Asymptotic behaviour of solutions of semilinear hyperbolic systems in arbitrary domains

F.Jochmann
Institut für angewandte Mathematik,
Humboldt Universität Berlin
Unter den Linden 6 10099 Berlin

Abstract:
In this paper the long time asymptotic behavior of solutions of semilinear symmetric hyperbolic system including Maxwell's equations and the scalar wave-equation in an arbitraty domain are investigated. The possibly nonlinear damping term may vanish on a certain subset of the domain. It is shown that the solution decays weakly to zero if and only if the initial-state is orthogonal to all stationary states. In the case that the nonlinear damping is in addition montone, also strong local L^q-convergence is shown.
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1 Introduction

The subject of this paper the long time asymptotic behavior of solutions of semilinear hyperbolic systems of the form

$$\partial_t \mathbf{E} = E^{(1)} \cdot \left[\left(\sum_{k=1}^{3} H_k^* \partial_k \mathbf{F} \right) - \mathbf{S}(t,x,\mathbf{E}) \right], \tag{1.1}$$

$$\partial_t \mathbf{F} = E^{(2)} \cdot \sum_{k=1}^{3} H_k \partial_k \mathbf{E}, \tag{1.2}$$

with the initial-condition

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), \mathbf{F}(0,x) = \mathbf{F}_0(x). \tag{1.3}$$

Here $\mathbf{E} \in C([0,\infty), L^2(\Omega, I\!R^M))$ and $\mathbf{F} \in C([0,\infty), L^2(\Omega, I\!R^N))$ are the unknown functions depending on the time $t \geq 0$ and the space-variable $x \in \Omega$. $\Omega \subset I\!R^3$ is an arbitrary domain. $H_k \in I\!R^{N\times M}$ are constant matrices, $E^{(1)} \in L^\infty(\Omega, I\!R^{M\times M})$ and $E^{(2)} \in L^\infty(\Omega, I\!R^{N\times N})$ are positive symmetric variable matrices, which depend on the space-variable $x \in \Omega$ and satisfy $E^{(1)} = 1$ and $E^{(2)} = 1$ on $\Omega_0 \stackrel{\text{def}}{=} \Omega \setminus G$ with some subset $G \subset \Omega$.

The generally nonlinear function $\mathbf{S} : [0, \infty) \times \Omega \times I\!R^M \to I\!R^M$ satisfies

$$\mathbf{S}(t, x, \mathbf{y}) = 0 \text{ for all } x \in \Omega_0 = \Omega \setminus G$$

$$\text{and } \mathbf{S}(t, x, 0) = 0 \text{ for all } x \in \Omega, t \in (0, \infty).$$

That meams that the damping-term $\mathbf{S}(t, x, \mathbf{E})$ is only present on a certain subset $G \subset \Omega$. The following coerciveness-assumption is imposed.

$$\mathbf{y}\mathbf{S}(x, \mathbf{y}) \geq \gamma(x) \min \{|\mathbf{y}|^p, |\mathbf{y}|\} \text{ for all } \mathbf{y} \in I\!R^M, x \in G.$$

Here $p \in [2, \infty)$ and $\gamma \in L^\infty(G)$ is a positive function on G, which does not necessarily have a uniform positive lower bound on G.
This means that $\mathbf{S}(t, x, \mathbf{y})$ is allowed to be bounded as $|y| \to \infty$ and $|\mathbf{S}(t, x, \mathbf{y})|$ behaves like $|y|^{p-1}$ for small $|y|$. For example $\mathbf{S}(t, x, \mathbf{E}) \stackrel{\text{def}}{=} \gamma(x)|\mathbf{E}|^q(1 + |\mathbf{E}|^q)^{-1}\mathbf{E}$ with $q \in [0, \infty)$ is possible.

A domain $D(B) \subset L^2(\Omega, I\!R^{M+N})$ containing $C_0^\infty(\Omega, I\!R^{M+N})$ is chosen, such that the operartor

$$B(\mathbf{E}, \mathbf{F}) \stackrel{\text{def}}{=} \left(E^{(1)} \left(\sum_{k=1}^{3} H_k^* \partial_k \mathbf{F} \right), E^{(2)} \left(\sum_{k=1}^{3} H_k \partial_k \mathbf{E} \right) \right)$$

is skew-adjoint on $D(B)$, i.e. $B^* = -B$ with respect to a weighted scalar-product. The choice of $D(B)$ involves boundary-conditions on $\partial\Omega$ supplementing 1.1-1.2.

A physically important example for this system are Maxwell'sequations describing the propagation of the electromagnetic field

$$\varepsilon \partial_t \mathbf{E} = \text{ curl } \mathbf{H} - \mathbf{S}(t, x, \mathbf{E}), \text{ and } \mu \partial_t \mathbf{H} = - \text{ curl } \mathbf{E}, \tag{1.4}$$

supplemented by the initial-boundary conditions

$$\vec{n} \wedge \mathbf{E} = 0 \text{ on } (0, \infty) \times \Gamma_1, \vec{n} \wedge \mathbf{H} = 0 \text{ on } (0, \infty) \times \Gamma_2, \tag{1.5}$$

$$\mathbf{E}(0, x) = \mathbf{E}_0(x), \mathbf{H}(0, x) = \mathbf{H}_0(x). \tag{1.6}$$

In 1.5 $\Gamma_1 \subset \partial\Omega$ and $\Gamma_2 \stackrel{\text{def}}{=} \partial\Omega \setminus \Gamma_1$. $\mathbf{E}, \mathbf{H}$ denote the electric and magnetic field respectively which depend on the time $t \geq 0$ and the space-variable $x \in \Omega$ and $\mathbf{S}(t, x, \mathbf{E})$ describes a possibly nonlinear resistor. The dielectric and magnetic susceptibilities $\varepsilon, \mu \in L^\infty(\Omega)$ are assumed to be uniformly positive.
For 1.4, 1.5 the operator B is defined in the space $X \stackrel{\text{def}}{=} L^2(\Omega, \mathbb{C}^6)$ by

$$B(\mathbf{E}, \mathbf{F}) \stackrel{\text{def}}{=} (\varepsilon^{-1} \text{ curl } \mathbf{F}, -\mu^{-1} \text{ curl } \mathbf{E}) \text{ for } (\mathbf{E}, \mathbf{F}) \in D(B) \stackrel{\text{def}}{=} W_E \times W_H.$$

Here W_H is the closure of $C_0^\infty(\mathbb{R}^3 \setminus \overline{\Gamma_2}, \mathbb{C}^3)$ in $H_{curl}(\Omega)$, where $H_{curl}(\Omega)$, is the space of all $\mathbf{E} \in L^2(\Omega, \mathbb{C}^3)$ with curl $\mathbf{E} \in L^2(\Omega)$.
W_E denotes the set of all $\mathbf{E} \in H_{curl}(\Omega)$, such that

$$\int_\Omega \mathbf{E} \text{ curl } \mathbf{F} - \mathbf{F} \text{ curl } \mathbf{E} dx = 0 \text{ for all } \mathbf{F} \in W_H,$$

which includes a weak formulation of the boundary-condition $\vec{n} \wedge \mathbf{E} = 0$ on Γ_1, see [5].

