 - x_2|\ \|_e &\leq (r(A) + \varepsilon)^{m+1}\|\ |x_1 - x_2|\ \|_e \\ &= q^{m+1}\|\ |x_1 - x_2|\ \|_e, \end{aligned}$$

from which we further conclude

$$\|\Phi(x_1, y) - \Phi(x_2, y)\|_m \leq q\|x_1 - x_2\|_m,$$

i.e., the conditions for the principle of contracting mappings are given. The fixed point equation (A.1) has exactly one solution $x(y) \in I\!R^N$ to each $y \in I\!R^N$. Thus, Theorem 2 (i) has been proved.

Let us now turn to the claims (ii) and (iii). By $x_* = \{\xi_1^*, \ldots, \xi_N^*\}$ we denote the only solution of the system (4) according to Theorem 1.
If we subtract the equation $\xi_i^* = f_i\,(\xi_1^*, \ldots, \xi_N^*)$ from (10), we obtain

$$\xi_i^n - \xi_i^* = f_i(\mu_{i1}\xi_1^n + (1-\mu_{i1})\xi_1^{n-1}, \ldots, \mu_{iN}\xi_N^n + (1-\mu_{iN})\xi_N^{n-1}) - f_i(\xi_1^*, \ldots, \xi_N^*)$$

and, taking into account the Lipschitz-condition (5), this implies

$$|\xi_i^n - \xi_i^*| \leq a_{i1}|\mu_{i1}\xi_1^n + (1-\mu_{i1})\xi_1^{n-1} - \xi_1^*| \quad + \cdots$$
$$+ a_{iN}|\mu_{iN}\xi_N^n + (1-\mu_{iN})\xi_N^{n-1} - \xi_N^*|,$$

hence

$$|\xi_i^n - \xi_i^*| \leq \sum_{j=1}^{N} \left(a_{ij}\mu_{ij}|\xi_j^n - \xi_j^*| + a_{ij}(1-\mu_{ij})|\xi_j^{n-1} - \xi_j^*|\right), \qquad i = 1, \ldots, N. \tag{A.5}$$

Written in a compact way, (A.5) is of the form

$$|x_n - x_*| \leq B|x_n - x_*| + (A-B)|x_{n-1} - x_*|$$

and

$$(I-B)|x_n - x_*| \leq (A-B)|x_{n-1} - x_*|, \tag{A.6}$$

respectively, with the matrix $B = (b_{ij}),\ b_{ij} = a_{ij}\mu_{ij}$ introduced in 2.2. As also $r(B) \leq r(A) < 1$ has to be true because of $0 \leq B \leq A$, the matrix $I - B$ is regular and its inverse $(I-B)^{-1}$ can be represented by the Neumann sequence as

$$(I-B)^{-1} = I + B + \cdots + B^k + \cdots \quad .$$

¿From $B \geq 0$ it follows that $(I-B)^{-1} \geq 0$. If the positive matrix $(I-B)^{-1}$ is applied to both sides of (A.6), we obtain $|x_n - x_*| \leq A_0|x_{n-1} - x_*|$ with $A_0 = (I-B)^{-1}(A-B) \geq 0$.
Thus, we have

$$|x_n - x_*| \leq A_0|x_{n-1} - x_*| \leq A_0^2|x_{n-2} - x_*| \leq \cdots \leq A_0^n|x_0 - x_*|,$$

and since $\|z\|_e = \||z|\|_e$ always holds for $z \in I\!R^N$, and $\|z_1\| \leq \|z_2\|$ for $0 \leq z_1 \leq z_2$, it results immediately that

$$\|x_n - x_*\|_e = \||x_n - x_*|\|_e \leq \|A_0^n|x_0 - x_*|\|_e \leq \|A_0^n\|_e\|x_0 - x_*\|_e\,. \tag{A.7}$$

