Evaluation of the epi-convergence [1] [2]

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Abstract

The paper presents an equivalent characterization of the epi-convergence of lower semicontinuous functions. The proposed 'measures' naturally estimate the distance between optimal values of two optimization programs. A comparison of optimal solutions is more complex. We propose an estimation for a given function which can be taken as the excess of a set over another, being in metric space. Hence, we receive an estimate of the distance between optimal solutions.

1 The concept of the epi-convergence

Looking for the global minimum of a given function is the central problem discussed in the literature and solved in optimization theory. The lack of complete information on the objective function is the main trouble met in the practice. We have to work with approximations and, therefore, we naturally ask about the stability of our optimization problem provided varying objective function. The epi-convergence looks to be the most efficient tool for that, see [1], [2] or [8]. We present an equivalent description of the epi-convergence which can be used to derive estimate of the distance between optimal values. Treating of the optimal solutions is more difficult. We offer a procedure which is natural in the case of metric spaces, as the example illustrates.

Let us specify the subject of our treatment. We work on the Hausdorff topological space $\mathcal{X}$. Therefore, we will employ open, closed and compact sets of $\mathcal{X}$, the notion of nets, cluster points, limit points, Kuratowski-Painlevé convergence of sets, etc. For convenience, we are giving the list of the used notation in Appendix.

We consider the function $f : \mathcal{X} \to \overline{\mathbb{R}} = [-\infty, +\infty]$. Our interest is focused in continuity property of its optimal value $\varphi(f) = \inf_{x\in\mathcal{X}} f(x)$ and its set of optimal solutions $\psi(f) = \{x \in \mathcal{X} : f(x) = \varphi(f)\}$. Let us note that the problem of ε-optimal solutions do not need special treating. It is sufficient to consider truncated function $f_\varepsilon = \max\{f, \varphi(f) + \varepsilon\}$, as we do in the example.

Known observation is that $\psi(f)$ is closed set provided f is l.s.c. (lower semicontinuous). To avoid any misunderstanding let us recall that $f : \mathcal{X} \to \overline{\mathbb{R}}$ is l.s.c. if $\liminf_{y\to x} f(y) \geq f(x)$ for each $x \in \mathcal{X}$. Let us denote the set of all l.s.c. functions on $\mathcal{X}$ by $\mathsf{LSC}(\mathcal{X})$.

We consider the space $\mathsf{LSC}(\mathcal{X})$ with the epi-convergence.

Definition 1 *Let Λ be a directed set, $f_\lambda \in \mathsf{LSC}(\mathcal{X})$ for each $\lambda \in \Lambda$ and $f \in \mathsf{LSC}(\mathcal{X})$. We say that the net $< f_\lambda >_{\lambda\in\Lambda}$ epi-converges to f, provided* $\mathsf{epi}(f) = \mathsf{K\text{-}lim}_{\lambda\in\Lambda}\, \mathsf{epi}(f_\lambda)$.

Recall $\mathsf{epi}(f) = \{(x,\gamma) \in \mathcal{X} \times \mathbb{R} : \gamma \geq f(x)\}$ *and the definition of* $\mathsf{K\text{-}lim}$ *is remembered in Appendix.*

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[2] The paper has partially been written during a stay at the Humboldt University of Berlin.

The epi-convergence admits a helpful characterization by means of the nets.

Proposition 1 *Let $f_\lambda \in \mathsf{LSC}(\mathcal{X})$ for each λ belonging to the directed set Λ and $f \in \mathsf{LSC}(\mathcal{X})$. The net $< f_\lambda >_{\lambda\in\Lambda}$ epi-converges to f if and only if at each point $x \in \mathcal{X}$ both of the following conditions hold:*

1. *Let Ψ be a directed set and $\omega : \Psi \to \Lambda$ be monotone (i.e. $\psi_1, \psi_2 \in \Psi$, $\psi_1 \leq \psi_2 \Rightarrow \omega(\psi_1) \leq \omega(\psi_2)$) and confinal (i.e. for each $\lambda \in \Lambda$ there exists $\psi \in \Psi$ such that $\lambda \leq \omega(\psi)$). Then we have $\liminf_{\psi\in\Psi} f_{\omega(\psi)}(x_\psi) \geq f(x)$ whenever $\lim_{\psi\in\Psi} x_\psi = x$.*

2. *There exists directed set Ψ and $x_{(\lambda,\psi)} \in \mathcal{X}$ for each $\lambda \in \Lambda$, $\psi \in \Psi$ such that $\lim_{(\lambda,\psi)\in\Lambda\times\Psi} x_{(\lambda,\psi)} = x$ and $\lim_{(\lambda,\psi)\in\Lambda\times\Psi} f_\lambda(x_{(\lambda,\psi)}) = f(x)$.*

Proof: Fix the point $x \in \mathcal{X}$

1. Let $< f_\lambda >_{\lambda\in\Lambda}$ epi-converges to f.

   a) Let Ψ be a directed set and $\omega : \Psi \to \Lambda$ be monotone and confinal. Assume $\lim_{\psi\in\Psi} x_\psi = x$ and let us denote $\gamma = \liminf_{\psi\in\Psi} f_{\omega(\psi)}(x_\psi)$.
   Take the set $G \in \mathcal{G}(\mathcal{X} \times \overline{\mathbb{R}})$ with $(x, \gamma) \in G$. According to the definition of the $\liminf$ we have for each $\lambda \in \Lambda$ some $\psi \in \Psi$ such that $\lambda \leq \omega(\psi)$ and $(x_\psi, f_{\omega(\psi)}(x_\psi)) \in G$. Then $(x, \gamma) \in \mathsf{Ls}_{\lambda\in\Lambda}\, \mathsf{epi}\,(f_\lambda) = \mathsf{epi}\,(f)$ because of the epi-convergence. Therefore, $f(x) \leq \gamma$.

   b) According to the definition of the epi-convergence, $\mathsf{Li}_{\lambda\in\Lambda}\, \mathsf{epi}\,(f_\lambda) = \mathsf{epi}\,(f)$ and, hence, $(x, f(x)) \in \mathsf{Li}_{\lambda\in\Lambda}\, \mathsf{epi}\,(f_\lambda)$.
   Therefore, for each $G \in \mathcal{G}(\mathcal{X} \times \overline{\mathbb{R}})$ with $(x, f(x)) \in G$ there is $\lambda_0 \in \Lambda$ such that $G \cap \mathsf{epi}\,(f_\lambda) \neq \emptyset$ for each $\lambda_0 \leq \lambda$. Let us take $(x_{(\lambda,G)}, \gamma_{(\lambda,G)}) \in G \cap \mathsf{epi}\,(f_\lambda)$ for $\lambda_0 \leq \lambda$ and define $x_{(\lambda,G)} = x$, $\gamma_{(\lambda,G)} = f_\lambda(x)$ whenever $\lambda_0 \not\leq \lambda$.
   The set $\Psi = \Big\{G \in \mathcal{G}(\mathcal{X} \times \overline{\mathbb{R}}), (x, f(x)) \in G\Big\}$ is directed by inclusion, i.e.