Another example for 1.1-1.2 is the first-order system corresponding to the initial-boundary-value-problem of the scalar wave-equation with nonlinear damping, see [3], [4], [7].

$$\partial_t^2 \varphi = \text{ div } (E \nabla \varphi) - S(x, \partial_t \varphi) \tag{1.7}$$

supplemented by the initial-boundary-onditions

$$\varphi = 0 \text{ on } (0, \infty) \times \partial\Omega \tag{1.8}$$

$$\varphi(0, x) = f_0(x) \text{ and } \partial_t \varphi(0, x) = f_1(x) \tag{1.9}$$

for initial-data $f_0 \in \overset{0}{H^1}(\Omega)$ and $f_1 \in L^2(\Omega)$. Here $E \in L^\infty(\Omega, \mathbb{R}^{3\times 3})$ is a symmetric matrix-valued function satisfying $E = 1$ on $\Omega_0 = \Omega \setminus G$.
Note that $\mathbf{u} \overset{\text{def}}{=} (\partial_t \varphi, E \nabla \varphi) \in C([0, \infty), L^2(\Omega, \mathbb{R}^4))$ solves the system

$$\partial_t \mathbf{u} = (\text{ div } (\mathbf{u}_2, .., \mathbf{u}_4), E \nabla \mathbf{u}_1) - (S(t, x, \mathbf{u}_1), 0, 0, 0) \tag{1.10}$$

which is of the form 1.1-1.3.

The set $\mathcal{N}$ of stationary states for 1.1-1.3 is the set of all $\mathbf{u} \in D(B)$ with $B\mathbf{u} + F(t, \mathbf{u}) = 0$ for all $t \geq 0$, where the nonlinear operator $F : (0, \infty) \times X \to X$ is defined by

$$F(t, \mathbf{u}) \overset{\text{def}}{=} -\left(E^{(1)} \mathbf{S}(t, \cdot, \underline{\mathbf{u}}_1(\cdot)), 0\right).$$

From the assumptions on $\mathbf{S}$ it follows

$$\{(\mathbf{E}, \mathbf{F}) \in \text{ ker } B : \mathbf{E} = 0 \text{ on } G\} \subset \mathcal{N}.$$

Conversely assume $\mathbf{u} = (\mathbf{E}, \mathbf{F}) \in D(B)$ with $B\mathbf{u} + F(t, \mathbf{u}) = 0$. Since B is skew-adjoint this yields $0 =< \mathbf{u}, B\mathbf{u} >_X = - < \mathbf{u}, F(t, \mathbf{u}) >_X = \int_\Omega \mathbf{u} \mathbf{S}(t, x, \mathbf{E}) dx$ which implies $\mathbf{E} = 0$ on G by the coerciveness assumption. Hence the set of stationary states is given by

$$\mathcal{N} = \{(\mathbf{E}, \mathbf{F}) \in \text{ ker } B : \mathbf{E} = 0 \text{ on } G\}.$$

The aim of this paper is to show

$$(\mathbf{E}(t), \mathbf{F}(t)) \overset{t \to \infty}{\longrightarrow} 0 \text{ in } L^2(\Omega) \text{ weakly} \tag{1.11}$$

if and only if the initial-data $(\mathbf{E}_0, \mathbf{F}_0) \in L^2(\Omega)$ obey

$$\int_\Omega \left(E^{(1)-1}\mathbf{E}_0\mathbf{e} + E^{(2)-1}\mathbf{F}_0\mathbf{f} \right) dx = 0 \text{ for all } (\mathbf{e}, \mathbf{f}) \in \mathcal{N}. \tag{1.12}$$

In the case of Maxwell's equations 1.4-1.6 the condition 1.12 on $(\mathbf{E}_0, \mathbf{F}_0)$ implies

$$\text{div } (\varepsilon \mathbf{E}_0) = 0 \text{ on } \Omega_0 \text{ and } \text{div } (\mu \mathbf{H}_0) = 0 \text{ on } \Omega \tag{1.13}$$

since $\mathcal{N}$ contains all elements of the form $(\nabla\varphi, \nabla\psi)$ with $\varphi \in C_0^\infty(\Omega_0)$ and $\psi \in C_0^\infty(\Omega)$.

The proof of 1.11 is based on a suitable modification of the approach in [3], [11] for the case that the operator B does not necessarily have purely discrete spectrum. The basic idea is to show that for each $f \in C_0^\infty(\mathbb{R} \setminus \{0\})$ and $\mathbf{g} \in \omega_0(\mathbf{E}_0, \mathbf{F}_0)$ the function $f(iB)\mathbf{g}$ is real-analytic and vanishes on G, where $\omega_0(\mathbf{E}_0, \mathbf{F}_0)$ denotes the ω-limit-set with respect to the weak topology of the orbit belonging to the initial-state $(\mathbf{E}_0, \mathbf{F}_0)$. This implies $f(iB)\mathbf{g} = 0$ for all $f \in C_0^\infty(\mathbb{R} \setminus \{0\})$ and hence $\mathbf{g} \in \ker B$. (Here the operator $f(iB)$ can be defined by the spectral-theorem, since iB is self-adjoint in $L^2(\Omega, \mathbb{C}^{M+N})$.)

If $\mathbf{S}$ is independent of t and monotone with respect to $\mathbf{E}$ strong L^r-convergence is shown, i.e.

$$||\mathbf{E}(t)||_{L^r(K)} + ||\mathbf{F}(t)||_{L^2(K)} \overset{t\to\infty}{\longrightarrow} 0 \text{ for all } 1 \le r < 2, \text{ and compact sets } K \subset \Omega \tag{1.14}$$

if the initial-data $(\mathbf{E}_0, \mathbf{F}_0) \in L^2(\Omega)$ obey condition 1.12.