Because of $\sqrt[n]{\|A_0^n\|_e} \to r(A_0)$ for $n \to \infty$, there exists a number n_ε to each $\varepsilon > 0$ such that

$$\text{(A.8)} \qquad \|x_n - x_*\|_e \le (r(A_0)+\varepsilon)^n \|x_0 - x_*\|_e$$
$$\text{for } n = n_\varepsilon,\ n_\varepsilon + 1, \dots$$

is fulfilled. Then we also find a $C_e(\varepsilon) < \infty$ with the property

$$\|x_n - x_*\|_e \le C_e(\varepsilon)(r(A_0)+\varepsilon)^n \|x_0 - x_*\|_e$$

$$\text{for } n = 1,\, 2, \dots$$

By (15) and sufficiently small $\varepsilon > 0$ we obtain $r(A_0) + \varepsilon \le r(A) + \varepsilon < 1$, consequently $\|x_n - x_*\|_e \to 0$ holds for $n \to \infty$. Since all $I\!R^N$-norms are equivalent to each other, x_n converges in any norm to x_* and even the estimation (21) is proved.
Next we show that, under the additional assumption (a), the estimation

$$\text{(A.9)} \qquad \|x_n - x_*\|_e \le C(r(A_0))^n \|x_0 - x_*\|,\ n = 1, 2, ...$$

which is stronger than (21), becomes valid. Let $\lambda_1, \dots, \lambda_m$ be those eigenvalues of the matrix A_0 for which $|\lambda_j| = r(A_0)$ holds, while the inequality $|\lambda_j| < r(A_0)$, $j = m+1, \dots, N$, holds for the remaining eigenvalues $\lambda_{m+1}, \dots, \lambda_N$. This implies e.g.

$$\text{(A.10)} \qquad \begin{aligned} \lambda_{2j-1} &= r(A_0)e^{i\phi_j}, \quad \lambda_{2j} = r(A_0)e^{-i\phi_j},\ j = 1, \dots, k, \\ \lambda_{2k+1} &= \cdots = \lambda_\ell = r(A_0), \lambda_{\ell+1} = \cdots = \lambda_m = -r(A_0)\ , \end{aligned}$$

with $\phi_j \ne 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3}{2}\pi, \dots$.
Let condition (a) be given, i.e., there are no principal vectors to the eigenvalues (A.10). Now we can construct a basis $\{e_1, \dots, e_N\}$ for $I\!R^N$, where A_0 is transformed into

$$A_1 = r(A_0) \begin{pmatrix} R_{\phi_1} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & R_{\phi_2} & & \vdots & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & R_{\phi_k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_{\ell-2k} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -I_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & G \end{pmatrix},$$

i.e., $A_1 = r(A_0)$ diag $\{R_{\phi_1}, R_{\phi_2}, \cdots, R_{\phi_k}, I_{\ell-2k}, -I_{m-1}, G\}$.
We denote

$$R_{\phi_j} = \begin{bmatrix} \cos\phi_j & \sin\phi_j \\ -\sin\phi_j & \cos\phi_j \end{bmatrix} \quad ,$$

and $I_{\ell-2k}, I_{m-1}$ are unit matrices of size $\ell - 2k$ and $m-1$. G is a matrix of order $N-m$ with the spectrum $\lambda_{m+1}/r(A_0), \ldots, \lambda_N/r(A_0)$ and the spectral radius

$$r(G) = \max\{|\lambda_j/r(A_0)| : j = m+1, \ldots, N\} < 1 \ .$$

Because of $\sqrt[n]{\|G^n\|_e} \to r(G) < 1$ for $n \to \infty$, $\|G^n\|_e < 1$ holds for all n larger than a certain n_0.

The property $R^n_{\phi_j} = R_{n\phi_j}$ provides the representation

$$A_1^n = (r(A_0))^n \text{ diag } \{R_{n\phi_1}, \ldots, R_{n\phi_k}, I_{\ell-2k}, (-1)^n I_{m-\ell}, G^n\},$$

and, moreover,

$$(A_1^n)^T A_1^n = (r(A_0))^{2n} \text{ diag } \{I_m, (G^n)^T G^n\}.$$

¿From the well-known relation $\|A_1^n\|_e = \sqrt{r((A_1^n)^T A_1^n)}$ it becomes obvious that $\|A_1^n\|_e = (r(A_0))^n$ holds for all $n > n_0$.

