Stability of periodic solutions of index-2 differential algebraic systems

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Abstract

This paper deals with periodic index-2 differential algebraic equations and the question whether a periodic solution is stable in the sense of Lyapunov. As the main result, a stability criterion is proved.This criterion is formulated in terms of the original data so that it may be used in practical computations.

Introduction

This paper deals with periodic index-2 differential algebraic equations (DAEs) of the form

$$A(x,t)x' + b(x,t) = 0,$$

and the question whether a periodic solution is stable in the sense of Lyapunov. As the main result, a stability criterion is proved. It sounds as nice as the well-known original model for regular ordinary differential equations (ODEs).
This criterion is formulated in terms of the original data so that it may be used in practical computations, too.
In view of various applications we try to do with smoothness conditions as low as possible.

The notion of stability to be used should reflect the geometrical meaning of Lyapunov stability properly. In the case of index-2 DAEs we have to consider also the so-called hidden constraints. However, in practice, we cannot proceed on the assumption that the state manifold and its tangent bundle are explicitly available. This is why we use special projectors to catch the neighbouring solutions on that manifold properly in order to compared with the given solution (e. g. März [9]).

We follow the lines of the standard ODE theory that combines linearization and Lyapunov reduction. Hence, what we have to do in essence is

- to clarify what Lyapunov reduction means for index-2 DAEs and to construct the respective transformations and

- to make sure that linearization works as expected.

The paper is organized as follows. Fundamentals on linear continuous coefficient index-2 DAEs and on linear transformations of them are given in Section 1 and 2. In Section 3, we construct special regular periodic matrix functions that transform a given periodic index-2 DAE into a constant coefficient Kronecker normal form. By this we prove a kind of Floquet-Theorem and a Lyapunov-reduction for index-2 DAEs (Theorems 3.1 and 3.2). Section 4 concerns nonlinear DAEs. There, the main result of the present paper, the stability criterion for periodic solutions, is given by Theorem 4.2. In Section 5, we discuss an application to multibody systems. Finally, we show the practical use by checking the stability of an oscillator circuit numerically.

With the present paper we continue and complete, for the time being, our attempts to generalize standard stability results known for regular ODEs to low-index DAEs.
In Lamour, März and Winkler [12] a respective reduction theorem and stability criterion were obtained for index-1 DAEs. The Perron-Theorem for index-2 DAEs proved in März [8] provides an appropriate theoretical background for Theorem 4.2 of the present paper. In this context, it should be pointed out once more that index-2 DAEs are much more complex than those having index 1, mainly in the particular case of non-autonomous equations.

The authors are of the opinion that the stability results obtained are sufficient, for the moment, for non-stationary solutions of DAEs. As far as the stability of stationary solutions of easier autonomous DAEs is concerned, this problem has been under consideration for a longer time, (e.g. Griepentrog and März [2]).
It should be mentioned that there are nice results in a more general geometric context (e. g. Reich [13]),which provides a good theoretical insight into the case of smooth systems.

1 Linear continuous coefficient equations

Consider the linear equation

$$A(t)x'(t) + B(t)x(t) = q(t), \quad t \in J \subset \mathbb{R}, \tag{1.1}$$

with continuous coefficients. Introduce the basic subspaces

$$\begin{aligned} N(t) &:= \ker A(t) \subset \mathbb{R}^m, \\ S(t) &:= \{z \in \mathbb{R}^m : B(t)z \in \operatorname{im} A(t)\} \subset \mathbb{R}^m, \end{aligned}$$

and assume $N(t)$ to be nontrivial as well as to vary smoothly with t, i.e., to be spanned by continuously differentiable basis functions $n_1, \ldots, n_{m-r} \in C^1(J, \mathbb{R}^m)$. Then, $A(t)$ has constant rank r.
The smoothness of $N(t)$ is equivalent (see e.g. Griepentrog and März (1989)) to the existence of a projector function $Q \in C^1(J, L(\mathbb{R}^m))$ such that

$$Q(t)^2 = Q(t), \quad \operatorname{im} Q(t) = N(t), \quad t \in J.$$

Further, let $P(t) := I - Q(t)$.

The nullspace $N(t)$ determines what kind of functions we should accept for solutions of (1.1). Namely, the trivial identity $A(t)Q(t) = 0$ implies

$$A(t)x'(t) = A(t)P(t)x'(t) = A(t)(Px)'(t) - A(t)P'(t)x(t)$$

and, therefore, we use Ax' as an abbreviation of $A(Px)' - AP'x$ in the following. Thus, (1.1) may be rewritten as

$$A(t)(Px)'(t) + (B(t) - A(t)P'(t))x(t) = q(t), \tag{1.2}$$

which shows the function space

$$C^1_N(J, I\!R^m) := \{y \in C(J, I\!R^m) : Py \in C^1(J, I\!R^m)\}$$

to become the appropriate one for (1.1). The realization of both the expression Ax' and the space C^1_N is independent of the special choice of the projector function.
Hence, we should ask for C^1_N-solutions, but not necessarily for C^1-solutions.
Obviously, $S(t)$ is the subspace in which the homogeneous equation solution proceeds. Recall the condition

$$S(t) \oplus N(t) = I\!R^m, \quad t \in J, \tag{1.3}$$

to characterize the class of **index-**1 DAEs (Griepentrog and März (1986)). (1.3) implies the matrix

$$A_1(t) := A(t) + (B(t) - A(t)P'(t))Q(t) \tag{1.4}$$

to be nonsingular for all $t \in J$. Multiplying (1.2) by PA_1^{-1} and QA_1^{-1} we decouple this equation into the system

$$\left.\begin{array}{rcl} (Px)' - P'Px + PA_1^{-1}BPx & = & PA_1^{-1}q \\ Qx + QA_1^{-1}BPx & = & QA_1^{-1}q \end{array}\right\}, \tag{1.5}$$

which immediately provides a solution expression. We have

$$x = Px + Qx = (I - QA_1^{-1}B)y + QA_1^{-1}q =: \Pi_{can(1)}y + QA_1^{-1}q,$$

where y solves the regular linear ODE

$$y' - P'y + PA_1^{-1}By = PA_1^{-1}q$$

and starts at $y(t_o) \in \operatorname{im} P(t_o)$ for some $t_o \in J$.
Since $\Pi_{can(1)}(t) = (I - (QA_1^{-1}BP)(t))P(t)$ represents the canonical projector onto $N(t)$ along $S(t)$, we know $S(t)$ to be filled by the homogeneous equation solution. On the other hand, nontrivial parts $QA_1^{-1}q$ of the inhomogeneity cause the solution to bulge from the subspace $S(t)$, and to cover the whole $I\!R^m$. Of course, such effects do not occur in regular ODEs.

For higher index DAEs, in particular for those having index 2, the condition (1.3) gets

lost. Consequently, different subspaces are relevant for those equations. In contrary to the above index-1 case, now a certain subspace of $S(t)$ is only filled by the homogeneous equation solution.
Introduce the two additional subspaces

$$\begin{aligned} N_1(t) &:= \ker A_1(t), \\ S_1(t) &:= \{z \in I\!R^m : B(t)P(t)z \in \operatorname{im} A_1(t)\}. \end{aligned}$$

Definition: The DAE (1.1) is said to be index-2 tractable if the conditions

$$\left.\begin{aligned} \dim(N(t) \cap S(t)) &= \text{const} > 0 \\ N_1(t) \oplus S_1(t) &= I\!R^m,\ t \in J \end{aligned}\right\} \tag{1.6}$$

are valid.
Remarks:

1) It holds that $N_1 = (I - PA^+(B - AP')Q)(N \cap S)$, and, consequently, $N_1(t)$ has the same dimension as $N(t) \cap S(t)$. Therefore, (1.6) implies $A_1(t)$ to have constant rank.
2) (1.6) implies both the matrices

$$G_2(t) := A_1(t) + B(t)P(t)Q_1(t)$$

and

$$A_2(t) := A_1(t) + (B(t) - A_1(t)(PP_1)'(t))P(t)Q_1(t)$$

$$= G_2(t)(I - P_1(t)(PP_1)'(t)PQ_1(t))$$

to become nonsingular, but $A_1(t)$ to be singular now. Thereby, $Q_1(t)$ denotes the projector onto $N_1(t)$ along $S_1(t)$, $P_1(t) := I - Q_1(t)$. By construction, Q_1 is continuous. In the following Q_1 is assumed to be C^1_N.
3) With $B_1 := (B - A_1(PP_1)')P$ the subspace $S_1(t)$ rewrites

$$S_1(t) = \{z \in I\!R^m : B_1(t)z \in \operatorname{im} A_1(t)\}.$$

We obtain the identities

$$Q_1 = Q_1 A_2^{-1} B_1 = Q_1 A_2^{-1} BP = Q_1 G_2^{-1} BP, \quad Q_1 Q = 0. \tag{1.7}$$

4) Each DAE (1.1) having Kronecker index–2 is index–2 tractable (März [6]).

The index-2 conditions (1.6) imply the decompositions

$$I\!R^m \;=\; N(t) \oplus P(t)S_1(t) \oplus P(t)N_1(t),$$

which are relevant now instead of (1.3), which was true in the index-1 case.