   $$G \leq H \iff G \supset H\ .$$

   Then we have $\lim_{(\lambda,G)\in\Lambda\times\Psi} x_{(\lambda,G)} = x$ and $\lim_{(\lambda,G)\in\Lambda\times\Psi} \gamma_{(\lambda,G)} = f(x)$.
   That implies $\lim_{(\lambda,G)\in\Lambda\times\Psi} f_\lambda(x_{(\lambda,G)}) = f(x)$ since always $f_\lambda(x_{(\lambda,G)}) \leq \gamma_{(\lambda,G)}$ and $\liminf_{(\lambda,G)\in\Lambda\times\Psi} f_\lambda(x_{(\lambda,G)}) \geq f(x)$, according to the first part of the proof.

2. Let the functions f_λ, $\lambda \in \Lambda$ and f fulfill the conditions.

   a) Accordingly to the second property, we have the directed set Ψ and the convergent net $\lim_{(\lambda,\psi)\in\Lambda\times\Psi} x_{(\lambda,\psi)} = x$ with $\lim_{(\lambda,\psi)\in\Lambda\times\Psi} f_\lambda(x_{(\lambda,\psi)}) = f(x)$.
   Take the set $G \in \mathcal{G}(\mathcal{X} \times \overline{\mathbb{R}})$ with $(x, f(x)) \in G$. There are $\lambda_0 \in \Lambda$ and $\psi_0 \in \Psi$ such that $\Big(x_{(\lambda,\psi)}, f_\lambda(x_{(\lambda,\psi)})\Big) \in G \cap \mathsf{epi}\,(f_\lambda)$ for each $\lambda_0 \leq \lambda$ and each $\psi_0 \leq \psi$. That means $(x, f(x)) \in \mathsf{Li}_{\lambda\in\Lambda}\, \mathsf{epi}\,(f_\lambda)$.

   b) Take $(x, \gamma) \in \mathsf{Ls}_{\lambda\in\Lambda}\, \mathsf{epi}\,(f_\lambda)$. We set

   $$\Psi = \Big\{(\lambda, G) \in \Lambda \times \mathcal{G}(\mathcal{X} \times \overline{\mathbb{R}}) : \mathsf{epi}\,(f_\lambda) \cap G \neq \emptyset, (x, \gamma) \in G\Big\}\ .$$

The set Ψ is directed by the natural ordering

$$(\lambda_1, G_1) \leq (\lambda_2, G_2) \quad \Leftrightarrow \quad \lambda_1 \leq \lambda_2 \text{ and } G_1 \supset G_2$$

since (x,γ) is a cluster point. By the definition of Ψ we have the point $(x_{(\lambda,G)}, \gamma_{(\lambda,G)}) \in \mathsf{epi}\,(f_\lambda) \cap G$ for each $(\lambda, G) \in \Psi$. Then $\lim_{(\lambda,G)\in\Psi} x_{(\lambda,G)} = x$ and $\lim_{(\lambda,G)\in\Psi} \gamma_{(\lambda,G)} = \gamma$. According to the first property we conclude $\gamma \geq f(x)$ and therefore $(x,\gamma) \in \mathsf{epi}\,(f)$.

We proved $\mathsf{epi}\,(f) = \mathsf{Li}_{\lambda\in\Lambda}\, \mathsf{epi}\,(f_\lambda) = \mathsf{Ls}_{\lambda\in\Lambda}\, \mathsf{epi}\,(f_\lambda)$, which is the epi-convergence.

Q.E.D.

We write the paper to introduce "distances" between l.s.c. functions giving an equivalent characterization of the epi-convergence. These "distances" can be defined for each couple of real functions $f, g : \mathcal{X} \to \mathbb{R}$ and $A \subset \mathcal{X}$, $x \in \mathcal{X}$ by

$$\rho_1\,(g, f; A, x) = \sup\left\{(f(x) - g(y))_+ : y \in A\right\} \ ,$$

$$\rho_2\,(g, f; A, x) = \inf\left\{|f(x) - g(y)| : y \in A\right\} \ .$$

Proposition 2 *Let $u : \overline{\mathbb{R}} \to \mathbb{R}$ be increasing continuous, Λ be a directed set, $f_\lambda \in \mathsf{LSC}(\mathcal{X})$ for each $\lambda \in \Lambda$, $f \in \mathsf{LSC}(\mathcal{X})$ and for each point $x \in \mathcal{X}$ we have given the base $\mathcal{G}_x$.*

Then the net $< f_\lambda >_{\lambda\in\Lambda}$ epi-converges to f if and only if

$$\lim_{G\in\mathcal{G}_x} \limsup_{\lambda\in\Lambda} \rho_1\,(u\circ f_\lambda, u\circ f; G, x) = 0 \quad \textit{for each} \quad x \in \mathcal{X} \ ,$$

$$\lim_{\lambda\in\Lambda} \rho_2\,(u\circ f_\lambda, u\circ f; G, x) = 0 \quad \textit{for each } G \in \mathcal{G}_x \ , \ x \in \mathcal{X} \ .$$

Proof: The function u naturally gives the bijection between $\mathsf{LSC}(\mathcal{X})$ and $\mathsf{LSC}(\mathcal{X}) \cap \left\{f : \mathcal{X} \to u(\overline{\mathbb{R}})\right\}$ preserving the epi-convergence. Therefore, it is sufficient to consider the nets of l.s.c. functions with real values, only, and show for them the characterization by nets given in the proposition 1. Fix the point $x \in \mathcal{X}$ for that.