Denote by $\{\eta_1, \ldots, \eta_N\}$ the coordinats of the vectors $x \in I\!R^N$ with respect to the basis $\{e_1, \ldots, e_N\}$ and let the norm $\|.\|_0$ be determined as $\|x\|_0 = \sqrt{\eta_1^2 + \cdots + \eta_N^2}$, then the norm $\|A_0^n\|_0$ of the matrix A_0^n coincides with $\|A_1^n\|_e$ by construction and it holds that $\|A_0^n\|_0 = \|A_1^n\|_e = (r(A_0))^n$ for $n > n_0$.
Because of the equivalence of the norms there are constants c_1, $c_2 > 0$ with $\|x\|_e \leq c_1\|x\|_0 \leq c_2\|x\|_e$, $x \in I\!R^N$, and we have e.g. the inequality $\|A_0^n\|_e \leq c_2\|A_0^n\|_0$. Thus, we obtain $\|A_0^n\|_e \leq c_2(r(A_0))^n$ for $n > n_0$ and by (A.7) also

$$\|x_n - x_*\|_e \leq c_2(r(A_0))^n \|x_0 - x_*\|_e.$$

If $r(A_0) > 0$, then there exists a $c < \infty$ such that the inequality

$$\|x_n - x_*\|_e \leq c(r(A_0))^n \|x_0 - x_*\|_e \text{ for } n = 1,\, 2, \ldots$$

holds true for arbitrary starting values $x_0 \in I\!R^N$. Then, according to the equivalence of norms, the inequality (A.9) holds, too.
In case $r(A_0)$ vanishes, i.e., $r(A_0) = 0$, all eigenvalues of the matrix A_0 are equal to zero. As there do not exists any principal vectors, $A_0 = 0$ has to be true, consequently $\|A_0^n\|_e = 0$, and $x_n = x_*$ for $n = 1,\, 2, \ldots$ because of (A.7). In this case (A.9) is trivially fulfilled.
Thus, Theorem 2 is proved. □

Proof of Theorem 3:
First we formulate equation (20) in a compact way as $x_n = Bx_n + (A-B)x_{n-1} + d$, and, with $x_* = (I-A)^{-1}d$ as $x_n = Bx_n + (A-B)x_{n-1} + x_* - Ax_*$.
We can also write

$$x_n - x_* = B(x_n - x_*) + (A-B)(x_{n-1} - x_*)$$

or $x_n - x_* = (I-B)^{-1}(A-B)(x_{n-1} - x_*)$, i.e., $x_n - x_* = A_0(x_{n-1} - x_*)$. Hence, it holds that

$$x_n - x_* = A_0^n(x_0 - x_*), \quad n = 0, 1, 2, \cdots . \tag{A.11}$$

Due to the positivity of the matrix A_0, $r(A_0)$ is an eigenvalue A_0 to which there exists an eigenvector $e \in K_+$. For $x_0 = x_* + e$ this provides $x_n - x_* = (r(A_0))^n(x_0 - x_*)$, consequently $r(A_0) \leq \inf Q$, where Q is, per definitionem, the set of all $q > 0$ for which the estimation (18) holds true. On the other hand, the inequality (21) provides the relation $r(A_0) \geq \inf Q$, i.e., $r(A_0) = \inf Q$ holds.
Since Q consists of positive numbers, $r(A_0) \notin Q$ has to be fulfilled trivially for $r(A_0) = 0$. If $r(A_0) > 0$ and the condition (a) is fulfilled, then $r(A_0) \in Q$ by Theorem 2 and the estimation (A.9). Let us now consider the case that there exists a principal vector h to at least one of the eigenvalues (A.10) of the matrix A_0. Let λ_0 be this eigenvalue. If λ_0 is a real number, i.e. either $\lambda_0 = r(A_0)$ or $\lambda_0 = -r(A_0)$, then vectors $e, h \in I\!R^N$ with $A_0 e = \lambda_0 e\,, A_0 h = \lambda_0 h + e$ and, furthermore, $A_0^n h = \lambda_0^n h + n\,\lambda_0^{n-1} e$ correspond to λ_0. Taking into account (A.11) we obtain for the initial guess $x_0 = x_* + h$ that

$$\|x_n - x_*\| = \|\lambda_0^n h + n\,\lambda_0^{n-1} e\| \geq n(r(A_0))^n \left(\frac{\|e\|}{r(A_0)} - \frac{\|h\|}{n} \right) ,$$

hence, $\|x_n - x_*\| \geq n(r(A_0))^n \frac{\|e\|}{2r(A_0)}$ for all sufficiently large n. Consequently, the estimation (18) can only apply for $q > r(A_0)$ and it holds that $r(A_0) \notin Q$.