Let us introduce further projectors T, which projects pointwisely onto $S(t) \cap N(t) = \operatorname{im} Q(t)Q_1(t)$ and $U := I - T$. Taking this into account, we decompose the DAE solution $x \in C_N^1(J, \mathbb{R}^m)$ into

$$x = TQx + PP_1x + (PQ_1 + UQ)x =: w + u + v. \tag{1.8}$$

Multiplying (1.2) by A_2^{-1} forms (1.2) into

$$P_1P(Px)' + A_2^{-1}BPP_1(I + P_1(PP_1)'PQ_1)Q_1 + Q = A_2^{-1}q. \tag{1.9}$$

Multiplying (1.9) by PP_1, TQ and $(PQ_1 + UQ)$, respectively, and carrying out a few technical computations, we decouple the index-2 DAE into the system

$$\begin{aligned} u' - (PP_1)'u + PP_1A_2^{-1}Bu &= PP_1A_2^{-1}q, & (1.10)\\ QQ_1(PP_1)'u - QQ_1(Pv)' + TQP_1A_2^{-1}Bu + w &= TQP_1A_2^{-1}q, & (1.11)\\ UQA_2^{-1}Bu + v &= (PQ_1 + UQ)A_2^{-1}q. & (1.12) \end{aligned}$$

Looking at system (1.10)-(1.12) we know the index–2 DAE (1.1) to become solvable if $PQ_1A_2^{-1}q$ belongs to C^1.

Remarks:

1) We ask for C_N^1 solutions again. Any higher regularity of solutions, say C^1, needs additional smoothness of the coefficients, projectors and sources involved. Again, the decoupled system provides some help to state right conditions. In particular, for C^1 solutions at least $Q_1A_2^{-1}q \in C^2$, $QP_1A_2^{-1}q \in C^1$ have to be valid additionally.
2) The inherent regular ODE (1.10) is affected by the complete coefficient matrix $PP_1A_2^{-1}B - (PP_1)'$, but not only by the first term $PP_1A_2^{-1}B$. If $(PP_1)(t)$ varies quickly, the second term $(PP_1)'$ may be the dominant one. This should be taken into account when considering the asymptotic behaviour.

Next we turn shortly to the homogeneous equation. For $q = 0$ the system (1.10) – (1.12) yields $v = 0$ and

$$\begin{aligned} x &= (I + QQ_1(PP_1)' - QP_1A_2^{-1}B)u\\ &= (I + (QQ_1(PP_1)' - QP_1A_2^{-1}B)PP_1)u \quad =: \quad ku. \end{aligned}$$

The matrix $k(t)$ is nonsingular. This defines the canonical projector for the index–2 case

$$\Pi_{can(2)} := kPP_1,$$

which projects on the solution space. Clearly, not the whole space $S(t)$ is filled by solutions of the homogeneous equation, as in the index–1 case, but a proper subspace of $S(t)$ only. The fundamental matrix $X(t)$ as a matrix solution of the homogeneous equation with the initial values

$$(PP_1)(t_0)(X(t_0) - I) = 0$$

has the structure

$$X(t) = \Pi_{can(2)}(t)Y(t)(PP_1)(t_0),$$

where $Y(t)$ represents the ordinary fundamental matrix of the ODE (1.10).
In the following we simply use Π_{can} for $\Pi_{can(2)}$.

2 General linear transformations

We have characterized the index–2 condition by (1.6). Do linear nonsingular transformations $x(t) = F(t)\bar{x}(t)$ of the unknown function keep this condition invariant? It is adequate to choose $F \in C_N^1$.
The coefficients of (1.1) are transformed by

$$\bar{A} = AF, \quad \bar{B} = BF + AF'. \tag{2.1}$$

In this context AF' is used as an abbreviation of $A((PF)' - FP')$ (see [12]).
The spaces N and S are transformed into

$$\bar{N} = F^{-1}N \text{ and } \bar{S} = F^{-1}S,$$

hence,

$$\bar{N} \cap \bar{S} = F^{-1}(N \cap S).$$

The nullspace $\bar{N}(t)$ varies smoothly with t if $N(t)$ does so ([12], Lemma 2.1). Let $\bar{Q}$ denote a C^1 projector function onto $\ker \bar{A}$, but $\bar{A}_1, \bar{S}_1$ etc. the respective matrices and subspaces formed by $\bar{A}, \bar{B}$.

Lemma 2.1 *:*

$$\begin{aligned}
\bar{A}_1 &= A_1F(I - F^{-1}QF\bar{P}) \\
\bar{S}_1 &= F^{-1}S_1 \textit{ and} \\
\bar{N}_1 &= (I - F^{-1}QF\bar{P})F^{-1}N_1 \\
\bar{S}_1 &= (I - F^{-1}QF\bar{P})F^{-1}S_1.
\end{aligned}$$

Proof: It holds that $PF\bar{Q} = 0$ and $\bar{P}F^{-1}Q = 0$ because $\bar{A}\bar{Q} = 0 (= APF\bar{Q})$ and $AQ = 0 (= \bar{A}\bar{P}F^{-1}Q)$. The transformed chain matrix $\bar{A}_1$ is

$$\begin{aligned}
\bar{A}_1 = \bar{A} + \bar{B}_0\bar{Q} &= AF + (BF + A\{(PF)' - P'F\} - AF\bar{P}')\bar{Q} && (2.2)\\
&\overset{F\bar{Q}=QF\bar{Q}}{=} AF + BQF\bar{Q} - AP'QF\bar{Q} && (2.3)\\
&= (A + (B - AP')Q)(PF + F\bar{Q}) && (2.4)\\
&= A_1F(F^{-1}PF + \bar{Q}) = A_1F(F^{-1}PF\bar{P} + \bar{Q}) && (2.5)\\
&= A_1F(I - F^{-1}QF\bar{P}) && (2.6)
\end{aligned}$$

with nonsingular $(I - F^{-1}QF\bar{P})$.
This shows that $\operatorname{im} \bar{A}_1 = \operatorname{im} A_1$ and $\bar{N}_1 = (I - F^{-1}QF\bar{P})F^{-1}N_1$. Further

$$\begin{aligned}
\bar{S}_1 &:= \{\bar{z} : \bar{B}\bar{P}\bar{z} \in \operatorname{im} \bar{A}_1\} && (2.7)\\
&= \{\bar{z} : (BF + AF')\bar{P}\bar{z} \in \operatorname{im} \bar{A}_1\} && (2.8)\\
&= \{\bar{z} : B(P + Q)F\bar{P}\bar{z} \in \operatorname{im} \bar{A}_1\} && (2.9)\\
&= \{\bar{z} : BPF\bar{P}F^{-1}F\bar{z} + ((B - AP')Q + AP'Q)F\bar{P}\bar{z} \in \operatorname{im} \bar{A}_1\} && (2.10)\\
&= \{\bar{z} : BPF\bar{z} \in \operatorname{im} \bar{A}_1\} && (2.11)
\end{aligned}$$

i.e., $\bar{S}_1 = F^{-1}S_1$. Finally, it holds that

$$\begin{aligned}(I - F^{-1}QF\bar{P})F^{-1}S_1 &= \{\bar{z} : BPF(I - F^{-1}QF\bar{P})\bar{z} \in \mathrm{im}\,\bar{A}_1\} \\ &= \{\bar{z} : BPF\bar{z} \in \mathrm{im}\,\bar{A}_1\} \\ &= F^{-1}S_1 = \bar{S}_1.\end{aligned}$$

Theorem 2.2 *The tractability index 2 is invariant under transformations $F \in C^1_N$ and it holds that $\bar{P}\bar{Q}_1 \in C^1$ iff $PQ_1 \in C^1$.*

Proof: The relations of Lemma 2.1 lead to $\bar{N}_1 \cap \bar{S}_1 = (I - F^{-1}QF\bar{P})F^{-1}(N_1 \cap S_1)$. Because of the non-singularity of $I - F^{-1}QF\bar{P}$, the relations $\bar{N}_1 \cap \bar{S}_1 = \{0\}$ and $N_1 \cap S_1 = \{0\}$ are equivalent. Taking into account that $\bar{N} \cap \bar{S} = F^{-1}(N \cap S)$, we know the invariance of index–2 tractability. The transformed projector Q_1 is given by

$$\bar{Q}_1 = (I + F^{-1}QF\bar{P})F^{-1}Q_1F(I - F^{-1}QF\bar{P}),$$

therefore $\bar{P}\bar{Q}_1 = \bar{P}F^{-1}Q_1F = \underbrace{\bar{P}F^{-1}}_{\in C^1}\underbrace{PQ_1}_{\in C^1}\underbrace{PF}_{\in C^1} \in C^1$.

Definition. Two linear DAEs given on $I\!R$ are said to be kinematically equivalent if there are nonsingular matrix functions $F \in C^1_N$, $E \in C$ which transform the coefficients by (2.1), and if $\sup\limits_{t \in I\!R} |F(t)| < \infty$, $\sup\limits_{t \in I\!R} |F(t)^{-1}| < \infty$.

