A NEW APPROACH TO LIPSCHITZ SPACES OF PERIODIC INTEGRABLE FUNCTIONS*†

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Abstract:

The usual definition of Lipschitz subspaces of $L_{2\pi}^p$,$1 \leq p < \infty$, is modified in order to obtain homogeneous Banach spaces and a Hilbert space for $p = 2$. In the latter case it is shown that the trigonometric system is an orthogonal basis.

1. Introduction

To introduce the Lipschitz spaces we have restricted ourselves to the linear space $\mathcal{F}_{2\pi}$, of all real Lebesgue measurable $2\pi-$periodic functions defined on the real space $I\!R$ with the usual identification of points modulo 2π.

The continuous functions in $\mathcal{F}_{2\pi}$, form a particular space denoted by $C_{2\pi}$, which becomes a Banach space under the sup-norm $\|\cdot\|_\infty$. This is also the space of all continuous real functions on the interval $[0, 2\pi[$, equipped with the metric

$$\forall x, y \in [0, 2\pi[\,,\ d(x, y) := \min \{|x-y|, 2\pi - |x-y|\}\,. \tag{1.1}$$

The other well known Banach spaces $L_{2\pi}^p$, $1 \leq p < \infty$, consist of all functions f for which

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$$(1.2)\qquad \|f\|_p := \left(\frac{1}{\pi}\int_0^{2\pi} |f(x)|^p\, dx\right)^{\frac{1}{p}} < \infty.$$

Here, as usual, two functions that are equal a.e. (i.e. equally Lebesgue almost everywhere) are identified.

For a function f, we denote

$$(1.3)\qquad \triangle_t(f,x) := f(x+t) - f(x)\,;\ t > 0$$

and for $0 < \alpha \leq 1$, $1 \leq p \leq \infty$, the Lipschitz space Lip_α^p is the class of all functions $f \in L_{2\pi}^p$, if $1 \leq p < \infty$, or $f \in C_{2\pi}$, if $p = \infty$, such that

$$(1.4)\qquad \varphi_\alpha^p(f) := \sup\left\{t^{-\alpha}\left\|\triangle_t(f,x)\right\|_p : t > 0\right\} < \infty.$$

For $\alpha > 1$ and each $p \geq 1$, the only functions that (1.4) holds for are constant.

Since Lip_α^p is a linear space and φ_α^p is a semi-norm, a natural norm on Lip_α^p is usually given by

$$(1.5)\qquad \|f\|_{p,\alpha}^* := \|f\|_p + \varphi_\alpha^p(f)\,.$$

Then one proves that Lip_α^p *is* a Banach space.

Now, let us denote by $\mathcal{T}_n$, the finite dimensional linear space of all trigonometric polynomials of degree $\leq n$,

$$(1.6)\qquad T_n(x) := \tfrac{a_0}{2} + \textstyle\sum_{k=1}^n (a_k \cos kx + b_k \sin kx)\,.$$

For any Banach space B , such that $\cup_n \mathcal{T}_n \subset B \subset \mathcal{F}_{2\pi}$, we denote the *best approximation* of $f \in B$ to $\mathcal{T}_n$ in the norm of B, by

$$(1.7)\qquad E_n(f) := E_n(f,B) := \inf\left\{\|f - T_n\|_B : T_n \in \mathcal{T}_n\right\}.$$

In the above frame, with $\|.\|_B = \|.\|_p$, $1 \leq p \leq \infty$, a series of typical problems in Approximation Theory have been well studied for functions in Lip_α^p and, at present, they form an important part of the basis of Approximation Theory. Here we only quote the representative advanced books [2], [4], [6], [7].

