# On Convolution Operators in the Spaces of Almost Periodic Functions and $L^p$ Spaces

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**Abstract.** We consider operators of convolution generated by $L^1$ functions in $L^p$ and various spaces of almost periodic functions. It turns out to be that if such an operator is invertible in one of these spaces, then it is invertible in all the spaces we consider. Further, we prove that any convolution has identical norms in many natural couples of functional spaces.

**Key words:** convolution operators, almost periodic functions

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# 1 Introduction

We consider convolution operators acting in the spaces $L^p$ and the spaces of Bohr, Stepanov, and Besicovich almost periodic (a.p.) functions. We prove that in all the spaces we consider the invertibility for convolutions take place, or do not take place, simultanuously.

Next, we study norms of convolutions in natural couples of functional spaces: $L^p$ and $B^p$, $BS^p$ and $S^p$, $C_b$ and $CAP$. We prove that convolution operators have identical norms in each of two members of any such couple.

Our study is motivated by the results of M. Shubin [1] and [2]. In those papers the results on norms and invertibility (spectra) were proved for a wide class of a.p. pseudo differential operators, but only in the $L^2$-$B^2$ setting. The case $p \neq 2$, as well as the case of Stepanov spaces, are not considered there. We attempt here to enlarge the range of spaces in which such "coincideness" results take place. However, since $L^p$-theory and, all the more, $B^p$- and $S^p$-theories of pseudo differential operators are not well-developed, we restrict ourself to the case of convolutions only. Even in this case a rich picture appears. We remark also that $L^p$-$B^p$ setting was studied in [3] and [4] for nonlinear differential operators. However, in these works the value of $p$ depends on the structure of operators under consideration and plays the same rôle as $p = 2$ in the linear theory.

In the present paper we deal with convolutions defined on the real line only. Nevertheless, all the results and techniques may be extended to the case of convolutions on $\mathbb{R}^n$ without any difficulties.

# 2 Preliminaries

We use the standard notations $L^p$ (resp. $L^p_{loc}$), $1 \leq p \leq \infty$, for the (resp. local) Lebesgue spaces of complex valued measurable functions on the real line $\mathbb{R}$. Let $\varphi \in L^1$ and $\lambda \in \mathbb{C}$. It is well-known [6] that the operator $A$ defined by the formula

$$Au = \lambda u + \varphi * u, \tag{1}$$

where $*$ stands for the convolution operation, acts continuously in each space $L^p$, $1 \leq p \leq \infty$. All such operators form an algebra which is denoted by $\mathcal{A}$.

Now we introduce other spaces in which operator (1) will be considered. Let us denote by $C_b$ the closed subspace of $L^\infty$ formed by all bounded continuous functions. Futhermore, for any $p \geq 1$ let us denote by $BS^p$ the space of all functions $f \in L^p_{loc}$ for which

$$\|f\|_{S^p} = \sup_{t\in\mathbb{R}} \left[\int_t^{t+1} |f(x)|^p dx\right]^{1/p} < \infty. \tag{2}$$

This is the space of Stepanov bounded functions (with exponent $p$). The closure in $BS^p$ of the set of all trigonometric polynomials

$$\sum_{finite} a_k \mathrm{e}^{i\xi_k x}, \quad a_k \in \mathbb{C}, \quad \xi_k \in \mathbb{R},$$

is denoted by $S^p$. It consists of all Stepanov a.p. functions (with the exponent $p$). Similarly, the closure of all trigonometric polynomials in the space $C_b$ consists of all Bohr a.p. functions. The last space is denoted by $CAP$.

For any $f \in CAP$ it is well-defined the mean value

$$\mathbb{M}\{f\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt.$$

As a consequence, for any $f \in CAP$ one can define the norm

$$\|f\|_{B^p} = \mathbb{M}\{|f|^p\}^{1/p}.$$

We define the space $B^p$ of Besicovich a.p. functions (with the exponent $p$) as the completion of $CAP$ with respect to the norm $\|\cdot\|_{B^p}$.