Now we consider the case of complex eigenvalues λ_0, Re $\lambda_0 \neq 0\,,\ \lambda_0 = r(A_0)e^{i\phi}$, $\phi \neq 0\,, \pm\frac{\pi}{2}\ \pm\pi, \ldots$. There exist vectors $e_1, e_2, h_1, h_2 \in I\!R^N$ with the properties

$$\begin{aligned} A_0 e_1 &= r(A_0)(e_1 \cos\phi - e_2 \sin\phi) \\ A_0 h_1 &= r(A_0)(h_1 \cos\phi - h_2 \sin\phi) + e_1\,, \\ A_0 e_2 &= r(A_0)(e_1 \cos\phi - e_2 \sin\phi) \\ A_0 h_2 &= r(A_0)(h_1 \cos\phi - h_2 \sin\phi) + e_2\,. \end{aligned}$$

from which we compute

$$\begin{aligned} A_0^n h_1 = {} & (r(A_0))^n (h_1 \cos(n\phi) - h_2 \sin(n\phi)) + \\ & n(r(A_0))^{n-1}(e_1 \cos((n-1)\phi) - e_2 \sin((n-1)\phi)). \end{aligned}$$

With $x_0 = x_* + h_1$, formula (A.11) finally provides

$$\begin{aligned}\|x_n - x_*\| &\geq n(r(A_0))^n \left(\tfrac{1}{r(A_0)}\| \; e_1 \cos((n-1)\phi) - e_2 \sin((n-1)\phi)\| \right.\\ &\qquad \left. -\tfrac{1}{n}\|h_1 \cos(n\phi) - h_2 \sin(n\phi)\|\right)\\ &\geq n(r(A_0))^n \left(\tfrac{1}{r(A_0}\gamma - \tfrac{\|h_1\|+\|h_2\|}{n}\right) ,\end{aligned}$$

where we have introduced $\gamma = \min\{\|e_1 \cos\Theta - e_2 \sin\Theta\| : 0 \leq \Theta \leq 2\pi\} > 0$. Then $\|x_n - x_*\| \geq n(r(A_0))^n \cdot \frac{\gamma}{2r(A_0)}$ holds for large n, which implies $r(A_0) \notin Q$.
□

Proof of Theorem 4.:

We use the notation of 2.4. By $0 \leq B^* \leq A$ it holds that $r(B^*) \leq r(A)$, thus $r(B^*) < 1$ and $(I - B^*)^{-1} \geq 0$ hold, too. ¿From $M^* \leq M^{**}$ it follows that $0 \leq B^* \leq B^{**} \leq A$ and, furthermore,

$$0 \leq B^{**} - B^* \leq A - B^* \; . \tag{A.12}$$

For the matrix $D = (I - B^*)^{-1}(B^{**} - B^*)$ we have that $0 \leq D \leq A_0^*$, which becomes immediately obvious when multiplying (A.12) by $(I - B^*)^{-1}$. Because of $r(A_0^*) < 1$ we can form the matrix $G = (I - D)^{-1}(A_0^* - D)$, for the spectral radius of which the relation analogous to (15)

$$r(G) \leq r(A_0^*) \tag{A.13}$$

holds true.
On the other hand, we compute

$$\begin{aligned} G &= (I - D)^{-1}(A_0^* - D)\\ &= (I - (I - B^*)^{-1}(B^{**} - B^*))^{-1}((I - B^*)^{-1}(A - B^*) - (I - B^*)^{-1}(B^{**} - B^*))\\ &= ((I - B^*)^{-1}(I - B^{**}))^{-1}(I - B^*)^{-1}(A - B^{**})\\ &= (I - B^{**})^{-1}(A - B^{**}) = A_0^{**} \; ,\end{aligned}$$

which immediately implies the claimed inequality (24) by (A.13). □

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