# THE SEMISTATIC LIMIT FOR MAXWELL'S EQUATIONS IN AN EXTERIOR DOMAIN

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**Abstract:**
This paper provides a $L^p$-theory for the Maxwell-system in the semistatic limit case in an exterior domain. The problem under consideration is of mixed type, since the possibly nonlinear electric conductivity vanishes on a certain subset of the domain. The solution is obtained from a singular perturbation of the full Maxwell-system including the displacement current.

## 1 Introduction

This paper concerns the initial-boundary value problem for Maxwell's equations [6] in an exterior domain, that is the system

$$\text{curl }\mathbf{h}_\varepsilon = \varepsilon\partial_t\mathbf{E}_\varepsilon + \sigma(t,x,\mathbf{E}_\varepsilon) + \mathbf{j}_0 \tag{1.1}$$

$$\text{curl }\mathbf{E}_\varepsilon = -\mu\partial_t\mathbf{h}_\varepsilon \tag{1.2}$$

supplemented by the initial-boundary-conditions

$$\vec{n}\wedge\mathbf{E}_\varepsilon = 0 \text{ on } [0,\infty)\times\partial\Omega, \tag{1.3}$$

$$\mathbf{E}_\varepsilon(0) = \mathbf{E}_0\text{and } \mathbf{h}_\varepsilon(0) = \mathbf{h}_0. \tag{1.4}$$

Here $\Omega \subset I\!R^3$ is a domain with bounded complement, $\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon$ denote the electric and magnetic field respectively which depend on the time $t \geq 0$ and the space-variable $x \in \Omega$. On the perfectly conducting boundary $\partial\Omega$ the tangential-component of $\mathbf{E}$ must vanish, which is expressed by boundary condition 1.3. $\mathbf{j}_0$ is a prescribed external current, which is assumed to be divergence-free. The charge current $\sigma(t,x,\mathbf{E}_\varepsilon)$ may depend nonlinearely on the electric field. $\sigma$ is a monotone function (with respect to $\mathbf{E}_\varepsilon$) with the property that

$$\sigma(t,x,\mathbf{y}) = 0 \text{ for all } x \in \Omega_0 \stackrel{\text{def}}{=} \Omega\setminus G$$

with some bounded set $G \subset \Omega$. The dielectric susceptibility $\varepsilon > 0$ is considered as a parameter in the above system. $\mu \in L^\infty(I\!R^3)$ is the magnetic susceptibility, which is assumed to be uniformly positive. The initial-data must be compatible with Maxwell's equations, i.e.

$$\text{div } \mathbf{E}_0 = 0 \text{ on } \Omega_0 = \Omega \setminus G, \ \text{div } \mathbf{h}_0 = 0 \text{ on } \Omega \tag{1.5}$$

$$\text{and } \int_{\mathcal{C}_k} \vec{n}\mathbf{E}_0(x)dS = Q_k \text{ for all } k \in \{1, .., N\}. \tag{1.6}$$

If these compatibility conditions are fulfilled then every solution of the system 1.1-1.4 satisfies

$$\text{div } \mathbf{E}_\varepsilon = 0 \text{ on } [0, \infty) \times \Omega_0 \text{ and } \ \text{div } \mathbf{h}_\varepsilon = 0 \text{ on } [0, \infty) \times \Omega.$$

$$\text{and } \int_{\mathcal{C}_k} \vec{n}\mathbf{E}_\varepsilon(t, x)dS = Q_k \text{ for all } t \in [0, \infty), k \in \{1, .., N\}. \tag{1.7}$$

Here $\mathcal{C}_k$ ,$k = 1, .., N$ denote the connected components of $I\!R^3 \setminus \overline{\Omega_0}$. The physical meaning of 1.7 is that the total electric charge $Q_k$ on each $\mathcal{C}_k$ is invariant under the time evolution of $\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon$ governed by 1.1-1.4. This is a consequence of the assumptions $\sigma(t, x, \mathbf{y}) = 0$ for $\mathbf{y} \in \Omega_0$ and div $\mathbf{j}_0 = 0$ on $I\!R^3$. In many situations the displacement current $\varepsilon\partial_t\mathbf{E}_\varepsilon$ is small in comparison with the charge currents and is often neglected. Therefore, it is the aim of this paper to investigate the singular limit $\varepsilon \to 0$. By letting $\varepsilon \to 0$ one obtains fomally from 1.1-1.4 and 1.7 the quasi-stationary Maxwell-System

$$\text{curl } \mathbf{h} = \sigma(t, x, \mathbf{E}) + \mathbf{j}_0 \tag{1.8}$$

$$\text{curl } \mathbf{E} = -\mu\partial_t\mathbf{h} \tag{1.9}$$

$$\text{div } \mathbf{E} = 0 \text{ on } [0, \infty) \times \Omega_0 \tag{1.10}$$

supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \mathbf{E} = 0 \text{ on } [0, \infty) \times \partial\Omega, \tag{1.11}$$

$$\int_{\mathcal{C}_k} \vec{n}\mathbf{E}(t, x)dS = Q_k \text{ for all } t \in [0, \infty), k \in \{1, .., N\}. \tag{1.12}$$

$$\text{and } \mathbf{h}(0) = \mathbf{h}_0 \text{ on } \Omega \tag{1.13}$$

This problem has been investigated in [8] in the case that the domain $\Omega$ is bounded and $\sigma$ is linear with respect to $\mathbf{E}$ and uniformly positive. See also [3] and [4], where a temperature dependent electrical conductivity is considered. However, in this paper $\Omega$ is an exterior domain and the electrical conductivity vanishes on a subset $\Omega_0 = \Omega \setminus \overline{G}$. It is assumed that

$$\mathbf{y}\sigma(t,x,\mathbf{y}) \geq \gamma(x)|\mathbf{y}|^2 \text{ for all } x \in G, \mathbf{y} \in I\!R^3 \tag{1.14}$$

with some $\gamma \in L^\infty(G)$, $\gamma > 0$. $\sigma$ may have mild degeneracies on $G$, i.e. it is not assumed that $\gamma$ is uniformly positive. More precisely,

$$\gamma^{-1} \in L^{r_0/(2-r_0)}(G) \text{ with some } 6/5 < r_0 \leq 2. \tag{1.15}$$

Note that 1.8-1.13 can be considered as a problem of mixed type. In the case that $\mu$ is constant on $\Omega$ and $\sigma$ is linear and independent of $x \in G$ it is elliptic on $\Omega_0$ (with respect to the space variable $x$ for fixed time) and parabolic on $(0,\infty) \times G$. It is shown in this paper that 1.8-1.13 admits for given $\mathbf{h}_0$, $\mathbf{j}_0$ and $Q_k$ a unique global weak solution $(\mathbf{E},\mathbf{h})$ with

$$\mathbf{E} \in W^{-1}_{loc}([0,\infty), L^{r_0}(\Omega \cap B_{R_0})) \cap W^{-1}_{loc}([0,\infty), L^6(I\!R^3 \setminus B_{R_0})) \cap L^2_{loc}([0,\infty), L^2_\gamma(G))$$
$$\subset \mathcal{D}'((0,\infty) \times \Omega) \text{ and } \mathbf{h} \in C([0,\infty), L^2(\Omega)),$$

where $\gamma, r_0 \in (6/5, 2]$ as in 1.14, 1.15 and $L^2_\gamma(G)$ is the weighted $L^2$- space and $R_0 > 0$, such that $G \cup (I\!R^3 \setminus \Omega) \subset B_{R_0} = \{|x| < R_0\}$. Moreover, it is shown that

$$\mathbf{h}_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \mathbf{h} \text{ in } L^\infty((0,T), L^2(\Omega)) \text{ weak } -* \text{ for all } T \in (0,\infty)$$
$$\text{and } \mathbf{E}_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \mathbf{E} \text{ in } \mathcal{D}'\left((0,\infty) \times \Omega\right).$$

In the case that $\sigma$ is independent of t and $(\mathbf{E}_0, \mathbf{h}_0)$, $\mathbf{j}_0$ are sufficiently regular it is shown that 1.8-1.13 admits global strong solution $(\mathbf{E},\mathbf{h})$ with
$\mathbf{E} \in L^2_{loc}([0,\infty), L^{r_0}(\Omega \cap B_{R_0})) \cap L^2_{loc}([0,\infty), L^6(I\!R^3 \setminus B_{R_0})) \cap L^2_{loc}([0,\infty), L^2_\gamma(G))$
and $\mathbf{h} \in W^{1,\infty}_{loc}([0,\infty), L^2(\Omega))$.
Moreover, one has in this case

$$\mathbf{h}_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \mathbf{h} \text{ in } W^{1,\infty}((0,T), L^2(\Omega)) \text{ weak } -*$$
$$\text{curl } \mathbf{h}_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \text{ curl } \mathbf{h} \text{ in } L^2((0,T), L^2(\Omega)) \text{ weakly}$$
$$\text{and } \mathbf{E}_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \mathbf{E} \text{ in } L^2((0,T), L^{r_0}(\Omega \cap B_{R_0}) \cap L^6(I\!R^3 \setminus B_{R_0})) \text{ weakly}.$$

for all $T \in (0,\infty)$. In the case that also $\partial\Omega$ is sufficiently regular this implies also

$$\mathbf{h}_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \mathbf{h} \text{ in } C([0,T], L^2(\Omega \cap B_R)) \text{ strongly}$$

for all $R \in (0,\infty)$ and $T \in (0,\infty)$.
Finally, the asymptotic behavior of the solution to 1.8-1.13 for $t \to \infty$ is investigated in the case $\mathbf{j}_0 = 0$ .
It is shown that $||\mathbf{h}(t)||_{L^2}$ decays exponentially provided that

$$\int_\Omega \mu \mathbf{h}_0 \mathbf{g} dx = 0 \text{ for all } \mathbf{g} \in L^2(\Omega) \text{ with curl } \mathbf{g} = 0.$$

This condition includes div $(\mu\mathbf{h}_0) = 0$ on $\Omega$ and $\vec{n}\mathbf{h} = 0$ on $\partial\Omega$ weakly.

## 2 Notation, definitions and assumptions

For an arbitrary open set $K \subset I\!R^3$ the space of all infinetely differentiable functions with compact support contained in $K$ is denoted by $C_0^\infty(K)$. Moreover, $\mathcal{D}'(K)$ is the space of distributions on $K$.
For $p \in [1,\infty)$ we define $H^p_{curl}(K)$ as the space of all $\mathbf{E} \in L^p(K)$ with $\operatorname{curl}\mathbf{E} \in L^p(K)$. $\overset{0}{H^p}_{curl}(K)$ denotes the set of all $\mathbf{E} \in H^p_{curl}(K)$, such that

$$\int_K \mathbf{E} \operatorname{curl} \mathbf{h} - \mathbf{h} \operatorname{curl} \mathbf{E} dx = 0 \text{ for all } \mathbf{h} \in H^{p^*}_{curl}(K),$$

which includes a weak formulation of the boundary-condition $\vec{n} \wedge \mathbf{E} = 0$ on $\partial K$. Here $\frac{1}{p} + \frac{1}{p^*} = 1$. Let $W^{1,p}(K)$ is the usual first order Sobolev space consisting of all functions in $L^p(K)$ whose distributional gradient belongs to $L^p(K)$. $\overset{0}{W}{}^{1,p}(K)$ denotes the closure of $C_0^\infty(K)$ in $W^{1,p}(K)$.
In the sequel $\Omega \subset I\!R^3$ is an open set with bounded complement, $G \subset \Omega$ and $\Omega_0 \overset{\text{def}}{=} \Omega \setminus \overline{G}$. It is assumed that $\mathcal{C} \overset{\text{def}}{=} I\!R^3 \setminus \overline{\Omega_0} = \mathcal{C}_1 \cup .. \cup \mathcal{C}_N$ is a bounded Lipschitz-domain, where $\mathcal{C}_k$ are its connected components with $\overline{\mathcal{C}}_j \cap \overline{\mathcal{C}}_k = \emptyset$ if $j \neq k$.
$\mu \in L^\infty(I\!R^3)$ is an uniformly positive function.
The assumptions on $\sigma : [0,\infty) \times \Omega \times I\!R^3 \to I\!R^3$ are the following.

$$\sigma(t,x,\mathbf{y}) = 0 \text{ if } x \in \Omega_0 = \Omega \setminus \overline{G}, \tag{2.16}$$

$$\sigma(\cdot,\cdot,\mathbf{y}) \in L^\infty_{loc}([0,\infty), L^\infty(\Omega)) \text{ for fixed } \mathbf{y} \in I\!R^3, \tag{2.17}$$

and Lipschitz-continuous, i.e. there exists $L \in (0,\infty)$, such that

$$|\sigma(t,x,\mathbf{y}) - \sigma(t,x,\tilde{\mathbf{y}})| \leq L|\mathbf{y} - \tilde{\mathbf{y}}| \text{ for all } \mathbf{y},\tilde{\mathbf{y}} \in I\!R^3, t \in [0,\infty) \text{ and } x \in \Omega. \tag{2.18}$$

Moreover,

$$|\sigma(t,x,\mathbf{y})| \leq C_0\gamma(x)|\mathbf{y}| \text{ and } \mathbf{y}\sigma(t,x,\mathbf{y}) \geq \gamma(x)|\mathbf{y}|^2 \tag{2.19}$$

for all $\mathbf{y} \in I\!R^3$, $t \in [0,\infty)$ and $x \in G$ with some $\gamma \in L^\infty(G)$, $\gamma > 0$ and $C_0 \in (0,\infty)$.
Finally, $\sigma$ is assumed to be monotone, i.e.

$$(\mathbf{y}-\tilde{\mathbf{y}})(\sigma(t,x,\mathbf{y}) - \sigma(t,x,\tilde{\mathbf{y}})) > 0 \text{ for all } \mathbf{y} \neq \tilde{\mathbf{y}} \in I\!R^3, t \in [0,\infty) \text{ and } x \in G. \tag{2.20}$$

Next, let $p_0 \in (2,6)$, such that for all $p \in [2,p_0]$ the following holds.

$$\varphi \in \overset{0}{W}{}^{1,2}(\Omega_0 \cap B_{2R_0}) \text{ and } \Delta\varphi \in W^{-1,p}(\Omega_0 \cap B_{2R_0}) \overset{\text{def}}{=} \left(\overset{0}{W}{}^{1,p^*}(\Omega_0 \cap B_{2R_0})\right)^* \tag{2.21}$$

$$\Longrightarrow \nabla\varphi \in L^p(\Omega_0 \cap B_{2R_0}).$$

Since $\Omega_0 \cap B_{2R_0} = B_{2R_0} \setminus \overline{\mathcal{C}}$ is a Lipschitz-domain, it follows from the result in [2] that there exist such a $p_0 > 2$. The following compatibility conditions will be imposed on $p_0$ and $\gamma$.

$$\gamma^{-1} \in L^{r_0/(2-r_0)}(G) \text{ with } r_0 \stackrel{\text{def}}{=} p_0^* \tag{2.22}$$

Let $L^2_\gamma(G)$ be the weighted $L^2$-space consisting of all functions $\mathbf{f} : G \to \mathbb{C}^3$ with

$$||\mathbf{f}||^2_{L^2_\gamma(G)} \stackrel{\text{def}}{=} \int_G |\mathbf{f}|^2 \gamma dx < \infty.$$

It follows from 2.19, 2.22 and Hölder's inequality that

$$L^2_\gamma(G) \subset L^{r_0}(G) \tag{2.23}$$

and for $\mathbf{F} \in L^2_\gamma(G)$ one has

$$\sigma(t, x, \mathbf{F}) \in L^2_{\gamma^{-1}}(G) \subset L^2(G) \tag{2.24}$$

$$\text{with } ||\sigma(t, x, \mathbf{F})||_{L^2_{\gamma^{-1}}(G)} = ||\sigma(t, x, \mathbf{F})\gamma^{-1/2}||_{L^2(G)} \le K||\mathbf{F}||_{L^2_\gamma(G)}$$

with some $K \in (0, \infty)$ independent of $t, \mathbf{f}$ .
For $\mathbf{E} \in L^1_{loc}(\Omega)$ with $\operatorname{div} \mathbf{E} = 0$ on $\Omega_0$ define

$$q_j(\mathbf{E}) \stackrel{\text{def}}{=} \int_\Omega \mathbf{E}\nabla\chi_j dx = \int_{\Omega_0} \mathbf{E}\nabla\chi_j dx, \tag{2.25}$$

where $\chi_j \in C_0^\infty(\mathbb{R}^3)$ obey $\chi_j|_{\mathcal{C}_k} = \delta_{j,k}$. Due to the condition $\operatorname{div} \mathbf{E} = 0$ this defintion is independent of the choice of the functions $\chi_j$.
Now, the following function spaces are defined.
Let for $q \in [1, 2]$ $Y^q_0$ be the space of all measurable functions $\mathbf{E} : \Omega \to \mathbb{C}^3$,
such that $\mathbf{E} \in L^q(\Omega \cap B_{R_0})$ and $\mathbf{E} \in L^6(\mathbb{R}^3 \setminus B_{R_0})$. Here $B_{R_0} \stackrel{\text{def}}{=} \{|x| < R_0\}$ and $R_0 > 0$, such that $G \subset B_{R_0/2}$ and $\mathbb{R}^3 \setminus \Omega \subset B_{R_0/2}$.
$Y^q_1$ is defined as the space of all $\mathbf{E} \in Y^q_0$ with the property that $\operatorname{div} \mathbf{E} = 0$ on $\Omega_0$,
$\operatorname{curl} \mathbf{E} \in L^2(\Omega)$ and $\varphi\mathbf{E} \in \overset{0}{H^q}_{curl}(\Omega)$ for all $\varphi \in C_0^\infty(\mathbb{R}^3)$.
The following norms are introduced.

$$||\mathbf{E}||_{Y^q_0} \stackrel{\text{def}}{=} ||\mathbf{E}||_{L^6(\mathbb{R}^3 \setminus B_{R_0})} + ||\mathbf{E}||_{L^q(\Omega \cap B_{R_0})},$$

$$|\mathbf{E}|_{Y^q_1} \stackrel{\text{def}}{=} ||\operatorname{curl} \mathbf{E}||_{L^2(\Omega)} + ||\mathbf{E}||_{L^q(G)} + \sum_{k=1}^N |q_k(\mathbf{E})|$$

$$\text{and } ||\mathbf{E}||_{Y^q_1} \stackrel{\text{def}}{=} |\mathbf{E}|_{Y^q_1} + ||\mathbf{E}||_{Y^q_0}.$$

Next, let $\tilde{\mathcal{X}}$ be the space of all $\mathbf{h} \in H^2_{curl}(\Omega)$ with $\text{curl}\ \mathbf{h} \in L^{r_0^*}(\Omega_0) \cap L^2_{\gamma^{-1}}(G)$, such that the support of $\text{curl}\ \mathbf{h}$ is bounded.
Moreover, let $\mathcal{X}$ be the space of all $\mathbf{E} \in Y_1^{r_0} \cap L^2_\gamma(G)$, such that

$$\int_\Omega \mathbf{E}\ \text{curl}\ \mathbf{h} - \mathbf{h}\ \text{curl}\ \mathbf{E} dx = 0 \text{ for all } \mathbf{h} \in \tilde{\mathcal{X}}. \tag{2.26}$$

Let $\mathcal{W}$ be the space of all $T \in \mathcal{D}'\left((0,\infty) \times I\!R^3\right) \cap L^2_{loc}([0,\infty), L^2_\gamma(G))$,
such that $T = \partial_t \mathbf{A}$ with some $\mathbf{A} \in L^\infty_{loc}([0,\infty), \mathcal{X}) \cap W^{1,2}_{loc}([0,\infty), L^2_\gamma(G))$.
For $\mathbf{A} \in L^\infty_{loc}([0,\infty), \mathcal{X}) \cap W^{1,2}_{loc}([0,\infty), L^2_\gamma(G))$ and $T \stackrel{\text{def}}{=} \partial_t \mathbf{A} \in \mathcal{W}$ we define
$\tilde{q}_k(T) \in \mathcal{D}'\left((0,\infty)\right)$ by $\tilde{q}_k(T) \stackrel{\text{def}}{=} \frac{d}{dt} q_k\left(\mathbf{A}(t)\right)$.
Next, the notion of weak solutions to 1.8 - 1.13 is given precisely.