When $\|.\|_B = \|.\|_{p,\alpha}^*$, the main trouble is that translations are not continuous operators with respect to the parameter. To explain this situation and for further use let us recall (c. f. [10] and [14], for instance) that a Banach space $B \subset L_{2\pi}^1$ is *homogeneous* if there exits a constant $C > 0$ such that $\|f\|_1 \leq C\|f\|_B$ for every $f \in B$ and if the following two conditions concerning translations are satisfied

(H_1) $\quad f \in B$ and $h \in R$, imply $f(x+h) \in B$ and $\|f(x+h)\|_B = \|f(x)\|_B$;

(H_2) $\quad f \in B$, $h, h_0 \in R$, and $h \to h_0$ imply $\|f(x+h) - f(x+h_0)\|_B \longrightarrow 0$.

Many good properties of approximation by Fourier series have been proved for homogeneous Banach spaces. However, Lip_α^p are not homogeneous because they do not satisfy (H_2) and this is no good news. In particular, the sequence $E_n(f, Lip_\alpha^p)$ does not always converge to zero.

With this bad property at hands, the researches have been organized following several directions. We only quote here a few representative papers which together with the already mentioned books, give an idea of the State-of-the-Art in a neighbourhood of our subject (c.f. [1], [3], [5], [8], [9], [11], [12], [13]).

However, we will see in this paper that an appropiate modification of the definition of Lipschitz spaces for $1 \leq p < \infty$, provides us homogeneous Banach spaces. Moreover, for $p = 2$, the corresponding version leads to a Hilbert space where the *trigonometric system*

$$\tag{1.8} \tfrac{1}{2}, \cos(x), \sin(x), \cdots, \cos(kx), \sin(kx), \cdots$$

is orthogonal and complete.

Then several questions on Fourier series and on best approximation by trigonometric polynomials in these spaces could be viewed in a frame similar to this one in $\mathrm{L}_{2\pi}^p$ spaces. This is the goal of the paper.

I am indebted to my colleague Jorge Bustamante, who has supported me with valuable advice.

2. The spaces $\mathbf{B}_\alpha^p, 1 \leq p < \infty$

Our first objective is to extend the function d on $[0,1[^2$, given by (1.1), to the whole plane. We define

$$\tag{2.1} \forall x, y \in [0, 2\pi[, \quad \forall j, k \in \mathbb{Z}, \quad d(x + 2j\pi, y + 2k\pi) = d(x, y).$$

It is easy to prove the following result which is a corner stone for our purposes:

Proposition 1 *The function d is a pseudometric that is $2\pi-$periodic in each of its two variables and translation invariant, i.e.*

(2.2) $\qquad \forall x,y,h\in I\!R, \qquad d\left(x+h,y+h\right)=d\left(x,y\right).$

In the following we assume that $\alpha>0$ is fixed and denote by $\mathcal{F}_{(2\pi)^2}$ the space of all real Lebesgue measurable functions on $I\!R^2$ that are $2\pi-$periodic in each variable. We define the translation operator on $\mathcal{F}_{(2\pi)^m}$, $m=1,2$, by

(2.3) $\qquad (T_hf)(x)=f(x+h) \quad \text{and} \quad (T_hf)(x,y)=f(x+h,y+h), \qquad h\in I\!R,$

respectively. Then T_h denotes two different linear operators.

We introduce the operator $F_\alpha:\mathcal{F}_{2\pi}\longrightarrow\mathcal{F}_{(2\pi)^2}$ by

(2.4) $\;(F_\alpha f)\left(x,y\right)=\frac{f(x)-f(y)}{d(x,y)^\alpha},\ x\neq y \bmod (2\pi) \qquad (\text{or } 0 \text{ if } x=y \bmod (2\pi))$

A simple but important remark is that F_α is linear and antisymmetric. This property means that

(2.5) $\qquad \forall x,y\in I\!R, \quad \forall f\in\mathcal{F}_{2\pi}, \qquad (F_\alpha f)(x,y)=-(F_\alpha f)(y,x)$