1. We show the equivalence between the first property of the proposition 1 and the first property of the proposition 2.

   a) Let $\lim_{G\in\mathcal{G}_x} \limsup_{\lambda\in\Lambda} \rho_1\,(f_\lambda, f; G, x) = 0$.
   Let Ψ be directed set, $\omega : \Psi \to \Lambda$ be monotone and confinal, and $\lim_{\psi\in\Psi} x_\psi = x$. For $G \in \mathcal{G}_x$ we have $\psi_0 \in \Psi$ such that $x_\psi \in G$ and

   $$\rho_1\left(f_{\omega(\psi)}, f; G, x\right) \geq \left(f(x) - f_{\omega(\psi)}(x_\psi)\right)_+ \quad \text{for each} \quad \psi_0 \leq \psi \ .$$

   Accordingly to the assumption, we have

   $$\lim_{G\in\mathcal{G}_x} \limsup_{\psi\in\Psi} \rho_1\left(f_{\omega(\psi)}, f; G, x\right) = 0$$

   and, therefore,

   $$\lim_{\psi\in\Psi} \left(f(x) - f_{\omega(\psi)}(x_\psi)\right)_+ = 0 \ .$$

   That is nothing else than $\liminf_{\psi\in\Psi} f_{\omega(\psi)}(x_\psi) \geq f(x)$.

b) Let for each directed set Ψ, $\omega : \Psi \to \Lambda$ monotone and confinal, and $\lim_{\psi \in \Psi} x_\psi = x$, we have $\liminf_{\psi \in \Psi} f_{\omega(\psi)}(x_\psi) \geq f(x)$.
For each $\lambda \in \Lambda$, $G \in \mathcal{G}_x$ and $0 < \Delta < 1$ there exist $x_{(\lambda,G,\Delta)} \in G$ such that

$$\rho_1(f_\lambda, f; G, x) < \Big(f(x) - f_\lambda(x_{(\lambda,G,\Delta)})\Big)_+ + \Delta \ .$$

Setting $\Psi = \Lambda \times \mathcal{G}_x \times (0,1)$ we receive a set directed by the ordering

$$(\lambda_1, G_1, \Delta_1) \leq (\lambda_2, G_2, \Delta_2) \Longleftrightarrow \lambda_1 \leq \lambda_2, G_1 \supset G_2, \Delta_1 \geq \Delta_2$$

and $\lim_{(\lambda,G,\Delta)\in\Psi} x_{(\lambda,G,\Delta)} = x$. Therefore according to the assumption,

$$\liminf_{(\lambda,G,\Delta)\in\Psi} f_\lambda(x_{(\lambda,G,\Delta)}) \geq f(x) \ .$$

Then $\lim_{(\lambda,G,\Delta)\in\Psi} \Big(f(x) - f_\lambda(x_{(\lambda,G,\Delta)})\Big)_+ = 0$ and, consequently, $\lim_{G\in\mathcal{G}_x} \limsup_{\lambda\in\Lambda} \rho_1(f_\lambda, f; G, x) = 0$.

2. We show the equivalence between the second property of the proposition 1 and the second property of the proposition 2.

a) Let $\lim_{\lambda\in\Lambda} \rho_2(f_\lambda, f; G, x) = 0$ for each $G \in \mathcal{G}_x$.
Then for each $\lambda \in \Lambda$, $G \in \mathcal{G}_x$ and $0 < \Delta < 1$ there exist $x_{(\lambda,G,\Delta)} \in G$ such that

$$\rho_2(f_\lambda, f; G, x) > \Big|f(x) - f_\lambda(x_{(\lambda,G,\Delta)})\Big| - \Delta \ .$$

Consequently, $\lim_{(\lambda,G,\Delta)\in\Lambda\times\mathcal{G}_x\times(0,1)} x_{(\lambda,G,\Delta)} = x$ and

$$\lim_{(\lambda,G,\Delta)\in\Lambda\times\mathcal{G}_x\times(0,1)} f_\lambda(x_{(\lambda,G,\Delta)}) = f(x) \ .$$

b) Let Ψ be a directed set and $x_{(\lambda,\psi)} \in \mathcal{X}$ be such that $\lim_{(\lambda,\psi)\in\Lambda\times\Psi} x_{(\lambda,\psi)} = x$ and $\lim_{(\lambda,\psi)\in\Lambda\times\Psi} f_\lambda(x_{(\lambda,\psi)}) = f(x)$.
Let $G \in \mathcal{G}_x$. Then there are $\lambda_0 \in \Lambda$ and $\psi_0 \in \Psi$ such that $x_{(\lambda,\psi)} \in G$ for each $\lambda_0 \leq \lambda$ and $\psi_0 \leq \psi$. Hence,

$$\rho_2(f_\lambda, f; G, x) \leq \Big|f(x) - f_\lambda(x_{(\lambda,\psi)})\Big|$$

for each $\lambda_0 \leq \lambda$ and $\psi_0 \leq \psi$.
Consequently, $\lim_{\lambda\in\Lambda} \rho_2(f_\lambda, f; G, x) = 0$.

Q.E.D.

If the space $\mathcal{X}$ is a metric space then we can receive characterizations of the epi-convergence by means of the Hausdorff distance of closed sets, see [2] and [9].

In general, the epi-convergence on $\mathsf{LSC}(\mathcal{X})$ is not induced by any topology. If the space $\mathcal{X}$ is first countable Hausdorff space then epi-convergence coincides with the convergence induced by Fell topology on closed subsets in $\mathcal{X} \times \overline{\mathbb{R}}$, see [5], theorem 5.2.10, p.140. If the space $\mathcal{X}$ is a finite dimensional space then there is a metric inducing the epi-convergence, see [2] or [5], p.161.

2 Stability of the optimization program

The epi-convergence implies consistency of the optimization program.

Theorem 1 *Let $f, f_\lambda \in \mathsf{LSC}(\mathcal{X})$ for each $\lambda \in \Lambda$, the net $< f_\lambda >_{\lambda\in\Lambda}$ epi-converges to f and $\sup_{K\in\mathcal{K}(\mathcal{X})} \inf_{y\notin K} \inf_{\lambda\in\Lambda} f_\lambda(y) > \varphi(f)$.*

Then $\lim_{\lambda\in\Lambda} \varphi(f_\lambda) = \varphi(f)$ and there exist $\lambda_G \in \Lambda$ for any $G \in \mathcal{G}(\mathcal{X})$ with $\psi(f) \subset G$ such that $\psi(f_\lambda) \subset G$ for each $\lambda \geq \lambda_G$, consequently $\mathsf{Ls}_{\lambda\in\Lambda} \psi(f_\lambda) \subset \psi(f)$. Moreover, $\psi(f)$ is a non-empty compact and there is a $\lambda_0 \in \Lambda$ such that $\psi(f_\lambda)$ is a non-empty compact for each $\lambda_0 \leq \lambda \in \Lambda$.

Proof: See [1] or [5], theorem 5.3.6, p.160.

Q.E.D.

Provided compact space $\mathcal{X}$, the statement can be expressed in the following way: the function φ is continuous and the multifunction ψ is upper semicontinuous. The assumptions of the theorem 1 can be easily fulfilled for convex functions, see [6].