**Definition 1** *Let $\mathbf{j}_0 \in L^2_{loc}([0,\infty), L^2(\Omega))$, $\mathbf{h}_0 \in L^2(\Omega)$ and $Q_k \in I\!R$.*
*Then $(\mathbf{E}, \mathbf{h}) \in \mathcal{W} \times C([0,\infty), L^2(\Omega))$ is called a solution to 1.8 - 1.13, if*

$$\textit{curl}\ \mathbf{h} = \sigma(t,x,\mathbf{E}) + \mathbf{j}_0, \qquad \textit{curl}\ \mathbf{E} = -\mu\partial_t \mathbf{h}, \tag{2.27}$$

$$\tilde{q}_k(\mathbf{E}) = Q_k \textit{ for all } k \in \{1,..,N\}, \tag{2.28}$$

$$\textit{and } \mathbf{h}(0) = \mathbf{h}_0. \tag{2.29}$$

Note that by the definition of $\mathcal{W}$ it follows $\mathbf{E}|_{(0,\infty)\times G)} \cap L^2_{loc}([0,\infty), L^2_\gamma(G))$.
Therefore $\sigma(t,x,\mathbf{E}) \in L^2_{loc}([0,\infty), L^2_{\gamma^{-1}}(G))$.

# 3 Some auxiliary lemmata

The following lemma will be used frequently.

**Lemma 1** *Let $q \in (1,3)$ and $\mathbf{E} \in \mathcal{D}'(I\!R^3)$, such that*

$$\lim_{|x|\to\infty} (T * \varphi)(x) = 0 \textit{ for all } \varphi \in C_0^\infty(I\!R^3),$$

*where $*$ denotes the convolution.*
*Moreover suppose that $\textit{curl}\ \mathbf{E} \in L^q(I\!R^3)$ and $\textit{div}\ \mathbf{E} \in L^q(I\!R^3)$. Then $\mathbf{E} \in L^p(I\!R^3)$ with $p = \frac{3q}{3-q}$ and*

$$||\mathbf{E}||_{L^p} \leq C_q(||\ \textit{curl}\ \mathbf{E}||_{L^q(I\!R^3)} + ||\ \textit{div}\ \mathbf{E}||_{L^q(I\!R^3)})$$

*with some konstant $C_q \in (0,\infty)$ independent of $\mathbf{E}$.*

**Proof:**
Let $S(x) \stackrel{\text{def}}{=} \frac{1}{4\pi|x|}$ be the fundamental solution of $-\Delta$ on $I\!R^3$. Then

$$\nabla S = \frac{-x}{4\pi|x|^3} \in L_w^{3/2}(I\!R^3) \tag{3.30}$$

in the sense of distributions, where $L_w^p(I\!R^3)$ denotes the weak $L^p$-space or Marcinciewicz space, see [10].
It follows from 3.30, $\operatorname{div} \mathbf{E} \in L^q(I\!R^3)$ and the generalized Young inequality [10] that

$$\mathbf{F}_1 \stackrel{\text{def}}{=} (\nabla S) * \operatorname{div} \mathbf{E} \in L^p(I\!R^3) \tag{3.31}$$

For all $\mathbf{g} \in C_0^\infty(I\!R^3, C\!\!\!\!I^3)$ one has

$$< \Delta \mathbf{F}_1, \mathbf{g} >_{\mathcal{D}'} = \int_{I\!R^3} (\operatorname{div} \mathbf{E})(y) \int_{I\!R^3} (\nabla S)(x-y)(\Delta \mathbf{g})(x) dx dy$$

$$= - \int_{I\!R^3} (\operatorname{div} \mathbf{E})(y) \int_{I\!R^3} S(x-y)(\Delta \operatorname{div} \mathbf{g})(x) dx dy$$

$$= \int_{I\!R^3} (\operatorname{div} \mathbf{E})(y)(\operatorname{div} \mathbf{g})(y) dy = - < \nabla \operatorname{div} \mathbf{E}, \mathbf{g} >_{\mathcal{D}'},$$

i.e.

$$\Delta \mathbf{F}_1 = -\nabla \operatorname{div} \mathbf{E} \text{ in } \mathcal{D}'(I\!R^3). \tag{3.32}$$

Let

$$\mathbf{F}_2(x) \stackrel{\text{def}}{=} \int_{I\!R^3} (\operatorname{curl} \mathbf{E})(y) \wedge (\nabla S)(x-y) dy.$$

It follows from similar conclusions as before that $\mathbf{F}_2 \in L^p(I\!R^3)$ and

$$\Delta \mathbf{F}_2 = \operatorname{curl} \operatorname{curl} \mathbf{E} \text{ in } \mathcal{D}'(I\!R^3). \tag{3.33}$$

3.32 and 3.33 yield

$$\Delta(\mathbf{E} + \mathbf{F}_1 + \mathbf{F}_2) = 0 \text{ in } \mathcal{D}'(I\!R^3). \tag{3.34}$$

Now, let $\varphi \in C_0^\infty(I\!R^3)$. It follows from 3.34 that $(\mathbf{E} + \mathbf{F}_1 + \mathbf{F}_2) * \varphi$ is harmonic on $I\!R^3$. By the assumption on $\mathbf{E}$ it follows $\lim_{|x|\to\infty} (\mathbf{E} + \mathbf{F}_1 + \mathbf{F}_2) * \varphi = 0$, since $\mathbf{F}_k \in L^p(I\!R^3)$. By the maximum-principle (applied to each component) this implies $(\mathbf{E} + \mathbf{F}_1 + \mathbf{F}_2) * \varphi = 0$ for all $\varphi \in C_0^\infty(I\!R^3)$ and hence $-\mathbf{E} = \mathbf{F}_1 + \mathbf{F}_2 \in L^p(I\!R^3)$.
The norm-estimate follows from 3.30, the generalized Young- inequality and the definitions of $\mathbf{F}_1, \mathbf{F}_2 \in L^p(I\!R^3)$.

$\square$

For our purposes it is essential that only the $L^q$-norms of $\operatorname{curl} \mathbf{E}$ and $\operatorname{div} \mathbf{E}$ occur and not any norm of $\mathbf{E}$ itself.
Let $Z \stackrel{\text{def}}{=} \{\varphi \in L^6 | \nabla\varphi \in L^2(I\!R^3)\}$. It is a Hilbert-space endowed with the scalar-product $< \varphi, \psi >_Z \stackrel{\text{def}}{=} \int_{I\!R^3} \nabla\varphi\nabla\overline{\psi}dx$
Let $Z_0$ be the closed subspace consisting of all $\varphi \in Z$ with $\nabla\varphi = 0$ on $\mathcal{C}$, i.e. which are constant on each component $\mathcal{C}_k$.

**Lemma 2** *Let $p \in [2, p_0]$, $\mathbf{F} \in L^p(\Omega_0) \cap L^2(\Omega_0)$ and $\psi_0 \in Z_0$ with*

$$\int_{\Omega_0} \nabla\psi_0\nabla\psi dx = \int_{\Omega_0} \mathbf{F}\nabla\psi dx \text{ for all } \psi \in Z_0.$$

*Then $\psi_0 \in W^{1,p}(B_{2R_0})$ and*

$$||\nabla\psi_0||_{L^p(B_{2R_0})} + \sum_{k=1}^{N} |\psi_0|_{\mathcal{C}_k}| \leq K_3(||\mathbf{F}||_{L^p(\Omega_0)} + ||\mathbf{F}||_{L^2(\Omega_0)})$$

*with some $K_3 \in (0, \infty)$ independent of $\mathbf{F}$.*

**Proof:**
Let $\alpha_k \stackrel{\text{def}}{=} \psi_0|_{\mathcal{C}_k}$. Then the trace theorem yields

$$\sum_{k=1}^{N} |\alpha_k| \leq c_1||\psi_0||_Z \leq c_1||\mathbf{F}||_{L^2} \tag{3.35}$$

Let $\varphi_0 \stackrel{\text{def}}{=} \eta \cdot (\psi_0 - \sum_{k=1}^{N} \alpha_k\chi_k) \in \stackrel{0}{H^1} (\Omega_0 \cap B_{3R_0})$,
where $\chi_k$ as in the definition of $q_k$ and $\eta \in C_0^\infty(B_{3R_0})$ with $\eta(x) = 1$ on $B_{2R_0}$.
Now, for all $\varphi \in \stackrel{0}{H^1} (\Omega_0 \cap B_{3R_0})$ one has by 3.35 and the embedding $W^{1,6/5}(\Omega_0 \cap B_{3R_0}) \hookrightarrow L^2(\Omega_0 \cap B_{3R_0})$

$$\begin{aligned}
|\int_{\Omega_0\cap B_{3R_0}} \nabla\varphi_0\nabla\varphi dx| &\leq |\int_{\Omega_0\cap B_{3R_0}} \nabla(\eta\psi_0)\nabla\varphi dx| \\
&\quad + \sum_{k=1}^{N} |\alpha_k| \quad |\int_{\Omega_0\cap B_{3R_0}} \nabla(\eta\chi_k)\nabla\varphi dx| \\
&\leq |\int_{\Omega_0\cap B_{3R_0}} \nabla\psi_0\nabla(\eta\varphi)dx| + c_1 \int_{\Omega_0\cap B_{3R_0}} (|\psi_0| + \sum_{k=1}^{N} |\alpha_k|)|\nabla\varphi| + |\nabla\psi_0||\varphi|dx \\
&\leq |\int_{\Omega_0\cap B_{3R_0}} \mathbf{F}\nabla(\eta\varphi)dx| + c_2(||\psi_0||_{L^6} + ||\nabla\psi_0||_{L^2} + \sum_{k=1}^{N} |\alpha_k|)||\varphi||_{W^{1,6/5}} \\
&\leq c_3||\mathbf{F}||_{L^p}||\eta\varphi||_{W^{1,p^*}} + c_3||\psi_0||_Z||\varphi||_{W^{1,6/5}}
\end{aligned}$$

$$\leq c_4||\mathbf{F}||_{L^p}||\varphi||_{W^{1,p^*}} + c_3||\mathbf{F}||_{L^2}||\varphi||_{W^{1,6/5}}$$

$$\leq c_5(||\mathbf{F}||_{L^p} + ||\mathbf{F}||_{L^2})||\varphi||_{W^{1,p^*}},$$

i.e. $\Delta\varphi_0 \in W^{-1,p}(\Omega_0 \cap B_{3R_0})$. Since $p \leq p_0$, it follows $\varphi_0 \in W^{1,p}(\Omega_0 \cap B_{3R_0})$ by 2.21.

□

In the sequel let $\mathcal{U} \subset L^2(\Omega_0)$ be the space of all $\mathbf{E} \in L^2(\Omega_0)$ with $\int_{\Omega_0} \mathbf{E}\nabla\psi dx = 0$ for all $\psi \in Z_0$. This is the weak formulation of $\operatorname{div} \mathbf{E} = 0$ on $\Omega_0$ and $\int_{\partial\mathcal{C}_k} \vec{n}\mathbf{E}dS = 0$ for all $k = 1,..,N$.

**Lemma 3** *i) Let* $\mathbf{E} \in L^2(I\!R^3)$ *with* $\operatorname{div} \mathbf{E} = 0$ *on* $I\!R^3$. *Then there exist a unique* $\mathbf{h} \in L^6(I\!R^3)$ *with* $\operatorname{curl} \mathbf{h} = \mathbf{E}$ *and* $\operatorname{div} \mathbf{h} = 0$ *on* $I\!R^3$ *in the sense of distributions.*
*ii) There exists a bounded operator* $\mathcal{T} : \mathcal{U} \to L^6(I\!R^3, \mathbb{C}^3)$ *such that* $\operatorname{curl}(\mathcal{T}\mathbf{E}) = \mathbf{E}$ *on* $\Omega_0$ *for all* $\mathbf{E} \in \mathcal{U}$.

**Proof:**
i) Let $H$ be the space of all $\mathbf{h} \in L^6(I\!R^3)$ with $\operatorname{div} \mathbf{h} = 0$ on $I\!R^3$ and $\operatorname{curl} \mathbf{h} \in L^2(I\!R^3)$. It follows easily from lemma 1 that this space is a Hilbert-space endowed with the scalar-product
$< \mathbf{g}, \mathbf{h} >_H \stackrel{\text{def}}{=} \int_{I\!R^3} \operatorname{curl} \mathbf{g} \operatorname{curl} \overline{\mathbf{h}} dx$.
Now, let $\mathbf{E} \in L^2(I\!R^3)$ with $\operatorname{div} \mathbf{E} = 0$ on $I\!R^3$. Then there exists a unique $\mathbf{h} \in H$ with

$$\int_{I\!R^3} \operatorname{curl} \mathbf{h} \operatorname{curl} \mathbf{g} dx =< \mathbf{h}, \overline{\mathbf{g}} >_H = \int_{I\!R^3} \mathbf{E} \operatorname{curl} \mathbf{g} dx \text{ for all } \mathbf{g} \in H.$$

Then $\mathbf{F} \stackrel{\text{def}}{=} \mathbf{E} - \operatorname{curl} \mathbf{h} \in L^2$ satisfies $\operatorname{curl} \mathbf{F} = 0$ and $\operatorname{div} \mathbf{F} = 0$. Hence, lemma 1 yields $\mathbf{F} = 0$, i.e. $\mathbf{E} = \operatorname{curl} \mathbf{h}$. Uniqueness of $\mathbf{h} \in L^6$ follows again from lemma 1.
Proof of ii):
The idea is to extend $\mathbf{E}$ to a divergence-free field on $I\!R^3$ and to apply the first part of the lemma. Let $E : H^1(\mathcal{C}) \to H^1(I\!R^3)$ be an extension operator, see [12]. Since the $\mathcal{C}_k$ are connected, a norm equivalent to the $H^1$-norm can be defined by $||f||^2 \stackrel{\text{def}}{=} \int_{\mathcal{C}} |\nabla f|^2 dx$ on the space $X \stackrel{\text{def}}{=} \{f \in H^1(\mathcal{C}) : \int_{\mathcal{C}_k} f dx = 0\}$.
Let $\mathbf{E} \in \mathcal{U}$. Then there exists a unique $f_0 \in X$ with

$$\int_{\mathcal{C}} \nabla f_0 \nabla f dx =< f_0, \overline{f} >_X = \int_{\Omega_0} \mathbf{E}\nabla(Ef)dx \text{ for all } f \in X. \tag{3.36}$$

Next, define $F\mathbf{E} \in L^2(I\!R^3, \mathbb{C}^3)$ by

$$(F\mathbf{E})(x) \stackrel{\text{def}}{=} \mathbf{E}(x) \text{ if } x \in \Omega_0, \text{ and } (F\mathbf{E})(x) \stackrel{\text{def}}{=} -\nabla f_0(x) \text{ if } x \in \overline{\mathcal{C}} = I\!R^3 \setminus \Omega_0 \tag{3.37}$$

Now, assume $\varphi \in C_0^\infty(I\!R^3)$ and set $\alpha_k \stackrel{\text{def}}{=} |\mathcal{C}_k|^{-1} \int_{\mathcal{C}_k} \varphi dx$, where $|\mathcal{C}_k|$ denotes the Lebesgue measure of $\mathcal{C}_k$.