Finally 1.11 is used to prove that the solution the wave-equation 1.7-1.8 in an arbitrary domain $\Omega \subset \mathbb{R}^3$ decays with respect to the energy-norm on each bounded subdomain of Ω. The case of a bounded domain has been considered in [3] and [11]. Here the domain Ω is not necessarily bounded. For all $R \in (0, \infty)$, $f_0 \in \overset{0}{H^1}(\Omega))$ and $f_1 \in L^2(\Omega))$ it is shown that

$$\left(||\nabla\varphi(t)||_{L^2(\Omega\cap B_R)} + ||\partial_t\varphi(t)||_{L^2(\Omega\cap B_R)} \right) \overset{t\to\infty}{\longrightarrow} 0.$$

2 Notation, Assumptions

For an arbitrary open set $K \subset \mathbb{R}^3$ the space of all infinitely differentiable functions with compact support contained in K is denoted by $C_0^\infty(K)$.
Let $\Omega \subset \mathbb{R}^3$ be a (connected) domain and let $\Omega_0 \subset \Omega$ be an open subset, such that $G \overset{\text{def}}{=} \Omega \setminus \Omega_0$ has nonempty interior. The variable matrices $E^{(1)} \in L^\infty(\Omega, \mathbb{R}^{(M\times M)})$ and $E^{(2)} \in L^\infty(\Omega, \mathbb{R}^{(N\times N)})$ assumed to be symmetric and uniformly positive in the sense that

$$y^\perp \cdot E^{(1)}(x)y \ge c_0|y|^2 \text{ and } z^\perp \cdot E^{(2)}(x)z \ge c_0|z|^2 \tag{2.15}$$

for all $x \in \Omega, y \in \mathbb{R}^M$ and $z \in \mathbb{R}^N$ with some $c_0 \in (0, \infty)$ independent of x, y, z.
Next,

$$E^{(1)}(x) = 1 \text{ and } E^{(2)}(x) = 1 \text{ for all } x \in \Omega_0 = \Omega \setminus G. \tag{2.16}$$

The assumptions on $\mathbf{S} : [0,\infty) \times \Omega \times I\!R^M \to I\!R^M$ are the following.

$$\mathbf{S}(t, x, \mathbf{y}) = 0 \text{ if } x \in \Omega_0 = \Omega \setminus G, \tag{2.17}$$

$$\mathbf{S}(\cdot, \cdot, \mathbf{y}) \text{ measurable for fixed } \mathbf{y} \in I\!R^M, \tag{2.18}$$

and Lipschitz-continuous, i.e. there exists $L \in (0,\infty)$, such that

$$|\mathbf{S}(t, x, \mathbf{y}) - \mathbf{S}(t, x, \tilde{\mathbf{y}})| \leq L|\mathbf{y} - \tilde{\mathbf{y}}| \text{ for all } \mathbf{y}, \tilde{\mathbf{y}} \in I\!R^M \text{ and } x \in \Omega. \tag{2.19}$$

$$|\mathbf{S}(t, x, \mathbf{y})|^2 \leq C_0 < \mathbf{y}, \mathbf{S}(t, x, \mathbf{y}) > \text{ for all } t \geq 0, x \in G, \mathbf{y} \in I\!R^M \tag{2.20}$$

with some $C_0 \in (0,\infty)$. Moreover,

$$\mathbf{y}\mathbf{S}(x, \mathbf{y}) \geq \gamma(x) \min\{|\mathbf{y}|^p, |\mathbf{y}|\} \text{ for all } \mathbf{y} \in I\!R^M, x \in G. \tag{2.21}$$

Here $\gamma \in L^\infty(G)$ with $\gamma > 0$ and $p \in [2,\infty)$. The function γ does not necessarily have a uniform positive lower bound on G. It follows from the two latter assumptions that $\mathbf{S}(t, x, \mathbf{y}) = 0$ if and only if $\mathbf{y} = 0$ for all $x \in G$.

In the sequel $L^q_\gamma(K)$ denotes for a measurable subset $K \subset G$ the weighted L^q-space endowed with the norm

$$||u||_{L^q_\gamma(K)} \stackrel{\text{def}}{=} \left(\int_K |u|^q \gamma dx\right)^{1/q}$$

where $q \in [1,\infty)$ and γ as in 2.21.

The matrices $H_j \in I\!R^{N\times M}$ obey the following algebraic condition, which is fulfilled in the examples 1.4-1.6 and 1.7-1.9.

$$\left(\sum_{k=1}^{3} \xi_k H_k\right)\left(\sum_{k=1}^{3} \xi_k H_k^*\right)\left(\sum_{k=1}^{3} \xi_k H_k\right) = |\xi|^2 \left(\sum_{k=1}^{3} \xi_k H_k\right) \text{ for all } \xi \in I\!R^3 \tag{2.22}$$

Let $W_0 \subset L^2(\Omega, \mathbb{C}^M)$ be the space of all $\mathbf{e} \in L^2(\Omega, \mathbb{C}^M)$ with $\sum_{k=1}^3 \partial_k(H_k\mathbf{e}) \in L^2(\Omega)$ in the sense of distributions endowed with the norm

$$||\mathbf{e}||^2_{W_0} \stackrel{\text{def}}{=} ||\mathbf{e}||^2_{L^2} + ||\sum_{k=1}^{3} \partial_k(H_k\mathbf{e})||^2_{L^2}.$$

Furthermore, let $D(A)$ with $C_0^\infty(\Omega, \mathbb{C}^M) \subset D(A)$ be closed subspace of W_0 with respect to the above norm and

$$A\mathbf{e} \stackrel{\text{def}}{=} \sum_{k=1}^{3} \partial_k(H_k\mathbf{e}) \text{ for } \mathbf{e} \in D(A). \tag{2.23}$$

Then the adjoint operator A^* obeys $C_0^\infty(\Omega, \mathbb{C}^N) \subset D(A^*)$ and

$$A^*\mathbf{F} = -\sum_{k=1}^{3} \partial_k(H_k^*\mathbf{F}) \text{ for all } \mathbf{F} \in D(A^*). \tag{2.24}$$

For a vector $\mathbf{w} \in \mathbb{C}^{M+N}$ we denote by $\underline{\mathbf{w}}_1$ the first M and by $\underline{\mathbf{w}}_2$ the last N components of $\mathbf{w}$.