For more details on a.p. functions we refer to [4], [5], [7] and [8]. We remark only that all the spaces just introduced are Banach spaces and the following continuous and dense embeddings

$$C_b \subset BS^p, \quad p \geq 1,$$

$$CAP \subset S^p \subset S^q, \quad CAP \subset B^p \subset B^q, \quad p \geq q,$$

take place.

Now we show that the algebra $\mathcal{A}$ acts naturally in all spaces we introduced.

**Lemma 1** *Let $A \in \mathcal{A}$. Then*

**1)** *A acts continuously in $C_b$;*

**2)** *A acts continuously in $CAP$.*

*Proof*. The first part of the statement is well-known. Let us prove the second one. Since $CAP$ is a closed subspace in $C_b$, one need only to show that $Au \in CAP$ provided $u \in CAP$. By the Bochner criterion, $u \in CAP$ iff the family $\{u(\cdot + y)\}_{y\in\mathbb{R}}$ is precompact in $C_b$. Now let $u \in CAP$. Since $A$ is translation invariant, we have

$$(Au)(\cdot + y) = A[u(\cdot + y)], \quad y \in \mathbb{R}.$$

By continuity of $A$ in $C_b$, we see that the family $\{(Au)(\cdot + y)\}_{y\in\mathbb{R}}$ is precompact. □

**Lemma 2** *Any operator $A \in \mathcal{A}$ acts continuously in $BS^p$, $1 \le p < \infty$, and leaves the subspace $S^p$ invariant.*

*Proof*. Without loss of generality we can assume that $\lambda = 0$. Let $u \in BS^p$, $p > 1$, and $p'$ the dual exponent, $1/p' + 1/p = 1$. Using the Hölder inequality, we have the following inequalities

$$\int_\tau^{\tau+1} \left| \int_{\mathbb{R}} \varphi(t) u(x-t) dt \right|^p dx \le \int_\tau^{\tau+1} \left[ \int_{\mathbb{R}} |\varphi(t)^{1/p'} |\varphi(t)|^{1/p} |u(x-t)| dt \right] dx \le$$
$$\le \int_\tau^{\tau+1} \left[ \int_{\mathbb{R}} |\varphi(t) dt \right]^{1/p'} \left[ \int_{\mathbb{R}} |\varphi(t)||u(x-t)|^p dt \right] dx \le$$
$$\le C \int_\tau^{\tau+1} \left[ \int_{\mathbb{R}} |\varphi(t)||u(x-t)|^p dt \right] dx =$$
$$= C \int_{\mathbb{R}} \left[ |\varphi(t)| \int_\tau^{\tau+1} |u(x-t)|^p dx \right] dt \le C \cdot \|\varphi\|_{L^1} \cdot \|u\|^p_{S^p},$$

where $C > 0$ does not depend on $u$. Thus,

$$\|Au\|_{S^p} \le C \cdot \|u\|_{S^p}$$

and the operator $A$ acts continuously in $BS^p$, $p > 1$.

The case $p = 1$ may be considered in the similar way; it is even simpler.

The proof of the second part of the lemma may be carried out exactly as in Lemma 1, using the version of Bochner criterion for Stepanov a.p. functions. □.

**Lemma 3** *Let $A \in \mathcal{A}$ and $p \geq 1$. Then there exists a constant $C = C_p(A) > 0$ which depends only on $A$ and $p$ such that*

$$\|Au\|_{B^p} \leq C\|u\|_{B^p}$$

*for all $u \in CAP$.*

*Proof*. It goes along the same lines as in the proof of the first statement of Lemma 2. $\square$

Lemma 3 permits us to extend, by continuity, any operator $A \in \mathcal{A}$ to an operator acting in $B^p$. Such the extension is still denoted by $A$.

**Remark 1** *It is easy to see that the norms of all the operators we consider are estimated by $C \cdot \|\varphi\|_{L^1}$.*

**Remark 2** It is well-known that, given $A \in \mathcal{A}$, the adjoint operator $A^*$ in the sense of the scale $L^p$ also belongs to $\mathcal{A}$. Moreover, a simple calculation show us that the extension of $A^*$ to $B^p$ is in fact the adjoint operator of $A$ in the sense of the scale $B^p$.