Let $\psi \in X \subset H^1(\mathcal{C})$ be defined by $\psi(x) \stackrel{\text{def}}{=} \varphi(x) - \alpha_k$ for $x \in \mathcal{C}_k$.
Moreover, define $\tilde{\psi} \in Z_0$ by $\tilde{\psi}(x) \stackrel{\text{def}}{=} \varphi(x) - (E\psi)(x)$ if $x \in \Omega_0$ and $\tilde{\psi}(x) \stackrel{\text{def}}{=} \alpha_k$ for $x \in \mathcal{C}_k$ .
Then 3.36 and 3.37 yield

$$\int_{I\!R^3} (F\mathbf{E})\nabla\varphi dx = \int_{\Omega_0} \mathbf{E}\nabla\varphi dx - \int_{\mathcal{C}} \nabla f_0 \nabla\psi dx$$

$$= \int_{\Omega_0} \mathbf{E}\nabla[\varphi - E\psi]dx = \int_{\Omega_0} \mathbf{E}\nabla\tilde{\psi}dx = 0,$$

since $\mathbf{E} \in \mathcal{U}$. Therefore,

$$\int_{I\!R^3} (F\mathbf{E})\nabla\varphi dx = 0 \text{ for all } \varphi \in C_0^\infty(I\!R^3),$$

i.e. div $(F\mathbf{E}) = 0$ on $I\!R^3$ and by i) there exist a unique $\mathbf{h} \in L^6(I\!R^3)$ with curl $\mathbf{h} = F\mathbf{E}$ and div $\mathbf{h} = 0$ on $I\!R^3$ .
Finally, define $\mathcal{T}\mathbf{E} \stackrel{\text{def}}{=} \mathbf{h}$. Then curl $(\mathcal{T}\mathbf{E}) = F\mathbf{E}$ on $I\!R^3$ and in particular curl $(\mathcal{T}\mathbf{E}) = \mathbf{E}$ on $\Omega_0$.

□

Now,the main estimate can be proved.

**Theorem 1** *For $r \in [r_0, 2]$, i. e. $r^* \le p_0$ there exists a constant $K_r \in I\!R^+$, such that for all $\mathbf{E} \in Y_1^r$ the estimate*

$$||\mathbf{E}||_{Y_0^r} = ||\mathbf{E}||_{L^6(I\!R^3 \setminus B_{R_0})} + ||\mathbf{E}||_{L^r(\Omega \cap B_{R_0})},$$

$$\le K_r |\mathbf{E}|_{Y_1^r} = K_r(||\ curl\ \mathbf{E}||_{L^2(\Omega)} + ||\mathbf{E}||_{L^r(G)} + \sum_{k=1}^{N} |q_k(\mathbf{E})|)$$

*holds. In particular $|\cdot|_{Y_1^r}$ and $||\cdot||_{Y_1^r}$ are equivalent norms on $Y_1^r$.*

**Proof:**
Suppose $\mathbf{E} \in Y_1^r$. Let $\tilde{\mathbf{E}} \in L^r(I\!R^3) + L^6(I\!R^3)$ be defined by $\tilde{\mathbf{E}}(x) \stackrel{\text{def}}{=} \mathbf{E}(x)$ if $x \in \Omega$ and $\tilde{\mathbf{E}}(x) \stackrel{\text{def}}{=} 0$ if $x \in I\!R^3 \setminus \Omega$. Since curl $\mathbf{E} \in L^2(\Omega)$ and $\varphi\mathbf{E} \in \overset{0}{H}{}^r{}_{curl}\ (\Omega)$ for all $\varphi \in C_0^\infty(I\!R^3)$, one has curl $\tilde{\mathbf{E}} \in L^2(I\!R^3)$. By lemma 3 there exists a unique $\mathbf{E}_1 \in L^6(I\!R^3)$ with

$$\text{curl } \mathbf{E}_1 = \text{ curl } \tilde{\mathbf{E}},\ \text{div } \mathbf{E}_1 = 0 \text{ on } I\!R^3 \tag{3.38}$$

and $||\mathbf{E}_1||_{L^6} \le K_1||$ curl $\mathbf{E}||_{L^2} \le K_1|\mathbf{E}|_{Y_1^r}$.
Now, $\mathbf{E}_2 \stackrel{\text{def}}{=} \tilde{\mathbf{E}} - \mathbf{E}_1 \in L^r(I\!R^3) + L^6(I\!R^3)$ satisfies

$$\text{curl } \mathbf{E}_2 = 0 \text{ on } I\!R^3 \text{ and } \text{div } \mathbf{E}_2 = 0 \text{ on } \Omega_0. \tag{3.39}$$

Let $\chi \in C^\infty(I\!R^3)$ with

$$\chi(x) = 0 \text{ on } B_{R_0} \text{ and } \chi(x) = 1 \text{ outside } B_{2R_0}. \tag{3.40}$$

Since supp $\chi \subset \Omega_0$ and $r \geq 6/5$, 3.39 yields div $(\chi \mathbf{E}_2) = \mathbf{E}_2 \nabla \chi \in L^{6/5}(I\!R^3)$ and curl $(\chi \mathbf{E}_2) = -\mathbf{E}_2 \wedge \nabla \chi \in L^{6/5}(I\!R^3)$. Thus, lemma 1 yield $\chi \mathbf{E}_2 \in L^2(I\!R^3)$ and therefore

$$\mathbf{E}_2 \in L^2(I\!R^3 \setminus B_{R_0}). \tag{3.41}$$

Next, define $\mathbf{F} \in L^{r^*}(\Omega_0) \cap L^2(\Omega_0)$ by $\mathbf{F}(x) \stackrel{\text{def}}{=} |\mathbf{E}_2(x)|^{r-2}\mathbf{E}_2(x)$ if $x \in \Omega_0 \cap B_{2R_0}$ and $\mathbf{F}(x) \stackrel{\text{def}}{=} 0$ if $x \in I\!R^3 \setminus B_{2R_0}$. Then there exists some $\psi_0 \in Z_0$, such that

$$\int_{\Omega_0} \nabla \psi_0 \nabla \psi dx =< \psi_0, \overline{\psi} >_{Z_0}= \int_{\Omega_0} \mathbf{F} \nabla \psi dx \tag{3.42}$$

for all $\psi \in Z_0$, since $\mathbf{F} \in L^2(\Omega_0)$.
Recall that $Z_0$ be the space of all $\varphi \in L^6$ with $\nabla \varphi \in L^2$ and $\nabla \varphi = 0$ on $\mathcal{C}$, i.e. $\varphi$ is constant on each component $\mathcal{C}_k$. It follows from lemma 2 that $\nabla \psi_0 \in W^{1,r^*}(B_{2R_0})$ with

$$||\nabla \psi_0||_{L^{r^*}(B_{2R_0})} + \sum_{k=1}^{N} |\psi_0|_{\mathcal{C}_k}| \leq c_3 ||\mathbf{F}||_{L^{r^*}(\Omega_0)} = c_3 ||\mathbf{E}_2||^{r-1}_{L^r(\Omega_0 \cap B_{2R_0})}. \tag{3.43}$$

Let $\varphi(x) \stackrel{\text{def}}{=} \psi_0(x) - \sum_{k=1}^{N} \alpha_k \chi_k(x)$, where $\alpha_k \stackrel{\text{def}}{=} \psi_0|_{\mathcal{C}_k}$ and $\chi_k$ as in the definition of $q_k$. Moreover, let $h_n(x) \stackrel{\text{def}}{=} h(x/n), x \in I\!R^3$, where $h \in C_0^\infty(I\!R^3)$, such that $h(x) = 1$ on a neighbouhood of $B_1$. By 3.40 and 3.43 one has $(1-\chi)h_n \varphi \in \overset{0}{W}{}^{1,r^*} (\Omega_0 \cap B_{2R_0})$. Hence, 3.39 yields

$$\int_{\Omega_0} \mathbf{E}_2 \nabla [(1-\chi) h_n \varphi] dx = 0 \tag{3.44}$$

Moreover, by 3.40 one has $\chi h_n \varphi \in \overset{0}{W}{}^{1,2} (B_{2n} \setminus B_{R_0})$.
Therefore, it follows from 3.39 and $\mathbf{E}_2 \in L^2(B_{2n} \setminus B_{R_0})$ that
$\int_{\Omega_0} \mathbf{E}_2 \nabla [\chi h_n \varphi] dx = 0$ and hence according to 3.44

$$\int_{\Omega_0} \mathbf{E}_2 \nabla [h_n \varphi] dx = 0 \text{ for all } n \in I\!N \tag{3.45}$$

By 3.41 $\mathbf{E}_2 \nabla \psi_0$ is integrable on $\Omega_0$, and hence

$$\int_{\Omega_0} \mathbf{E}_2 \nabla \psi_0 dx = \lim_{n \to \infty} (\int_{\Omega_0} \mathbf{E}_2 \nabla [h_n \psi_0] dx + \beta_n),$$

where

$$|\beta_n| \leq |\int_{\Omega} \psi_0 \mathbf{E}_2 \nabla h_n dx| \leq K/n \int_{\{|x|>n\}} |\mathbf{E}_2||\psi_0| dx$$

$$\leq K/n||\mathbf{E}_2||_{L^2(\{|x|>n\})}||\psi_0||_{L^6(\{|x|>n\})}|B_{2n}|^{1/3} \leq C||\mathbf{E}_2||_{L^2(\{|x|>n\})}||\psi_0||_{L^6(\{|x|>n\})}$$

$$\stackrel{n\to\infty}{\longrightarrow} 0$$

by Hölder's inequality, since $\mathbf{E}_2 \in L^2(\{|x| > R_0\})$ and $\psi_0 \in L^6(\{|x| > R_0\})$. By 3.45 it follows

$$\int_{\Omega_0} \mathbf{E}_2 \nabla\psi_0 dx = \lim_{n\to\infty} \int_{\Omega_0} \mathbf{E}_2 \nabla[h_n\psi_0]dx \tag{3.46}$$

$$= \lim_{n\to\infty} \int_{\Omega_0} \mathbf{E}_2 \nabla[h_n(\varphi + \sum_{k=1}^{N} \alpha_k \chi_k)]dx = \sum_{k=1}^{N} \alpha_k q_k(\mathbf{E}_2).$$

Now, 3.38, 3.43 and 3.46 yield

$$|\int_{\Omega_0} \mathbf{E}_2 \nabla\psi_0 dx| \leq \sum_{k=1}^{N} |\psi_0|_{\mathcal{C}_k} q_k(\mathbf{E}_2)| \tag{3.47}$$

$$\leq c_3||\mathbf{E}_2||^{r-1}_{L^r(\Omega_0\cap B_{2R_0})}(||\mathbf{E}_1||_{L^6} + \sum_{k=1}^{N} |q_k(\mathbf{E})|) \leq c_4||\mathbf{E}_2||^{r-1}_{L^r(\Omega_0\cap B_{2R_0})}|\mathbf{E}|_{Y_1^r}.$$

By 3.42 one has $\mathbf{F} - \nabla\psi_0 \in \mathcal{U}$ and lemma 3 yields

$$\mathbf{F} - \nabla\psi_0 = \text{ curl } \mathbf{h} \text{ where } \mathbf{h} \stackrel{\text{def}}{=} \mathcal{T}(\mathbf{F} - \nabla\psi_0) \in L^6(I\!R^3) \tag{3.48}$$

with

$$||\mathbf{h}||_{L^6} \leq c_4||\mathbf{E}_2||^{r-1}_{L^r(\Omega_0\cap B_{2R_0})}.$$

Let $A : H^r_{curl}(\mathcal{C}) \to H^r_{curl}(I\!R^3)$ be an extnsion operator constructed in the appendix with the properties

$$(A\mathbf{w})(x) = \mathbf{w}(x) \text{ for all } \mathbf{w} \in H^r_{curl}(\mathcal{C}), x \in \mathcal{C}, \tag{3.49}$$

and $||A\mathbf{w}||_{H^r_{curl}(I\!R^3)} \leq C_r||\mathbf{w}||_{H^r_{curl}(\mathcal{C})}$ and supp $(A\mathbf{w}) \subset B_{R_0}$ for all $\mathbf{w} \in H^r_{curl}(\mathcal{C})$. Let

$$\mathbf{E}_3 \stackrel{\text{def}}{=} A(\mathbf{E}_2|_{\mathcal{C}}) \in H^r_{curl}(I\!R^3).$$

Then supp $\mathbf{E}_3 \subset B_{R_0}$ and 3.39 yields the estimate

$$||\mathbf{E}_3||_{H^r_{curl}} \leq C_r||\mathbf{E}_2||_{H^r_{curl}(\mathcal{C})} = C_r||\mathbf{E}_2||_{L^r(\mathcal{C})} \tag{3.50}$$

$$\leq c_5\left(||\tilde{\mathbf{E}}||_{L^r(\mathcal{C})} + ||\mathbf{E}_1||_{L^6(B_{R_0})}\right)$$

$$\leq c_6\left(||\mathbf{E}||_{L^r(G)} + ||\text{ curl }\mathbf{E}||_{L^2(\Omega)}\right) \leq c_7|\mathbf{E}|_{Y_1^r}.$$

By 3.39, 3.40 and 3.49 one has $(1-\chi)h_n(\mathbf{E}_2-\mathbf{E}_3) \in \overset{0}{H}{}^{r}_{curl}(\Omega_0 \cap B_{2R_0})$.
Moreover, $\mathbf{h} \in H^{r^*}_{curl}(\Omega_0 \cap B_{2R_0})$ by 3.48, since $\mathbf{F} - \nabla\psi_0 \in L^{r^*}(\Omega_0 \cap B_{2R_0})$ and $r^* \leq 6/5$.
Hence,

$$\int_{\Omega_0} (1-\chi)h_n[\mathbf{E}_2-\mathbf{E}_3] \text{ curl } \mathbf{h} dx = \int_{\Omega_0} \mathbf{h} \text{ curl } \left((1-\chi)h_n[\mathbf{E}_2-\mathbf{E}_3]\right) dx \tag{3.51}$$

By 3.40 one has $\chi h_n[\mathbf{E}_2-\mathbf{E}_3] \in \overset{0}{H}{}^{2}{}_{curl}(B_{2n} \setminus B_{R_0})$.
Therefore

$$\int_{\Omega_0} \chi h_n[\mathbf{E}_2-\mathbf{E}_3] \text{ curl } \mathbf{h} dx = \int_{\Omega_0} \mathbf{h} \text{ curl } (\chi h_n[\mathbf{E}_2-\mathbf{E}_3]) dx$$

and hence according to 3.39, 3.48 and 3.51

$$\int_{\Omega_0} h_n[\mathbf{E}_2-\mathbf{E}_3](\mathbf{F}-\nabla\psi_0) dx = \int_{\Omega_0} h_n[\mathbf{E}_2-\mathbf{E}_3] \text{ curl } \mathbf{h} dx \tag{3.52}$$

$$= \int_{\Omega_0} \mathbf{h} \text{ curl } (h_n[\mathbf{E}_2-\mathbf{E}_3]) dx = \gamma_n - \int_{\Omega_0} h_n \mathbf{h} \text{ curl } \mathbf{E}_3 dx.$$

Here

$$|\gamma_n| \leq \int_{\Omega} |\mathbf{h}||\mathbf{E}_2-\mathbf{E}_3||\nabla h_n| dx = \int_{\Omega} |\mathbf{h}||\mathbf{E}_2||\nabla h_n| dx \leq K/n \int_{\{|x|>n\}} |\mathbf{E}_2||\mathbf{h}| dx$$

$$\leq K/n ||\mathbf{E}_2||_{L^2(\{|x|>n\})} ||\mathbf{h}||_{L^6(\{|x|>n\})} |B_{2n}|^{1/3} \leq C ||\mathbf{E}_2||_{L^2(\{|x|>n\})} ||\mathbf{h}||_{L^6(\{|x|>n\})}$$

$$\stackrel{\text{n}\to\infty}{\longrightarrow} 0,$$

since $\mathbf{E}_2 \in L^2(\{|x| > R_0\})$ by 3.41 and $\mathbf{h} \in L^6(\{|x| > R_0\})$. It follows now from 3.43, 3.50, the estimate in 3.48 and 3.52 by letting $n \to \infty$ that

$$|\int_{\Omega_0} \mathbf{E}_2(\mathbf{F}-\nabla\psi_0) dx| = |\int_{\Omega_0} \mathbf{E}_3(\mathbf{F}-\nabla\psi_0) dx - \int_{\Omega_0} \mathbf{h} \text{ curl } \mathbf{E}_3 dx| \tag{3.53}$$

$$\leq ||\mathbf{E}_3||_{H^r_{curl}} (||\mathbf{F}-\nabla\psi_0||_{L^{r^*}(\Omega_0 \cap B_{2R_0})} + ||\mathbf{h}||_{L^{r^*}(B_{2R_0})})$$

$$\leq c_8 |\mathbf{E}|_{Y_1^r} ||\mathbf{E}_2||^{r-1}_{L^r(\Omega_0 \cap B_{2R_0})}.$$

Next, 3.47 and 3.53 yield

$$||\mathbf{E}_2||^r_{L^r(\Omega_0 \cap B_{2R_0})} = \int_{\Omega_0} \mathbf{E}_2 \mathbf{F} dx \leq c_9 ||\mathbf{E}_2||^{r-1}_{L^r(\Omega_0 \cap B_{2R_0})} |\mathbf{E}|_{Y_1^r}.$$

and hence

$$||\mathbf{E}_2||_{L^r(\Omega_0 \cap B_{2R_0})} \leq c_9 |\mathbf{E}|_{Y_1^r}.$$

Since $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ on $\Omega$, this yields together with 3.38

$$||\mathbf{E}||_{L^r(\Omega_0 \cap B_{2R_0})} \leq C|\mathbf{E}|_{Y_1^r}. \tag{3.54}$$

with some constant $C$ independent of $\mathbf{E}$.
Now, choose $\chi_a \in C_0^\infty(\Omega_0 \cap B_{2R_0})$ and $\chi_b \in C^\infty(I\!R^3)$ with $\chi_a(x) = 1$ on supp $\nabla \chi_b$ and $\chi_b(x) = 1$ for $|x| > R_0$. Then $\chi_a \mathbf{E} \in L^r(I\!R^3)$ (extended by zero outside $\Omega$) satisfies curl $(\chi_a \mathbf{E}) \in L^r(I\!R^3)$ and div $(\chi_a \mathbf{E}) \in L^r(I\!R^3)$, since $r \leq 2$, curl $\mathbf{E} \in L^2(\Omega)$ and div $\mathbf{E} = 0$ on $\Omega_0$. By 3.54 it follows

$$||\text{ curl } (\chi_a \mathbf{E})||_{L^r} + ||\text{ div } (\chi_a \mathbf{E})||_{L^r} \leq C_1(||\text{ curl } \mathbf{E}||_{L^2} + |\mathbf{E}|_{Y_1^r}) \leq (C_1 + 1)|\mathbf{E}|_{Y_1^r}.$$

Since $r \geq 6/5$ and $\chi_a$ has bounded support, lemma 1 yields $\chi_a \mathbf{E} \in L^2(I\!R^3)$ and

$$||\chi_a \mathbf{E}||_{L^2} \leq C_2|\mathbf{E}|_{Y_1^r}. \tag{3.55}$$

Since $\chi_a(x) = 1$ on supp $\nabla \chi_b$, curl $\mathbf{E} \in L^2(\Omega)$ and div $\mathbf{E} = 0$ on $\Omega_0$. it follows from 3.54 that curl $(\chi_b \mathbf{E}) \in L^2(I\!R^3)$ and div $(\chi_b \mathbf{E}) \in L^2(I\!R^3)$,

$$||\text{ curl } (\chi_b \mathbf{E})||_{L^2} + ||\text{ div } (\chi_b \mathbf{E})||_{L^2} \leq C_3(||\text{ curl } \mathbf{E}||_{L^2} + |\mathbf{E}|_{Y_1^r}) \leq (C_3 + 1)|\mathbf{E}|_{Y_1^r}.$$

and thus $\chi_b \mathbf{E} \in L^6(I\!R^3)$ and

$$||\chi_b \mathbf{E}||_{L^6} \leq C_4|\mathbf{E}|_{Y_1^r} \tag{3.56}$$

again from lemma 1. Finally, the desired estimate follows immediately from 3.54 and 3.56.