Now, the following operators are defined.
Let $D(B_0) \stackrel{\text{def}}{=} D(A) \times D(A^*)$ and

$$B_0\mathbf{w} \stackrel{\text{def}}{=} (-A^*\underline{\mathbf{w}}_2, A\underline{\mathbf{w}}_1) \text{ for } \mathbf{w} \in D(B_0) = D(A) \times D(A^*).$$

Next, $B \stackrel{\text{def}}{=} EB_0$ with $E \stackrel{\text{def}}{=} \text{diag}\,(E^{(1)}, E^{(2)})$, i.e. $D(B) \stackrel{\text{def}}{=} D(B_0)$ and

$$B\mathbf{w} \stackrel{\text{def}}{=} EB_0\mathbf{w} = \left(-E^{(1)}A^*\underline{\mathbf{w}}_2, E^{(2)}A\underline{\mathbf{w}}_1\right) \tag{2.25}$$

for $\mathbf{w} \in D(B)$. It turns out that B is a densely defined skew self-adjoint operator in the Hilbert-space $X \stackrel{\text{def}}{=} L^2(\Omega, \mathbb{C}^{M+N})$ endowed with the scalar-product

$$< \mathbf{F}, \mathbf{G} >_X \stackrel{\text{def}}{=} \int_\Omega E^{-1}\mathbf{F}\overline{\mathbf{G}}dx$$

This follows from the closedness of A, which implies that $A^{**} = \overline{A} = A$. (It is advantageous for following considerations to consider a complex space X. But whenever the $\mathbf{S}(t, x, \mathbf{E})$ occurs in an equation, the function $\mathbf{E}$ is of course assumed to be real-valued.)

Now, let $\mathcal{N}$ be the set of all $\mathbf{a} \in$ ker B with $\underline{\mathbf{a}}_1(x) = 0$ for all $x \in G$.
Moreover, let $X^0 \stackrel{\text{def}}{=} \mathcal{N}^\perp$ be the space of all $\mathbf{w} \in X$ with $< \mathbf{u}, \mathbf{w} >_X = 0$ for all $\mathbf{u} \in \mathcal{N}$.

For $\mathbf{w} \in L^2(\Omega, \mathbb{R}^{M+N})$ a function $\mathbf{u} \in C(\mathbb{R}, X)$ is called a weak soution to the problem 1.1-1.3, if

$$\frac{d}{dt}\langle \mathbf{u}(t), \mathbf{a}\rangle_X = -\langle \mathbf{u}(t), B\mathbf{a}\rangle_X + \langle F(t, \mathbf{u}(t)), \mathbf{a}\rangle_X \text{ for all } \mathbf{a} \in D(B) \tag{2.26}$$

Here $F : (0, \infty) \times X \to X$ is defined by

$$F(t, \mathbf{u}) \stackrel{\text{def}}{=} -\left(E^{(1)}\mathbf{S}(t, \cdot, \underline{\mathbf{u}}_1(\cdot)), 0\right).$$

2.26 is equivalent to the variation of constant formula

$$\mathbf{u}(t) = \exp{(tB)}\mathbf{w} + \int_0^t \exp{((t-s)B)}F(s, \mathbf{u}(s))ds \tag{2.27}$$

where $(\exp{(tB)})_{t\in\mathbb{R}}$ is the unitary group generated by B. Since $F(t, \cdot)$ is assumed to be Lipschitz-continuous in X by assumption 2.19, it follows from a standard result that this integal-equation has a unique solution $\mathbf{u} \in C([0, \infty), X)$, see [8], ch.7.

2.27 yields the energy-estimate

$$\frac{1}{2}\frac{d}{dt}||(t)||_X^2 = \langle F(t, \mathbf{u}(t)), \mathbf{u}(t)\rangle_X = -\int_G \mathbf{S}(t, x, \underline{\mathbf{u}(t)}_1) \cdot \underline{\mathbf{u}(t)}_1 dx \leq 0. \tag{2.28}$$

In the sequel $T(\cdot)\mathbf{w} \in C([0, \infty), X)$ denotes the unique solution to 1.1-1.3 in the sense of 2.26.

3 Weak convergence for $t \to \infty$

In the following lemma it is shown in particular that $T(\cdot)\mathbf{w} \in L^\infty((0,\infty),X)$, i. e. $||T(t)\mathbf{w}||_X$ is bounded as $t \to \infty$.

Lemma 1 *Suppose* $\mathbf{w} \in X$ *and* $\mathbf{u}(t) \stackrel{\text{def}}{=} T(t)\mathbf{w}$. *Then*

$$\int_0^\infty \langle \mathbf{u}(t), F(t,\mathbf{u}(t))\rangle dt + ||\mathbf{u}(t)||_X^2 \le ||\mathbf{w}||_X^2, \tag{3.29}$$

$$\int_0^\infty ||F(t,\mathbf{u}(t))||_X^2 dt \le C_0 ||\mathbf{w}||_X^2$$

with some $C_0 \in (0,\infty)$ *independent of* $\mathbf{w}$. *Moreover,*

$$\underline{\mathbf{u}}_1 \in L^p((0,\infty), L^1_\gamma(K)) \text{ for all bounded measurable subsets } K \subset G. \tag{3.30}$$

Proof: Let $\mathbf{u}(t) = (\mathbf{E}(t), \mathbf{F}(t)) \stackrel{\text{def}}{=} T(t)\mathbf{w}$. By the assumptions 2.20 on $\mathbf{S}$ one has

$$||F(t,\mathbf{f})||_X^2 \le C_0 \langle F(t,\mathbf{f}), \mathbf{f}\rangle \text{ for all } \mathbf{f} \in X$$

with some $C_0 > 0$ independent of $\mathbf{f}$. Therefore, the energy-estimate 2.28 yields

$$\frac{1}{2}\frac{d}{dt}||T(t)\mathbf{w}||_X^2 \le \langle F(t,\mathbf{f}), \mathbf{f}\rangle \le -C_0^{-1}||FT(t)\mathbf{w}||_X^2.$$

This implies 3.29 by Gronwall's lemma.

To prove 3.30 let $\mathbf{f} \in X$ and define $\mathbf{a}, \mathbf{b} \in L^2((G, I\!R^M)$ by $\mathbf{a}(x) \stackrel{\text{def}}{=} \underline{\mathbf{f}}_1(x)$ if $|\underline{\mathbf{f}}_1(x)| \le 1$ and $\mathbf{a}(x) \stackrel{\text{def}}{=} 0$ if $|\underline{\mathbf{f}}_1(x)| > 1$. Moreover, $\mathbf{b}(x) \stackrel{\text{def}}{=} \underline{\mathbf{f}}_1(x)$ if $|\underline{\mathbf{f}}_1(x)| > 1$ and $\mathbf{b}(x) \stackrel{\text{def}}{=} 0$ if $|\underline{\mathbf{f}}_1(x)| \le 1$.
Then it follows from assumption 2.21 that