# 3 Invertibility of Convolutions

To any operator $A \in \mathcal{A}$ of the form (1) one can associate its *symbol* $a(\xi)$ defined by the formula

$$a(\xi) = \lambda + \hat{\varphi}(\xi), \tag{3}$$

where $\hat{\varphi}$ is the Fourier transform of $\varphi$. It is well-known (see, e.g., [9]) that $a(\xi)$ is a continuous function on $\mathbb{R}$ and

$$\lim_{\xi \to \infty} a(\xi) = \lambda.$$

Therefore, $a(\xi)$ may be regarded as a continuous function on $\bar{\mathbb{R}}$, one-point compactification of the real line. The set $\mathcal{A}$ is a commutative algebra with the natural involution $A \mapsto A^*$. The map $A \mapsto a(\xi)$ is a homomophism $\mathcal{A} \longrightarrow C(\bar{\mathbb{R}})$ of algebras with involution. Here the algebra $C(\bar{\mathbb{R}})$ of continuous functions on $\bar{\mathbb{R}}$ is endowed with the natural involution, complex conjugation.

**Theorem 1** *Let $A \in \mathcal{A}$. The following statements are equivalent:*

*$(i)$ $a(\xi)$ is nowhere vanishing on $\bar{\mathbb{R}}$;*

*$(ii')$ $A$ is invertible in $L^{p_0}$ for some $p_0 \in [1, \infty)$;*

*$(ii)$ $A$ is invertible in $L^p$ for all $p \in [1, \infty)$;*

*$(iii')$ $A$ is invertible in $B^{p_0}$ for some $p_0 \in [1, \infty)$;*

*$(iii)$ $A$ is invertible for all $p \in [1, \infty)$;*

*$(iv)$ $A$ is invertible in $C_b$;*

*$(v)$ $A$ is invertible in $CAP$;*

*$(vi')$ $A$ is invertible in $BS^{p_0}$ for some $p_0 \in [1, \infty)$;*

*$(vi)$ $A$ is invertible in $BS^p$ for all $p \in [1, \infty)$;*

*$(vii')$ $A$ is invertible in $S^{p_0}$ for some $p_0 \in [1, \infty)$;*

*$(vii)$ $A$ is invertible in $S^p$ for all $p \in [1, \infty)$.*

*Proof.* Assume that $a(\xi) \neq 0$ for all $\xi \in \bar{\mathbb{R}}$. By the classical Wiener theorem the function $b(\xi) = 1/a(\xi)$ is of the form

$$b(\xi) = \frac{1}{\lambda} + \hat{\varphi}(\xi),$$

where $\varphi \in L^1$. Hence, the operator

$$Bu = \frac{1}{\lambda} u + \varphi * u$$

acts in all the spaces we consider and is the inverse to $A$. Thus, $(i)$ implies all other statements listed above. Moreover, it is easy to see that $(ii')$ with $p_0 = 1$ implies $(i)$.

Now let us suppose that $(ii')$ is fulfilled. Without loss of generality, we may assume $p_0 > 1$. Consider the operators

$$Tu = \bar{u}$$

and

$$(J(u))(x) = u(-x).$$

It is obvious that $T^2 = J^2 = I$, $J$ is linear and $T$ is antilinear. A direct calculation shows that

$$A = JTA^*TJ. \tag{4}$$

Since $A^*$ is invertible in $L^{p_0'}$, $p_0'$ being the dual exponent, (4) implies that $A$ is invertible in $L^{p_0'}$. By the Riesz–Thorin theorem, $A$ is invertible in $L^2$. Since

$$(\widehat{Au})(\xi) = a(\xi)\hat{u}(\xi),$$

we have proved $(i)$.