3 Linear periodic index–2 DAEs

Let us turn to linear homogeneous DAEs with periodic coefficients

$$A(t)x'(t) + B(t)x(t) = 0, \tag{3.1}$$

where $A, B \in C(I\!R, L(I\!R^m))$, $A(t) = A(t + \tau)$, $B(t) = B(t + \tau)$ for all $t \in I\!R$.
Note that the spaces $N(t)$ and $S(t)$ are τ-periodic since the coefficients $A(t)$ and $B(t)$ are so.
Does a smooth periodic basis of $N(t)$ exist ?
$N(t)$ is supposed to be smooth. Consequently the orthoprojector $Q_\perp(t) := I - A^+(t)A(t)$ depends continuously differentiable on t. Obviously, it holds that $Q_\perp(t) = Q_\perp(t + \tau)$. Given a basis $n^0_1, ..., n^0_{m-r} \in R^m$ of $N(0)$, then the solutions of the initial value problems

$$n' = P'_\perp n, \qquad n(0) = n^0_i, \qquad i = 1, ..., m - r$$

form a smooth basis of $N(.)$. These functions are periodic, namely

$$\begin{aligned} n_i(t + \tau) &= exp(\int_0^{t+\tau} P'_\perp(s)ds)n^0_i \\ &= exp(P_\perp(t + \tau) - P_\perp(0))n^0_i \\ &= exp(P_\perp(t) - P_\perp(0))n^0_i = n_i(t).\end{aligned}$$

Let us agree to choose periodic smooth projectors Q, P in the following. Then the matrices A_1, etc. but also the subspaces N_1, S_1, are periodic, hence the projector Q_1 is periodic, too. Since PQ_1 is continuously differentiable , we find periodic C^1–functions $b_1, ..., b_\mu$ that span im PQ_1.
In this section, we show how to transform a linear periodic index–2 DAE into a kinematically equivalent one with constant coefficients A and B. To construct such a transformation we decompose $\mathbb{R}^m$ using the projectors. Note that

$$\begin{aligned} Q_1 &= QQ_1 + PQ_1 = (QQ_1 + I)PQ_1, \\ \operatorname{im} QQ_1 &= N \cap S = \operatorname{im} T = \operatorname{im} TQ. \end{aligned}$$

With $N = \mathrm{im}QQ_1 \oplus \mathrm{im}UQ$ we have the splitting $\mathbb{R}^m = \mathrm{im}PP_1 \oplus \mathrm{im}PQ_1 \oplus \mathrm{im}QQ_1 \oplus \mathrm{im}UQ$. We span im PQ_1 by τ-periodic functions $b_1(t) \dots b_\mu(t) \in C^1$. With $q_i := (I + QQ_1)b_i \in$ im Q_1 we have a basis $b_i = Pq_i$ for im PQ_1 and $n_i = Qq_i$ is a basis for im QQ_1. With im $PP_1 =: \mathbf{span}\{p_1 \dots p_{r-\mu}\}, p_i \in C^1$ and im $UQ =: \mathbf{span}\{n_{\mu+1} \dots n_{m-r}\}$ we introduce the nonsingular matrix

$$V(t) := (p_1, \dots p_{r-\mu}, b_1, \dots b_\mu, n_1, \dots n_\mu, n_{\mu+1}, \dots n_{m-r}).$$

With the aid of V the projectors can be represented by

$$P(t) = V \begin{pmatrix} I & & & \\ & I & & \\ & & 0 & \\ & & & 0 \end{pmatrix} V^{-1},\ PP_1(t) = V \begin{pmatrix} I & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} V^{-1}$$

and $PQ_1(t) = V \begin{pmatrix} 0 & & & \\ & I & & \\ & & 0 & \\ & & & 0 \end{pmatrix} V^{-1}$.

We are interested in a similar representation for the projector Q_1, too. By

$$V \begin{pmatrix} 0 & & & \\ & I & & \\ & I & 0 & \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & & 0 & 0 \\ \vdots & & \vdots & \vdots \\ \vdots & b_1 + n_1 \dots b_\mu + n_\mu & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ 0 & & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & & 0 & 0 \\ \vdots & & \vdots & \vdots \\ \vdots & q_1 \dots q_\mu & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ 0 & & 0 & 0 \end{pmatrix} = Q_1 V$$

we see

$$Q_1 = V \begin{pmatrix} 0 & & & \\ & I & & \\ & I & 0 & \\ & & & 0 \end{pmatrix} V^{-1}$$

We aim at constructing a transformation that transforms the time varying linear DAE into a constant one. Remember that, in the index–2 case, $\Pi_{can} = kPP_1$ with a nonsingular periodic k. The fundamental matrix given by $AX' + BX = 0, (PP_1)(0)(X(0) - I) = 0$ has the representation

$$
\begin{aligned}
X(t) &= \Pi_{can}(t) Y(t) (PP_1)(0) && (3.2) \\
&= k(t) V(t) \underbrace{\begin{pmatrix} I & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} V^{-1}(t) Y(t) V(0) \begin{pmatrix} I & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}}_{=: \begin{pmatrix} Z(t) & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}} V^{-1}(0) && (3.3)
\end{aligned}
$$

with $Z(0) = I$.
Also the so–called *monodromy* matrix $X(\tau)$ is given by

$$
X(\tau) = k(0) V(0) \begin{pmatrix} Z(\tau) & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} V^{-1}(0).
$$

From linear algebra (see e.g. [10]) it is known that every nonsingular matrix $C \in L(\mathbb{R}^r)$ can be represented in the form

$$
\begin{aligned}
& C = e^W && \text{with } W \in L(\mathbb{C}^r) \\
\text{and} \quad & C^2 = e^{\bar{W}} && \text{with } \bar{W} \in L(\mathbb{R}^r).
\end{aligned}
$$

Now, let

$$
Z(\tau) = e^{\tau W_0}, \quad W_0 \in L(\mathbb{C}^r) \tag{3.4}
$$

and

$$
Z(2\tau) = Z(\tau)^2 = e^{2\tau W_0}, \quad W_0 \in L(\mathbb{R}^r). \tag{3.5}
$$

We introduce the transformation

$$
\begin{aligned}
F(t) &:= k(t) V(t) \begin{pmatrix} Z(t) e^{-tW_0} & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix} && (3.6) \\
&= X(t) V(0) \begin{pmatrix} e^{-tW_0} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + k(t) V(t) \begin{pmatrix} 0 & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix}. && (3.7)
\end{aligned}
$$

From (3.6) we see that F is nonsingular and not smooth, but $PF \in C^1$.

Theorem 3.1 *The fundamental matrix $X(t)$ of the DAE* (3.1) *can be written in the form*

$$X(t) = F(t) \begin{pmatrix} e^{tW_0} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} F(0)^{-1},$$

where $F \in C^1_N(\mathbb{R}, L(\mathbb{C}^m))$ is nonsingular and τ-periodic.

Proof: We will show that F given by (3.6) realizes this representation, indeed. First, we look at the transformed spaces and projectors. The basis functions of the nullspace N are represented by $n_i = V(t)e_{i+r}, \quad i = 1 \ldots m - r$, where e_i are the unit vectors. What is the transformed nullspace $\bar{N} = F^{-1}N$. We consider

$$\begin{aligned} F^{-1}n_i &= \begin{pmatrix} e^{tW_0}Z^{-1}(t) & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix} V^{-1}(t)k^{-1}(t)n_i \\ &= \begin{pmatrix} e^{tW_0}Z^{-1}(t) & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix} V^{-1}(t)n_i \quad \text{since } k^{-1}n_i = n_i \\ &= \begin{pmatrix} e^{tW_0}Z^{-1}(t) & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix} e_{i+r} = e_{i+r}. \end{aligned}$$

It follows that

$$\bar{N} = \mathbf{span}\{e_{r+1}, \ldots, e_m\}.$$

Therefore, in the transformed nullspace we can choose the projectors $\bar{Q} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & I & \\ & & & I \end{pmatrix}$

and $\bar{P} = I - \bar{Q}$. What about $P\bar{Q}_1$?

$$
\begin{aligned}
P\bar{Q}_1 &= \bar{P}P\bar{Q}_1 = \bar{P}F^{-1}PQ_1F \\
&= \bar{P}\begin{pmatrix} e^{tW_0}Z^{-1}(t) & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix} \underbrace{V^{-1}(t)\,\underbrace{k^{-1}(t)PQ_1k}_{PQ_1}\,V}_{\begin{pmatrix} 0 & & & \\ & I & & \\ & & 0 & \\ & & & 0 \end{pmatrix}} \begin{pmatrix} Z(t)e^{-tW_0} & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix} \\
&= \begin{pmatrix} 0 & & & \\ & I & & \\ & & 0 & \\ & & & 0 \end{pmatrix}
\end{aligned}
$$

It follows that $P\bar{P}_1 = \bar{P} - P\bar{Q}_1 = \begin{pmatrix} I & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$.