□

By 2.23 and the previous theorem the space $\mathcal{X}$ is complete with respect to the norm

$$||\mathbf{E}||_{\mathcal{X}} \stackrel{\text{def}}{=} ||\text{ curl } \mathbf{E}||_{L^2(\Omega)} + ||\mathbf{E}|_G||_{L^2_\gamma(G)} + \sum_{k=1}^N |q_k(\mathbf{E})| \text{ for } \mathbf{E} \in \mathcal{X}.$$

The estimate

$$||\mathbf{E}||_{L^6(\{|x|>R_0\})} + ||\mathbf{E}||_{L^{r_0}(\Omega \cap B_{R_0})} \leq K||\mathbf{E}||_{\mathcal{X}} \text{ for all } \mathbf{E} \in \mathcal{X} \tag{3.57}$$

holds.

# 4 Uniquenss for the quasistaionary ME

In this section uniqueness for problem 2.27 - 2.29 is shown. Let $\mathbf{h}_0 \in H^2_{curl}(\Omega)$ and $Q_k \in I\!R$. Suppose $(\mathbf{E}^{(1)}, \mathbf{h}^{(1)})$ and $(\mathbf{E}^{(2)}, \mathbf{h}^{(2)})$ are solutions to 2.27 - 2.29 in the sense of definition 1 with $\mathbf{E}^{(k)} \in \mathcal{W}$ and $\mathbf{h}^{(k)} \in C([0,\infty), L^2(\Omega))$ and $\mathbf{E}^{(k)} = \partial_t \mathbf{A}^{(k)}$,
where $\mathbf{A}^{(k)} \in L^\infty_{loc}([0,\infty), \mathcal{X}) \cap W^{1,2}_{loc}([0,\infty), L^2_\gamma(G))$.
Let $\mathbf{A} \stackrel{\text{def}}{=} \mathbf{A}^{(1)} - \mathbf{A}^{(2)}$, $\mathbf{E} \stackrel{\text{def}}{=} \mathbf{E}^{(1)} - \mathbf{E}^{(2)}$, $\mathbf{h} \stackrel{\text{def}}{=} \mathbf{h}^{(1)} - \mathbf{h}^{(2)}$,
and $\mathbf{J}(t) \stackrel{\text{def}}{=} [\sigma(t,x,\mathbf{E}^{(1)}(t)) - \sigma(t,x,\mathbf{E}^{(2)}(t))] \in L^2_{\gamma^{-1}}(G)$ by 2.24.
A regularizing procedure with respect to the time-variable is used.
Let $\omega_n \in C_0^\infty((-\infty,0), n \in I\!N$ be an usual mollifier-sequence and define

$$\mathbf{A}_n(t) \stackrel{\text{def}}{=} \int_{I\!R} \omega_n(t-s)\mathbf{A}(s)ds \in \mathcal{X},$$

$$\text{and } \mathbf{h}_n(t) \stackrel{\text{def}}{=} \int_0^\infty \omega_n(t-s)\mathbf{h}(s)ds \in L^2(\Omega).$$

Since $\mathbf{h} \in C([0,\infty), L^2(\Omega)) = 0$ and $\mathbf{h}(0) = 0$ one has

$$||\mathbf{h}_n(0)||_{L^2} \stackrel{\text{n}\to\infty}{\longrightarrow} 0. \tag{4.58}$$

For $t \geq 0$ one has supp $\omega_n(t-\cdot) \subset (0,\infty)$. Since $\partial_t \mathbf{A} = \mathbf{E}$ and $\partial_t(\mu \mathbf{h}) = -$ curl $\mathbf{E}$ in $\mathcal{D}'((0,\infty)\times\Omega)$ it follows easily that

$$\mu \partial_t \mathbf{h}_n(t) = -\text{ curl } \partial_t \mathbf{A}_n(t). \tag{4.59}$$

By 2.27 and 2.24 it follows that

$$\text{curl } \mathbf{h}_n(t) = \int_0^\infty \omega_n(t-s) \text{ curl } \mathbf{h}(s)ds = \mathbf{J}_n(s) \stackrel{\text{def}}{=} \int_0^\infty \omega_n(t-s)\mathbf{J}(s)ds \tag{4.60}$$

$\in L^2_{\gamma^{-1}}(G)$ and curl $\mathbf{h}_n(t,x) = 0$ if $x \in \Omega_0 = \Omega \setminus \overline{G}$. In particular

$$\mathbf{h}_n(t)ds \in \tilde{\mathbf{X}}. \tag{4.61}$$

Now, 2.26 and 4.59 - 4.61 yield

$$\frac{1}{2}\frac{d}{dt}||\mu^{1/2}\mathbf{h}_n(t)||^2_{L^2} = -\int_\Omega \mathbf{h}_n(t) \text{ curl } \partial_t \mathbf{A}_n(t)dx \tag{4.62}$$

$$= -\int_G \partial_t \mathbf{A}_n(t) \text{ curl } \mathbf{h}_n(t)dx = -\int_G \partial_t \mathbf{A}_n(t)\mathbf{J}_n(t)dx.$$

Since $\partial_t \mathbf{A}|_{[0,\infty)\times G} = \mathbf{E} \in L^2_{loc}([0,\infty), L^2_\gamma(G))$ and supp$\omega_n(t-\cdot) \subset (0,\infty)$ one has

$$\partial_t \mathbf{A}_n(t)|_G = \int_{I\!R} \omega_n(t-s)\mathbf{E}(s)ds \in L^2_\gamma(G) \tag{4.63}$$

and $\partial_t \mathbf{A}_n|_{(0,T)\times G} \overset{\text{n}\to\infty}{\longrightarrow} \mathbf{E}$ in $L^2((0,T), L^2_\gamma(G))$

strongly for all $T > 0$. Moreover, it follows from 4.60 that

$$\mathbf{J}_n = \text{ curl } \mathbf{h}_n \overset{\text{n}\to\infty}{\longrightarrow} \mathbf{J} \text{ in } L^2((0,T), L^2_{\gamma^{-1}}(G)) \text{ strongly .} \tag{4.64}$$

Now, 4.58 and 4.62-4.63 yield

$$\frac{1}{2}||\mu^{1/2}\mathbf{h}_n(T)||^2_{L^2} = -\int_0^T \int_G \partial_t \mathbf{A}_n(t) \text{ curl } \mathbf{h}_n(t) dx dt \tag{4.65}$$

$$\overset{\text{n}\to\infty}{\longrightarrow} -\int_0^T \int_G \mathbf{E}(t)\mathbf{J}(t) dx dt$$

$$= \int_0^T \int_G (\mathbf{E}^{(2)}(t)) - \mathbf{E}^{(1)}(t))[\sigma(t,x,\mathbf{E}^{(1)}(t)) - \sigma(t,x,\mathbf{E}^{(2)}(t))] dx dt$$

for all $T > 0$. By the monotonicity of $\sigma$, 2.20, this implies

$$\mathbf{h}(t) = \lim_{n\to\infty} ||\mathbf{h}_n(t)||_{L^2} = 0 \text{ for all } t \in (0,\infty) \tag{4.66}$$

$$\text{and } \partial_t \mathbf{A}|_{[0,\infty)\times G} = \mathbf{E}|_{[0,\infty)\times G} = 0 \text{ on } (0,\infty)\times G.$$

Since $\mathbf{E} = \partial_t \mathbf{A}$ it follows

$$\partial_t \text{ curl } \mathbf{A} = \text{ curl } \mathbf{E} = \partial_t(\mu \mathbf{h}) = 0. \tag{4.67}$$

$$\text{Moreover, } \frac{d}{dt} q_k(\mathbf{A}(t)) = \tilde{q}(\mathbf{E}(t)) = \tilde{q}(\mathbf{E}^{(1)}(t)) - \tilde{q}(\mathbf{E}^{(2)}(t)) = 0. \tag{4.68}$$

Since $\mathbf{A} \in L^\infty_{loc}(I\!R, Y_1^{r_0})$, it follows from 4.66 - 4.68 and lemma 1 that $\mathbf{A}$ is constant with respect to $t$ and hence $\mathbf{E} = \partial_t \mathbf{A} = 0$. This completes the proof.

□

# 5 The semistatic limit for ME

In the sequel let $\mathbf{j}_0 \in L^2_{loc}(I\!R, L^2(\Omega))$, such that there exists some
$\mathbf{g}_0 \in L^2_{loc}(I\!R, H^2_{curl}(\Omega)) \cap W^{1,1}_{loc}(I\!R, L^2(\Omega))$, with

$$\mathbf{j}_0(t) = \text{ curl } \mathbf{g}_0(t), \tag{5.69}$$

in particular $\mathbf{j}_0(t)$ is assumed to be divergence-free .
Moreover let $(\mathbf{E}_0, \mathbf{h}_0) \in X_0 = L^2(\Omega)$ with

$$\mathbf{g}_0(0) - \mathbf{h}_0 \in \overline{\tilde{\mathcal{X}}_0}, \tag{5.70}$$

where the closure is taken in the $L^2(\Omega)$-topology.
Here $\tilde{\mathcal{X}}_0$ denotes the space of all $\mathbf{h} \in H^2_{curl}(\Omega)$
with curl $\mathbf{h} = 0$ on $\Omega_0$ and curl $\mathbf{h} \in L^2_{\gamma^{-1}}(G)$.
It is assumed that $\mathbf{E}_0$ satisfies

$$\text{div } \mathbf{E}_0 = 0 \text{ on } \Omega_0, \quad Q_k \stackrel{\text{def}}{=} q_k(\mathbf{E}_0) \tag{5.71}$$

Now, Maxwell's equations

$$\varepsilon\partial_t \mathbf{E}_\varepsilon = \text{ curl } \mathbf{h}_\varepsilon - \mathbf{j}_0 - \sigma(t, x, \mathbf{E}_\varepsilon), \tag{5.72}$$

$$\mu\partial_t \mathbf{h}_\varepsilon = - \text{ curl } \mathbf{E}_\varepsilon, \tag{5.73}$$

are considered, where $\varepsilon > 0$ is a parameter. 5.72, 5.73 is supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \mathbf{E}_\varepsilon = 0 \text{ on } (0, \infty) \times \partial\Omega, \tag{5.74}$$

$$\mathbf{E}_\varepsilon(0, x) = \mathbf{E}_0(x), \mathbf{h}_\varepsilon(0, x) = \mathbf{h}_0(x). \tag{5.75}$$

For the weak formulation of 5.72 - 5.75 the operator $B$ defined by

$$B(\mathbf{E}, \mathbf{h}) \stackrel{\text{def}}{=} (\text{ curl } \mathbf{h}, -\mu^{-1} \text{ curl } \mathbf{E})$$

for $(\mathbf{E}, \mathbf{h}) \in D(B) \stackrel{\text{def}}{=} \overset{0}{H^2}_{curl}(\Omega) \times H^2_{curl}(\Omega)$ is introduced. It turns out that $B$ is a densely defined skew self-adjoint operator in the Hilbert-space $X_0 \stackrel{\text{def}}{=} L^2(\Omega, \mathbb{C}^6)$ endowed with the scalar-product
$< (\mathbf{E}, \mathbf{h}), (\mathbf{F}, \mathbf{g}) >_{X_0} \stackrel{\text{def}}{=} \int_\Omega (\mathbf{E}\overline{\mathbf{F}} + \mu\mathbf{h}\overline{\mathbf{g}})dx$ .
Setting $\mathbf{w}_\varepsilon \stackrel{\text{def}}{=} (\varepsilon^{1/2}\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon - \mathbf{g}_0)$ 5.72-5.75 reads as

$$\partial_t \mathbf{w}_\varepsilon = \varepsilon^{-1/2} B\mathbf{w}_\varepsilon + \varepsilon^{-1/2} F_\sigma(t, \mathbf{E}_\varepsilon) + \mathbf{f}_0, \quad \mathbf{w}_\varepsilon(0) = \mathbf{w}_{\varepsilon,0} \stackrel{\text{def}}{=} (\varepsilon^{1/2}\mathbf{E}_0, \mathbf{h}_0 - \mathbf{g}_0(0)) \tag{5.76}$$

where $\mathbf{f}_0 \stackrel{\text{def}}{=} -(0, \partial_t \mathbf{g}_0)$ and $F_\sigma : [0, \infty) \times L^2(\Omega, \mathbb{R}^3) \to X_0 = L^2(\Omega, \mathbb{C}^6)$ is defined by

$$(F_\sigma(t, \mathbf{E}))(t, x) \stackrel{\text{def}}{=} - (\sigma(t, x, \mathbf{E}(x)), 0) \text{ for } \mathbf{E} \in L^2(\Omega).$$

Now, $\mathbf{w}_\varepsilon \in C([0, \infty), X_0)$ is called a weak solution to 5.76, if $\mathbf{w}_\varepsilon \stackrel{\text{def}}{=} (\varepsilon^{1/2}\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon)$ satisfies for all $\mathbf{u} \in D(B)$

$$\frac{d}{dt} < \mathbf{w}_\varepsilon(t), \mathbf{u} >_{X_0} = -\varepsilon^{-1/2} < \mathbf{w}_\varepsilon(t), B\mathbf{u} >_{X_0} \tag{5.77}$$

$$+ < \varepsilon^{-1/2} F_\sigma(t, \mathbf{E}_\varepsilon(t)) + \mathbf{f}_0(t), \mathbf{u} >_{X_0}$$

This is equivalent to the variation of constant formula

$$\mathbf{w}_\varepsilon(t) = \exp(\varepsilon^{-1/2}tB)\mathbf{w}_{\varepsilon,0} \tag{5.78}$$

$$+\int_0^t \exp(\varepsilon^{-1/2}(t-s)B)[\varepsilon^{-1/2}F_\sigma(s,\mathbf{E}(s))+\mathbf{f}_0(s)]ds,$$

where $\exp(tB), t \in I\!R$ is the unitary group generated by $B$. Since $F_\sigma$ is Lipschitz-continuous with respect to $\mathbf{E} \in L^2(\Omega)$, it follows from a standard result that this integral equation has a unique solution $\mathbf{w}_\varepsilon = (\varepsilon^{1/2}\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon) \in C([0,\infty), X_0)$, see [7], chapter 6.
This yields the energy balance

$$\frac{1}{2}\frac{d}{dt}||\mathbf{w}_\varepsilon(t)||^2_{X_0} =< \varepsilon^{-1/2}F_\sigma(t,\mathbf{E}_\varepsilon(t))+\mathbf{f}_0(t), \mathbf{w}_\varepsilon(t) >_{X_0} \tag{5.79}$$

$$= -\int_G \mathbf{E}_\varepsilon(t)\sigma(t,x,\mathbf{E}_\varepsilon(t))dx + \int_\Omega \mu(\mathbf{g}_0(t)-\mathbf{h}_\varepsilon(t))\partial_t\mathbf{g}_0(t)dx.$$

The aim of the following considerations is the investigation of the limit $\varepsilon \to 0$.
Let $\mathbf{A}_\varepsilon \in C(I\!R, L^2(\Omega))$ be defined by

$$\mathbf{A}_\varepsilon(t) \stackrel{\text{def}}{=} \int_0^t \mathbf{E}_\varepsilon(s)ds \text{ for } t \geq 0. \tag{5.80}$$

**Lemma 4** *The family*

$$(\mathbf{w}_\varepsilon = (\varepsilon^{1/2}\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon - \mathbf{g}_0))_{\varepsilon>0} \textit{ is bounded in } L^\infty((0,T), X_0), \tag{5.81}$$

$$(\mathbf{E}_\varepsilon)_{\varepsilon>0} = (\partial_t\mathbf{A}_\varepsilon)_{\varepsilon>0} \textit{ is bounded in } L^2((0,T), L^2_\gamma(G)) \tag{5.82}$$

$$\textit{and } (\mathbf{A}_\varepsilon)_{\varepsilon>0} \textit{ is bounded in } L^\infty((0,T),\mathcal{X}) \subset L^\infty((0,T), Y_1^{r_0} \cap L^2_\gamma(G)), \tag{5.83}$$

*for every $T > 0$. Moreover,*

$$\textit{div } \mathbf{E}_\varepsilon(t) = 0 \textit{ on } \Omega_0, \quad q_k(\mathbf{E}_\varepsilon(t)) = Q_k. \tag{5.84}$$

$$\textit{and } \textit{ curl } \mathbf{A}_\varepsilon(t) = \mu[\mathbf{h}_0 - \mathbf{h}_\varepsilon(t)] \in L^2(\Omega). \tag{5.85}$$

**Proof:** The energy balance 5.79 and assumption 2.19 on $\sigma$ yield

$$\frac{1}{2}\frac{d}{dt}||\mathbf{w}_\varepsilon(t)||^2_{X_0} \leq -\int_G \gamma|\mathbf{E}_\varepsilon(t)|^2dx + ||\mathbf{w}_\varepsilon(t)||^2_{X_0} + ||\partial_t\mathbf{g}_0(t)||^2_{L^2}.$$