We are interested not only in consistency, we would like to estimate the rate of consistency. To receive an estimate of the distance between the optimal values we need the following simple lemma.

Lemma 1 *If $f, g \in \mathsf{LSC}(\mathcal{X})$ and $u : \overline{\mathbb{R}} \to \mathbb{R}$ is increasing continuous then we have the estimates*

$$-\rho_1(u\circ g, u\circ f; A, \hat{x}) \leq u\circ\varphi(g) - u\circ\varphi(f) \leq \rho_2(u\circ g, u\circ f; \mathcal{X}, \hat{x}) \ ,$$

whenever $\hat{x} \in \psi(f)$ and $A \cap \psi(g) \neq \emptyset$.

Proof: Let $\hat{x} \in \psi(f)$ and $\hat{y} \in A \cap \psi(g)$. Then we have the estimates

$$u\circ\varphi(f) - u\circ\varphi(g) = u\circ f(\hat{x}) - u\circ g(\hat{y}) \leq \rho_1(u\circ g, u\circ f; A, \hat{x})$$

and for each $x \in \mathcal{X}$

$$u\circ\varphi(g) - u\circ\varphi(f) = u\circ g(\hat{y}) - u\circ f(\hat{x}) \leq$$

$$\leq u\circ g(x) - u\circ f(\hat{x}) \leq |u\circ g(x) - u\circ f(\hat{x})| \ .$$

Minimizing along $x \in \mathcal{X}$ we receive the estimate

$$u\circ\varphi(g) - u\circ\varphi(f) \leq \rho_2(u\circ g, u\circ f; \mathcal{X}, \hat{x}) \ .$$

Q.E.D.

Theorem 2 *Let $f, f_\lambda \in \mathsf{LSC}(\mathcal{X})$ for each $\lambda \in \Lambda$, $\sup_{K\in\mathcal{K}(\mathcal{X})} \inf_{y\notin K} \inf_{\lambda\in\Lambda} f_\lambda(y) > \varphi(f)$, $\mathcal{G}_x$ be a given base at each point $x \in \mathcal{X}$ and $u : \overline{\mathbb{R}} \to \mathbb{R}$ be increasing continuous.*

Let

$$\lim_{G\in\mathcal{G}_x} \limsup_{\lambda\in\Lambda} \rho_1(u\circ f_\lambda, u\circ f; G, x) = 0 \quad \textit{for each} \quad x \in \mathcal{X} \ ,$$

$$\lim_{\lambda\in\Lambda}\rho_2\left(u\circ f_\lambda, u\circ f; G, x\right)=0 \quad \textit{for each} \quad x\in\mathcal{X},\ G\in\mathcal{G}_x$$

and let $G_x\in\mathcal{G}_x$, $\lambda_x\in\Lambda$ *for* $x\in\psi(f)$, $0\le D_\lambda<+\infty$ *for* $\lambda\in\Lambda$ *be such that*

$$\rho_1\left(u\circ f_\lambda, u\circ f; G_x, x\right)\le D_\lambda\ ,\quad \rho_2\left(u\circ f_\lambda, u\circ f; \mathcal{X}, x\right)\le D_\lambda$$

for each $\lambda\ge\lambda_x$, $x\in\psi(f)$.

Then there is $\lambda_0\in\Lambda$ *such that* $|u\circ\varphi(f_\lambda)-u\circ\varphi(f)|\le D_\lambda$ *for each* $\lambda\ge\lambda_0$ *and there exist* $\lambda_G\in\Lambda$ *for any* $G\in\mathcal{G}(\mathcal{X})$ *with* $\psi(f)\subset G$ *such that* $\psi(f_\lambda)\subset G$ *for each* $\lambda\ge\lambda_G$, *consequently* $\mathsf{Ls}_{\lambda\in\Lambda}\,\psi(f_\lambda)\subset\psi(f)$. *Moreover,* $\psi(f)$ *is a non-empty compact and there is* $\lambda_1\in\Lambda$ *such that* $\psi(f_\lambda)$ *is a non-empty compact for each* $\lambda_1\le\lambda\in\Lambda$.

Proof: According to the proposition 2, the net of functions epi-converges and, therefore, almost all assertions of the theorem are contained in the theorem 1. We have to prove the estimation of the distance between optimal values, only. From the theorem 1 we know that the set $\psi(f)$ is a non-empty compact. Then we can select a finite set $I\subset\psi(f)$ such that $\psi(f)\subset\bigcup_{x\in I}\operatorname{int}G_x$ since $\psi(f)\subset\bigcup_{x\in\psi(f)}\operatorname{int}G_x$. Then there is $\lambda_0\in\Lambda$ such that $\psi(f_\lambda)\subset\bigcup_{x\in I}\operatorname{int}G_x$ for each $\lambda_0\le\lambda$, according to the theorem 1. The stated assertion is straightforward, now, since the set I is finite and

$$-\max_{x\in I}\rho_1\left(u\circ f_\lambda, u\circ f; G_x, \hat{x}\right)\le u\circ\varphi(f_\lambda)-u\circ\varphi(f)\le\rho_2\left(u\circ f_\lambda, u\circ f; \mathcal{X}, \hat{x}\right)\ ,$$

accordingly to the lemma 1. The point $\hat{x}\in\psi(f)$ can be chosen arbitrarily.

Q.E.D.

The shown statement is better than the statement based on the uniform topology, assuming an open set $G\supset\psi(f)$ and $\lambda_0\in\Lambda$ such that

$$\sup_{y\in G}|f_\lambda(y)-f(y)|\le D_\lambda \quad \text{for each } \lambda\ge\lambda_0.$$

Then for each $x\in\psi(f)$ there is $G_x\in\mathcal{G}_x$ such that $G_x\subset G$. For each $y\in G_x$ we have the estimate

$$\left(f(x)-f_\lambda(y)\right)_+\le\left(f(y)-f_\lambda(y)\right)_+\le|f(y)-f_\lambda(y)| \text{ since } x\in\psi(f)$$

and, hence,

$$\rho_1\left(f_\lambda, f; G_x, x\right)\le\sup_{y\in G}|f_\lambda(y)-f(y)|\le D_\lambda\ .$$

Further,

$$\rho_2\left(f_\lambda, f; \mathcal{X}, x\right)\le|f_\lambda(x)-f(x)|\le\sup_{y\in G}|f_\lambda(y)-f(y)|\le D_\lambda\ .$$

The distance of the optimal solution can be estimated using the following theorem.