The rest of the proof is simple. For example, assume $(v)$ to hold. Let

$$e_\xi(x) = \exp(i\xi x), \quad \xi \in \mathbb{R}.$$

Then $Ae_\xi = a(\xi)e_\xi$ and $\|e_\xi\|_{CAP} = 1$. If there exists a bounded inverse operator $A^{-1}$, then we easily have

$$1 \leq a(\xi)\|A^{-1}\|, \quad \xi \in \mathbb{R}.$$

By continuity of $a(\xi)$, the same inequality holds for all $\xi \in \bar{\mathbb{R}}$. Hence, $a(\xi) \neq 0$ for all $\xi \in \bar{\mathbb{R}}$ and $(v) \Rightarrow (i)$.

The same argument proves that each of the statements $(iii')$, $(iv)$, $(vi')$, $(vii')$ implies $(i)$ and the proof is complete . $\square$

**Remark 3** The implications $(i) \Leftrightarrow (ii') \Leftrightarrow (ii)$ are well-known (see e.g. [10]. We have included here the proof for the sake of completeness.

# 4 Norms of Convolutions

Now we want to study connections between norms of convolution operators acting in various function spaces. As usual, $L(E)$ stands for the space of all bounded linear operators in a Banach space $E$.

We start with the spaces $BS^p$ and $S^p$. We need the following additional continuity property of convolutions.

**Lemma 4** *Let $A \in \mathcal{A}$. If $u_k$ is bounded in $BS^p$ and $u_k \to u$ in $L^p_{loc}$, then $Au_k \to Au$ in $L^p_{loc}$.*

*Proof*. Without loss of generality, we may assume that $\lambda = 0$ and $u = 0$. Fix $x_0 > 0$. By assumption,

$$\int_{-x_0}^{x_0} |u_k(x-t)|^p dx \leq C, \tag{5}$$

where $C > 0$ is independent on $k$ and $t$. As in the proof of Lemma 2, we have

$$\int_{|x|\leq x_0} |Au_k(x)|^p dx \leq C \int_{|x|\leq x_0} \left[\int_{\mathbb{R}} |\varphi(t)||u_k(x-t)|^p dt\right] dx.$$

Then, for any $t_0 > 0$,

$$\begin{aligned}\int_{|x|\leq x_0} |Au_k(x)|^p dx &\leq C\left[\int_{|t|\geq t_0} + \int_{|t|\leq t_0}\right]\left[|\varphi(t)| \int_{|x|\leq x_0} |u_k(x-t)|^p dx\right] dt = \\ &= I_1 + I_2.\end{aligned}$$

Let $\varepsilon > 0$. Due to (5),

$$I_1 \leq C \int_{|t|\geq t_0} |\varphi(t)| dt$$

and $I_1 \leq \varepsilon$ if $t_0$ is choosen large enough. Now

$$\begin{aligned}I_2 &\leq C \int_{|t|\leq t_0} |\varphi(t)| dt \int_{|y|\leq x_0+t_0} |u_k(y)|^p dy \leq \\ &\leq C\|\varphi\|_{L^1} \int_{|y|\leq x_0+t_0} |u_k(y)|^p dy.\end{aligned}$$

Since $u_k \to 0$ in $L^p_{loc}$, we see that $I_2 \leq \varepsilon$ if $k$ is large enough. The proof is complete. $\square$

**Remark 4** It is easy to verify that if $u_k$ is bounded in $BS^p$ and $u_k \to u$ in $L^p_{loc}$, then $u \in BS^p$ and

$$\liminf \|u_k\|_{S^p} \geq \|u\|_{S^p}.$$

**Theorem 2** *Let $A \in \mathcal{A}$. Then for any $p \in [1, \infty)$*

$$\|A\|_{L(BS^p)} = \|A|_{S^p}\|_{L(S^p)}. \tag{6}$$

*Proof*. Let $a$ and $b$ stand for the left- and right-hand sides of (6), respectively. Since $S^p$ is a closed subspace of $BS^p$, we have $b \leq a$, and we need only to prove that

$$a \leq b. \tag{7}$$

By definition,

$$a = \sup\{\|Au\|_{S^p} : u \in BS^p, \|u\|_{S^p} \leq 1\},$$
$$b = \sup\{\|Av\|_{S^p} : v \in S^p, \|v\|_{S^p} \leq 1\}.$$