This implies 5.81 and 5.82.
Next, 5.84 is proved. For this purpose assume that $\varphi \in C_0^\infty(I\!R^3)$ with $\nabla\varphi = 0$ on $\mathcal{C} = I\!R^3 \setminus \overline{\Omega_0}$. Then one has for all $\mathbf{g} \in C_0^\infty(I\!R^3) \subset H^2_{curl}(\Omega)$

$$\int_\Omega (\nabla\varphi) \text{ curl } \mathbf{g}dx = \int_{I\!R^3} (\nabla\varphi) \text{ curl } \mathbf{g}dx = 0.$$

Hence, $\nabla\varphi \in \overset{0}{H^2}_{curl}(\Omega)$, i. e. $(\nabla\varphi, 0) \in \ker B$. Therefore 5.69 and 5.78 yield

$$\int_\Omega \mathbf{E}_\varepsilon(t)\nabla\varphi dx = \varepsilon^{-1/2} < \mathbf{w}_n(t), (\nabla\varphi, 0) >_{X_0}$$

$$= \varepsilon^{-1/2}\left[< \mathbf{w}_{\varepsilon,0}, (\nabla\varphi, 0) >_{X_0} + \int_0^t < (\varepsilon^{-1/2}F_\sigma(t, \mathbf{E}_\varepsilon(s)) + \mathbf{f}_0(s)), (\nabla\varphi, 0) >_{X_0} ds\right]$$

$$= \int_\Omega \mathbf{E}_0\nabla\varphi dx,$$

since $\nabla\varphi = 0$ on $G$ whereas $\sigma(t, x, \mathbf{E}_\varepsilon(t, x)) = 0$ on $\Omega_0 = \Omega \setminus \overline{G}$. By the definition of $q_k$ 5.84 is shown.
In order to prove 5.83 note that 5.84 implies

$$\text{div } \mathbf{A}_\varepsilon(t) = 0 \text{ on } \Omega_0 \text{ and } q_k(\mathbf{A}_\varepsilon(t)) = tQ_k. \tag{5.86}$$

Next, it is shown that $\mathbf{A}_\varepsilon(t) \in \overset{0}{H^2}_{curl}(\Omega)$ and that 5.85 holds.
For this purpose let $\mathbf{g} \in C_0^\infty(I\!R^3) \subset H^2_{curl}(\Omega)$. Then one has $(0, \mathbf{g}) \in D(B)$ and hence

$$\int_\Omega \mathbf{A}_\varepsilon(t) \text{ curl } \mathbf{g} dx = \int_0^t \int_\Omega \mathbf{E}_\varepsilon(s) \text{ curl } \mathbf{g} dx ds$$

$$= \varepsilon^{-1/2} \int_0^t < \mathbf{w}_\varepsilon(s), B(0, \mathbf{g}) >_{X_0} ds$$

$$= \int_0^t \left[< \varepsilon^{-1/2}F_\sigma(s, \mathbf{E}_\varepsilon(s)) + \mathbf{f}_0(s), (0, \mathbf{g}) >_{X_0} - \frac{d}{ds} < \mathbf{w}_\varepsilon(s), (0, \mathbf{g}) >_{X_0}\right] ds$$

$$= -\int_0^t \frac{d}{ds} \int_\Omega \mu \mathbf{h}_\varepsilon(s)\mathbf{g} dx ds = \int_\Omega \mu[\mathbf{h}_0 - \mathbf{h}_\varepsilon(t)]\mathbf{g} dx.$$

Hence 5.85 is proved.
Next, let $\chi \in C_0^\infty(I\!R^3)$ with $\chi(x) = 1$ outside some bounded set and $\chi(x) = 0$ on $\mathcal{C}$ .
Since supp $\chi \subset \Omega_0$, 5.86 and 5.85 yield

$$\text{div } (\chi\mathbf{A}_\varepsilon(t)) = \mathbf{A}_\varepsilon(t)\nabla\chi \in L^2(I\!R^3)$$

and

$$\text{curl } (\chi\mathbf{A}_\varepsilon(t)) = -\mathbf{A}_\varepsilon(t) \wedge \nabla\chi + \chi\mu[\mathbf{h}_0 - \mathbf{h}_\varepsilon(t)] \in L^2(I\!R^3)$$

This implies $\chi\mathbf{A}_\varepsilon(t)) \in H^1(I\!R^3) \subset L^6(I\!R^3)$. Hence $\mathbf{A}_\varepsilon(t) \in \overset{0}{H^2}_{curl}(\Omega) \cap L^6(\{|x| > R_0\})$. By 5.86 this yields $\mathbf{A}_\varepsilon(t) \in \mathcal{X}$. Finally, 5.83 follows from estimate 3.57 (as a consequence of lemma 1), 5.82, 5.85, 5.86 and the boundedness of $(\mathbf{h}_\varepsilon)_{\varepsilon>0}$ in $L^\infty((0,T),X_0)$ which follows from 5.81.

$\square$

By lemma 4 there exist $\mathbf{A} \in L^\infty_{loc}(I\!R,\mathcal{X}) \cap W^{1,2}_{loc}([0,\infty),L^2_\gamma(G))$ and a sequence $\varepsilon_n, n \in I\!N$ with $\lim_{n\to\infty}\varepsilon_n = 0$, such that

$$\mathbf{A}_{\varepsilon_n} \overset{\text{n}\to\infty}{\longrightarrow} \mathbf{A} \text{ in } L^\infty((0,T),L^6(\{|x|>R_0\})) \text{ weak } -*, \tag{5.87}$$

$$L^2((0,T),L^{r_0}(\Omega_0 \cap B_{R_0})) \text{ weak } -* \text{ and in } W^{1,2}((0,T),L^2_\gamma(G)) \text{ weakly}$$

$$\text{with curl } \mathbf{A}_{\varepsilon_n} \overset{\text{n}\to\infty}{\longrightarrow} \text{curl } \mathbf{A} \text{ in } L^\infty((0,T),L^2(\Omega)) \text{ weak } -*,$$

Since $\sigma(\cdot,\mathbf{E}_{\varepsilon_n})_{n\in I\!N}$ is bounded in $L^2((0,T),L^2_{\gamma^{-1}}(G))$ by 2.24 and5.82, it can be also assumed that

$$\sigma(\cdot,\mathbf{E}_{\varepsilon_n}) \overset{\text{n}\to\infty}{\longrightarrow} \mathbf{J} \text{ in } L^2((0,T),L^2_{\gamma^{-1}}(G)) \text{ weakly }. \tag{5.88}$$

From 5.80, 5.84 and 5.87 it follows that

$$q_k(\mathbf{A}(t)) = tQ_k \text{ for all } k \in \{1,..,N\} \text{ and} \tag{5.89}$$

$$\mathbf{E}_{\varepsilon_n}|_{(0,\infty)\times G} = \partial_t \mathbf{A}_{\varepsilon_n}|_{(0,\infty)\times G} \overset{\text{n}\to\infty}{\longrightarrow} \mathbf{e} \overset{\text{def}}{=} \partial_t \mathbf{A}|_{(0,\infty)\times G} \tag{5.90}$$

in $L^2((0,T),L^2_\gamma(G))$ weakly, in particular $\mathbf{A}(t)|_G = \int_0^t \mathbf{e}(s)ds \in L^2_\gamma(G)$.
Moreover, it follows from 5.85 and 5.87 that

$$\mathbf{h}_{\varepsilon_n} = \mathbf{h}_0 - \mu^{-1} \text{ curl } \mathbf{A}_{\varepsilon_n} \overset{\text{n}\to\infty}{\longrightarrow} \mathbf{h} \overset{\text{def}}{=} \mathbf{h}_0 - \mu^{-1} \text{ curl } \mathbf{A} \tag{5.91}$$

in $L^\infty((0,T),L^2(\Omega))$ weak -$*$.
Since $||\varepsilon_n \mathbf{E}_{\varepsilon_n}||_{L^\infty((0,T),L^2(\Omega))} \overset{\text{n}\to\infty}{\longrightarrow} 0$ by 5.81, it follows easily from 5.72 that

$$\text{curl } \mathbf{h} = \mathbf{J} + \mathbf{j}_0 \in L^2_{loc}([0,\infty),L^2(\Omega)), \tag{5.92}$$

in particular $\mathbf{h}(t) - \mathbf{g}_0(t) \in \tilde{\mathcal{X}}$ with

$$\text{curl } (\mathbf{h} - \mathbf{g}_0) = \mathbf{J} \in L^2_{loc}([0,\infty),L^2_{\gamma^{-1}}(G)).$$

**Lemma 5** *i)*

$$\int_\Omega \mu \mathbf{h}(t)\mathbf{G}dx = \int_\Omega \mu \mathbf{h}_0 \mathbf{G}dx - \int_0^t \int_G \mathbf{e}(r) \textit{ curl } \mathbf{G}dxdr$$

*for all* $\mathbf{G} \in \tilde{\mathcal{X}}_0$*, that means* $\mathbf{G} \in H^2_{curl}(\Omega)$ *with curl* $\mathbf{G} = 0$ *on* $\Omega_0$ *and curl* $\mathbf{G} \in L_{\gamma^{-1}}(G)$.

$$\textit{ii)} \int_\Omega \mu|\mathbf{h}(t) - \mathbf{g}_0(t)|^2 dx = \int_\Omega \mu|\mathbf{h}_0 - \mathbf{g}_0(0)|^2 dx$$

$$-2\int_0^t \left(\int_G \mathbf{e}(r)\mathbf{J}(r)dx + \int_\Omega \mu[\mathbf{h}(r) - \mathbf{g}_0(r)]\partial_t \mathbf{g}_0(r)dx\right) dr$$

**Proof:**
i) By 2.26, 5.90 and 5.91 one has for all $\mathbf{G} \in \tilde{\mathcal{X}}_0$

$$\int_\Omega \mu \mathbf{h}(t)\mathbf{G}dx = \int_\Omega [\mu \mathbf{h}_0 - \text{ curl } \mathbf{A}(t)]\mathbf{G}dx = \int_\Omega [\mu \mathbf{h}_0 \mathbf{G} - \mathbf{A}(t) \text{ curl } \mathbf{G}]\, dx$$

$$= \int_\Omega \mu \mathbf{h}_0 \mathbf{G} dx - \int_0^t \int_G \mathbf{e}(r) \text{ curl } \mathbf{G} dx dr.$$

Proof of ii)
Let $\omega_n \in C_0^\infty(0,\infty), n \in I\!N$ be a mollifier-sequence and define

$$\mathbf{F}_n(t) \stackrel{\text{def}}{=} \int_0^\infty \omega_n(t-s)\mathbf{A}(s)ds \in \mathcal{X} \text{ and}$$

$$\mathbf{H}_n(t) \stackrel{\text{def}}{=} \int_0^\infty \omega_n(t-s)[\mathbf{h}(s) - \mathbf{g}_0(s)]ds$$

Then 5.71 and 5.92 yield

$$\text{curl } \mathbf{H}_n(t) = \mathbf{J}_n(t) \stackrel{\text{def}}{=} \int_0^\infty \omega_n(t-s)\mathbf{J}(s)ds \in L^2_{\gamma^{-1}}(G). \tag{5.93}$$

In particular $\mathbf{H}_n(t) \in \tilde{\mathcal{X}}_0$.
Since $\omega_n(-s) = 0$ for $s \geq 0$, one has

$$\mathbf{H}_n(0) = 0. \tag{5.94}$$

Next, 5.90 yields

$$\partial_t \mathbf{F}_n(t)|_G = \mathbf{e}_n(t) \stackrel{\text{def}}{=} \int_0^\infty \omega_n(t-s)\mathbf{e}(s)ds. \tag{5.95}$$

Moreover, it follows from5.85 that

$$\partial_t \mathbf{H}_n = \int_0^\infty \omega_n'(t-s)\left[\mathbf{h}_0 - \mu^{-1} \text{ curl } \mathbf{A}(s) - \mathbf{g}_0(s)\right] dx \tag{5.96}$$

$$= \omega_n(t)\left[\mathbf{h}_0 - \mathbf{g}_0(0)\right] - \mu^{-1} \text{ curl } \partial_t \mathbf{F}_n(t) - \omega_n * \partial_t \mathbf{g}_0(t)$$

Since $\partial_t \mathbf{F}_n(t) \in \mathcal{X}$, 5.93, 5.95 5.96 and 2.26 yield

$$\frac{1}{2}\frac{d}{dt}||\sqrt{\mu}\mathbf{H}_n(t)||^2_{L^2} \tag{5.97}$$

$$= \omega_n(t) \int_\Omega \mu \mathbf{H}_n(t)\left[\mathbf{h}_0 - \mathbf{g}_0(0)\right] dx$$

$$- \int_\Omega \mathbf{H}_n(t) \text{ curl } \partial_t \mathbf{F}_n(t) dx - \int_\Omega \mu \mathbf{H}_n(t)\left(\omega_n * \partial_t \mathbf{g}_0\right)(t) dx$$

$$= \omega_n(t) \int_0^\infty \omega_n(t-s) f(s) ds - \int_G \mathbf{J}_n(t) \mathbf{e}_n(t) dx - \int_\Omega \mu \mathbf{H}_n(t) \left(\omega_n * \partial_t \mathbf{g}_0\right)(t) dx,$$

where $f(t) \stackrel{\text{def}}{=} \int_\Omega \mu \left(\mathbf{h}(t) - \mathbf{g}_0(t)\right) [\mathbf{h}_0 - \mathbf{g}_0(0)]\, dx$. Here the fact is used that the support of $\operatorname{curl} \mathbf{H}_n(t) = \mathbf{J}_n(t)$ is contained in $\overline{G}$ by 5.93. By the definition of $\mathbf{F}_n$, $\mathbf{H}_n$, $\mathbf{e}_n$ and $\mathbf{J}_n$ it follows that

$$\mathbf{H}_n \stackrel{\text{n}\to\infty}{\longrightarrow} \mathbf{h} - \mathbf{g}_0 \text{ in } L^\infty((0,T), L^2(\Omega)) \text{ weak} - *, \tag{5.98}$$

$$\mathbf{e}_n \stackrel{\text{n}\to\infty}{\longrightarrow} \mathbf{e} \text{ in } L^2((0,T), L^2_\gamma(G)) \text{ strongly }, \tag{5.99}$$

$$\mathbf{J}_n|_{(0,T)\times G} \stackrel{\text{n}\to\infty}{\longrightarrow} \mathbf{J} \text{ in } L^2((0,T), L^2_{\gamma^{-1}}(G)) \text{ strongly }. \tag{5.100}$$

$$\text{and } \omega_n * \partial_t \mathbf{g}_0 \stackrel{\text{n}\to\infty}{\longrightarrow} \partial_t \mathbf{g}_0 \text{ in } L^1((0,T), L^2(\Omega)) \text{ strongly }. \tag{5.101}$$

Now, 5.97-5.101 and 5.94 yield

$$\frac{1}{2} ||\sqrt{\mu} \mathbf{H}_n(\tau)||^2_{L^2} - \int_0^\tau \omega_n(t) \int_0^\infty \omega_n(t-s) f(s) ds dt \tag{5.102}$$

$$\stackrel{\text{n}\to\infty}{\longrightarrow} - \int_0^\tau \int_G \mathbf{e}(t) \mathbf{J}(t) dx dt - \int_0^\tau \int_\Omega \mu [\mathbf{h}(t) - \mathbf{g}_0(t)] \partial_t \mathbf{g}_0(t) dx dt.$$

By 5.70 there exists a sequence $(\mathbf{G}_m)_{m\in I\!N}$ in $\tilde{\mathcal{X}}_0$ with

$$||\mathbf{h}_0 - \mathbf{g}_0(0) - \mathbf{G}_m||_{L^2(\Omega)} \stackrel{\text{m}\to\infty}{\longrightarrow} 0.$$

Let $f_m(t) \stackrel{\text{def}}{=} \int_\Omega \mu \left(\mathbf{h}(t) - \mathbf{g}_0(t)\right) \mathbf{G}_m dx$. Then

$$||f_m - f||_{L^\infty(0,T)} \stackrel{\text{m}\to\infty}{\longrightarrow} 0.$$

$$\text{and } f_m(t) = \int_\Omega \mu \left(\mathbf{h}_0 - \mathbf{g}_0(t)\right) \mathbf{G}_m dx - \int_0^t \int_G \mathbf{e}(r) \operatorname{curl} \mathbf{G}_m dx dr$$

by lemma 5.
This implies that $f$ is continuous and

$$f(0) = \lim_{m\to\infty} f_m(0) = \lim_{m\to\infty} \int_\Omega \mu \left(\mathbf{h}_0 - \mathbf{g}_0(0)\right) \mathbf{G}_m dx = \int_\Omega \mu |\mathbf{h}_0 - \mathbf{g}_0(0)|^2 dx.$$

Hence

$$\left| \int_0^\tau \omega_n(t) \int_0^\infty \omega_n(t-s) f(s) ds dt - 1/2 \int_\Omega \mu |\mathbf{h}_0 - \mathbf{g}_0(0)|^2 dx \right|$$

$$= \left| \int_0^\tau \omega_n(t) \int_0^t \omega_n(u) f(t-u) du dt - 1/2 f(0) \right|$$

$$= |\int_0^T \omega_n(t) \int_0^t \omega_n(u)[f(t-u) - f(0)]dudt| \leq \sup_{0 \leq t \leq 1/n} |f(t-u) - f(0)|dudt$$

$$\overset{m\to\infty}{\longrightarrow} 0.$$

Finally, assertion ii) follows from 5.102.

$\square$

Next, let $\mathbf{E} \overset{\text{def}}{=} \partial_t \mathbf{A} \in \mathcal{W} \subset \mathcal{D}'((0,\infty) \times \Omega)$.
It follows from 5.89 and 5.91 that $\partial_t(\mu \mathbf{h}) = -$ curl $\mathbf{E}$ and $\tilde{q}_t(\mathbf{E}) = Q_k$.