The general transformation rules for the coefficients A and B are given by (2.1). Hence, by the special transformation (3.6) the coefficients

$$\bar{A} = AF, \bar{B} = BF + AF'$$

are well defined. As we have constant projectors $\bar{P}, \bar{Q}, P\bar{Q}_1$ etc., the following relations become true

$$
\begin{aligned}
\bar{A}_2^{-1}\bar{A} &= \bar{P}_1\bar{P} = I - \bar{Q} - \bar{Q}_1 \\
\bar{A}_2^{-1}\bar{B} &= \bar{A}_2^{-1}\bar{B}\bar{P}\bar{P}_1 + \bar{Q}_1 + \bar{Q}.
\end{aligned}
$$

In particular, we have now

$$
\begin{aligned}
\bar{B}\bar{P}\bar{P}_1 &= (BF + A\{(PF)' - P'F\})P\bar{P}_1 \\
&= (BXV(0)\begin{pmatrix} e^{-tW_0} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + A\{(PF)'P\bar{P}_1 - P'XV(0)\begin{pmatrix} e^{-tW_0} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}\} \\
&= A\{(PF)'P\bar{P}_1 - (PX)'V(0)\begin{pmatrix} e^{-tW_0} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}\} \\
&= APXV(0)\begin{pmatrix} e^{-tW_0}(-W_0) & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \\
&= AF\begin{pmatrix} -W_0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} = \bar{A}\begin{pmatrix} -W_0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}
\end{aligned}
$$

Using the structure of our transformed projectors in more detail yields

$$
\hat{A} = \bar{A}_2^{-1}\bar{A} = \begin{pmatrix} I & & & \\ & 0 & & \\ & -I & 0 & \\ & & & 0 \end{pmatrix}
$$

and

$$
\hat{B} = \bar{A}_2^{-1}\bar{B} = \bar{A}_2^{-1}\bar{B}\bar{P}\bar{P}_1 + \bar{Q}_1 + \bar{Q}.
$$

Now it becomes clear that scaling by $\bar{A}_2^{-1}$ leads to

$$
\begin{aligned}
\hat{B} = \bar{A}_2^{-1}\bar{B} &= \bar{A}_2^{-1}\bar{A}\begin{pmatrix} -W_0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \bar{Q}_1 + \bar{Q} \\
&= \bar{P}_1 P\bar{P}_1\begin{pmatrix} -W_0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \bar{Q}_1 + \bar{Q} = \begin{pmatrix} -W_0 & 0 & & \\ & I & & \\ & I & I & \\ & & & I \end{pmatrix}.
\end{aligned}
$$

Finally, we know that using the transformation given by (3.6) and then scaling by $\bar{A}^{-1}$ we succeed in reducing the variable coefficient DAE (3.1) to a DAE that has the constant coefficients

$$
\hat{A} = \begin{pmatrix} I & & & \\ & 0 & & \\ & -I & 0 & \\ & & & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} -W_0 & 0 & & \\ & I & & \\ & I & I & \\ & & & I \end{pmatrix},
$$

and the fundamental solution matrix

$$\hat{X}(t) = \begin{pmatrix} e^{W_0 t} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

Definition. *Two linear, homogeneous, τ-periodic DAEs are said to be (periodically) equivalent iff the relation*

$$\bar{A} = EAF \quad \text{and} \quad \bar{B} = E(BF + AF'), \tag{3.8}$$

where $F \in C^1_N$, $E \in C$ are τ-periodic and nonsingular matrix functions, is true for their coefficients.
Periodic equivalence means kinematic equivalence by periodic transformations.

Verifying Theorem 3.1 we have proved, in fact, the following generalization of Lyapunov's Reduction Theorem.

Theorem 3.2 (i) *If two linear homogeneous τ-periodic index–2 DAEs are (periodically) equivalent, then their monodromy matrices are similar and, hence, their characteristic multipliers coincide.*

(ii) *If the monodromy matrices of two linear τ-periodic index–2 DAEs are similar, then the DAEs are (periodically) equivalent.*

(iii) *Each index–2 DAE with periodic coefficients is (periodically) equivalent to a T-periodic complex (2τ-periodic real) linear system with constant coefficients.*

Remark: Let $\Phi(t) := X(t,0)V(0)$, where we choose $V(t)$ with $\Pi(t) = V(t)\begin{pmatrix} I & \\ & 0 \end{pmatrix} V^{-1}(t)$ and $D(t) := \Phi(t)e^{-Wt}$ with $W := \begin{pmatrix} w_0 & \\ & 0 \end{pmatrix}$.
Denote by X^- the reflexive general inverse of X with

$$\begin{aligned} XX^- &= \Pi_{can}(t) \quad \text{and} \\ X^-X &= \Pi(0). \end{aligned}$$

It follows that

$$\Phi\Phi^- = \Pi_{can}(t), \quad \Phi^-\Phi = \begin{pmatrix} I & \\ & 0 \end{pmatrix}$$

and

$$DD^- = \Pi_{can}(t), \quad D^-D = \begin{pmatrix} I & \\ & 0 \end{pmatrix}$$

, and Φ remains a periodic function. The transformation F is given by

$$F := D + (I - \Pi)V \quad (= D + kV \begin{pmatrix} 0 & \\ & I \end{pmatrix})$$

and its inverse by

$$F^{-1} = D^{-} + V^{-1}(I - \Pi)k^{-1}.$$

This representation of F seems to be the direct generalization of the ODE-case one and it is valid at least for the cases

index 0	:	$\Pi \equiv I$
index 1	:	$\Pi \equiv P$ and
index 2	:	$\Pi \equiv PP_1$.

4 Quasilinear periodic index-2 DAEs

We consider the quasilinear DAE

$$f(x'(t), x(t), t) := A(x(t), t)x'(t) + b(x(t), t) = 0, \tag{4.1}$$

where the coefficients A and b are continuous, continuously differentiable with respect to the variable x, and τ - periodical, i.e., $A(x,t) = A(x, t+\tau)$, $b(x,t) = b(x, t+\tau)$. We suppose here, as in Chapter 2, that $\ker A(x,t) =: N(t)$ is independent of x and smooth, and, additionally, that also $\operatorname{im} A(x,t)$ is independent of x and smooth. This allows us, analogously to Chapter 2, to work with the corresponding smooth and periodic projectors. Let us denote

$Q(t)$	a smooth, periodic projector onto $N(t)$,
$P(t)$	$:= I - Q(t)$
$R(t)$	a smooth, periodic projector onto $\operatorname{im} A(x,t)$.

Then we have for the space tangential to the constraint manifold

$$\begin{aligned} S(x,t) : &= \{z \in \mathbb{R}^m : b'_x(x,t)z \in \operatorname{im} A(x,t)\} \\ &= \{z \in \mathbb{R}^m : (I - R(t))b'_x(x,t)z = 0\}. \end{aligned}$$

Now, let $x_\star \in C^1_N$ be the periodic solution of (4.1), whose stability we want to check. We linearize (4.1) in this solution and rewrite the nonlinear DAE (4.1) in the form

$$\begin{aligned} 0 &= f(x'(t), x(t), t) - f(x'_\star(t), x_\star(t), t) \\ &= A(x_\star(t), t)(x'(t) - x'_\star(t)) + B(x'_\star(t), x_\star(t), t)(x(t) - x_\star(t)) \\ &\quad + h(x'(t) - x'_\star(t), x(t) - x_\star(t), t), \end{aligned}$$

where

$$B(y, x, t) := f'_x(y, x, t) = b'_x(x, t) + [A(x, t)y]'_x.$$

Shifting the solution and writing $x(t)$ for $x(t)-x_\star(t)$ and $x'(t)$ for $x'(t)-x'_\star(t)$ we obtain

$$0 = \underbrace{A(x_\star(t),t)}_{=:A(t)}\,x'(t) + \underbrace{B(x'_\star(t),x_\star(t),t)}_{=:B(t)}\,x(t) + h(x'(t),x(t),t) \tag{4.2}$$

with

$$\begin{aligned} h(y,x,t) &:= f(x'_\star(t)+y, x_\star(t)+x,t) - A(t)y - B(t)x, \\ &= A(x_\star(t)+x,t)(x'_\star(t)+y) + b(x_\star(t)+x,t) - A(t)y - B(t)x, \end{aligned} \tag{4.3}$$

where we have to check the stability of the trivial solution $x=0$. By construction the function h describes a small nonlinearity. It holds that

$$\begin{aligned} h(0,0,t) &= A(x_\star(t),t)x'_\star(t) + b(x_\star(t),t) \;=\; 0, \\ h'_y(y,x,t) &= A(x_\star(t)+x,t) - A(t), \\ h'_y(y,x,t)z &\in \operatorname{im} A(x,t) = \operatorname{im} A(0,t) \quad \text{for all} \quad z \in I\!R^m, \\ h'_y(y,x,t)z &= 0 \quad \text{for all} \quad z \in N(t), \\ h(y,x,t) &= h(P(t)y,x,t), \\ h'_x(y,x,t) &= b'_x(x_\star(t)+x,t) + [A(x_\star(t)+x,t)(x_\star(t)+y)]'_x - B(t). \end{aligned}$$

To prove that the trivial solution is stable under certain conditions we will work with linearizations. Firstly, we suppose that the linear part

$$A(t)x'(t) + B(t)x(t) = 0 \tag{4.4}$$

is of index 2. This index-2 property of the linear part (4.4) does not automatically imply the index-2 property for neighbouring equations like (4.2), too. Additional structural conditions are necessary. Illustrating examples of this phenomenon are given in [7], for a more detailed discussion we refer to [14]. In our situation these structural conditions can be formulated in terms of that part c of the small nonlinearity h that corresponds to the derivative-free equations of (4.1).
Therefore, we consider

$$\begin{aligned} c(x,t) &:= (I-R(t))h(0,x,t) \\ &= (I-R(t))[b(x_\star(t)+x,t) - b'_x(x_\star(t)+x,t)x], \end{aligned} \tag{4.5}$$

where we stress that c depends only on parts of b, and suppose that at least one of the following structural conditions shall be true:

(S1) $c(x,t) = c(P(t)x,t)$, or

(S2) $c(x,t) = c((P+UQ)(t)x,t)$,where $U(t)$ is a projector along $S(0,t)\cap N(t)$, or

(S3) $c(x,t) - c(P(t)x,t) \in \operatorname{im} A_1(t)$, or

(S4) $S(x,t)\cap N(t) = S(0,t)\cap N(t)$.

In case of index-2 Hessenberg systems or linear index-2 systems each of these conditions is fulfilled.