Let $u \in BS^p$. Given $T > 0$, let $u_T$ be the $2T$-periodic extension of $u|_{[-T,T]}$ to $\mathbb{R}$. It is easy to verify that

$$\|u\|_{S^p} = \lim_{T\to\infty} \|u_T\|_{S^p},$$

$$\|u\|_{S^p} \leq \|u\|_{S^p} \leq 1.$$

Also it is obvious that $u_T \to u$ in $L^p_{loc}$, as $T \to \infty$. Since $A$ is a bounded operator, we see that $Au_T$ is uniformly bounded in $BS^p$ and $2T$-periodic (hence, a.p.). By Lemma 4, $Au_T \to Au$ in $L^p_{loc}$. By Remark 4,

$$\liminf_{T\to\infty} \|Au_T\|_{S^p} \geq \|Au\|_{S^p}.$$

This implies (7), and we conclude. □

**Remark 5** A similar approach was used in the proof of Proposition 2.2 in [11].

In the same way, using uniform convergence on compact sets instead of $L^p_{loc}$-convergence, one can prove the following

**Theorem 3** *Let $A \in \mathcal{A}$. Then*

$$\|A\|_{L(C_b)} = \|A|_{CAP}\|_{L(CAP)} = \|A|_{C_0}\|_{L(C_0)},$$

*where $C_0 = \{u \in C_b : \lim_{x\to\infty} u(x) = 0\}$ is the closed subspace of $C_b$.*

Now we consider the spaces $L^p$ and $B^p$.

**Theorem 4** *Let $A \in \mathcal{A}$. Then*

$$\|A\|_{L(L^p)} = \|A\|_{L(B^p)}. \tag{8}$$

*Proof*. First we give a separate proof for the case $p = 1$, since it clarifies the duality between this case and Theorem 3. Then we consider the general case. It is well-known (see e.g [6] or [12]) that

$$\|u\|_{L^1} = \sup \left\{ \frac{|(u,v)|}{\|v\|_{L^\infty}} : v \in L^\infty, v \neq 0 \right\}.$$

By the classical Lusin Theorem ([6], [12]) one can replace here $L^\infty$ by $C_b$. Therefore,

$$\|A\|_{L(L^1)} = \sup \left\{ \frac{|(Au,v)|}{\|u\|_{L^1}\|v\|_{C_b}} : u \in L^1, v \in C_b, u \neq 0, v \neq 0 \right\}. \tag{9}$$

Now we recall that $B^1$ and $CAP$ are naturally isomorphic to $L^1(\mathbb{R}_B)$ and $C(\mathbb{R}_B)$ respectively, where $\mathbb{R}_B$ is the so-called Bohr compactification [4], [5] and $L^1(\mathbb{R}_B)$ is regarded with respect to the Haar measure on $\mathbb{R}_B$. Hence, we have, as above,

$$\|A\|_{L(B^1)} = \sup \left\{ \frac{|(Au,v)_B|}{\|u\|_{B^1}\|v\|_{CAP}} : u \in B^1, v \in CAP, u \neq 0, v \neq 0. \right\} \tag{10}$$

Since

$$(Au, v) = (u, A^*v),$$

$$(Au, v)_B = (u, A^*v)_B,$$

we conclude that

$$\|A\|_{L(L^1)} = \|A^*\|_{L(C_b)},$$

$$\|A\|_{L(B^1)} = \|A^*\|_{L(CAB)},$$

and Theorem 3 implies the required.

Now we consider the general case assuming for definitness that $p > 1$. For the sake of brevity, let us denote by $a$ and $b$ the left- and right-hand sides of (8), respectively. We have

$$a = \sup \left\{ \frac{\|Au\|_{L^p}}{\|u\|_{L^p}} : u \in C, \operatorname{supp} u \subset\subset \mathbb{R}, u \neq 0 \right\},$$

$$b = \sup \left\{ \frac{\|Av\|_{B^p}}{\|v\|_{B^p}} : v \in CAP, v \neq 0 \right\}.$$

Recall that "$X \subset\subset Y$" means that $X$ is a compact subset of $Y$.