**Theorem 2** $(\mathbf{E}, \mathbf{h})$ *with* $\mathbf{E} \in \mathcal{W} \subset \mathcal{D}'((0,\infty) \times \Omega)$ *is the unique solution of problem 2.27 - 2.29 in the sense of definition 1. Moreover,*

$$\mathbf{A}_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \mathbf{A} \ \textit{in} \ L^\infty((0,T), Y_1^{r_0}) \ \textit{weak} \ * \ \textit{and in} \ W^{1,2}((0,T), L^2_\gamma(G)) \ \textit{weakly} \ , \tag{5.103}$$

*in particular*

$$\mathbf{E}_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \mathbf{E} = \partial_t \mathbf{A} \ \textit{in} \ \mathcal{D}'((0,\infty) \times \Omega) \ \textit{and in} \ L^2((0,T), L^2_\gamma(G)) \ \textit{weakly} \tag{5.104}$$

$$\textit{and} \ \mathbf{h}_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \mathbf{h} = \mathbf{h}_0 - \mu^{-1} \ \textit{curl} \ \mathbf{A} \ \textit{in} \ L^\infty((0,T), L^2(\Omega)) \ \textit{weak} \ - *. \tag{5.105}$$

*Finally,*

$$\sigma(t, x, \mathbf{E}_\varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} \sigma(t, x, \mathbf{E}) \ \textit{in} \ L^2((0,T), L^2_{\gamma^{-1}}(G)) \ \textit{weakly} \tag{5.106}$$

**Proof:**
First it is shown that

$$\mathbf{h} \in C([0,\infty), L^2(\Omega)) \tag{5.107}$$

$$\text{and } \mathbf{h} = (0). \tag{5.108}$$

Let $T > 0$ and $\omega_n \in C_0^\infty(-1/n, 0), n \in I\!N$ be a mollifier-sequence and define

$$\mathbf{h}_n(t) \overset{\text{def}}{=} (\mathbf{h} * \omega_n)(t) = \int_{-\infty}^0 \omega_n(u) \mathbf{h}(t-u) du.$$

$$\text{Then } ||\mathbf{h}_n - \mathbf{h}||_{L^p((0,T), L^2(\Omega))} \overset{n\to\infty}{\longrightarrow} 0.$$

Hence there exist a set $\mathcal{N} \subset [0,T]$ of measure zero and a subsequence still labelled by $(\mathbf{h}_n)_{n \in I\!N}$, such that

$$||\mathbf{h}_n(t) - \mathbf{h}(t)||_{L^2(\Omega)} \overset{n\to\infty}{\longrightarrow} 0 \text{ for all } t \in (0,T) \setminus \mathcal{N}. \tag{5.109}$$

In particular, assertion i) of lemma 5 holds for all $t \in (0,T)\backslash\mathcal{N}$ and all $\mathbf{G} \in \tilde{\mathcal{X}}_0$. Moreover, it follows from 5.92 that $\mathcal{N}$ can be chosen such that

$$\mathbf{h}(t) - \mathbf{g}_0(t) \in \tilde{\mathcal{X}} \text{ with } \text{ curl } (\mathbf{h}(t) - \mathbf{g}_0(t)) = \mathbf{J}(t) \in L^2_{\gamma^{-1}}(G) \tag{5.110}$$

and assertion ii) of lemma 5 holds for all $t \in (0,T) \setminus \mathcal{N}$ . Now, it follows from 5.109, 5.110 and lemma 5 that for all $s, t \in (0,T) \setminus \mathcal{N}$

$$\begin{aligned}
&||\mu^{1/2} (\mathbf{h}(t) - \mathbf{g}_0(t)) - \mu^{1/2} (\mathbf{h}(s) - \mathbf{g}_0(s)) ||^2_{L^2} \\
&= ||\mu^{1/2} (\mathbf{h}(t) - \mathbf{g}_0(t)) ||^2_{L^2} - ||\mu^{1/2} (\mathbf{h}(s) - \mathbf{g}_0(s)) ||^2_{L^2} \\
&+2 \int_\Omega \mu (\mathbf{h}(s) - \mathbf{g}_0(s)) [\mathbf{h}(s) - \mathbf{h}(t) - \mathbf{g}_0(s) + \mathbf{g}_0(t)] \, dx \\
&= -2 \int_s^t \int_G \mathbf{e}(r)\mathbf{J}(r) dx dr - 2 \int_s^t \int_\Omega \mu[\mathbf{h}(r) - \mathbf{g}_0(r)] \partial_t \mathbf{g}_0(r) dx dr \\
&+2 \int_s^t \int_G \mathbf{e}(r) \text{ curl } (\mathbf{h}(s) - \mathbf{g}_0(s)) \, dx dr - 2 \int_\Omega \mu (\mathbf{h}(s) - \mathbf{g}_0(s)) [\mathbf{g}_0(s) - \mathbf{g}_0(t)] \, dx \\
&\leq 2 \int_s^t ||\mathbf{e}(r)||_{L^2_\gamma(G)} ||\mathbf{J}(r)||_{L^2_{\gamma^{-1}}(G)} dr + 2M \int_s^t ||\partial_t \mathbf{g}_0(r)||_{L^2} dr \\
&+2 \int_s^t \int_G \mathbf{e}(r)\mathbf{J}(s) dx dr + 2M ||\mathbf{g}_0(t) - \mathbf{g}_0(s)||_{L^2},
\end{aligned}$$

where $M \stackrel{\text{def}}{=} ||\mathbf{h} - \mathbf{g}_0||_{L^\infty((0,T),L^2(\Omega))}$.
Let $\alpha_n \stackrel{\text{def}}{=} \sup_{t\in(0,T)} \left( \int_t^{t+1/n} ||\mathbf{e}(r)||^2_{L^2_\gamma(G)} dr \right)^{1/2}$ and $\beta_n \stackrel{\text{def}}{=} \sup_{t\in(0,T)} \left( \int_t^{t+1/n} ||\partial_t \mathbf{g}_0(r)||_{L^2} dr \right)$.
Then the previous estimate yields

$$\begin{aligned}
&||\mu^{1/2} (\mathbf{h}(t) - \mathbf{g}_0(t)) - \mu^{1/2} (\mathbf{h}(s) - \mathbf{g}_0(s)) ||^2_{L^2} \\
&\leq 2\alpha_n ||\mathbf{J}||_{L^2((0,T),L^2_{\gamma^{-1}}(G))} + 4M\beta_n + 2\alpha_n |t-s|^{1/2} ||\mathbf{J}(s)||_{L^2_{\gamma^{-1}}(G)} \\
&\quad \leq C_1(\alpha_n + \beta_n) + 2\alpha_n |t-s|^{1/2} ||\mathbf{J}(s)||_{L^2_{\gamma^{-1}}(G)}
\end{aligned}$$

with $C_1 \in (0,\infty)$ independent of $s, t, n$. Now,

$$\begin{aligned}
&||\mu^{1/2} (\mathbf{h}(t) - \mathbf{g}_0(t)) - \mu^{1/2} (\mathbf{h}_n(t) - (\omega_n * \mathbf{g}_0)(t)) ||^2_{L^2} \\
&= ||\mu^{1/2} \int_{-\infty}^0 \omega_n(u) [(\mathbf{h}(t) - \mathbf{g}_0(t)) - (\mathbf{h}(t-u) - \mathbf{g}_0(t-u))] \, du||^2 \\
&\leq \int_{-\infty}^0 \omega_n(u) ||\mu^{1/2} (\mathbf{h}(t) - \mathbf{g}_0(t)) - \mu^{1/2} (\mathbf{h}(t-u) - \mathbf{g}_0(t-u)) ||^2 du.
\end{aligned}$$

$$\leq \int_{-\infty}^{0} \omega_n(u) \left( C_1(\alpha_n + \beta_n) + 2\alpha_n |u|^{1/2} ||\mathbf{J}(t-u)||_{L^2_{\gamma^{-1}}(G)} \right) du.$$

$$= C_1(\alpha_n + \beta_n) + \alpha_n \left( \int_{-\infty}^{0} \omega_n(u)^2 |u| du \right)^{1/2} \left( \int_0^T ||\mathbf{J}(r)||^2_{L^2_{\gamma^{-1}}(G)} dr \right)^{1/2}$$

$\overset{n\to\infty}{\longrightarrow} 0$ uniformly with respect to $t \in (0,T)$.

This proves 5.107.
In order to show 5.108 let $(\mathbf{G}_m)_{m\in I\!N}$ a sequence in $\tilde{\mathcal{X}}_0$ with

$$||\mathbf{h}_0 - \mathbf{g}_0(0) - \mathbf{G}_m||_{L^2(\Omega)} \overset{m\to\infty}{\longrightarrow} 0,$$

see 5.70. Let $f(t) \overset{\text{def}}{=} \int_\Omega \mu\,(\mathbf{h}(t) - \mathbf{g}_0(t))\,[\mathbf{h}_0 - \mathbf{g}_0(0)]\,dx$.
and $f_m(t) \overset{\text{def}}{=} \int_\Omega \mu\,(\mathbf{h}(t) - \mathbf{g}_0(t))\,\mathbf{G}_m dx$. Then

$$||f_m - f||_{L^\infty(0,T)} \overset{m\to\infty}{\longrightarrow} 0.$$

and $f_m(t) = \int_\Omega \mu\,(\mathbf{h}_0 - \mathbf{g}_0(t))\,\mathbf{G}_m dx - \int_0^t \int_G \mathbf{e}(r) \text{ curl } \mathbf{G}_m dx dr$

by lemma 5.
This implies that $f$ is continuous and

$$f(0) = \lim_{m\to\infty} f_m(0) = \lim_{m\to\infty} \int_\Omega \mu\,(\mathbf{h}_0 - \mathbf{g}_0(0))\,\mathbf{G}_m dx = \int_\Omega \mu |\mathbf{h}_0 - \mathbf{g}_0(0)|^2 dx.$$

Hence $||\,(\mathbf{h}(t) - \mathbf{g}_0(t)) - (\mathbf{h}_0 - \mathbf{g}_0(0))\,||^2_{L^2}$

$$= ||\,(\mathbf{h}(t) - \mathbf{g}_0(t))\,||^2_{L^2} + ||\,(\mathbf{h}_0 - \mathbf{g}_0(0))\,||^2_{L^2} - 2f(t) \overset{t\to 0}{\longrightarrow} 0.$$

Since $\mathbf{e} = \mathbf{E}|_{(0,\infty)\times G} \in L^2_{loc}([0,\infty), L^2_\gamma(G))$ by 5.90, it remains to show

$$\mathbf{J} = \sigma(x, \mathbf{e}). \tag{5.111}$$

This is will be done using a monotonicity-argument. For this purpose let $\mathcal{T} : X \to X^*$ with $X \overset{\text{def}}{=} L^2((0,T), L^2_\gamma(G))$ and $X^* \overset{\text{def}}{=} L^2((0,T), L^2_{\gamma^{-1}}(G))$ be defined by

$$(\mathcal{T}\mathbf{f})(t,x) \overset{\text{def}}{=} \chi(t)\sigma(t,x,\mathbf{f}(t,x)), \mathbf{f} \in X,$$

where $\chi \in C^\infty(I\!R)$ with $\chi \geq 0$, $\chi' \leq 0$, $\chi(0) = 1$ and $\chi(T) = 0$. The aim of the following considerations is to show

$$\chi\mathbf{J} = \mathcal{T}(\mathbf{e}) \tag{5.112}$$

which implies 5.111, since $T > 0$ can be choosen arbitrary large. Let

$$\mathbf{e}_n \stackrel{\text{def}}{=} \mathbf{E}_{\varepsilon_n}|_{(0,T)\times G} \in L^2((0,T), L^2_\gamma(G)).$$

Then 5.88 and 5.90 yield

$$\mathbf{e}_n \stackrel{\text{n}\to\infty}{\longrightarrow} \mathbf{e} \text{ in } X \text{ weakly and } \mathcal{T}(\mathbf{e}_n) \stackrel{\text{n}\to\infty}{\longrightarrow} \chi\mathbf{J} \text{ in } X^* \text{ weakly .}$$

Since $\mathcal{T}$ is monotone and Lipschitz-continuous by the assumptions on $\sigma$, it suffices to show

$$\overline{\lim_{n\to\infty}} < \mathcal{T}(\mathbf{e}_n), \mathbf{e}_n >_X \leq < \chi\mathbf{J}, \mathbf{e} >_X \tag{5.113}$$

The energy balance 5.79 yields

$$< \mathcal{T}(\mathbf{e}_n), \mathbf{e}_n >_X = \int_0^T \chi(t) \int_G \mathbf{E}_{\varepsilon_n}(t)\sigma(t,x,\mathbf{E}_{\varepsilon_n}(t))dxdt$$

$$= \int_0^T \chi(t) \left[ \int_\Omega \mu(\mathbf{g}_0(t) - \mathbf{h}_{\varepsilon_n}(t))\partial_t \mathbf{g}_0(t)dx - \frac{1}{2}\frac{d}{dt}||\mathbf{w}_{\varepsilon_n}(t)||^2_{X_0} \right] dt$$

$$\leq \int_0^T \chi(t) \int_\Omega \mu(\mathbf{g}_0(t) - \mathbf{h}_{\varepsilon_n}(t))\partial_t \mathbf{g}_0(t)dxdt + \frac{1}{2}||\mathbf{w}_{\varepsilon_n}(0)||^2_{X_0}$$

$$+\frac{1}{2}\int_0^T \chi'(t)||\sqrt{\mu}(\mathbf{h}_{\varepsilon_n}(t) - \mathbf{g}_0(t))||^2_{L^2}dt$$

Since $\chi' \leq 0$, it follows from 5.91 that

$$\overline{\lim_{n\to\infty}} < \mathcal{T}(\mathbf{e}_n), \mathbf{e}_n >_X$$

$$\leq \int_0^T \chi(t) \int_\Omega \mu(\mathbf{g}_0(t) - \mathbf{h}(t))\partial_t \mathbf{g}_0(t)dxdt + \frac{1}{2}||\sqrt{\mu}(\mathbf{h}_0 - \mathbf{g}_0(0))||^2_{L^2}$$

$$+\frac{1}{2}\int_0^T \chi'(t)||\sqrt{\mu}(\mathbf{h}(t) - \mathbf{g}_0(t))||^2_{L^2}dt$$

In order to prove 5.113 it suffices now to show that

$$< \chi\mathbf{J}, \mathbf{e} >_X = \int_0^T \chi(t) \int_\Omega \mu(\mathbf{g}_0(t) - \mathbf{h}(t))\partial_t \mathbf{g}_0(t)dxdt \tag{5.114}$$

$$+\frac{1}{2}||\sqrt{\mu}(\mathbf{h}_0 - \mathbf{g}_0(0))||^2_{L^2} + \frac{1}{2}\int_0^T \chi'(t)||\sqrt{\mu}(\mathbf{h}(t) - \mathbf{g}_0(t))||^2_{L^2}dt.$$

Lemma 5 ii) yields

$$\frac{1}{2}\frac{d}{dt}||\sqrt{\mu}(\mathbf{h}(t) - \mathbf{g}_0(t))||^2_{L^2} = -\int_G \mathbf{e}(t)\mathbf{J}(t)dx$$

$$-\int_\Omega \mu[\mathbf{h}(t)-\mathbf{g}_0(t)]\partial_t\mathbf{g}_0(t)dx.$$

Hence

$$< \chi\mathbf{J},\mathbf{e} >_X = \int_0^T \chi(t)\int_G \mathbf{e}(t)\mathbf{J}(t)dxdt$$

$$= -\int_0^T \chi(t)[\frac{1}{2}\frac{d}{dt}||\sqrt{\mu}(\mathbf{h}(t)-\mathbf{g}_0(t))||^2_{L^2} + \int_\Omega \mu[\mathbf{h}(t)-\mathbf{g}_0(t)]\partial_t\mathbf{g}_0(t)dx]dt$$

$$= \frac{1}{2}||\mu^{1/2}(\mathbf{h}_0-\mathbf{g}_n(0))||^2_{L^2} + \frac{1}{2}\int_0^T \chi'(t)||\sqrt{\mu}(\mathbf{h}(t)-\mathbf{g}_0(t))||^2_{L^2}dt$$

$$-\int_0^T \chi(t)\int_\Omega \mu[\mathbf{h}(t)-\mathbf{g}_0(t)]\partial_t\mathbf{g}_0(t)dxdt$$

whence 5.114.
Hence, is proved that $(\mathbf{E},\mathbf{h}) \in \mathcal{W}\times C([0,\infty),L^2(\Omega))$ solve 2.27 - 2.29. Thus, the proof of existence and uniqueness of solutions to 2.27-2.29 is complete.
The above considerations show that each accumulation point of $(\mathbf{E}_\varepsilon,\mathbf{h}_\varepsilon),\varepsilon>0$ in the weak topologies described in 5.103 - 5.106 provides a solution to 2.27-2.29. Since the solution of this problem is unique, one obtains also assertions 5.103 - 5.106.

$\square$

# 6 Asymptptic behavior: Decay of the magnetic field

In this section the asymptotic behavior for $t\to\infty$ of solutions to 2.27-2.29 in the case

$$\mathbf{j}_0 = 0 \tag{6.115}$$

is investigated. In the sequel let $H_{curl,0}(\Omega)$ be the space of all $\mathbf{h}\in H^2_{curl}(\Omega)$ with $\operatorname{curl}\mathbf{h}=0$ on $\Omega$. It is a closed subspace of $L^2(\Omega)$. Its orthogonal complement with respect to the scalar-product
$< \mathbf{h},\mathbf{g} >_\mu \overset{\text{def}}{=} \int_\Omega \mu\mathbf{h}\overline{\mathbf{g}}dx$ is denoted by $X_h$. Let $P$ be the orthogonal projection on $X_h\subset L^2(\Omega)$ with respect to $< \cdot,\cdot >$.
Since $\nabla\psi\in H_{curl,0}(\Omega)$ for all $\psi\in H^1(\Omega)\overset{\text{def}}{=}W^{1,2}(\Omega)$, it follows that each $\mathbf{h}\in X_h$ obeys

$$\text{div }(\mu\mathbf{h}) = 0 \text{ on } \Omega \text{ and } \vec{n}\mathbf{h} = 0 \text{ on } \partial\Omega \text{ weakly }. \tag{6.116}$$

in the sense that

$$-\int_\Omega \mu\mathbf{h}\nabla\psi dx = 0 \text{ for all } \varphi\in H^1(\Omega).$$

Next, let $(\mathbf{E},\mathbf{h})\in\mathcal{W}\times C([0,\infty),L^2(\Omega))$
be the solution to 2.27-2.29. In order to apply a compactness theorem, it is assumed that $I\!R^3\setminus\overline{\Omega}$ is also a Lipschitz-domain.