To prove the desired stability theorem we will transform the DAE (4.2) by means of a nonsingular $F \in C^1_N$ for the transformation of variables and a nonsingular $E \in C$ for the scaling of the equations. In this way we obtain a transformed DAE

$$\hat{A}\bar{x}'(t) + \hat{B}\bar{x}(t) + \hat{h}(\bar{x}'(t), \bar{x}(t), t) = 0, \tag{4.6}$$

where

$$\begin{aligned} x &= F(t)\bar{x} \\ \hat{A}(t) &= E(t)A(t)F(t) \\ \hat{B}(t) &= E(t)(BF + AF')(t) \\ \hat{h}(\bar{y}, \bar{x}, t) &= E(t)h(F(t)\bar{y} + F'(t)\bar{x}, F(t)\bar{x}, t). \end{aligned}$$

For the small nonlinearity $\hat{h}$ we compute

$$\begin{aligned} \hat{h}'_{\bar{y}}(\bar{y}, \bar{x}, t)\bar{z} &= E(t)h'_y(F'(t)\bar{x} + F(t)\bar{y}, F(t)\bar{x}, t)F(t)\bar{z}, \\ \hat{h}'_{\bar{y}}(\bar{y}, \bar{x}, t)\bar{z} &\in E(t)\mathrm{im}\, A(t) = \mathrm{im}\, \hat{A} \quad \text{for all} \quad \bar{z} \in I\!R^m, \\ \hat{h}'_{\bar{y}}(\bar{y}, \bar{x}, t)\bar{z} &= 0 \quad \text{for} \quad \bar{z} \in \bar{N} = F(t)^{-1}N(t) \quad \text{and} \\ \hat{h}(\bar{y}, \bar{x}, t) &= \hat{h}(\hat{P}(t)\bar{y}, \bar{x}, t) \quad \text{for any projector} \quad \hat{P}(t) \quad \text{along} \quad \bar{N}. \end{aligned} \tag{4.7}$$

Further, we will see that each of the structural conditions (S1),(S2),(S3),(S4) for the original problem carries over to the transformed one. For the transformed equations we have

$$\begin{aligned} \hat{c}(\bar{x}, t) &= (I - \hat{R})E(t)h(F'(t)\bar{x}, F(t)\bar{x}, t) \\ &= E(t)(I - E(t)^{-1}\hat{R}E(t))h(0, F(t)\bar{x}, t) \\ &= E(t)c(F(t)\bar{x}, t), \end{aligned}$$

where $R(t) := E(t)^{-1}\hat{R}E(t)$ is used as a special projector onto $\mathrm{im}\, A(t)$, and it holds:

Lemma 4.1 *For quasilinear DAEs (4.1) with only time-dependent, smooth spaces* $\ker A(x,t)$ *and* $\mathrm{im}\, A(x,t)$ *any of the structural conditions (S1),(S2),(S3),(S4) is invariant under a nonsingular transformation of variables* $F \in C^1_N$ *and a scaling of the equations* $E \in C$*.*

Proof: Suppose that one of the structural conditions (S1),(S2),(S3),(S4) is true. Then we have for the conditions:

(S1): For the special projector $\hat{P}(t) := F(t)^{-1}P(t)F(t)$ along $\bar{N}$ we compute

$$\begin{aligned} \hat{c}(\hat{P}(t)\bar{x}, t) &= E(t)c(F(t)\hat{P}(t)\bar{x}, t) \\ &= E(t)c(P(t)F(t)\bar{x}, t) \\ &= E(t)c(F(t)\bar{x}, t) \; = \; \hat{c}(\bar{x}, t) \end{aligned}$$

and, hence, it follows for any projector $\bar{P}$ along $\bar{N}$ that

$$\hat{c}(\bar{P}\bar{x}, t) \; = \; \hat{c}(\hat{P}(t)\bar{P}\bar{x}, t) \; = \; \hat{c}(\hat{P}(t)\bar{x}, t) \; = \; \hat{c}(\bar{x}, t).$$

(S2): First, we mention that also condition (S2) is independent of the special choice of the projectors $Q(t)$ and $U(t)$. To see this let $Q(t)$ and $\breve{Q}(t)$ be projectors onto $N(t)$, and $U(t)$ and $\breve{U}(t)$ be projectors along $N(t) \cap S(0,t)$. If (S2) is true for the projectors Q and U, then (S2) is also true for $\breve{Q}$ and $\breve{U}$, since

$$\begin{array}{rclcccccc} (P+UQ)(\breve{P}+\breve{U}\breve{Q}) & = & P\breve{P} & + & P\breve{U}\breve{Q} & + & UQ\breve{P} & + & UQ\breve{U}\breve{Q} \\ & = & P & - & P(I-\breve{U})\breve{Q} & + & UQ\breve{P} & + & U\breve{U}\breve{Q} - UP\breve{U}\breve{Q} \\ & = & P & + & 0 & + & UQ\breve{P} & + & U\breve{Q} - U0 \\ & = & P & + & & & UQ\breve{P} & + & UQ\breve{Q} \\ & = & P & + & UQ & & & & \end{array}$$

and, hence,

$$c((\breve{P}+\breve{U}\breve{Q})x,t) = c((P+UQ)(\breve{P}+\breve{U}\breve{Q})x,t) = c((P+UQ)x,t) = c(x,t).$$

Now, considering $\hat{c}((\hat{P}+\hat{U}\hat{Q})\bar{x},t)$, where $\hat{P} = F^{-1}PF$, and $\hat{U} = F^{-1}UF$ with the dropped argument t, we obtain

$$\begin{array}{rcl} \hat{c}((\hat{P}+\hat{U}\hat{Q})\bar{x},t) & = & Ec(F(\hat{P}+\hat{U}\hat{Q})\bar{x},t) \\ & = & Ec((PF+UQF)\bar{x},t) \;=\; Ec((P+UQ)F\bar{x},t) \\ & = & Ec(F\bar{x},t) \;=\; \hat{c}(\bar{x},t). \end{array}$$

(S3): Like (S1) and (S2) also (S3) is independent of the special choice of the projector P and we see that

$$\begin{array}{rcl} \hat{c}(\bar{x},t) - \hat{c}(\bar{P}\bar{x},t) & = & E(t)[c(F(t)\bar{x},t) - c(F(t)\bar{P}\bar{x},t)] \\ & = & E(t)[c(F(t)\bar{x},t) - c((F(t)\bar{P}F(t)^{-1})F(t)\bar{x},t)] \\ & \in & E(t)\mathrm{im}\, A_1(t) = \mathrm{im}\, \hat{A}_1. \end{array}$$

(S4): (S4) implies

$$\bar{S}(\bar{y},\bar{x},t) \cap \bar{N} = \bar{S}(0,0,t) \cap \bar{N}, \quad \text{where}$$

$$\bar{S}(\bar{y},\bar{x},t) := \{\bar{z} : \hat{B}\bar{z} + \hat{h}'_{\bar{x}}(\bar{y},\bar{x},t)\bar{z} \in \mathrm{im}\, \hat{A}\}.$$

Namely, we have

$$\begin{array}{rcl} \hat{A} & = & EAF, \\ \hat{B} + \hat{h}'_{\bar{x}} & = & E(BF + AF') + E(h'_x F + h'_y F') \quad , \end{array}$$

hence

$$\begin{array}{rcl} \bar{S}(\bar{y},\bar{x},t) & = & \{\bar{z} : [B(t) + h'_x(F(t)\bar{y} + F'(t)\bar{x}, F(t)\bar{x},t)]F(t)\bar{z} \in \mathrm{im}\, A(t)\} \\ & = & \{\bar{z} : [b'_x(x_\star(t) + F(t)\bar{x},t)]F(t)\bar{z} \in \mathrm{im}\, A(t)\} \\ & = & F(t)^{-1}S(F(t)\bar{x},t), \end{array}$$

thus

$$\begin{aligned}\bar{S}(\bar{y},\bar{x},t)\cap\bar{N} &= F(t)^{-1}(S(F(t)\bar{x},t)\cap N(t))\\ &= F(t)^{-1}(S(0,t)\cap N(t))\\ &= \bar{S}(0,0,t)\cap\bar{N} \quad .\end{aligned}$$

q.e.d.

As in [12] we now follow the lines of the well-known Floquet-theory for ODEs and look for a transformation of the linear part (4.4) to a linear DAE with constant coefficients firstly. Therefore, we apply Theorem 3.2, which guarantees (4.4) to be periodically equivalent to a system with constant coefficients. More precisely, there exists a special τ-periodic nonsingular $F \in C^1_N$ for the transformation of variables and a special τ-periodic nonsingular $E \in C$ for the scaling of the equations such that

$$\hat{A} = E(t)A(t)F(t) = \begin{pmatrix} I & & & \\ & 0 & & \\ & -I & 0 & \\ & & & 0 \end{pmatrix} \text{ and } \hat{B} = E(t)(BF+AF')(t) = \begin{pmatrix} -W_0 & 0 & & \\ & I & & \\ & I & I & \\ & & & I \end{pmatrix}$$

with a constant matrix $W_0 \in L(\mathbb{C}^{m-r-\mu})$. The system

$$\hat{A}\bar{x}'(t) + \hat{B}\bar{x}(t) = 0 \tag{4.8}$$

possesses the same characteristic multipliers as (4.4) since the monodromy matrices of the systems are similar.