Since compactly supported continuous functions are dense in $L^1$, we can choose a sequence $\varphi_n$ of such functions such that $\varphi_n \to \varphi$ in $L^1$. By Remark 1, $\|A_n\|_{B^p} \to \|A\|_{B^p}$ and $\|A_n\|_{L^p} \to \|A\|_{L^p}$, where $A_n$ stands for the operator generated by $\varphi_n$. Therefore, we can assume that the kernel $\varphi$ itself is continuous and compactly supported.

Now let $u \in C$ and $\operatorname{supp} u \subset [-T_0, T_0]$. For $T \geq T_0$ we denote by $\bar{u}_T$ the $2T$-periodic extention of $u|_{[-T_0,T_0]}$ to $\mathbb{R}$, and set

$$u_T = (2T)^{1/p}\bar{u}_T.$$

It is easy to verify that

$$\|u_T\|_{B^p} = \|u\|_{L^p}. \tag{11}$$

Since $\varphi$ is compactly supported, we see that

$$A(u_T) = (Au)_T,$$

provided $T$ is large enough. Due to (11) we have

$$\|Au_T\|_{B^p} = \|Au\|_{L^p} \tag{12}$$

for such $T$.

Now using (11) and (12), we have

$$\|Au\|_{L^p} = \|Au_T\|_{B^p} \leq b\|u_T\|_{B^p} \leq b\|u\|_{L^p}$$

for $T$ being large enouph. This implies that $a \leq b$.

Now let us prove that $b \leq a$. For any $T > 0$, let $\chi_T$ be the characteristic function of the interval $[-T, T]$. For $v \in CAP$ it is easy to see that

$$\lim_{T\to\infty} \frac{1}{2T}\|\chi_T v\|_{L^p}^p = \|v\|_{B^p}^p. \tag{13}$$

We want to show that

$$\lim_{T\to\infty} \frac{1}{2T}\|A(\chi_T v)\|_{L^p}^p = \|Av\|_{B^p}^p. \tag{14}$$

In view of (13), to do this it suffices to prove that

$$\lim_{T\to\infty} \frac{1}{2T}\|\chi_T Av - A(\chi_T v)\|_{L^p}^p = 0. \tag{15}$$

First of all, we observe that the operator

$$B_T = \chi_T A - A\chi_T$$

is an integral operator with the kernel

$$\varphi_T(x,t) = \chi_T(x)\varphi(x-t) - \varphi(x-t)\chi_T(t).$$

Moreover, $B_T$ acts in $L^\infty$ and is uniformly (with respect to $T$) bounded in this space. In particular, we see that for $\kappa \in (0,1)$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{T-T^\kappa \le |x| \le T+T^\kappa} |B_T v(x)|^p dx = 0 \tag{16}$$

since the domain of integration has Lebesgue measure of order $T^\kappa$.

Now we have

$$\int_{|x|\ge T+T^\kappa} |B_T v(x)|^p dx \le \int_{|x|\ge T+T^\kappa} dx \left[ \int_{|t|\le T} |\varphi(x-t)||v(t)|dt \right]^p .$$

Here $|x-t| \ge T^\kappa$. Since the support of $\varphi$ is compact, we see that in the right-hand side $\varphi(x-t) = 0$ if $T$ is large enough. Hence,

$$\frac{1}{2T} \int_{|x|\ge T+T^\kappa} |B_T v(x)|^p dx = 0. \tag{17}$$

Similarly,

$$\frac{1}{2T} \int_{|x|\le T-T^\kappa} |B_T v(x)|^p dx = 0. \tag{18}$$

Puting together (16)–(18), we get (15) and, hence, (14).

Now

$$\begin{aligned} \|Av\|_{B^p}^p &= \lim_{T\to\infty} \frac{1}{2T} \|A(\chi_T v)\|_{L^p}^p \le a^p \lim_{T\to\infty} \frac{1}{2T} \|\chi_T v\|_{L^p}^p = \\ &= a^p \|v\|_{B^p}^p . \end{aligned}$$

Therefore, $b \le a$. Hence, $a = b$. $\Box$

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