$$\mathbf{a}(x)\mathbf{S}(t,x,\mathbf{a}(x)) \ge \gamma(x)|\mathbf{a}(x)|^p \text{ and } \mathbf{b}(x)\mathbf{S}(t,x,\mathbf{b}(x)) \ge \gamma(x)|\mathbf{b}(x)|$$

for all $x \in G$. Hölder's inequality yields

$$||\underline{\mathbf{f}}_1||_{L^1_\gamma(K)} \le ||\mathbf{a}||_{L^1_\gamma(K)} + ||\mathbf{b}||_{L^1_\gamma(K)} \tag{3.31}$$

$$\le C_{K,1}||\mathbf{a}||_{L^p_\gamma(K)} + ||\mathbf{b}||_{L^1_\gamma(K)} = C_{K,1}\left(\int_G |\mathbf{a}(x)|^p \gamma dx\right)^{1/p} + \int_G |\mathbf{b}(x)|\gamma dx$$

$$\le C_{K,1}\left(\int_G \mathbf{a}(x)\mathbf{S}(t,x,\mathbf{a}(x))dx\right)^{1/p} + \int_G \mathbf{b}(x)\mathbf{S}(t,x,\mathbf{b}(x))dx$$

$$\le C_{K,1}\left(\int_G \mathbf{f}(x)\mathbf{S}(t,x,\mathbf{f}(x))dx\right)^{1/p} + \int_G \mathbf{f}(x)\mathbf{S}(t,x,\mathbf{f}(x))dx$$

$$= C_{K,1}\left(\langle \mathbf{f}, F(t,\mathbf{f})\rangle\right)^{1/p} + \langle \mathbf{f}, F(t,\mathbf{f})\rangle \le C_{K,2}\left(1 + ||\mathbf{f}||_X^{2-2/p}\right)\left(\langle \mathbf{f}, F(t,\mathbf{f})\rangle\right)^{1/p}$$

Finally, the assertion follows from 3.29 and 3.31

□

Lemma 2 *$X^0 \cap D(B^n)$ is dense in $X^0 \cap D(B^m)$ for all $m, n \in I\!N$ with $m < n$.*

Proof:
Let $\mathbf{w} \in X^0 \cap D(B^m)$ and define $\mathbf{w}_\tau \stackrel{\text{def}}{=} \tau^n(\tau - B)^{-n}\mathbf{w} \in D(B^n)$ for $\tau > 0$. Then

$$||B^k(\mathbf{w}_\tau - \mathbf{w})||_X \tag{3.32}$$

$$= ||B^k\mathbf{w} - [\tau(\tau - B)^{-1}]^n B^k\mathbf{w}||_X \stackrel{\tau\to\infty}{\longrightarrow} 0 \text{ for all } k \in \{0, 1, .., m\}.$$

Suppose $\mathbf{a} \in \mathcal{N}$. Then

$$< \mathbf{w}_\tau, \mathbf{a} >_X = < \mathbf{w}, \tau^n(\tau + B)^{-n}\mathbf{a} >_X = < \mathbf{w}, \mathbf{a} >_X = 0.$$

Hence $\mathbf{w}_\tau \in X^0$. By 3.32 the proof is complete.

□

Lemma 3 *i) $\Delta\mathbf{w} = B_0^2\mathbf{w}$ on Ω for all $\mathbf{w} \in$ (ran $B_0) \cap D(B_0^2)$, in particular $-\Delta\mathbf{e} = A^*A\mathbf{e}$ and $-\Delta\mathbf{f} = AA^*\mathbf{f}$ on Ω for all $\mathbf{e} \in$ (ran $A^*)\cap D(A)$ and $\mathbf{f} \in$ (ran $A)\cap D(A^*)$. with $A\mathbf{e} \in D(A^*)$ and $A^*\mathbf{f} \in D(A)$.*
ii) $\Delta\mathbf{w} = B^2\mathbf{w}$ on $\Omega_0 = \Omega \setminus G$ for all $\mathbf{w} \in X^0 \cap D(B^2)$.

Proof:
Let $\mathbf{u} \in C_0^\infty(\Omega, \mathbb{C}^{M+N}) \subset D(B_0^n)$ for all $n \in I\!N$. Then it follows from 2.22 using Fourier-transform that

$$\mathcal{F}\underline{(B_0^3\mathbf{u})}_1(\xi) = -i\left(\sum_{j=1}^3 \xi_j H_j^*\right)\left(\sum_{k=1}^3 \xi_k H_k\right)\left(\sum_{l=1}^3 \xi_l H_l^*\right)\mathcal{F}(\underline{\mathbf{u}}_2)(\xi)$$

$$= -i|\xi|^2\left(\sum_{l=1}^3 \xi_l H_l^*\right)\mathcal{F}(\underline{\mathbf{u}}_2)(\xi)$$

Analogously,

$$\mathcal{F}\underline{(B_0^3\mathbf{u})}_2(\xi) = -i|\xi|^2\left(\sum_{l=1}^3 \xi_l H_l\right)\mathcal{F}(\underline{\mathbf{u}}_1)(\xi)$$

and hence

$$B_0^3\mathbf{u} = B_0\Delta\mathbf{u} \text{ for all } \mathbf{u} \in C_0^\infty(\Omega, \mathbb{C}^{M+N}). \tag{3.33}$$

Now, assume $\mathbf{w} \in$ (ran $B_0) \cap D(B_0^2)$, i.e. $\mathbf{w} = B_0\mathbf{v}$ with some $\mathbf{v} \in D(B_0^3)$. Then

$$\int_\Omega (B_0^2\mathbf{w})\mathbf{u}dx = \langle B_0^3\mathbf{v}, \overline{\mathbf{u}}\rangle_{L^2} = \langle \mathbf{w}, B_0^3\overline{\mathbf{u}}\rangle_{L^2}$$

$$= \langle \mathbf{v}, B_0 \Delta \overline{\mathbf{u}} \rangle_{L^2} = \langle \mathbf{w}, \Delta \overline{\mathbf{u}} \rangle_{L^2} = \int_\Omega \mathbf{w} \Delta \mathbf{u} dx$$

for all $\mathbf{u} \in C_0^\infty(\Omega)$, which means $B_0^2 \mathbf{w} = \Delta \mathbf{w}$ in the sense of distributions.