In the next step we apply the special transformation F and scaling E to the nonlinear system (4.2) and obtain :

$$\hat{A}\bar{x}'(t) + \hat{B}\bar{x}(t) + \hat{h}(\bar{x}'(t),\bar{x}(t),t) = 0, \tag{4.9}$$

which is by construction a DAE with a small nonlinearity and a constant linear part, which is of index-2 even in Kronecker-like normal form. It has the following block structure:

$$\begin{array}{rcccccl} \bar{x}_1' & - & W_0\bar{x}_1 & + & \hat{h}_1((\bar{x}_1',\bar{x}_2',0,0),(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4),t) & = & 0,\\ & & \bar{x}_2 & + & \hat{h}_2(0,(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4),t) & = & 0,\\ -\bar{x}_2' & + & \bar{x}_2+\bar{x}_3 & + & \hat{h}_3((\bar{x}_1',\bar{x}_2',0,0),(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4),t) & = & 0,\\ & & \bar{x}_4 & + & \hat{h}_4(0,(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4),t) & = & 0, \end{array} \tag{4.10}$$

where $\hat{h} = \begin{pmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \\ \hat{h}_4 \end{pmatrix}$.

For this specially structured equation we can also have a closer look at the structural conditions mentioned before. In our case, with $\hat{R} = \begin{pmatrix} I & & & \\ & 0 & & \\ & & I & \\ & & & 0 \end{pmatrix}$ as a projector onto $\mathrm{im}\,\hat{A} = \mathbb{R}^{r-\mu} \times \{0\}^{\mu} \times \mathbb{R}^{\mu} \times \{0\}^{m-r-\mu}$ we have

$$\hat{c}(\bar{x},t) \;=\; (I-\hat{R})\hat{h}(0,\bar{x},t) = \begin{pmatrix} 0 \\ \hat{h}_2(0,\bar{x},t) \\ 0 \\ \hat{h}_4(0,\bar{x},t) \end{pmatrix}.$$

Choosing $\bar{P} = \begin{pmatrix} I & & & \\ & I & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$ and $\bar{U}\bar{Q} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & I \end{pmatrix}$ and taking into account that $im\hat{A}_1 = im \begin{pmatrix} I & & & \\ & 0 & & \\ & -I & I & \\ & & & I \end{pmatrix} = \mathbb{R}^{r-\mu} \times \{0\}^{\mu} \times \mathbb{R}^{\mu} \times \mathbb{R}^{m-r-\mu}$, we see that the structural conditions for (4.9/4.10) mean the following:

(S1) $\hat{h}_2$ and $\hat{h}_4$ are independent of $\bar{x}_3$ and $\bar{x}_4$

(S2) $\hat{h}_2$ and $\hat{h}_4$ are independent of $\bar{x}_3$

(S3) $\hat{h}_2$ is independent of $\bar{x}_3$ and $\bar{x}_4$

(S4) $$\bar{N} \cap \bar{S}(\bar{x},t) = \{z : z_1 = z_2 = 0\,, \begin{array}{rcrcl} \hat{h}'_{2\bar{x}_3} z_3 & + & \hat{h}'_{2\bar{x}_4} z_4 & = & 0, \\ \hat{h}'_{4\bar{x}_3} z_3 & + & (I+\hat{h}'_{4\bar{x}_4}) z_4 & = & 0 \end{array} \}$$
$$= \bar{N} \cap \bar{S}(0,t) = \{z : \quad z_1 = z_2 = z_4 = 0\}.$$

Now, we will use a result of [8] to prove that under certain smoothness conditions the trivial solution of (4.9) is stable in the sense of Lyapunov if all eigenvalues of the monodromy matrix $\hat{X}$ lie in $\{z \in \mathbb{C} : |z| < 1\}$ or, equivalently, if the finite spectrum $\sigma(\hat{A}, \hat{B})$ is contained in the left side $\mathbb{C}^-$ of the complex plane. Using the transformation $x = F(t)\bar{x}$ we will derive the following main theorem:

Theorem 4.2 *Let* $\ker A(x,t)$ *and* $\mathrm{im}\, A(x,t)$ *be only time-dependent and smooth and let* $x_\star$ *be a* τ*-periodic solution of (4.1), let the linearized equation (4.4) be of index-2 and let one of the structural conditions (S1), (S2), (S3), (S4) be true. Suppose that (4.1) is sufficiently smooth, which will be specified later in the proof, and suppose that all eigenvalues of the monodromy matrix* X *of (4.4) lie inside the complex unit circle, i.e., in* $\{z \in \mathbb{C} : |z| < 1\}$*. Then the periodic solution* $x_\star$ *is stable in the sense of Lyapunov.*

Proof: We will prove that the trivial solution of (4.9) is stable in the sense of Lyapunov since then the assertion of Theorem 4.2 follows by the transformation of variables $x =$

$F(t)\bar{x}$. We know that all eigenvalues of the monodromy matrix $\hat{X}$ of (4.8) lie inside the complex unit circle since the corresponding property for the original monodromy matrix X also applies to $\hat{X}$. Now, we look for properties of the small nonlinearity $\hat{h}$. From (4.7) we see that

$$\begin{aligned}\operatorname{im} \hat{h}'_{\bar{y}}(\bar{y},\bar{x},t) &\subseteq \operatorname{im} \hat{A} \quad \text{, and}\\ \ker \hat{A} &\subseteq \ker \hat{h}'_{\bar{y}}(\bar{y},\bar{x},t).\end{aligned}$$

Further, we know by Lemma 4.1 that the structural conditions (S1), (S2), (S3), (S4) carry over to the transformed problem.
Next, by construction we have that $\hat{h}$ is continuous together with its partial Jacobians $\hat{h}'_{\bar{y}}$, $\hat{h}'_{\bar{x}}$,

$$\hat{h}(0,0,t) \;=\; E(t)h(0,0,t) \;=\; 0 \quad \text{for} \quad t \in I\!R,$$

and, to each small $\varepsilon > 0$, a $\delta(\varepsilon) > 0$ can be found such that $|\bar{x}| \leq \delta(\varepsilon)$, $|\bar{y}| \leq \delta(\varepsilon)$ yield

$$\hat{h}'_{\bar{y}}(\bar{y},\bar{x},t)| \leq \varepsilon, \quad |\hat{h}'_{\bar{x}}(\bar{y},\bar{x},t)| \leq \varepsilon$$

uniformly for all $t \in I\!R$.
To apply Theorem 3.1 of [8] we finally need that the part $\hat{c}$ additionally has continuous derivatives $\hat{c}'_t$, $\hat{c}''_{\bar{x}t}$, $\hat{c}''_{\bar{x}\bar{x}}$ and

$$\hat{c}'_t(0,t) = 0 \quad \text{for all} \quad t \in I\!R$$

$$\hat{c}''_{\bar{x}t}(\bar{x},t) \leq \kappa\varepsilon \quad \text{and} \quad \hat{c}''_{\bar{x}\bar{x}}(\bar{x},t) \leq \bar{\kappa} \quad \text{for} |x| \leq \delta(\varepsilon), \quad t \in I\!R,$$

where $\kappa, \bar{\kappa}$ are constants.
These smoothness and smallness conditions for $\hat{c}$ lead to smoothness assumptions for the corresponding derivative-free part of the original problem after a suitable scaling of the equations. We compute:

$$\begin{aligned}\hat{c}(\bar{x},t) &= (I-\hat{R})\hat{h}(0,\bar{x},t)\\ &= (I-\hat{R})Eh(0,F\bar{x},t)\\ &= (I-\hat{R})E(I-R)h(0,F\bar{x},t) \quad \text{for any projector} \quad R \quad \text{onto} \quad \operatorname{im} A\\ &= (I-\hat{R})Ec(F\bar{x},t)\end{aligned}$$

since

$$(I-\hat{R})ERh(0,F\bar{x},t) \;=\; (I-\hat{R})\underbrace{\underbrace{EA}_{=\hat{A}F^{-1}} A^{+}Rh(0,F\bar{x},t)}_{\in \operatorname{im} \hat{A}} \;=\; 0,$$

and for the special choice of $R = E^{-1}\hat{R}E$ or $\hat{R} = ERE^{-1}$ we obtain

$$\hat{c}(\bar{x},t) \;=\; E(t)c(F(t)\bar{x},t).$$

Now, if the function $\tilde{c}(x,t) = E(t)c(x,t)$ is continuous and possesses continuous derivatives $\tilde{c}'_t, \tilde{c}'_x, \tilde{c}'_{xt}, \tilde{c}'_{xx}$ and if c does not depend on the components $Q(t)x$, i.e., the structural condition (S1) is fulfilled , we see by

$$\begin{aligned}
\hat{c}'_t(\bar{x},t) &:= \tilde{c}'_t((PF)(t)\bar{x},t) + \tilde{c}'_x((PF)(t)\bar{x},t)(PF)'(t)\bar{x},\\
\hat{c}'_{\bar{x}}(\bar{x},t) &:= \tilde{c}'_x((PF)(t)\bar{x},t)(PF)(t)\\
\hat{c}''_{\bar{x}t}(\bar{x},t) &:= \tilde{c}'_x((PF)(t)\bar{x},t)(PF)'(t)\\
&\quad + \tilde{c}''_{xx}((PF)(t)\bar{x},t)(PF)'(t)\bar{x}(PF)'(t)\\
&\quad + \tilde{c}''_{xt}((PF)(t)\bar{x},t)(PF)(t).
\end{aligned}$$

that the required smoothness and smallness conditions for $\hat{c}$ are fulfilled then, and summarizing we see that all suppositions of Theorem 3.1 of [8] are satisfied. Finally, applying this Theorem completes the proof. Without condition (S1) we might additionally have to guarantee that the transformation F itself is smooth. That would mean smoothness for the solution $x_\star$ and the associated subspaces N_1 and S_1. q.e.d.