Proposition 2 *The operators T_h and F_α commute in the following sense:*

(2.6) $\qquad \forall f\in\mathcal{F}_{2\pi}, \quad \forall x,y,h\in I\!R, \qquad F_\alpha(T_hf)(x,y)=T_h(F_\alpha f)(x,y).$

Proof We use proposition 1. For $x\neq y \bmod (2\pi)$

$$F_\alpha(T_hf)\left(x,y\right)=\frac{T_hf(x)-T_hf(y)}{d(x,y)^\alpha}=\frac{f(x+h)-f(y+h)}{d(x+h,y+h)^\alpha}=T_h(F_\alpha f)\left(x,y\right) \quad \blacksquare$$

Let $L^p_{(2\pi)^2}$ be the Banach spaces of functions f$\in\mathcal{F}_{(2\pi)^2}$ for which

(2.7) $$\|f\|_p:=\left(\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left|f\left(x,y\right)\right|^p dxdy\right)^{\frac{1}{p}}<\infty; \qquad 1\leq p<\infty$$

Definition 1 *Fix $1\leq p<\infty$ and $\alpha>0$. The space B^p_α is the class of all functions $f\in L^p_{2\pi}$ for which $F_\alpha f\in L^p_{(2\pi)^2}$.*

Clearly, $\left\|F_\alpha\left(.\right)\right\|_p$ is a semi-norm on B^p_α. Then B^p_α is a normed space with

$$\|f\|_{p,\alpha} := \left(\|f\|_p^p + \|F_\alpha(f)\|_p^p\right)^{1/p}. \tag{2.8}$$

We will prove in Remark 1 after formula (2.23) that $\cap_{\alpha>1} B_\alpha^p$ is reduced to constant functions for every $1 \leq p < \infty$. This is the only reason for which we restrict ourselves to the bound $\alpha \leq 1$.

Theorem 1 *For every 1*$\leq p < \infty$ *and* $0 < \alpha \leq 1$, *the space* B_α^p *is a homogeneous Banach Space.*

Proof We begin with the proof that the space is complete.

Let (f_n) be a Cauchy sequence in B_α^p. In particular, (f_n) is a Cauchy sequence in $L_{2\pi}^p$. Then there exists $f \in L_{2\pi}^p$ such that

$$\|f_n - f\|_p \longrightarrow 0, \quad \text{if} \quad n \longrightarrow \infty. \tag{2.9}$$

We need to prove that $f \in B_\alpha^p$ and that

$$\|F_\alpha(f_n - f)\|_p \longrightarrow 0 \qquad \text{if} \qquad n \longrightarrow \infty. \tag{2.10}$$

Since $f \in L_{2\pi}^p$ and $\|F_\alpha f\|_p \leq \|F_\alpha (f_n - f)\|_p + \|F_\alpha f_n\|_p$, the assertion that $f \in B_\alpha^p$ automatically follows from (2.10).

To prove this last property, observe that $(F_\alpha f_n)$ is also a Cauchy sequence in $L_{(2\pi)^2}^p$. Then there is a $g \in L_{(2\pi)^2}^p$, such that

$$\|F_\alpha f_n - g\|_p \longrightarrow 0, \qquad \text{if} \qquad n \longrightarrow \infty. \tag{2.11}$$

On the other hand $\|F_\alpha (f_n - f)\|_p = \|F_\alpha f_n - F_\alpha f\|_p$. So we only have to prove that

$$F_\alpha f = g \text{ a.e.} \tag{2.12}$$

By (2.9), there exists a subsequence (f_{n_j}) converging to f a.e. on $[0, 2\pi[$ and by (2.11), another sub-sequence $(F_\alpha f_{n_{j_k}})$ converges to g a.e. on $[0, 2\pi[^2$. Then (2.12) holds.