Theorem 3 *Let* $f, f_\lambda\in\mathsf{LSC}(\mathcal{X})$ *for each* $\lambda\in\Lambda$, $\sup_{K\in\mathcal{K}(\mathcal{X})}\inf_{y\notin K}\inf_{\lambda\in\Lambda}f_\lambda(y)>\varphi(f)$, $\mathcal{G}_x$ *be a given base at each point* $x\in\mathcal{X}$ *and* $u:\overline{\mathbb{R}}\to\mathbb{R}$ *be increasing continuous.*

Let

$$\lim_{G\in\mathcal{G}_x}\limsup_{\lambda\in\Lambda}\rho_1\left(u\circ f_\lambda, u\circ f; G, x\right)=0 \quad \textit{for each} \quad x\in\mathcal{X}\ ,$$

$$\lim_{\lambda\in\Lambda} \rho_2 \left(u\circ f_\lambda, u\circ f; G, x\right) = 0 \quad \textit{for each} \quad x\in\mathcal{X},\ G\in\mathcal{G}_x\ .$$

Let us consider a given function $\mathsf{d} : \bigcup_{x\in\mathcal{X}} (\mathcal{G}_x \times \{x\}) \to [0,+\infty)$. *Suppose, we have functions* $\beta : [0,+\infty) \to [0,+\infty)$, $\gamma : [0,+\infty)\times\Lambda \to [0,+\infty)$, $\lambda_0 \in \Lambda$, *an open set* $Q \supset \psi(f)$ *and* $G_{x,\lambda} \in \mathcal{G}_x$ *for each* $x \in Q$, $\lambda \geq \lambda_0$, *such that* $G_{x,\lambda} \cap \psi(f) \neq \emptyset$,

$$u\circ f(x) - u\circ\varphi(f) \geq \beta\left(\mathsf{d}\left(G_{x,\lambda}; x\right)\right)\ ,$$

$$u\circ f(x) - u\circ\varphi(f) - u\circ f_\lambda(x) + u\circ\varphi(f_\lambda) \leq \gamma\left(\mathsf{d}\left(G_{x,\lambda}; x\right); \lambda\right)$$

for each $x\in Q$, $\lambda\geq\lambda_0$.

Then $\lim_{\lambda\in\Lambda}\varphi(f_\lambda) = \varphi(f)$ *and there exist* $\lambda_G\in\Lambda$ *for any* $G\in\mathcal{G}(\mathcal{X})$ *with* $\psi(f)\subset G$ *such that* $\psi(f_\lambda)\subset G$ *for each* $\lambda\geq\lambda_G$, *consequently* $\mathsf{Ls}_{\lambda\in\Lambda}\psi(f_\lambda)\subset\psi(f)$. *The set* $\psi(f)$ *is a non-empty compact and there is* $\lambda_1\in\Lambda$ *such that* $\psi(f_\lambda)$ *is a non-empty compact for each* $\lambda_1\leq\lambda\in\Lambda$. *Moreover, there is* $\lambda_2\in\Lambda$ *such that*

$$\beta\left(\mathsf{d}\left(G_{x,\lambda}; x\right)\right) \leq \gamma\left(\mathsf{d}\left(G_{x,\lambda}; x\right); \lambda\right)$$

for each $x\in\psi(f_\lambda)$, $\lambda\geq\lambda_2$.

Proof: According to the proposition 2, the net of functions epi-convergences and, consequently, the assumptions of the theorem 1 are fulfilled. Therefore, almost all assertions of the theorem are contained in the theorem 1 and we have to show the estimate for the function d, only.

There is $\lambda_2\in\Lambda$ such that $\psi(f_\lambda)\subset Q$ for each $\lambda\geq\lambda_2$. Taking $x\in\psi(f_\lambda)$, $\lambda\geq\lambda_2$, we receive the estimate

$$0 = u\circ f(x) - u\circ\varphi(f) - \Big(u\circ f(x) - u\circ\varphi(f) - u\circ f_\lambda(x) + u\circ\varphi(f_\lambda)\Big) \geq$$

$$\geq \beta\left(\mathsf{d}\left(G_{x,\lambda}; x\right)\right) - \gamma\left(\mathsf{d}\left(G_{x,\lambda}; x\right); \lambda\right)\ .$$

The received estimate coincides with the statement of the theorem.

Q.E.D.

The additional assumptions of the theorem are realistic. The function β represents so called "growth condition" on the limit function. Such a condition is probably almost necessary. The second assumption of the theorem 3 can be fulfilled, for example, using 'Lipschitz norm'

$$\mathsf{Lip}\,(g; A) = \inf\left\{L\in[0,+\infty] : |g(x)-g(y)| \leq L\gamma\left(\mathsf{d}\left(G; x\right)\right), x\in A, G\in\mathcal{G}_x, y\in G\right\}\ .$$

Let $Q\supset\psi(f)$ be an open set and $\lambda_0\in\Lambda$ be such that $Q\supset\psi(f_\lambda)$ for each $\lambda\geq\lambda_0$. Taking $x\in Q$, $G\in\mathcal{G}_x$, $G\subset Q$ and $\hat{x}\in\psi(f)\cap G$, we receive the estimate

$$u\circ f(x) - u\circ\varphi(f) - u\circ f_\lambda(x) + u\circ\varphi(f_\lambda) \leq$$

$$\leq u\circ f(x) - u\circ f(\hat{x}) + u\circ f_\lambda(\hat{x}) - u\circ f_\lambda(x) \leq \mathsf{Lip}\,(u\circ f_\lambda - u\circ f; Q)\,\gamma\left(\mathsf{d}\left(G; x\right)\right)\ .$$

3 Metric spaces

This chapter is written to show the meaning of the theorem 3. Let $\mathcal{X}$ be a metric space with the metric ρ. We take the collection of closed balls $\mathcal{V}(x,\varepsilon) = \{y \in \mathcal{X} : d(x,y) \leq \varepsilon\}$ for $\varepsilon > 0$ as the base at the point x and $\mathsf{d}(\mathcal{V}(x,\varepsilon);x) = \varepsilon$. Further, we employ the excess of the set A over the set B given by $\mathsf{excess}(A,B) = \sup_{a\in A} \inf_{b\in B} \rho(a,b)$.