Remark: In the proof of Theorem 4.2 we have seen that the function $\hat{c}(\bar{x},t) = E(t)c(F(t)\bar{x},t)$, for which we had to suppose special smoothness properties, depends on the used scaling $E = \bar{A}_2^{-1}$. To get a deeper understanding of which parts of the original DAE have to be smooth we now aim at expressing $\hat{c}$ in terms of the original equation.
For this we exploit

$$E = \bar{A}_2^{-1} = F^{-1}[I + QF\bar{P}F^{-1}PP_1 + QQ_1(PF)'\bar{P}F^{-1}PQ_1]\, A_2^{-1}$$

and use $R := I - A_2(PQ_1 + UQ)A_2^{-1}$ as a special projector onto $\operatorname{im} A$. Thus we obtain:

$$\begin{aligned}
\hat{c}(\bar{x},t) &= (I-\hat{R})\bar{A}_2^{-1}(I-R)h(0,F\bar{x},t)\\
&= (I-\hat{R})F^{-1}[I + QF\bar{P}F^{-1}PP_1 + QQ_1(PF)'\bar{P}F^{-1}PQ_1]\, A_2^{-1}\\
&\qquad A_2(PQ_1+UQ)A_2^{-1}h(0,F\bar{x},t)\\
&= (I-\hat{R})F^{-1}[PQ_1 + UQ + QQ_1(PF)'\bar{P}F^{-1}PQ_1]A_2^{-1}h(0,F\bar{x},t)\\
&= (I-\hat{R})F^{-1}(I-PP_1)[PQ_1 + UQ + QQ_1(PF)'\bar{P}F^{-1}PQ_1]A_2^{-1}h(0,F\bar{x},t).
\end{aligned}$$

Taking into account the applied transformation of variables

$$F(t) = k(t)V(t)\begin{pmatrix} Z(t)e^{-tW_0} & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix}$$

and using the representation of the projectors by means of the basis

$$V(t) = (p_1, ..., p_{r-\mu}, b_1, ..., b_\mu, n_1, ..., n_\mu, n_{\mu+1}, ..., n_{m-r})$$

as in Chapter 3, we further obtain

$$
\begin{aligned}
\hat{c}(\bar{x},t) &= \begin{pmatrix} 0 & & & \\ & I & & \\ & & 0 & \\ & & & I \end{pmatrix} \begin{pmatrix} e^{tW_0}Z(t)^{-1} & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix} (k(t)V(t))^{-1}(I-PP_1)\cdot \\
&\qquad \cdot[PQ_1 + UQ + QQ_1(PF)'\bar{P}F^{-1}PQ_1]A_2^{-1}h(0,F\bar{x},t).
\end{aligned}
$$

Now, with

$$
\begin{aligned}
(kV)^{-1}(I-PP_1) &= V^{-1}k^{-1}(I-PP_1) \\
&= V^{-1}(I-QQ_1(PQ_1)'PP_1 - QP_1A_2^{-1}BPP_1)^{-1}(I-PP_1) \\
&= V^{-1}(I+QQ_1(PQ_1)'PP_1 + QP_1A_2^{-1}BPP_1)(I-PP_1) \\
&= V^{-1}(I-PP_1)
\end{aligned}
$$

we have, in terms of the original equation,

$$
\begin{aligned}
\hat{c}(\bar{x},t) &= \begin{pmatrix} 0 & & & \\ & I & & \\ & & 0 & \\ & & & I \end{pmatrix} V^{-1}A_2^{-1}h(0,F\bar{x},t) \\
&= \begin{pmatrix} 0 & & & \\ & I & & \\ & & 0 & \\ & & & I \end{pmatrix} V^{-1}A_2^{-1}(I-R)h(0,F\bar{x},t) \\
&= \begin{pmatrix} 0 & & & \\ & I & & \\ & & 0 & \\ & & & I \end{pmatrix} V^{-1}A_2^{-1}c(F\bar{x},t).
\end{aligned}
$$

5 Application to index-3 Euler-Lagrangian equations

Having dealt with the Floquet-theory for index-1 and index-2 DAEs one will naturally ask for corresponding theorems for higher index DAEs, too. The main difficulties here are caused by the necessary linearizations.
Here we show how Theorem 4.2 also applies to index-3 Euler-Lagrangian equations arising in the modelling of multibody systems in mechanics. Consider the Euler-Lagrangian equation

$$
\begin{aligned}
p' &= v && (5.1)\\
M(p)v' &= f(p,v,t) \; + \; G(p,t)^T\lambda && (5.2)\\
0 &= g(p,t), && (5.3)
\end{aligned}
$$

where $p, v \in \mathbb{R}^n$ are the position and velocity coordinates, $\lambda \in \mathbb{R}^k$, $k \leq n$, represents the Lagrangian multipliers, and $G(p,t) := g_p'(p,t)$. Assuming that $M(p)$ is positive definite, and $G(p,t)$ has full rank k, the system (5.1) - (5.3) constitutes an index-3 differential algebraic equation (see e.g. [5]). Since this index-3 equation may meet serious numerical difficulties (cf. [4]), Gear, Gupta and Leimkuhler [1] proposed to solve, instead of (5.1) - (5.3), the extended system

$$
\begin{aligned}
p' &= v + G(p,t)^T \mu && (5.4)\\
M(p)v' &= f(p,v,t) + G(p,t)^T \lambda && (5.5)\\
0 &= g(p,t), && (5.6)\\
0 &= G(p,t)v + g_t'(p,t), && (5.7)
\end{aligned}
$$

which is obtained by introducing the additional (artificial) Lagrangian multiplier μ as well as the constraint on velocity level (5.7).
Under the assumption above that $G(p,t)$ has full rank, the system (5.4) - (5.7) represents an index-2 differential algebraic equation. Moreover, (5.4) - (5.7) is equivalent to (5.1) - (5.3) in the sense that for each solution of (5.4) - (5.7) the component μ vanishes identically. Hence, there is a one-to-one correspondence of the solution and the solution spaces of (5.4) - (5.7) and (5.1) - (5.3) .The dimension of the inherent dynamics is $2(n-k)$ in both cases.
This one-to-one correspondence of the two systems was also pointed out in [11]. There, the authors have shown that the eigenvalues of corresponding autonomous systems linearized in some point (p_0, v_0, λ_0) resp. $((p_0, v_0, \lambda_0, 0)$ coincide such that the stability behaviour in a stationary solution is the same. Here we aim at showing that the fundamental solution matrices of the two systems correspond to each other in the same way as the solutions of the nonlinear systems themself. Therefore, let $(p_\star, v_\star, \lambda_\star) \in C_n^1 \times C_n^1 \times C_k$ and $(p_\star, v_\star, \lambda_\star, 0) \in C_n^1 \times C_n^1 \times C_k \times C_k$ be a solution of (5.1) - (5.3) and (5.4) - (5.7), respectively. We now look at the systems linearized in this solution:

$$
\begin{aligned}
p' &= v && (5.8)\\
M(p_\star)v' &= K(p_\star, v_\star, v_\star', \lambda_\star, t)p + f_v'(p_\star, v_\star, t)v + G(p_\star, t)^T \lambda && (5.9)\\
0 &= G(p_\star, t)p, && (5.10)
\end{aligned}
$$

and

$$
\begin{aligned}
p' &= v + G(p_\star, t)^T \mu && (5.11)\\
M(p_\star)v' &= K(p_\star, v_\star, v_\star', \lambda_\star, t)p + f_v'(p_\star, v_\star, t)v + G(p_\star, t)^T \lambda && (5.12)\\
0 &= G(p_\star, t)p, && (5.13)\\
0 &= G(p_\star, t)_p'[v_\star, p] + G_t'(p_\star, t)p + G(p_\star, t)v, , && (5.14)
\end{aligned}
$$

where $K(p_\star, v_\star, v_\star', \lambda_\star, t) = f_p'(p_\star, v_\star, t) - [M(p_\star)v_\star']_p' + G_p'(p_\star, t)^T \lambda_\star$.

Next, we show that the solutions of the two linearized systems (5.8) - (5.10) resp.(5.11) - (5.14) correspond to each other in the same way as the solutions of the original nonlinear DAEs.
First, let $(\tilde{p}, \tilde{v}, \tilde{\lambda}) \in C_n^1 \times C_n^1 \times C_k$ be a solution of (5.8) - (5.10). From (5.10) we have

$$0 = G(p_\star(t), t)\tilde{p}(t) \qquad \text{for all } t$$

and, therefore, if g is C^2, then

$$\begin{aligned}
0 &= \frac{d}{dt}[G(p_\star(t),t)\tilde{p}(t)] \\
&= \frac{d}{dt}[G(p_\star(t),t)]\tilde{p}(t) \;+\; G(p_\star(t),t)\tilde{p}'(t) \\
&= [G(p_\star(t),t)\tilde{p}(t)]'_p p'_\star(t) \;+\; G'_t(p_\star(t),t)\tilde{p}(t) \;+\; G(p_\star(t),t)\tilde{p}'(t) \\
&= [G(p_\star(t),t)\tilde{p}(t)]'_p v_\star(t) \;+\; G'_t(p_\star(t),t)\tilde{p}(t) \;+\; G(p_\star(t),t)\tilde{v}(t) \\
&= [G(p_\star(t),t)v_\star(t)]'_p \tilde{p}(t) \;+\; G(p_\star(t),t)\tilde{v}(t).
\end{aligned}$$

Hence, $(\tilde{p},\tilde{v},\tilde{\lambda},0) \in C^1_n \times C^1_n \times C_k \times C_k$ is a solution of .(5.11) - (5.14).