To prove the properties (H_1) and (H_2) in B_α^p, we will utilize the fact that both of them are satisfied in $L_{(2\pi)^m}^p$, $m = 1, 2$, as well as proposition 2. Let $f \in B_\alpha^p$ be given and $h > 0$. Then

$$\|T_h f\|_{p,\alpha}^p = \|T_h f\|_p^p + \|F_\alpha (T_h f)\|_p^p = \|T_h f\|_p^p + \|T_h(F_\alpha f)\|_p^p$$
$$= \|f\|_p^p + \|F_\alpha f\|_p^p = \|f\|_{p,\alpha}^p$$

So (H_1) holds in B_α^p. Further, it is enough to consider the case $h_0 = 0$ in (H_2):

$$\|T_h f - f\|_{p,\alpha}^p = \|T_h f - f\|_p^p + \|F_\alpha (T_h f - f)\|_p^p = \|T_h f - f\|_p^p + \|T_h(F_\alpha f) - F_\alpha f\|_p^p$$

that converges to 0 if h tends to 0. ■

Corollary 1 *For each $f \in B_\alpha^p$ and each summability kernel of $2\pi-$periodic continuous functions (K_n) in $L_{2\pi}^1$, one has*

(2.13) $$\|K_n * f - f\|_{p,\alpha} \longrightarrow 0, \quad \text{if} \quad n \longrightarrow \infty.$$

In particular, the Féjer's sums of the Fourier series

(2.14) $$\tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

of f, where

(2.15) $$a_n := \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\ dt \quad \text{and} \quad b_n := \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\ dt, \quad n = 0, 1, 2, \cdots$$

converge to f in the norm $\|\cdot\|_{p,\alpha}$ and the trigonometric polynomials are everywhere dense in B_α^p.

Proof See paragraph 2, Chapter 1, of [10], that also includes the definition of summability kernels. ■

Now one might wish to simplify the double integrals that appear in our approach. We define the domains:

$D\ :=\ [0, 2\pi[^2$

$D_1\ :=\ \{(x, y) \in D : 0 \le x \le \pi \ \text{ and } \ x + \pi \le y \le 2\pi\}$

$D_2\ :=\ \{(x, y) \in D : 0 \le x \le 2\pi \ \text{ and } \ x \le y \le x + \pi\}$

$D_3\ :=\ \{(x, y) \in D :\ (y, x) \in D_2\}$

$D_4\ :=\ \{(x, y) \in D :\ (y, x) \in D_1\}$

$D_5\ :=\ \{(x, y) \in R^2 : \pi \le x \le 2\pi \ \text{ and } \ 2\pi \le y \le x + \pi\}$

Then, for every $f \in L^1(D) = L^1_{(2\pi)^2}$, we have

(2.16) $$\int_D f = \sum_{i=0}^{4} \int_{D_i} f$$

and using (2.5), for $g = |F_\alpha f|^p$ if $f \in B^p_\alpha$ or $g = F_\alpha u\ F_\beta v$ if $u \in B^p_\alpha$ and $v \in B^q_\beta$, $0 < \alpha, \beta \leq 1$, $1/p + 1/q = 1$:

(2.17) $$\int_{D_2 \cup D_3} g = 2 \int_{D_i} g, \qquad i = 2, 3$$

(2.18) $$\int_{D_1 \cup D_4} g = 2 \int_{D_i} g, \qquad i = 1, 4$$

Now, since g is $2\pi-$ periodic in each variable, it follows from $(2.16 - 17 - 18)$ that

(2.19) $$\int_D g = 2 \int_{D_2 \cup D_5} g.$$

With techniques of Measure Theory, one can write

(2.20) $$\int_{D_2 \cup D_5} f = \int_0^\pi \left[\int_0^{2\pi} f(x, x+t)\, dx\right] dt \quad \text{for} \quad f \in L^1(D_2 \cup D_5).$$

Then, from $(2.19 - 20)$, we rediscover the most familiar formulas

(2.21) $$\int_0^{2\pi} \int_0^{2\pi} |F_\alpha f(x, y)^p|\, dx\ dy = 2 \int_0^\pi \int_0^{2\pi} \left|\frac{f(x) - f(x+t)}{t^\alpha}\right|^p dx dt$$