Theorem 4 *Let $f, f_\lambda \in \mathsf{LSC}(\mathcal{X})$ for each $\lambda \in \Lambda$, $\sup_{K\in\mathcal{K}(\mathcal{X})} \inf_{y\notin K} \inf_{\lambda\in\Lambda} f_\lambda(y) > \varphi(f)$ and $u : \overline{\mathbb{R}} \to \mathbb{R}$ be increasing continuous. Let*

$$\lim_{\varepsilon\to 0+} \limsup_{\lambda\in\Lambda} \rho_1(u\circ f_\lambda, u\circ f; \mathcal{V}(x,\varepsilon), x) = 0 \quad \textit{for each} \quad x \in \mathcal{X} \ ,$$

$$\lim_{\lambda\in\Lambda} \rho_2(u\circ f_\lambda, u\circ f; \mathcal{V}(x,\varepsilon), x) = 0 \quad \textit{for each} \quad x \in \mathcal{X},\ \varepsilon > 0 \ .$$

Suppose, we have functions $\beta, \gamma : [0,+\infty) \to [0,+\infty)$, $\lambda_0 \in \Lambda$ and an open set $Q \supset \psi(f)$ such that

$$u\circ f(x) - u\circ\varphi(f) \geq \beta(\mathsf{excess}(\{x\},\psi(f))) \ ,$$

$$u\circ f(x) - u\circ\varphi(f) - u\circ f_\lambda(x) + u\circ\varphi(f_\lambda) \leq \gamma(\mathsf{excess}(\{x\},\psi(f));\lambda)$$

for each $x \in Q$, $\lambda \geq \lambda_0$.

Then $\lim_{\lambda\in\Lambda} \varphi(f_\lambda) = \varphi(f)$ and there exist $\lambda_G \in \Lambda$ for any $G \in \mathcal{G}(\mathcal{X})$ with $\psi(f) \subset G$ such that $\psi(f_\lambda) \subset G$ for each $\lambda \geq \lambda_G$, consequently $\mathsf{Ls}_{\lambda\in\Lambda}\psi(f_\lambda) \subset \psi(f)$. The set $\psi(f)$ is a non-empty compact and there is $\lambda_1 \in \Lambda$ such that $\psi(f_\lambda)$ is a non-empty compact for each $\lambda_1 \leq \lambda \in \Lambda$. Moreover, there is $\lambda_2 \in \Lambda$ such that

$$\beta(\mathsf{excess}(\psi(f_\lambda),\psi(f))) \leq \gamma(\mathsf{excess}(\psi(f_\lambda),\psi(f));\lambda)$$

for each $\lambda \geq \lambda_2$.

Proof: The assertion is a easy consequence of the theorem 3. It is sufficient to set $G_{x,\lambda} = \mathcal{V}(x, \mathsf{excess}(\{x\},\psi(f)))$ and $\mathsf{d}(\mathcal{V}(x,\varepsilon);x) = \varepsilon$ and consider that always there is a point $x \in \psi(f_\lambda)$ such that $\mathsf{excess}(\{x\},\psi(f)) = \mathsf{excess}(\psi(f_\lambda),\psi(f))$.

Q.E.D.

4 Example

Let us give a simple example illustrating the subject. Consider the functions $f_n(x) = (x-\alpha_n)^2 + \beta_n$, $n \in \mathbb{N}$ and $f(x) = x^2$, where $\alpha_n \to 0$ and $\beta_n \to 0$.

Evidently, f, f_n are l.s.c. and $\varphi(f_n) = \beta_n$, $\psi(f_n) = \{\alpha_n\}$, $\varphi(f) = 0$, $\psi(f) = \{0\}$.

Our 'measures' are

$$\rho_1(f_n, f; [x-\varepsilon, x+\varepsilon], x) = \begin{cases} \left(x^2 - (|x-\alpha_n| - \varepsilon)^2 - \beta_n\right)_+ & \text{if} \quad |x-\alpha_n| > \varepsilon, \\ \left(x^2 - \beta_n\right)_+ & \text{if} \quad |x-\alpha_n| \leq \varepsilon \end{cases}$$

and

$$\rho_2(f_n, f; [x-\varepsilon, x+\varepsilon], x) \leq |f(x) - f_n(x)| = |x^2 - (x-\alpha_n)^2 - \beta_n| \ .$$

$$\lim_{n\to+\infty} \rho_1\left(f_n, f; [x-\varepsilon, x+\varepsilon], x\right) = \begin{cases} x^2 - (|x|-\varepsilon)^2 & \text{if} \quad |x| > \varepsilon, \\ x^2 & \text{if} \quad |x| \le \varepsilon, \end{cases}$$

Hence,

$$\lim_{\varepsilon\to 0+} \limsup_{n\to+\infty} \rho_1\left(f_n, f; [x-\varepsilon, x+\varepsilon], x\right) = 0 \quad \forall x \in \mathbb{R},$$

$$\lim_{n\to+\infty} \rho_2\left(f_n, f; [x-\varepsilon, x+\varepsilon], x\right) = 0 \quad \forall x \in \mathbb{R}, \varepsilon > 0.$$

Consequently, the functions f_n epi-converges to f.

Accordingly to the theorem 2, we have $|\varphi\left(f_n\right) - \varphi\left(f\right)| \le |\beta_n|$ for each $n \in \mathbb{N}$ since

$$\sup_{t>0} \inf_{|y|>t} \inf_{n\in\mathbb{N}} f_n(y) \ge \sup_{t\ge 2A} \left(\frac{1}{4}t^2 + \inf_{n\in\mathbb{N}} \beta_n\right) = +\infty \ , \ \text{where } A = \sup_{n\in\mathbb{N}} |\alpha_n| \ ,$$

$$\rho_1\left(f_n, f; [-\varepsilon, \varepsilon], 0\right) \le \max\left\{(-(|\alpha_n|-\varepsilon)^2 - \beta_n)_+, (-\beta_n)_+\right\} = (\beta_n)_- \ ,$$

$$\rho_2\left(f_n, f; \mathbb{R}, 0\right) \le |f(0) - f_n(\alpha_n)| = |\beta_n|$$

for each $n \in \mathbb{N}$ and $\varepsilon > 0$.

Accordingly to the theorem 3, we have $\mathsf{excess}(\psi\left(f_n\right), \psi\left(f\right)) \le 2|\alpha_n|$ for each $n \in \mathbb{N}$ since

$$f(x) - \varphi\left(f\right) = x^2 = \left(\mathsf{excess}(\{x\}, \psi\left(f\right))\right)^2 \ ,$$

$$f(x) - \varphi\left(f\right) - f_n(x) + \varphi\left(f_n\right) = x^2 - (x-\alpha_n)^2 = 2\alpha_n x - \alpha_n^2 \le 2|\alpha_n|\mathsf{excess}(\{x\}, \psi\left(f\right))$$

for each $n \in \mathbb{N}$ and each $x \in \mathbb{R}$. Here, $\beta(t) = t^2$ and $\gamma(t, n) = 2|\alpha_n|t$ are the functions assumed in the theorem 3.