To prove ii) let $\mathbf{w} \in X^0 \cap D(B^2)$. Suppose $\mathbf{u} \in C_0^\infty(\Omega_0, \mathbb{C}^{M+N})$. and define $\tilde{\mathbf{u}} \stackrel{\text{def}}{=} (B_0^2 - \Delta)\mathbf{u} \in C_0^\infty(\Omega_0, \mathbb{C}^{M+N}) \subset D(B_0^n)$. Then 3.33 yields $B_0 \tilde{\mathbf{u}} = 0$ and hence $\tilde{\mathbf{u}} \in \mathcal{N}$. In particular $0 = \langle \mathbf{w}, \tilde{\mathbf{u}} \rangle$, because $\mathbf{w} \in X^0$. Since $E = 1$ on Ω_0, it follows $B\tilde{\mathbf{u}} = B_0 \tilde{\mathbf{u}} \in D(B)$ and $\tilde{\mathbf{u}} = (B^2 - \Delta)\mathbf{u}$. With $\mathbf{w} \in X^0$ and $\tilde{\mathbf{u}} \in \mathcal{N}$ one obtains

$$0 = \langle \mathbf{w}, \tilde{\mathbf{u}} \rangle_X = \langle \mathbf{w}, B^2 \mathbf{u} \rangle_X - \langle \mathbf{w}, \Delta \mathbf{u} \rangle_X = \langle B^2 \mathbf{w}, \mathbf{u} \rangle_X - \langle \mathbf{w}, \Delta \mathbf{u} \rangle_X$$

$$= \int_\Omega \left([B^2 \mathbf{w}] \overline{\mathbf{u}} - \mathbf{w} \Delta \overline{\mathbf{u}} \right) dx$$

Since for all $\mathbf{u} \in C_0^\infty(\Omega_0, \mathbb{C}^{M+N})$ is arbitrary, the assertion follows.

$\square$

Remark 1 *Due to the facts that generally $E^{(j)} \neq 1$ and $\underline{\mathbf{a}}_1 = 0$ on G for all $\mathbf{a} \in \mathcal{N}$ we have $\Delta \underline{\mathbf{w}}_1 \neq \underline{(B^2 \mathbf{w})}_1$ on G for all $\mathbf{w} \in X^0 \cap D(B^2)$ in general.*
For example is the case of Maxwell's equations 1.4-1.6 all $\mathbf{w} \in X^0 \cap D(B^2)$ obey $\underline{(B^2 \mathbf{w})}_1 = -\varepsilon^{-1}$ curl $(\mu^{-1}$ curl $\underline{\mathbf{w}}_1)$. The condition $\mathbf{w} \in X^0$ implies div $(\varepsilon \underline{\mathbf{w}}_1) = 0$ on Ω_0 and div $(\mu \underline{\mathbf{w}}_2) = 0$ on Ω, as mentioned in the introduction, but it does not provide any information on the divergence of $\underline{\mathbf{w}}_1$ on the set G, since $\underline{\mathbf{a}}_1 = 0$ on G for all $\mathbf{a} \in \mathcal{N}$.

In the next lemma it is shown that X^0 is an invariant space of $T(t)$.

Lemma 4 *i) $< T(t)\mathbf{w}, \mathbf{a} >_X = < \mathbf{w}, \mathbf{a} >_X$ for all $\mathbf{w} \in X$, $\mathbf{a} \in \mathcal{N}$ and $t \geq 0$.*
ii) $T(t)\mathbf{w} \in X^0$ for all $\mathbf{w} \in X^0$ and $t \geq 0$.

Proof:
Suppose $\mathbf{w} \in X^0$ and $\mathbf{a} \in \mathcal{N}$, that means $\mathbf{a} \in$ ker B and $\underline{\mathbf{a}}_1 = 0$ on G. Then 3.51 and 2.27 yield

$$< T(t)\mathbf{w}, \mathbf{a} >_X = < \exp(tB)\mathbf{w} - \int_0^t \exp((t-s)B) F(s, T(s)\mathbf{w}) ds, \mathbf{a} >_X$$

$$= < \mathbf{w}, \exp(-tB)\mathbf{a} >_X - \int_0^t < F(s, T(s)\mathbf{w}), \exp((s-t)B)\mathbf{a} >_X ds$$

$$= < \mathbf{w}, \mathbf{a} >_X - \int_0^t < F(s, T(s)\mathbf{w}), \mathbf{a} >_X ds = < \mathbf{w}, \mathbf{a} >_X .$$

Hence, i) is proved. ii) follows from i) and the definition $X^0 \stackrel{\text{def}}{=} \mathcal{N}^\perp$.

$\square$

In the sequel let $\omega_0(\mathbf{w})$ denote the ω-limit-set of the solution $T(\cdot)\mathbf{w}$ with respect to the weak topology of X, i.e. the set of all $\mathbf{g} \in X$, such that there exists a sequence $t_n \overset{n\to\infty}{\longrightarrow} \infty$ with $T(t_n)\mathbf{w} \overset{n\to\infty}{\longrightarrow} \mathbf{g}$ in X weakly, that means with $< T(t_n)\mathbf{w}, \mathbf{f} >_X \overset{n\to\infty}{\longrightarrow} < \mathbf{g}, \mathbf{f} >_X$ for all $\mathbf{f} \in X$.

Since the $T(\cdot)\mathbf{w} \in L^\infty((0,\infty), X)$ by lemma 1 the weak ω-limit-set $\omega_0(\mathbf{w})$ in nonempty for all $\mathbf{w} \in X$.

Theorem 1 *i) Let* $\mathbf{w} \in X$ *. Then* $\omega_0(\mathbf{w}) \subset \mathcal{N}$*.*

Proof:
Let $\mathbf{u}(t) \overset{\text{def}}{=} T(t)\mathbf{w}$ for $t \in I\!R$. Suppose $\mathbf{g} \in X$ and $t_n \overset{n\to\infty}{\longrightarrow} \infty$ with $T(t_n)\mathbf{w} \overset{n\to\infty}{\longrightarrow} \mathbf{g}$ in X^0 weakly. Since

$$\mathbf{u}(t_n + t) = \exp(tB)\mathbf{u}(t_n) + \int_{t_n}^{t_n+t} \exp((t_n + t - \tau)B)F\mathbf{u}(\tau)d\tau,$$

by 2.27, it follows from lemma 1, 3.29 that

$$||\mathbf{u}(t_n + t) - \exp(tB)\mathbf{u}(t_n)||_X \leq \int_{t_n}^{t_n+t} ||F\mathbf{u}(\tau)||_X d\tau$$

$$\leq t^{1/2} \left(\int_{t_n}^{t_n+t} ||F\mathbf{u}(\tau)||_X^2 d\tau \right)^{1/2} \overset{n\to\infty}{\longrightarrow} 0$$

for all $t \in I\!R$ and hence

$$\mathbf{u}(t_n + t) \overset{n\to\infty}{\longrightarrow} \exp(tB)\mathbf{g} \text{ in } X \text{ weakly for all } t \in I\!R. \tag{3.34}$$