**Lemma 6** *There exists some $\delta > 0$ (depending only on $\sigma$), such that*

$$\frac{d}{dt}\frac{1}{2}||\sqrt{\mu}\mathbf{h}(t)||_{L^2}^2 \leq -\delta|| \; curl \; \mathbf{h}(t)||_{L^2(\Omega)}^2.$$

**Proof:** By 6.115 one has

$$\text{curl } \mathbf{h}(t) = \mathbf{J}(t) \in L_{\gamma^{-1}}(G), \tag{6.117}$$

and curl $\mathbf{h}(t,x) = 0$ for $x \in \Omega_0 = \Omega \setminus \overline{G}$, in particular $\mathbf{h}(t) \in \tilde{\mathcal{X}}_0$. Therefore it follows from the same arguments as in the proof of 4.62 that

$$\frac{1}{2}\frac{d}{dt}||\sqrt{\mu}\mathbf{h}(t)||_{L^2}^2 = -\int_G \sigma(t,x,\mathbf{E}(t))\mathbf{E}(t)dx.$$

By the assumptions on $\sigma$ one has

$$-\frac{1}{2}\frac{d}{dt}||\sqrt{\mu}\mathbf{h}(t)||_{L^2}^2 = \int_G \mathbf{E}(t,x)\sigma(t,x,\mathbf{E}(t,x))dx \geq \int_G \gamma(x)|\mathbf{E}(t,x)|^2 dx$$

$$\geq \frac{1}{C^2}\int_G \gamma(x)^{-1}|\sigma(t,x,\mathbf{E}(t,x))|^2 dx \geq \delta \int_G |\sigma(t,x,\mathbf{E}(t,x))|^2 dx$$

$$= \delta \int_G | \text{ curl } \mathbf{h}(t,x)|^2 dx = \delta \int_\Omega | \text{ curl } \mathbf{h}(t,x)|^2 dx.$$

with $\delta \stackrel{\text{def}}{=} (C^2||\gamma||_{L^\infty})^{-1}$. This completes the proof.

□

The aim of the following considerations is to estimate $||\mathbf{h}||_{L^2}$ by $||$ curl $\mathbf{h}||_{L^2}$ for $\mathbf{h} \in X_h$.

**Lemma 7** *Let $\mathbf{h} \in L^2(I\!R^3)$ with curl $\mathbf{h} \in L^{6/5}(I\!R^3)$ and div $(\mu\mathbf{h}) \in L^{6/5}(I\!R^3)$. Then*

$$||\mathbf{h}||_{L^2(I\!R^3)} \leq K(|| \; curl \; \mathbf{h}||_{L^{6/5}(I\!R^3)} + || \; div \; (\mu\mathbf{h})||_{L^{6/5}(I\!R^3)})$$

*with some $K \in (0,\infty)$ indepndent of* $\mathbf{h}$.

**Proof:**
Recall that $Z \stackrel{\text{def}}{=} \{\varphi \in L^6 | \nabla\varphi \in L^2(I\!R^3)\}$ is a Hilbert-space endowed with the scalar-product
$< \varphi, \psi >_Z \stackrel{\text{def}}{=} \int_{I\!R^3} \mu\nabla\varphi\nabla\overline{\psi}dx$. Thus, there exists a unique $\psi_0 \in Z$, such that

$$\int_{I\!R^3} \mu\nabla\psi_0\nabla\varphi dx =< \psi_0, \overline{\varphi} >_{Z_0}= \int_{I\!R^3} \varphi \text{ div } (\mu\mathbf{h})dx$$

for all $\varphi \in Z$, since div $(\mu\mathbf{h}) \in L^{6/5}(I\!R^3)$, such that the last term defines a continuous linear functional on $Z \subset L^6$.

$$||\mu^{1/2}\nabla\psi_0||_{L^2(I\!R^3)} = ||\psi_0||_Z \leq K_1|| \text{ div } (\mu\mathbf{h})||_{L^{6/5}(I\!R^3)} \tag{6.118}$$

Let $\mathbf{h}_1 \stackrel{\text{def}}{=} \mathbf{h} + \nabla\psi_0 \in L^2(I\!R^3)$. Then curl $\mathbf{h}_1 =$ curl $\mathbf{h} \in L^{6/5}(I\!R^3)$ and div $(\mu\mathbf{h}_1) = 0$ on $I\!R^3$. By lemma 3 i) there exists a unique $\mathbf{F} \in L^6(I\!R^3)$ with curl $\mathbf{F} = \mu\mathbf{h}_1 \in L^2(I\!R^3)$ and div $\mathbf{h} = 0$ on $I\!R^3$. The estimate

$$||\mathbf{F}||_{L^6(I\!R^3)} \leq K_2||\mathbf{h}_1||_{L^2(I\!R^3)} \tag{6.119}$$

holds with constants $K_1, K_2 \in (0,\infty)$ independent of $\mathbf{h}$. Let $\psi_1 \in C_0^\infty(B_2)$ with $\psi_1 = 1$ on $B_1$ and $\psi_n \stackrel{\text{def}}{=} \psi_1(x/n)$. Then

$$||\mu^{1/2}\mathbf{h}_1||^2_{L^2(I\!R^3)} = \int_{I\!R^3} \mathbf{h}_1 \text{ curl } \mathbf{F} dx = \lim_{n\to\infty} \int_{I\!R^3} (\mathbf{h}_1 \text{ curl } \mathbf{F})\psi_n dx$$

$$= \lim_{n\to\infty} \int_{I\!R^3} [(\mathbf{F} \wedge \mathbf{h}_1)\nabla\psi_n + \psi_n \mathbf{F} \text{ curl } \mathbf{h}_1] dx$$

Since supp$(\nabla\psi_n) \subset \{n < |x| < 2n\}$, $\mathbf{h}_0 \in L^2(I\!R^3)$ and $\mathbf{F} \in L^6(I\!R^3)$, Hölder's inequality yields

$$|\int_{I\!R^3} (\mathbf{F} \wedge \mathbf{h}_1)\nabla\psi_n dx| \leq ||\mathbf{F}||_{L^6(\{|x|>n\})}||\mathbf{h}_1||_{L^2(\{|x|>n\})}||\nabla\psi_n||_{L^\infty(I\!R^3)}|B_{2n}|^{1/3}$$

$$\leq C||\mathbf{F}||_{L^6(\{|x|>n\})} \stackrel{\text{n}\to\infty}{\longrightarrow} 0.$$

Therefore 6.119 yields

$$||\mu^{1/2}\mathbf{h}_1||^2_{L^2(I\!R^3)} = \int_{I\!R^3} \mathbf{F} \text{ curl } \mathbf{h}_1 dx \leq ||\mathbf{F}||_{L^6(I\!R^3)}|| \text{ curl } \mathbf{h}||_{L^{6/5}(I\!R^3)} \tag{6.120}$$

$$\leq K_2||\mathbf{h}_1||_{L^2(I\!R^3)}|| \text{ curl } \mathbf{h}||_{L^{6/5}(I\!R^3)}.$$

By the definition of $\mathbf{h}_1$ the assertion follows from 6.118 and 6.120.

□

**Lemma 8** *Let $(\mathbf{h}_n)_{n\in I\!N}$ be a sequence in $X_h$, such that ( curl $\mathbf{h}_n)_{n\in I\!N}$ is bounded in $L^2(\Omega)$, supp ( curl $\mathbf{h}_n) \subset B_R$ for some $R \in (R_0,\infty)$ in dependent of $n$.*
*Then $(\mathbf{h}_n)_{n\in I\!N}$ is precompact in $L^2(\Omega)$.*

**Proof:**
Let $\tilde{\Omega} \stackrel{\text{def}}{=} B_{3R} \cap \Omega$ and choose $\chi_b \in C^\infty(I\!R^3)$ with $\chi_b(x) = 0$ on $B_R$, $\chi_b(x) = 1$ for $|x| > 2R$ and let $\chi_a \in C_0^\infty(B_{3R})$ with $\chi_a(x) = 1$ on $B_{2R}$, in particular

$$\chi_a(x) = 1 \text{ on supp } (\nabla\chi_b) \tag{6.121}$$

It follows from the assumptions that

$$(\chi_a\mathbf{h}_n)_{n\in I\!N} \text{ is bounded in } H^2_{curl}(B_{3R} \cap \Omega) = H^2_{curl}(\tilde{\Omega}) \tag{6.122}$$

Moreover, it follows from 6.116 that

$$( \text{ div } [\mu\chi_a \mathbf{h}_n])_{n\in I\!N} \text{ is bounded in } L^2(\tilde{\Omega}) \tag{6.123}$$

and $\chi_a \vec{n}\mathbf{h} = 0$ on $\partial\tilde{\Omega}$, in the sense that

$$-\int_{\tilde{\Omega}} \mu\chi_a \mathbf{h}_n \nabla\psi dx = \int_{\tilde{\Omega}} ( \text{ div } [\mu\chi_a \mathbf{h}_n])\psi dx = \int_{\tilde{\Omega}} \mu(\mathbf{h}_n \nabla\chi_a)\psi dx$$

for all $\varphi \in H^1(\tilde{\Omega})$.
Since $\tilde{\Omega} = B_{3R} \cap \Omega$ is a Lipschitz-domain, it follows from 6.122, 6.123 and the result in [9], [11] or [5] that the sequence

$$(\chi_a \mathbf{h}_n)_{n\in I\!N} \text{ is precompact in } L^2(B_{3R} \cap \Omega) = L^2(\tilde{\Omega}). \tag{6.124}$$

Let $\mathbf{g}_n(x) \stackrel{\text{def}}{=} \chi_b \mathbf{h}_n(x)$ if $x \in I\!R^3 \setminus B_R = \Omega \setminus B_R$ and $\mathbf{g}_n(x) \stackrel{\text{def}}{=} 0$ if $x \in B_R$.
Since supp curl $\mathbf{h}_n \subset B_R$ and $\chi_b(x) = 0$ on $B_R$, it follows from 6.116, 6.122 and 6.124 that

$$( \text{ curl } \mathbf{g}_n)_{n\in I\!N} = ((\nabla\chi_b) \wedge \mathbf{h}_n)_{n\in I\!N} \text{ is precompact in } L^{6/5}(I\!R^3) \tag{6.125}$$

and

$$( \text{ div } (\mu\mathbf{g}_n))_{n\in I\!N} = (\mu\mathbf{h}_n \nabla\chi_b)_{n\in I\!N} \text{ is precompact in } L^{6/5}(I\!R^3). \tag{6.126}$$

By 6.125 and 6.126 it follows from lemma 7 that the sequence $(\mathbf{g}_n)_{n\in I\!N}$ is precompact in $L^2(I\!R^3)$ and hence

$$(\mathbf{h}_n)_{n\in I\!N} \text{ is precompact in } L^2(\Omega \setminus B_{2R}). \tag{6.127}$$

since $\chi_b(x) = 1$ for $|x| > 2R$. Finally, the precompactness of $(\mathbf{h}_n)_{n\in I\!N}$ in $L^2(\Omega)$ follows from 6.124 and 6.127.

$\square$

Now, the following estimate can be proved.

**Lemma 9** *Let $R \in (0, \infty)$. Then there exists a constant $K_R \in (0, \infty)$, such that for all $\mathbf{h} \in X_h \cap H^2_{curl}(\Omega)$ with curl $\mathbf{h}(x) = 0$ if $|x| > R$ the estimate*

$$||\mathbf{h}||_{L^2(\Omega)} \leq K_R|| \textit{ curl } \mathbf{h}||_{L^2(\Omega\cap B_R)}$$

*holds.*

**Proof:**
Suppose that the estimate were not correct, i.e. would exist a sequence $\mathbf{h}_n \in X_h \cap H^2_{curl}(\Omega), n \in I\!N$ with

$$1 = ||\mathbf{h}_n||_{L^2} \geq n|| \text{ curl } \mathbf{h}_n||_{L^2} \text{ for all } n \in I\!N \tag{6.128}$$

By theorem 8 there exist $\mathbf{h} \in X_h$ and a subsequence $\mathbf{h}_{n_k}, k \in I\!N$ with

$$||\mathbf{h}_{n_k} - \mathbf{h}||_{L^2} \overset{k\to\infty}{\longrightarrow} 0 \text{ in particular } ||\mathbf{E}||_{L^2} = 1. \tag{6.129}$$

From 6.128 and 6.129 it follows that $(\mathbf{E}_{n_k})_{k\in I\!N}$ is convergent in $X_h \cap H^2_{curl}(\Omega)$.
Hence, $\mathbf{E} \in X_h \cap H^2_{curl}(\Omega)$,
Moreover, $||\text{ curl } \mathbf{E}||_{L^2} = \lim_{k\to\infty} ||\mathbf{E}_{n_k}||_{L^2} = 0$, i.e. $\mathbf{E} \in H_{curl,0}(\Omega) \cap X_h = \{0\}$. This contradicts 6.129.

□

Now the asymptotic behavior of the solution $(\mathbf{E}, \mathbf{h})$ is investigated.
Let $\mathbf{h}^{(1)}(t) \overset{\text{def}}{=} P\mathbf{h}(t) \in X_h$.

**Theorem 3** *There exists some $C, d > 0$, such that*

$$||\mathbf{h}^{(1)}(t)||_{L^2(\Omega)} \leq C \exp(-dt).$$

**Proof:** Suppose that $\mathbf{E} = \partial_t \mathbf{A}$ with some $\mathbf{A} \in L^\infty_{loc}([0,\infty), \mathcal{X})$.
Then $\mu\mathbf{h} + \text{ curl } \mathbf{A} \in L^\infty_{loc}([0,\infty), L^2(\Omega))$ satisfies

$$\partial_t [\mu\mathbf{h} + \text{ curl } \mathbf{A}] = 0 \text{ and hence}$$

$$\mathbf{h}(t) + \mu^{-1}\mathbf{A}(t) = \mathbf{h}_2 \text{ for all } t \in (0,\infty) \tag{6.130}$$

with some $\mathbf{h}_2 \in L^2(\Omega)$.
Let $P$ be the orthogonal projection on $X_h \subset L^2(\Omega)$ with respect to the scalar-product $< \mathbf{h}, \mathbf{g} >_\mu = \int_\Omega \mu\mathbf{h}\overline{\mathbf{g}}dx$.
Since $H_{curl,0}(\Omega) \subset \tilde{\mathcal{X}}$, it follows from 2.26 that $\mu^{-1}$ curl $\mathbf{A}(t) \in X_h$. Hence 6.130 yields

$$\mathbf{h}(t) = \mathbf{h}^{(1)}(t) + (1-P)\left(\mathbf{h}_2 - \mu^{-1}\mathbf{A}(t)\right) = \mathbf{h}^{(1)}(t) + (1-P)\mathbf{h}_2$$

and therefore

$$||\sqrt{\mu}\mathbf{h}(t)||^2_{L^2} = ||\sqrt{\mu}\mathbf{h}^{(1)}(t)||^2_{L^2} + ||\sqrt{\mu}(1-P)\mathbf{h}_2||^2_{L^2} \tag{6.131}$$

Since curl $\mathbf{h}^{(1)}(t) =$ curl $\mathbf{h}(t)$, it follows from 6.131, lemma 6 and 9 that

$$\frac{1}{2}\frac{d}{dt}||\sqrt{\mu}\mathbf{h}^{(1)}(t)||^2_{L^2} = \frac{1}{2}\frac{d}{dt}||\sqrt{\mu}\mathbf{h}(t)||^2_{L^2}$$

$$\leq -\delta|| \text{ curl } \mathbf{h}(t)||^2_{L^2(G)} = -\delta|| \text{ curl } \mathbf{h}^{(1)}(t)||^2_{L^2(G)} \leq -K^{-2}_{R_0}\delta||\mathbf{h}^{(1)}(t)||^2_{L^2(G)}.$$

This completes the proof

□

# 7 Strong solutions

In this section a sufficient condition for the existence of strong soutions $(\mathbf{E}, \mathbf{h})$ with

$$\mathbf{E} \in L^2_{loc}([0,\infty), \mathcal{X}) \subset L^2_{loc}([0,\infty), Y_1^{r_0} \cap L^2_\gamma(G))$$

and

$$\mathbf{h} \in W^{1,2}_{loc}([0,\infty), L^2(\Omega))$$

is given. For this purpose it is assumed that $\sigma$ is independent of t, i.e. $\sigma(t,x,\mathbf{y}) = \sigma_0(x,\mathbf{y})$. Moreover, additional regularity properties of the prescribed current $\mathbf{j}_0$ and the initial-data $\mathbf{E}_0, \mathbf{h}_0$ is imposed, namely

$$\mathbf{g}_0 \in L^2_{loc}(I\!R, H^2_{curl}(\Omega)) \cap W^{2,2}_{loc}(I\!R, L^2(\Omega)), \tag{7.132}$$

$$\mathbf{E}_0 \in \overset{0}{H^2}_{curl}(\Omega), \quad \mathbf{h}_0 \in H^2_{curl}(\Omega) \text{ with} \tag{7.133}$$

$$\text{curl } \mathbf{h}_0 = \mathbf{j}_0(0) + \sigma_0(x, \mathbf{E}_0) \text{ on } \Omega. \tag{7.134}$$

**Lemma 10** *The family*

$$(\mathbf{w}_\varepsilon = (\varepsilon^{1/2}\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon - \mathbf{g}_0))_{\varepsilon>0} \textit{ is bounded in } W^{1,\infty}((0,T), X_0), \tag{7.135}$$

$$(\mathbf{E}_\varepsilon)_{\varepsilon>0} \textit{ is bounded in } L^2((0,T),\mathcal{X}) \subset L^2((0,T), Y_1^{r_0} \cap L^2_\gamma(G)), \tag{7.136}$$

$$\textit{and } (\textit{ curl } \mathbf{h}_\varepsilon)_{\varepsilon>0} \textit{ is bounded in } L^2((0,T), L^2(\Omega)), \tag{7.137}$$

*for every* $T > 0$.