Now, we are interested in behavior of the Δ_n-optimal sets

$$\psi\left(f_n, \Delta_n\right) = \{x \in \mathbb{R} : f_n(x) \le \varphi\left(f_n\right) + \Delta_n\} \ ,$$

provided $0 \le \Delta_n < +\infty$ and $\Delta_n \to \Delta < +\infty$.

Let us define $g(x) = \max\{f(x), \varphi\left(f\right) + \Delta\}$ and $g_n(x) = \max\{f_n(x), \varphi\left(f_n\right) + \Delta_n\}$. Then our 'measures' are

$$\rho_1\left(g_n, g; [x-\varepsilon, x+\varepsilon], x\right) = \begin{cases} \left(x^2 - (|x-\alpha_n|-\varepsilon)^2 - \beta_n\right)_+ & \text{if} \quad |x-\alpha_n| > \varepsilon + \sqrt{\Delta_n} \\ & \qquad \text{and } |x| > \sqrt{\Delta}, \\ \left(x^2 - \Delta_n - \beta_n\right)_+ & \text{if} \quad |x-\alpha_n| \le \varepsilon + \sqrt{\Delta_n} \\ & \qquad \text{and } |x| > \sqrt{\Delta}, \\ \left(\Delta - (|x-\alpha_n|-\varepsilon)^2 - \beta_n\right)_+ & \text{if} \quad |x-\alpha_n| > \varepsilon + \sqrt{\Delta_n} \\ & \qquad \text{and } |x| \le \sqrt{\Delta}, \\ \left(\Delta - \Delta_n - \beta_n\right)_+ & \text{if} \quad |x-\alpha_n| \le \varepsilon + \sqrt{\Delta_n} \\ & \qquad \text{and } |x| \le \sqrt{\Delta}, \end{cases}$$

and

$$\rho_2\left(g_n, g; [x-\varepsilon, x+\varepsilon], x\right) \le |g(x) - g_n(x)| =$$

$$= \begin{cases} |x^2 - (x-\alpha_n)^2 - \beta_n| & \text{if} \quad |x-\alpha_n| > \sqrt{\Delta_n}, \ |x| > \sqrt{\Delta}, \\ |x^2 - \Delta_n - \beta_n| & \text{if} \quad |x-\alpha_n| \le \sqrt{\Delta_n}, \ |x| > \sqrt{\Delta}, \\ |\Delta - (x-\alpha_n)^2 - \beta_n| & \text{if} \quad |x-\alpha_n| > \sqrt{\Delta_n}, \ |x| \le \sqrt{\Delta}, \\ |\Delta - \Delta_n - \beta_n| & \text{if} \quad |x-\alpha_n| \le \sqrt{\Delta_n}, \ |x| \le \sqrt{\Delta}. \end{cases}$$

Hence,

$$\lim_{n\to+\infty}\rho_1\left(g_n,g;[x-\varepsilon,x+\varepsilon],x\right)=\begin{cases} x^2-(|x|-\varepsilon)^2 & \text{if} \quad |x|>\varepsilon+\sqrt{\Delta},\\ x^2-\Delta & \text{if} \quad \sqrt{\Delta}<|x|\le\varepsilon+\sqrt{\Delta},\\ 0 & \text{if} \quad |x|\le\sqrt{\Delta},\end{cases}$$

Hence,

$$\lim_{\varepsilon\to 0+}\limsup_{n\to+\infty}\rho_1\left(g_n,g;[x-\varepsilon,x+\varepsilon],x\right)=0 \quad \forall x\in\mathbb{R},$$

$$\lim_{n\to+\infty}\rho_2\left(g_n,g;[x-\varepsilon,x+\varepsilon],x\right)=0 \quad \forall x\in\mathbb{R},\varepsilon>0.$$

Consequently, the functions g_n epi-converges to g.

Accordingly to the theorem 2, we have $|\varphi(g_n)-\varphi(g)|\le|\Delta-\Delta_n-\beta_n|$ for each $n\in\mathbb{N}$ since

$$\sup_{t>0}\inf_{|y|>t}\inf_{n\in\mathbb{N}}g_n(y)\ge\sup_{t\ge 2A}\left(\frac{1}{4}t^2+\inf_{n\in\mathbb{N}}\beta_n\right)=+\infty\ ,$$

where $A=\sup_{n\in\mathbb{N}}\left\{|\alpha_n|+\sqrt{\Delta_n}\right\}$,

$$\rho_1\left(g_n,g;[-\varepsilon,\varepsilon],x\right)\le(\Delta-\Delta_n-\beta_n)_+\ ,$$

$$\rho_2\left(g_n,g;\mathbb{R},x\right)\le|g(x)-g_n(\alpha_n)|=|\Delta-\Delta_n-\beta_n|$$

for each $n\in\mathbb{N}$, $x\in\psi(g)$, i.e. $|x|\le\sqrt{\Delta}$, and $\varepsilon>0$.

Estimating the excess between the sets of optimal solutions we have to establish the functions β and γ required in the theorem 4.

Let $x\notin\psi(g)$, i.e. $|x|>\sqrt{\Delta}$ and $\mathsf{excess}(\{x\},\psi(g))=|x|-\sqrt{\Delta}$. That we have

$$g(x)-\varphi(g)=x^2-\Delta=\beta(\mathsf{excess}(\{x\},\psi(g)))\ ,$$

where $\beta(t)=t^2+2t\sqrt{\Delta}$. The other difference is

$$g(x)-\varphi(g)-g_n(x)+\varphi(g_n)=\begin{cases} x^2-\Delta & \text{if} \quad |x-\alpha_n|\le\sqrt{\Delta_n},\\ x^2-(x-\alpha_n)^2-\Delta+\Delta_n & \text{if} \quad |x-\alpha_n|>\sqrt{\Delta_n}.\end{cases}$$

Hence, we estimate

$$g(x)-\varphi(g)-g_n(x)+\varphi(g_n)\le\begin{cases} x^2-\Delta & \text{if} \quad ||x|-|\alpha_n||\le\sqrt{\Delta_n},\\ x^2-(|x|-|\alpha_n|)^2-\Delta+\Delta_n & \text{if} \quad ||x|-|\alpha_n||>\sqrt{\Delta_n},\end{cases}$$

and

$$g(x)-\varphi(g)-g_n(x)+\varphi(g_n)\le\begin{cases} x^2-\Delta & \text{if} \quad |x|\le|\alpha_n|+\sqrt{\Delta_n},\\ 2|x||\alpha_n|-\Delta+\Delta_n & \text{if} \quad |x|>|\alpha_n|+\sqrt{\Delta_n}.\end{cases}$$