(2.22) $$\int_0^{2\pi} \int_0^{2\pi} F_\alpha u(x, y)\, F_\beta v(x, y)\, dx dy = 2 \int_0^{\pi} \int_0^{2\pi} \frac{(u(x) - u(x+t))(v(x) - v(x+t))}{t^{\alpha+\beta}} dx dt$$

if $u \in B^p_\alpha$, $v \in B^q_\beta$, $0 < \alpha, \beta \leq 1 \ < p, q < \infty$ and $1/p + 1/q = 1$

Finally we also has

(2.23) $$\left[\int_D = \int_{D_2 \cup D_3} = \int_{D_1 \cup D_4}\right] F_\alpha f = 0, \quad \text{for} \quad f \in B^1_\alpha$$

Remark 1 The equation (2.21) above shows that B^p_α is not necessarily reduced to constant functions when $\alpha > 1$. In fact, from this equality and under the optimal assumption on f that there exists a constant $C := C(f) > 0$ such that $|\triangle_t(f, x)| \leq C\ t$ for every $t > 0$, we deduce that $F_\alpha f \in L^p_{(2\pi)^2}$ whenever $\alpha < (p+1)/p$. However, since $(p+1)/p \longrightarrow 1$ if p tends to ∞ , we have that $\alpha \leq 1$ represents the common case for every $1 \leq p < \infty$. In other words, if f is not a constant function and $\alpha > 1$, then there exists $p < \infty$ such that $f \notin B^p_\alpha$.

Proposition 3 *For every $0 < \alpha \leq 1$ and $1 \leq p < \infty$, the classical Lipschitz spaces Lip_α^p defined in Section 1 are continuously embedded in B_α^p by the identity operator.*

Proof For any positive finite measure space (X, μ) and $1 \leq p < \infty$, there exits a constant $C := C_{\mu,p} > 0$, such that $\|.\|_p \leq C\,\|.\|_\infty$. On the other hand, the semi-norm φ_α^p in (1.4) could be equivalently defined with $0 < t \leq \pi$ (see Sec. 4.1 of [2]). Then, the proposition follows from (2.22). ■

Proposition 4 *For every $0 < \alpha \leq 1$ and $1 < p < \infty$, the spaces B_α^p are strictly convex.*

Proof Let $f, g \in B_\alpha^p$ be such that $\|f\|_{p,\alpha} = \|g\|_{p,\alpha} = 1$. It follows that

$$\begin{aligned}\|f+g\|_{p,\alpha} &= \left(\|f+g\|_p^p + \|F_\alpha(f) + F_\alpha(g)\|_p^p\right)^{1/p} \\ &\leq \left(\left(\|f\|_p + \|g\|_p\right)^p + \left(\|F_\alpha f\|_p + \|F_\alpha g\|_p\right)^p\right)^{1/p} \\ &\leq \left(\|f\|_p^p + \|F_\alpha(f)\|_p^p\right)^{\frac{1}{p}} + \left(\|g\|_p^p + \|F_\alpha(g)\|_p^p\right)^{1/p} = 2.\end{aligned}$$

Then $\|f+g\|_{p,\alpha} = 2$ is possible only if $f = g$ a. e. ■

As a consequence we have the non-trivial result:

Corollary 2 *For every $f \in B_\alpha^p$, $0 < \alpha \leq 1$, $1 < p < \infty$ and $n = 1, 2, ..$ there is a unique polynomial of best approximation of f to $\mathcal{T}_n$ in B_α^p.*

3. The Hilbert Space $\mathbf{B}_\alpha$

In this section we usually write B_α instead of B_α^2. We shall prove that the trigonometric system is an orthogonal basis of this space. In fact, one easily proves:

Proposition 5 *The bilinear functional*

$$(f \mid g) := (f \mid g)_{L^2_{2\pi}} + (F_\alpha f \mid F_\alpha g)_{L^2_{(2\pi)^2}}, \qquad f, g \in B_\alpha, \tag{3.1}$$

is an inner product whose associated norm $\|f\|_\alpha = (f \mid g)^{\frac{1}{2}}$ *is equal to* $\|f\|_{2,\alpha}$. *Here*

$$(f \mid g)_{L^2_{2\pi}} = \frac{1}{\pi}\int_0^{2\pi} f(x)g(x)dx$$

$$(F_\alpha f \mid F_\alpha g)_{L^2_{(2\pi)^2}} = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_\alpha f(x,y)F_\alpha g(x,y)dxdy.$$

Then the general results of Hilbert spaces hold in B_α.

Theorem 2 *The trigonometric system* (1.8) *is an orthogonal basis of* B_α *whose elements have the norms:*

$$\left\|\frac{1}{2}\right\|_\alpha = 1\ ,\ \|\cos(kx)\|^2_\alpha = \|\sin(kx)\|^2_\alpha = N_\alpha(k)^2 = 1 + \frac{4}{\pi}\int_0^\pi \frac{1-\cos(mt)}{t^{2\alpha}}dt \tag{3.2}$$

for $k = 1, 2, \cdots$

Proof The trigonometric system is a set of B_α whose finite linear combinations (i. e. the trigonometric polynomials) are everywhere dense in B_α as we stated in Corollary 4. In order to prove that is an orthogonal basis, we only need to check the orthogonality condition. But this, as well as (3.2), are straightforward tasks accomplished by means of (2.21-22) and using that (1.8) is an orthonormal basis in $L^2_{2\pi}$. ■

I have not found any reference to the following striking result. Then the proof is given here:

Theorem 3 *Let* H *be any Hilbert space, with inner product* $(\cdot \mid \cdot)_H$ *and norm* $\|.\|_H$ *. Let* F *be a linear subspace of* H *that becomes a Hilbert space under the inner product* $(\cdot \mid \cdot)_F$ *and such that* $\|.\|_H \leq \|.\|_F$ *on* F. *If* $\{u_j : j = 1, 2, ..\}$ *is an orthonormal basis of* H *that simultaneously is an orthogonal basis of* F, *then for every* $f \in F$, *the Fourier series of* f *are formally equal for both spaces.*

Proof Put $C_j := \|u_j\|_F$. Since the Fourier series $\sum_j \left(f \mid \frac{u_j}{C_j}\right)_F \frac{u_j}{C_j}$ converges to f in $\|.\|_F$, it also converges to f in H due to the hypothesis on the norms. Then $(f \mid u_j)_H = \frac{1}{C_j^2}(f \mid u_j)_F$ for $j = 1, 2, ...$ ■

Corollary 3 *For every $f \in B_\alpha$, the Fourier series of f in this space is given by*

(3.3) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \quad (a_n \cos(nx) + b_n \sin\,(nx)),$$

where a_n and b_n , n = 0,1,2, are calculated in the usual form remembered in (2.15) !

Then a function

$$f\,(x) = \frac{A_o}{2} + \sum_{n=1}^{\infty}(A_n \cos(nx) + B_n \sin(nx)) \in L^2_{2\pi}$$

is in B_α if and only if

$$\sum_{n=1}^{\infty} N_\alpha(n)^2({A_n}^2 + B_n^2) < \infty.$$

Proof Combine the last two theorems . ■

Corollary 4 *For every $f \in B_\alpha$ and $n = 1, 2, ..,$ the polynomials of best approximation of f to $\mathcal{T}_n$ in B_α are the partial sums of the Fourier series of f given by* (3.3) *and*

$$E_n\,(f, B_\alpha)^2 = \sum_{k=n+1}^{\infty} N_\alpha(k)^2\,(a_k^2 + b_k^2).$$

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