Suppose $\chi \in C_0^\infty(I\!R)$ and define $\mathbf{f} \overset{\text{def}}{=} \int_{I\!R} \chi(t)\exp(tB)\mathbf{g}dt$ and $\mathbf{f}^{(n)} \overset{\text{def}}{=} \int_{I\!R} \chi(t)\mathbf{u}(t_n + t)dt$ Then 3.34 yields by the dominated convergence-theorem

$$\langle \mathbf{f}^{(n)}, \mathbf{h} \rangle_X = \int_{I\!R} \chi(t)\langle \mathbf{u}(t_n + t), \mathbf{h} \rangle_X dt$$

$$\overset{n\to\infty}{\longrightarrow} \int_{I\!R} \chi(t)\langle \exp(tB)\mathbf{g}, \mathbf{h} \rangle_X dt = \langle \mathbf{f}, \mathbf{h} \rangle_X$$

for all $\mathbf{h} \in X$, i.e. $\mathbf{f}^{(n)} \overset{n\to\infty}{\longrightarrow} \mathbf{f}$ weakly. In particular

$$\underline{\mathbf{f}^{(n)}}_1 \overset{n\to\infty}{\longrightarrow} \underline{\mathbf{f}}_1 \text{ in } L^2(G) \subset L^1_\gamma(K) \text{ weakly for all bounded } K \subset G. \tag{3.35}$$

On the other hand it follows from lemma 1 iii) that

$$||\underline{\mathbf{f}^{(n)}}_1||_{L^1_\gamma(K)} \leq ||\chi||_{L^{p^*}(I\!R)} \left(\int_{a+t_n}^{b+t_n} ||\underline{\mathbf{u}}_1(t)||^p_{L^1_\gamma(K)} dt \right)^{1/p} \overset{n\to\infty}{\longrightarrow} 0 \tag{3.36}$$

for all $t \in I\!R$. Here $a, b \in I\!R$ with supp $\chi \subset (a, b)$. 3.35 and 3.36 yield

$$\underline{\mathbf{f}}_1 = 0 \text{ on } K \text{ for all bounded } K \subset G \text{ and all } \chi \in C_0^\infty(I\!R),$$

where $\mathbf{f} \stackrel{\text{def}}{=} \int_{I\!R} \chi(t) \exp(tB)\mathbf{g}dt$. This implies

$$\underline{(\exp(tB)\mathbf{g})}_1 = 0 \text{ on } G \text{ for all } t \in I\!R. \tag{3.37}$$

Since iB is self-adjoint in X, $f(iB) = \int_{I\!R} f(\lambda) dE_\lambda$ can be defined by the spectral-theorem for a Borel-measurable function $f : I\!R \to \mathbb{C}$. Here $(E_\lambda)_{\lambda \in I\!R}$ denotes the family of spectral-projectors of iB. If $f \in C_0^\infty(I\!R)$, then bounded operator $f(iB)$ has the representation

$$f(iB)\mathbf{u} = \int_{I\!R} \hat{f}(t) \exp(-tB)\mathbf{u}dt \text{ for all } \mathbf{u} \in X. \tag{3.38}$$

Here $\hat{f}$ denotes the Fourier-transform of f. To see this let $\mathbf{u}, \mathbf{v} \in X$. Then

$$\langle f(iB)\mathbf{u}, \mathbf{v}\rangle = \int_{I\!R} f(\lambda) d\langle E_\lambda \mathbf{u}, \mathbf{v}\rangle$$

$$= (2\pi)^{-1/2} \int_{I\!R} \int_{I\!R} \hat{f}(t) \exp(it\lambda) dt d\langle E_\lambda \mathbf{u}, \mathbf{v}\rangle$$

$$= (2\pi)^{-1/2} \int_{I\!R} \hat{f}(t) \int_{I\!R} \exp(it\lambda) d\langle E_\lambda \mathbf{u}, \mathbf{v}\rangle dt$$

$$= (2\pi)^{-1/2} \int_{I\!R} \hat{f}(t) \langle \exp(-tB)\mathbf{u}, \mathbf{v}\rangle dt$$

$$= (2\pi)^{-1/2} \langle \int_{I\!R} \hat{f}(t) \exp(-tB)\mathbf{u}dt, \mathbf{v}\rangle$$

Suppose $f \in C_0^\infty(I\!R \setminus \{0\})$. Then 3.37 and 3.38 yield

$$\underline{(f(iB)\mathbf{g})}_1 = 0 \text{ on } G. \tag{3.39}$$

Moreover,

$$\tilde{f}(iB)\mathbf{g} = Bf(iB)\mathbf{g} = \left(-E^{(1)}A^* \underline{(f(iB)\mathbf{g})}_2, E^{(2)}A \underline{(f(iB)\mathbf{g})}_1\right) \text{ on } \Omega, \tag{3.40}$$

where $\tilde{f}(\lambda) = \lambda f(\lambda)$. In particular 3.39 and 3.40 yield in the case that f is replaced by $g(\lambda) \stackrel{\text{def}}{=} \lambda^{-1} f(\lambda) \in C_0^\infty(I\!R \setminus \{0\})$ that

$$\underline{(f(iB)\mathbf{g})}_2 = E^{(2)}A \underline{(g(iB)\mathbf{g})}_1 = 0 \text{ on } G$$

and hence by 3.39

$$f(iB)\mathbf{g} = 0 \text{ on } G \tag{3.41}$$

Since $E(x) = 1$ on $\Omega \setminus G$, 3.39 - 3.41 yield

$$B_0 f(iB)\mathbf{g} = B(f(iB)\mathbf{g}) = \tilde{f}(iB)\mathbf{g} \text{ for all } f \in C_0^\infty(I\!R \setminus \{0\}) \tag{3.42}$$

with $\tilde{f}(\lambda) = \lambda f(\lambda)$.
In particular it follows by induction

$$f(iB)\mathbf{g} \in (\text{ ran } B_0) \cap D(B_0^n) \text{ with } B_0^n f(iB)\mathbf{g} = B^n(f(iB)\mathbf{g}) \tag{3.43}$$

for all $f \in C_0^\infty(I\!R \setminus \{0\})$ and $n \in I\!N$.