On the other hand, let $(\tilde{p},\tilde{v},\tilde{\lambda},\tilde{\mu}) \in C^1_n \times C^1_n \times C_k \times C_k$ be a solution of .(5.11) - (5.14). Then it follows with the same argument that

$$\begin{aligned}
0 &= \frac{d}{dt}[G(p_\star(t),t)\tilde{p}(t)] \\
&= \frac{d}{dt}[G(p_\star(t),t)]\tilde{p}(t) \;+\; G(p_\star(t),t)\tilde{p}'(t) \\
&= [G(p_\star(t),t)\tilde{p}(t)]'_p p'_\star(t) \;+\; G'_t(p_\star(t),t)\tilde{p}(t) \;+\; G(p_\star(t),t)\tilde{p}'(t) \\
&= [G(p_\star(t),t)\tilde{p}(t)]'_p v_\star(t) \;+\; G'_t(p_\star(t),t)\tilde{p}(t) \;+\; G(p_\star(t),t)(\tilde{v}(t) + G(p_\star(t),t)^T\tilde{\mu}(t)) \\
&= [G(p_\star(t),t)v_\star(t)]'_p \tilde{p}(t) \;+\; G'_t(p_\star(t),t)\tilde{p}(t) \;+\; G(p_\star(t),t)(\tilde{v}(t) + G(p_\star(t),t)^T\tilde{\mu}(t)) \\
&= G(p_\star(t),t)G(p_\star(t),t)^T\tilde{\mu}(t) \quad \text{because of (5.14).}
\end{aligned}$$

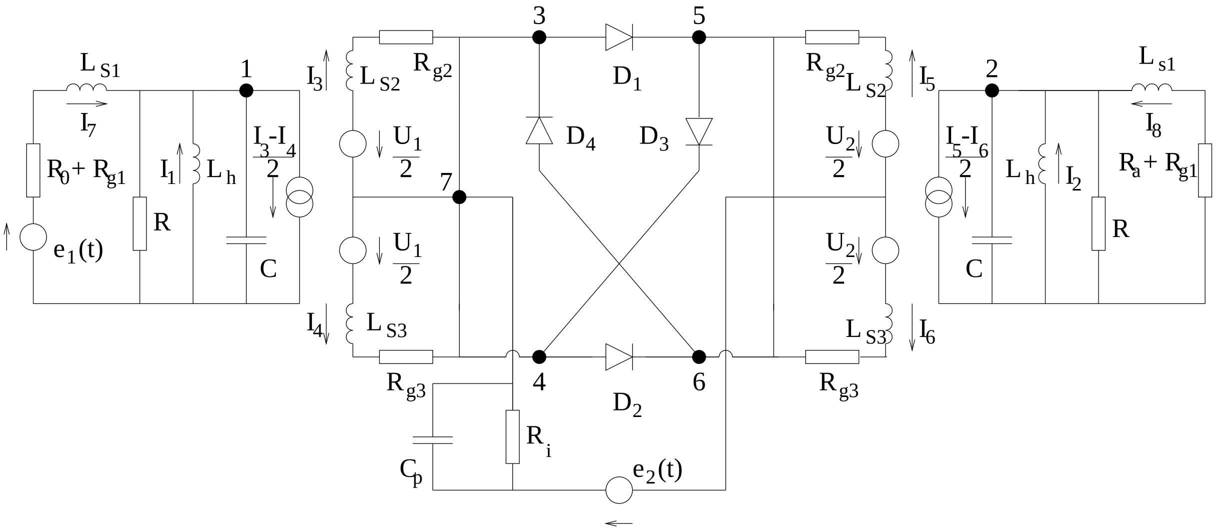
Hence, since G has full rank, it follows that $\tilde{\mu}(t) \;=\; 0$ and $(\tilde{p},\tilde{v},\tilde{\lambda}) \in C^1_n \times C^1_n \times C_k$ is a solution of (5.8) - (5.10).
Now, let X_{EL} resp. X_{GGL} denote the fundamental solution matrix of the original Euler-Lagrangian system (5.1) - (5.3) resp. of the extended index-2 system (5.4) - (5.7) . Then we can summarize

$$X_{GGL} = \begin{pmatrix} X_{EL} & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, the eigenvalues of the monodromy matrices $X_{EL}(\tau)$ and $X_{GGL}(\tau)$ coincide with the exception of k additional zero eigenvalues in $\sigma(X_{GGL}(\tau))$.

6 Numerical example

As a real example we present the so–called ring-modulator, the electrical network of which is given by



This circuit was modelled by Horneber [3].

$$C\dot{u}_1 = I_1 - I_3 \cdot 0.5 + I_4 \cdot 0.5 + I_7 - u_1/R \tag{6.1}$$
$$C\dot{u}_2 = I_2 - I_5 \cdot 0.5 + I_6 \cdot 0.5 + I_8 - u_2/R \tag{6.2}$$
$$0 = I_3 - G(UD_1) + G(UD_4) \tag{6.3}$$
$$0 = -I_4 + G(UD_2) - G(UD_3) \tag{6.4}$$
$$0 = I_5 + G(UD_1) - G(UD_3) \tag{6.5}$$
$$0 = -I_6 - G(UD_2) + G(UD_4) \tag{6.6}$$
$$C_P\dot{u}_7 = u_7/R_i + G(UD_1) + G(UD_2) - G(UD_3) - G(UD_4) \tag{6.7}$$
$$L_h\dot{I}_1 = -u_1 \tag{6.8}$$
$$L_h\dot{I}_2 = -u_2 \tag{6.9}$$
$$L_{S2}\dot{I}_3 = u_1 \cdot 0.5 - u_3 - R_{g2} \cdot I_3 \tag{6.10}$$
$$L_{S3}\dot{I}_4 = -u_1 \cdot 0.5 + u_4 - R_{g3} \cdot I_4 \tag{6.11}$$
$$L_{S2}\dot{I}_5 = u_2 \cdot 0.5 - u_5 - R_{g2} \cdot I_5 \tag{6.12}$$
$$L_{S3}\dot{I}_6 = -u_2 \cdot 0.5 + u_6 - R_{g3} \cdot I_6 \tag{6.13}$$
$$L_{S1}\dot{I}_7 = -u_1 + e_1(t) - (R_0 + R_{g1}) \cdot I_7 \tag{6.14}$$
$$L_{S1}\dot{I}_8 = -u_2 - (R_a + R_{g1}) \cdot I_8, \tag{6.15}$$

the diode-functions are given by

$$G(UD) = 40.67286402 \cdot 10^{-9} \cdot [exp(17.7493332 \cdot UD) - 1],$$

the voltages at the different diodes are expressed by

$$UD_1 = u_3 - u_5 - u_7 - e_2(t)$$
$$UD_2 = -u_4 + u_6 - u_7 - e_2(t)$$
$$UD_3 = u_4 + u_5 + u_7 + e_2(t)$$
$$UD_4 = -u_3 - u_6 + u_7 + e_2(t).$$

For the technical parameters, it holds that

$$
\begin{aligned}
R_{g1} &= 36.3\Omega, \quad R_{g2} = R_{g3} = 17.3\Omega \\
R_0 &= R_i = 50\Omega \\
R_a &= 600\Omega, \quad R = 25000\Omega \\
C &= 16 \cdot 10^{-9} F, \quad C_P = 10 \cdot 10^{-9} F \\
L_h &= 4.45H, \quad L_{S1} = 2 \cdot 10^{-3} H, \quad L_{S2} = L_{S3} = 0.5 \cdot 10^{-3} H
\end{aligned}
$$

The input signals are as follows:

$$
\begin{aligned}
e_2(t) &= 2 \cdot \sin(2\pi \cdot 10^4 \cdot t) \\
e_1(t) &= 0.5 \cdot \sin(2\pi \cdot 10^3 \cdot t).
\end{aligned}
$$

The system (6.1)–(6.15) becomes an index-2 tractable DAE. The fundamental matrix was computed by the simultaneously solution of the system

$$
\begin{array}{rcll}
f(x', x, t) & = & 0 & \qquad ((6.1) - (6.15)) \\
f'_{x'}(x', x, t)X'(t) + f'_x(x', x, t)X(t) & = & 0 & \\
\text{with the initial conditions} & & & \\
PP_1(t_0)(x(t_0) - x_{0,periodic}) & = & 0 & \\
PP_1(t_0)(X(t_0) - I) & = & 0, &
\end{array}
$$

where $x_{0,periodic}$ represents the initial value of the periodic solution. Since we have four constraints and one hidden constraint, the rank of the fundamental matrix $X(\tau)$ is $15 - 4 - 1 = 10$ so that the monodromy matrix should have zero as an eigenvalue with multiplicity 5. The eigenvalues of the approximated $X(\tau)$ computed by Mathematica are given by

$$
\begin{pmatrix}
0.9829 \\
0.9536 \\
-5.6314e - 14 \\
4.01576e - 14 \\
2.2897e - 14 \\
-6.8540e - 15 + 3.8462e - 15i \\
-6.8540e - 15 - 3.8462e - 15i \\
2.6327e - 15 + 2.9971e - 15i \\
2.6327e - 15 - 2.9971e - 15i \\
5.8078e - 16 \\
2.78159e - 19 \\
0. \\
0. \\
0. \\
0.
\end{pmatrix}
$$

All eigenvalues lie inside the unit circle. This shows that the ring-modulator has a stable periodic solution.

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