Consequently,

$$g(x)-\varphi(g)-g_n(x)+\varphi(g_n)\le\gamma(\mathsf{excess}(\{x\},\psi(g)),n)\ ,$$

where

$$\gamma(t,n)=\begin{cases} t^2+2t\sqrt{\Delta} & \text{if} \quad 0\le t\le\max\left\{0,|\alpha_n|+\sqrt{\Delta_n}-\sqrt{\Delta}\right\},\\ 2t|\alpha_n|+2|\alpha_n|\sqrt{\Delta}-\Delta+\Delta_n & \text{if} \quad t>\max\left\{0,|\alpha_n|+\sqrt{\Delta_n}-\sqrt{\Delta}\right\}.\end{cases}$$

Accordingly to the theorem 4, we have

$$\beta(\mathsf{excess}(\psi(g_n), \psi(g))) \leq \gamma(\mathsf{excess}(\psi(g_n), \psi(g)), n)$$

and solving this inequality we get

$$\mathsf{excess}(\psi(g_n), \psi(g)) \leq \max\left\{0, |\alpha_n| + \sqrt{\Delta_n} - \sqrt{\Delta}, |\alpha_n| + \sqrt{\Delta_n + \alpha_n^2} - \sqrt{\Delta}\right\} =$$

$$= \max\left\{0, |\alpha_n| + \sqrt{\Delta_n + \alpha_n^2} - \sqrt{\Delta}\right\} \leq \max\left\{0, 2|\alpha_n| + \sqrt{\Delta_n} - \sqrt{\Delta}\right\}$$

for each $n \in \mathbb{N}$.

5 Appendix

The appendix is giving a list of the notation used in the paper. We combine the notations used in [10] and in [5].

Let $\mathcal{X}$ be a topological space. The set of all open (resp. closed, compact) sets is denoted $\mathcal{G}(\mathcal{X})$ (resp. $\mathcal{F}(\mathcal{X})$, $\mathcal{K}(\mathcal{X})$). The collection of sets $\mathcal{G} \subset \mathcal{G}(\mathcal{X})$ is called the (topological) base if each open set of $\mathcal{X}$ can be written as the union of the sets belonging to $\mathcal{G}$. The collection of sets $\mathcal{G} \subset \mathcal{G}(\mathcal{X})$ is called the (topological) subbase if the collection of all finite intersections of the sets from $\mathcal{G}$ forms the (topological) base. The collection of sets $\mathcal{G}_x \subset \exp \mathcal{X}$ is called the (topological) base at the point $x \in \mathcal{X}$ if each set from $\mathcal{G}_x$ contains an open set containing the point x and each open set containing the point x contains a set from $\mathcal{G}_x$. The space $\mathcal{X}$ is called first countable provided a countable base at each point of $\mathcal{X}$. The space $\mathcal{X}$ having a countable base is called separable.

We say the set $\Lambda = (\Lambda, \leq)$ is directed provided the relation $\leq$ is reflexive (i.e. $\lambda \leq \lambda$), transitive (i.e. $\lambda_1 \leq \lambda_2$ & $\lambda_2 \leq \lambda_3 \Rightarrow \lambda_1 \leq \lambda_3$) and for each pair $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.

Let X be a non-empty set. A net in a set X based on a directed set Λ is a function $a : \Lambda \to X$. We use the notation $< a_\lambda >_{\lambda \in \Lambda}$.

Let $\mathcal{X}$ be a topological space and $< a_\lambda >_{\lambda \in \Lambda}$ be a net in $\mathcal{X}$. We say the net $< a_\lambda >_{\lambda \in \Lambda}$ converges to the point $a \in \mathcal{X}$, provided $\lambda_G \in \Lambda$ for each $G \in \mathcal{G}(\mathcal{X})$, $a \in G$ such that $a_\lambda \in G$ for each $\lambda_G \leq \lambda \in \Lambda$. Traditional notation is $\lim_{\lambda \in \Lambda} a_\lambda = a$. Treating the net in generalized real line $\overline{\mathbb{R}}$, we use $\limsup_{\lambda \in \Lambda} a_\lambda = \lim_{\lambda \in \Lambda} \sup_{\lambda \leq \phi \in \Lambda} a_\phi$ and $\liminf_{\lambda \in \Lambda} a_\lambda = \lim_{\lambda \in \Lambda} \inf_{\lambda \leq \phi \in \Lambda} a_\phi$. Treating the function f from a topological space $\mathcal{X}$ to $\overline{\mathbb{R}}$ we use $\liminf_{y \to x} f(y) = \lim_{G \in \mathcal{G}_x} \inf_{y \in G} f(y)$, $\limsup_{y \to x} f(y) = \lim_{G \in \mathcal{G}_x} \sup_{y \in G} f(y)$, the result is, of course, independent on the choice of the bases at the point x. In the case lim sup coincides with lim inf we set $\lim_{y \to x} f(y) = \limsup_{y \to x} f(y) = \liminf_{y \to x} f(y)$.

Let $\mathcal{X}$ be a topological space and $< A_\lambda >_{\lambda \in \Lambda}$ be a net of the subsets of $\mathcal{X}$. We say the point $x \in \mathcal{X}$ is the limit point of $< A_\lambda >_{\lambda \in \Lambda}$ if for each $G \in \mathcal{G}(\mathcal{X})$, $x \in G$ there is $\lambda_G \in \Lambda$ such that $G \cap A_\lambda \neq \emptyset$ for each $\lambda_G \leq \lambda \in \Lambda$. The set of all limit points is denoted $\mathsf{Li}_{\lambda \in \Lambda} A_\lambda$.

We say the point $x \in \mathcal{X}$ is the cluster point of $< A_\lambda >_{\lambda \in \Lambda}$ if for each $G \in \mathcal{G}(\mathcal{X})$, $x \in G$ and each $\lambda \in \Lambda$ there is $\phi \in \Lambda$, $\lambda \leq \phi$ such that $G \cap A_\phi \neq \emptyset$. The set of all cluster points is denoted $\mathsf{Ls}_{\lambda \in \Lambda} A_\lambda$.

We declare $< A_\lambda >_{\lambda\in\Lambda}$ Kuratowski-Painlevé convergent to $A \subset \mathcal{X}$, we write $A = \mathsf{K\text{-}lim}_{\lambda\in\Lambda} A_\lambda$, provided $\mathsf{Li}_{\lambda\in\Lambda} A_\lambda = \mathsf{Ls}_{\lambda\in\Lambda} A_\lambda = A$. Recall the set A must be closed since $\mathsf{Li}_{\lambda\in\Lambda} A$ as well as $\mathsf{Ls}_{\lambda\in\Lambda} A$ are always closed.

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