**Proof:** By lemma 4 we have

$$(\mathbf{w}_\varepsilon = (\varepsilon^{1/2}\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon - \mathbf{g}_0))_{\varepsilon>0} \text{ is bounded in } L^\infty((0,T), X_0), \tag{7.138}$$

$$\text{and } (\mathbf{E}_\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^2(0,T, L^2_\gamma(G)). \tag{7.139}$$

Next, it follws from 5.78 that

$$\mathbf{w}_\varepsilon(t+\tau) - \mathbf{w}_\varepsilon(t) = \exp(\varepsilon^{-1/2}tB)\mathbf{u}_{\varepsilon,\tau,0} + \int_0^t \exp(\varepsilon^{-1/2}(t-s)B)\mathbf{G}_{\varepsilon,\tau}(s)ds, \tag{7.140}$$

where

$$\mathbf{u}_{\varepsilon,\tau,0} \overset{\text{def}}{=} (\exp(\varepsilon^{-1/2}\tau B) - 1)\mathbf{w}_{\varepsilon,0}$$
$$+ \int_0^\tau \exp(\varepsilon^{-1/2}(\tau - s)B)\left[\varepsilon^{-1/2}F_{\sigma_0}(\mathbf{E}_\varepsilon(s)) + \mathbf{f}_0(s)\right] ds$$

and

$$\mathbf{G}_{\varepsilon,\tau} \stackrel{\text{def}}{=} \varepsilon^{-1/2}\left[F_{\sigma_0}(\mathbf{E}_\varepsilon(\tau+s)) - F_{\sigma_0}(\mathbf{E}_\varepsilon(s))\right] + \mathbf{f}_0(\tau+s) - \mathbf{f}_0(s).$$

Here $F_{\sigma_0} : L^2(\Omega, I\!R^3) \to X_0 = L^2(\Omega, \mathbb{C}^6)$ is defined by $F_{\sigma_0}(\mathbf{F})(x) \stackrel{\text{def}}{=} -(\sigma_0(x, \mathbf{F}(x)), 0)$ for $\mathbf{F} \in L^2(\Omega)$ .
By 7.134 it follows that $\mathbf{w}_{\varepsilon,0} \in D(B)$ and hence

$$\begin{aligned} &\tau^{-1}\mathbf{u}_{\varepsilon,\tau,0} \stackrel{\tau\to 0}{\longrightarrow} \varepsilon^{-1/2}B\mathbf{w}_{\varepsilon,0} + \varepsilon^{-1/2}F_{\sigma_0}(\mathbf{E}_0) + \mathbf{f}_0(0) \\ &= -(0, \mu^{-1} \text{ curl } \mathbf{E}_0 + \partial_t \mathbf{g}_0(0)). \end{aligned} \tag{7.141}$$

7.140 yields the estimate

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}||\mathbf{w}_\varepsilon(t+\tau) - \mathbf{w}_\varepsilon(t)||_{X_0}^2 =< G_{\varepsilon,\tau}(t), \mathbf{w}_\varepsilon(t+\tau) - \mathbf{w}_\varepsilon(t) >_{X_0} \\ &= -\int_G [\sigma_0(x\mathbf{E}_\varepsilon(t+\tau)) - \sigma_0(x\mathbf{E}_\varepsilon(t))](\mathbf{E}_\varepsilon(t+\tau) - \mathbf{E}_\varepsilon(t))dx \\ &\quad + < \mathbf{f}_0(t+\tau) - \mathbf{f}_0(t), \mathbf{w}_\varepsilon(t+\tau) - \mathbf{w}_\varepsilon(t) >_{X_0} \\ &\leq ||\mathbf{f}_0(t+\tau) - \mathbf{f}_0(t)||_{X_0}^2 + ||\mathbf{w}_\varepsilon(t+\tau) - \mathbf{w}_\varepsilon(t)||_{X_0}^2, \end{aligned}$$

since $\sigma_0$ is monotone. Now, Gronwall's lemma yields

$$||\mathbf{w}_\varepsilon(t+\tau) - \mathbf{w}_\varepsilon(t)||_{X_0}^2 \leq K_T||\mathbf{f}_0||_{W^{1,2}((0,T),X_0)}^2\tau^2 + ||\mathbf{u}_{\varepsilon,\tau,0}||_{X_0}^2 \tag{7.142}$$

with some $K_T \in (0, \infty)$ independent of $\varepsilon, \tau$. By 7.141 this yields $\mathbf{w}_\varepsilon \in W^{1,\infty}((0,T), X_0)$ and

$$||\partial_t \mathbf{w}_\varepsilon||_{L^\infty((0,T),X_0)}^2 \leq K_T||\mathbf{f}_0||_{W^{1,2}((0,T),X_0)}^2 + ||\mu^{-1} \text{ curl } \mathbf{E}_0 + \partial_t \mathbf{g}_0(0)||_{L^2}^2.$$

Now, 7.135 follows from 7.138 and the previous estimate.
7.135 and 5.77 yield $\mathbf{w}_\varepsilon \in L^2((0,T), D(B))$, in particular

$$\begin{aligned} &\mathbf{E}_\varepsilon(t) \in \overset{0}{H}{}^2{}_{curl}\,(\Omega),\ \text{curl } (\mathbf{E}_\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T,L^2(\Omega)) \\ &\text{ and curl } (\mathbf{h}_\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^2(0,T,L^2(\Omega)). \end{aligned} \tag{7.143}$$

Next, let $\chi \in C_0^\infty(I\!R^3)$ with $\chi(x) = 1$ outside some bounded set and $\chi(x) = 0$ on $\mathcal{C}$ .
Since supp $\chi \subset \Omega_0$, 5.84 and 7.143 yield

$$\begin{aligned} &\text{div } (\chi\mathbf{E}_\varepsilon(t)) = \mathbf{E}_\varepsilon(t)\nabla\chi \in L^2(I\!R^3) \\ &\text{and curl } (\chi\mathbf{E}_\varepsilon(t)) = -\mathbf{E}_\varepsilon(t) \wedge \nabla\chi + \chi \text{ curl } \mathbf{E}(t) \in L^2(I\!R^3) \end{aligned}$$

This implies $\chi\mathbf{E}_\varepsilon(t)) \in H^1(I\!R^3) \subset L^6(I\!R^3)$. Hence $\mathbf{E}_\varepsilon(t) \in \overset{0}{H}{}^2{}_{curl}\,(\Omega) \cap L^6(\{|x| > R_0\})$. By 5.84 this yields $\mathbf{E}_\varepsilon(t) \in \mathcal{X}$. Finally, 7.136 follows from estimate 3.57, 5.84, 7.139 and 7.143.

□

**Theorem 4** *The unique solution* $(\mathbf{E}, \mathbf{h})$ *of problem 2.27 - 2.29 obeys* $\mathbf{E} \in L^2_{loc}([0,\infty), \mathcal{X})$ *and* $\mathbf{h} \in W^{1,\infty}_{loc}([0,\infty), L^2(\Omega))$.
*Moreover,*

$$\mathbf{E}_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} \mathbf{E} \text{ in } L^2((0,T), Y_1^{r_0} \cap L^2_\gamma(G)) \text{ weakly }, \tag{7.144}$$

$$\sigma_0(x, \mathbf{E}_\varepsilon) \stackrel{\varepsilon\to 0}{\longrightarrow} \sigma_0(x, \mathbf{E}) \text{ in } L^2((0,T), L^2_{\gamma^{-1}}(G)) \text{ weakly} \tag{7.145}$$

*and*

$$\mathbf{h}_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} \mathbf{h} \text{ in } W^{1,\infty}((0,T), L^2(\Omega)) \text{ weak } -* \tag{7.146}$$

*and in* $L^2((0,T), H^2_{curl}(\Omega))$ *weakly* .

**Proof:**
This follows immediately from lemma 10 and theorem 2.

□

**Corollary 1** *Suppose that* $I\!R^3 \setminus \overline{\Omega}$ *is also a Lipschitz-domain Then*

$$\mathbf{h}_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} \mathbf{h} \text{ in } C([0,T], L^2(\Omega \cap B_R)) \text{ strongly}$$

*for all* $R \in (0,\infty)$ *and* $T \in (0,\infty)$.

**Proof:** Choose $\chi \in C_0^\infty(I\!R^3)$ with $\chi(x) = 1$ on $B_R$. Then it follows form 6.116 and the result in [11] that for each bounded set $G \subset X_h \cap H^2_{curl}(\Omega)$ the set $\{\chi \cdot \mathbf{g} : \mathbf{g} \in G\}$ is precompact in $L^2(\Omega)$.
By 7.146 the set $(P\mathbf{h}_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2((0,T), X_h \cap H^2_{curl}(\Omega)) \cap W^{1,\infty}((0,T), L^2(\Omega))$ Hence, it follows for Arzela's theorem that $(\chi P\mathbf{h}_\varepsilon)_{\varepsilon>0}$ is precompact in $C([0,T], L^2(\Omega))$ for all $T \in [0,\infty)$.
Since $\mu^{-1}$ curl $\mathbf{E}_\varepsilon(t) \in X_h$, it follows from 5.73 that $(1-P)\mathbf{h}_\varepsilon(t) = (1-P)\mathbf{h}_0$. Therefore, $(\chi\mathbf{h}_\varepsilon)_{\varepsilon>0}$ is precompact in $C([0,T], L^2(\Omega))$ for all $T \in [0,\infty)$. Finally, the assertion follows from 7.146.

□

# 8 Appendix

In this section the extension theorem for the spaces $H^p_{curl}(\Omega)$ used in the proof of theorem 1 is given.

**Theorem 5** *Let $\Omega_1 \subset \mathbb{R}^3$ be a bounded Lipschitz-domain. Then there exists a bounded operator $T : L^1(\Omega_1, \mathbb{C}^3) \to L^1(\mathbb{R}^3, \mathbb{C}^3)$ with the following properties.*
*i) $(T\mathbf{w})(x) = \mathbf{w}(x)$ for all $\mathbf{w} \in L^1(\Omega_1), x \in \Omega_1$.*
*ii) Let $p \in [1, \infty)$. Then $T\mathbf{w} \in L^p(\mathbb{R}^3, \mathbb{C}^3)$ and $||T\mathbf{w}||_{L^p(\mathbb{R}^3)} \leq C_p ||\mathbf{w}||_{L^p(\Omega_1)}$ for all $\mathbf{w} \in L^p(\Omega_1)$.*
*iii) Let $p \in [1, \infty)$. Then $T\mathbf{w} \in H^p_{curl}(\mathbb{R}^3)$ and $||T\mathbf{w}||_{H^p_{curl}(\mathbb{R}^3)} \leq C_p ||\mathbf{w}||_{H^p_{curl}(\Omega_1)}$ for all $\mathbf{w} \in H^p_{curl}(\Omega_1)$.*

**Proof:**
By the assumptions on $\Omega_1$ there exist $U_0, U_1, .., U_M \subset \mathbb{R}^3$ and Bi-Lipschitz transformations $T_k : (-1, 1)^3 \to U_k$ with the properties

$$\Omega_1 \subset U_1 \cup ... \cup U_N, \quad \overline{U_0} \subset \Omega_1,$$

$$U_k \cap \Omega_1 = T_k(V_k) \text{ with } V_k \stackrel{\text{def}}{=} \{x \in (-1, 1)^3 : x_3 < 0\}$$

$$\text{and } U_k \cap \partial\Omega_1 = T_k(\{x \in (-1, 1)^3 : x_3 = 0\}).$$

Let $\chi_k \in C^\infty_0(U_k), k \in \{0, .., M\}$ be a partition of unity subordinate to the covering $U_0, ..., U_M$ of $\overline{\Omega}_1$, i.e. $\sum_{k=0}^M \chi_k = 1$ on $\overline{\Omega}_1$.
Now, suppose $\mathbf{w} \in L^1(\Omega_1, \mathbb{C}^3)$.
Define for $k = 1, .., M$ $\mathbf{f}_k \in L^1(V_k, \mathbb{C}^3)$ by

$$\mathbf{f}_k(y) \stackrel{\text{def}}{=} DT_k(y)^* \mathbf{w}(T_k(y)) \text{ for } y \in V_k.$$

Let $\mathbf{F}_k \in L^1((-1, 1)^3, \mathbb{C}^3)$ be an extension defined on $(-1, 1)^3$ by reflection at the plane $y_3 = 0$, i.e.

$$\mathbf{F}_k(y) \stackrel{\text{def}}{=} (\mathbf{f}_{k,1}(y_1, y_2, -y_3), \mathbf{f}_{k,2}(y_1, y_2, -y_3), -\mathbf{f}_{k,3}(y_1, y_2, -y_3)) \text{ if } y_3 \geq 0$$

$$\text{and } \mathbf{F}_k(y) \stackrel{\text{def}}{=} \mathbf{f}_k(y) \text{ if } y_3 \leq 0.$$

Finally, set

$$(T\mathbf{w})(x) \stackrel{\text{def}}{=} \chi_0(x)\mathbf{w}(x) + \sum_{k=1}^{M} \chi_k(x) D(T_k^{-1})(x)^* \mathbf{F}_k(T_k^{-1}(x)) \qquad (8.147)$$

for $x \in U_0 \cup .. \cup U_M$ and

$$(T\mathbf{w})(x) \stackrel{\text{def}}{=} 0 \text{ for } y \in \mathbb{R}^3 \setminus (U_0 \cup .. \cup U_M).$$

Now, suppose $x \in \Omega_1$. Then it follows from the definitions of $\mathbf{f}_k, \mathbf{F}_k$ and $T(\mathbf{w}) \in L^1(I\!R^3, \mathbb{C}^3)$ that

$$(T\mathbf{w})(x) = \chi_0(x)\mathbf{w}(x) + \sum_{k=1}^{M} \chi_k(x) D(T_k^{-1})(x)^* \mathbf{f}_k(T_k^{-1}(x))$$

$$= \sum_{k=0}^{M} \chi_k(x)\mathbf{w}(x) = \mathbf{w}(x),$$

i.e. T obeys i). ii) follows from the substitution formula.
Now, suppose $\mathbf{w} \in H^p_{curl}(\Omega_1)$.
As a consequence of the next lemma one has $\mathbf{f}_k \in H^p_{curl}(V_k)$. Moreover, the extension $\mathbf{F}_k \in L^p((-1,1)^3)$ of $\mathbf{f}_k$ defined by reflection obeys $\mathbf{F}_k \in H^p_{curl}((-1,1)^3)$. Finally, it follows from 8.147 and again from the next lemma that $T\mathbf{w} \in H^p_{curl}(I\!R^3)$.

□

In the previos theorem the following lemma is used.

**Lemma 11** *Let $U, V \subset I\!R^3$ be open sets, $\mathbf{w} \in L^p_{loc}(U)$ with curl $\mathbf{w} \in L^p_{loc}(U)$. Moreover, let $T : V \to U$ be a Bi-Lipschitz transformation.*
*Define*

$$\mathbf{f}(y) \stackrel{\text{def}}{=} DT(y)^* \mathbf{w}(T(y)) \text{ for } y \in V.$$

*Then $\mathbf{f} \in L^p_{loc}(V)$ with curl $\mathbf{f} \in L^p_{loc}(V)$ and*

$$(\text{ curl } \mathbf{f})(y) = M_T(y)(\text{ curl } \mathbf{w})(T(y)) \text{ for } y \in V, \tag{8.148}$$

*where $M_T \in L^\infty_{loc}(V, I\!R^{3\times 3})$ has the line-vectors $\mathbf{a}_1 \stackrel{\text{def}}{=} \partial_2 T \wedge \partial_3 T$, $\mathbf{a}_2 \stackrel{\text{def}}{=} \partial_3 T \wedge \partial_1 T$ and $\mathbf{a}_3 \stackrel{\text{def}}{=} \partial_1 T \wedge \partial_2 T$ .*

**Proof:**
By 8.148 and standard density arguments it suffices to consider $\mathbf{w} \in C^\infty(U, \mathbb{C}^3)$. Suppose $V_0$ is open, bounded with $\overline{V_0} \subset V$. Let $\omega_n \in C_0^\infty(I\!R^3)$ be a mollifier-sequence and define $T_n \stackrel{\text{def}}{=} \omega_n * T \in C^\infty(V_0)$. Since $T \in W^{1,\infty}(V)$, one has

$$DT_n = \omega_n * DT \stackrel{n\to\infty}{\longrightarrow} DT \text{ in } L^1(V_0) \text{ strongly,} \tag{8.149}$$

$$||T_n - T||_{L^\infty(V_0)} \stackrel{n\to\infty}{\longrightarrow} 0 \text{ and } ||DT_n||_{L^\infty(V_0)} \le K. \tag{8.150}$$

Define $\mathbf{f}_n \in C^\infty(V_0)$ by

$$\mathbf{f}_n(y) \stackrel{\text{def}}{=} DT_n(y)^* \mathbf{w}(T_n(y)) \text{ for } y \in V_0. \tag{8.151}$$

Then

$$( \text{ curl } \mathbf{f}_n)(y) = M_{T_n}(y)( \text{ curl } \mathbf{w})(T_n(y)) \text{ for } y \in V_0. \tag{8.152}$$

From 8.149 - 8.152 it follows that

$$\mathbf{f}_n \overset{\text{n}\to\infty}{\longrightarrow} DT_n^* \cdot (\mathbf{w} \circ T_n) \text{ and } \text{ curl } \mathbf{f}_n \overset{\text{n}\to\infty}{\longrightarrow} M_T \cdot (( \text{ curl } \mathbf{w}) \circ T_n) \tag{8.153}$$

in $L^1(V_0)$ strongly. This yields

$$\text{curl } \mathbf{f} = M_T \cdot (( \text{ curl } \mathbf{w}) \circ T_n) \in L^p(V_0)$$

in the sense of distributions.

□

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