The aim of the following considerations is to show that $f(iB)\mathbf{g}$ is real analytic on Ω. This will be achieved by means of a local integral representation.
Let $f \in C_0^\infty(I\!R \setminus \{0\})$ and choose $\chi \in C_0^\infty(I\!R \setminus \{0\})$ with $\chi(\lambda) = 1$ on supp f. Define

$$\mathbf{F}(t) \stackrel{\text{def}}{=} \exp(-tB)\chi(iB)\mathbf{g} = (2\pi)^{-1/2} \int_{I\!R} \hat{\chi}(\xi) \exp((-t-\xi)B)\mathbf{g} d\xi.$$

Then 3.43 and lemma 3 i) yield

$$\partial_t^2 \mathbf{F}(t) = B^2 \mathbf{F}(t) = B_0^2 \mathbf{F}(t) = \Delta \mathbf{F}, \tag{3.44}$$

in particular

$$\partial_t^n \mathbf{F} = B^n \mathbf{F}(\cdot) \in L^\infty(I\!R, L^2(\Omega)) \text{ and}$$

$$\Delta^n \mathbf{F} = B_0^{2n} \mathbf{F}(\cdot) = B^{2n} \mathbf{F}(\cdot) \in L^\infty(I\!R, L^2(\Omega)) \text{ for all } n \in I\!N,$$

which implies $\mathbf{F} \in C^\infty(I\!R \times \Omega)$ and

$$\partial_t^j \partial^\alpha \mathbf{F} \in L^\infty(I\!R \times \mathcal{K}) \text{ for all compact } \mathcal{K} \subset \Omega, j \in I\!N_0 \text{ and } \alpha \in I\!N_0^3. \tag{3.45}$$

Suppose $x_0 \in \Omega$ and choose $R > 0$ with $B_{2R} \subset \Omega$. Let

$$K(x,\xi) \stackrel{\text{def}}{=} (4\pi|x|)^{-1} \hat{f}(\xi - |x|) \text{ for } \xi \in I\!R \text{ and } x \in I\!R^3$$

Then 3.45 yields for all $x \in B_{R/2}(x_0)$

$$\lim_{r \to 0} \int_{I\!R} \int_{\partial B_r(x)} \vec{n}(y) \left[K(x-y,\xi) \nabla_y \mathbf{F}_j(\xi, y) \right. \tag{3.46}$$

$$\left. - \mathbf{F}_j(\xi,y) \nabla_y K(x-y,\xi)\right] dS(y) d\xi$$

$$= (4\pi)^{-1} \lim_{r \to 0} \left(r^{-3} \int_{I\!R} \hat{f}(\xi - r) \int_{\partial B_r(x)} [\vec{n}(y)(x-y)] \mathbf{F}_j(\xi,y) dS(y) d\xi \right)$$

$$= \int_{I\!R} \hat{f}(\xi) \mathbf{F}_j(\xi, x) d\xi = \int_{I\!R} \hat{f}(\xi) \left(\exp(-\xi B)\chi(iB)\mathbf{g}\right)_j (x) d\xi$$

$$= (2\pi)^{1/2}(f(iB)\chi(iB)\mathbf{g})_j(x) = (2\pi)^{1/2}(f(iB)\mathbf{g})_j(x).$$

For all $x \in B_{R/2}(x_0)$ and all $y \in B_{2R}(x_0)$ with $y \neq x$ one has by 3.44

$$\text{div}_{\,y}\left[K(x-y,\xi)\nabla_y \mathbf{F}_j(\xi,y) - \mathbf{F}_j(\xi,y)\nabla_y K(x-y,\xi)\right]$$

$$= K(x-y,\xi)\Delta_y \mathbf{F}_j(\xi,y) - \mathbf{F}_j(\xi,y)\Delta_y K(x-y,\xi)$$

$$= K(x-y,\xi)\partial_\xi^2 \mathbf{F}_j(\xi,y) - \mathbf{F}_j(\xi,y)\partial_\xi^2 K(x-y,\xi)$$

$$= \partial_\xi\left[K(x-y,\xi)\partial_\xi \mathbf{F}_j(\xi,y) - \mathbf{F}_j(\xi,y)\partial_\xi K(x-y,\xi)\right]$$

and hence

$$\int_{I\!R}\int_{\partial B_R(x_0)} \vec{n}(y)\left[K(x-y,\xi)\nabla_y \mathbf{F}_j(\xi,y)\right. \tag{3.47}$$

$$\left. -\mathbf{F}_j(\xi,y)\nabla_y K(x-y,\xi)\right] dS(y)d\xi$$

$$-\int_{I\!R}\int_{\partial B_r(x)} \vec{n}(y)\left[K(x-y,\xi)\nabla_y \mathbf{F}_j(\xi,y) - \mathbf{F}_j(\xi,y)\nabla_y K(x-y,\xi)\right] dS(y)d\xi$$

$$= \int_{I\!R}\int_{B_R(x_0)\setminus B_r(x)} \text{div}_{\,y}\left[K(x-y,\xi)\nabla_y \mathbf{F}_j(\xi,y)\right.$$

$$\left. -\mathbf{F}_j(\xi,y)\nabla_y K(x-y,\xi)\right] dy d\xi$$

$$= \int_{B_R(x_0)\setminus B_r(x)}\int_{I\!R} \partial_\xi\left[K(x-y,\xi)\partial_\xi \mathbf{F}_j(\xi,y)\right.$$

$$\left. -\mathbf{F}_j(\xi,y)\partial_\xi K(x-y,\xi)\right] d\xi dy = 0,$$

since $K(x-y,\xi) \stackrel{|\xi|\to\infty}{\longrightarrow} 0$ and $\partial_\xi K(x-y,\xi) \stackrel{|\xi|\to\infty}{\longrightarrow} 0$, whereas $\mathbf{F}$ and $\partial_\xi \mathbf{F}$ remains bounded as $|\xi| \to \infty$ by 3.45 for fixed $y \neq x$.
Now, 3.46 and 3.47 yield for all $x \in B_{R/2}(x_0)$

$$(2\pi)^{1/2}(f(iB)\mathbf{g})_j(x) = \int_{I\!R}\int_{\partial B_R(x_0)} \vec{n}(y)\left[K(x-y,\xi)\nabla_y \mathbf{F}_j(\xi,y)\right. \tag{3.48}$$

$$\left. -\mathbf{F}_j(\xi,y)\nabla_y K(x-y,\xi)\right] dS(y)d\xi$$

Since $f \in C_0^\infty(I\!R)$, there exists a constant $C_1 \in (0,\infty)$ with

$$(1+\xi^2)|\hat{f}^{(k)}(\xi)| \leq C_1^k \text{ for all } \xi \in I\!R \text{ and } k \in I\!N.$$

Hence there exists a constant $C_2 \in (0, \infty)$ with

$$\int_{I\!R} \int_{\partial B_R(x_0)} \left(|\frac{d^k}{d\tau^k} K(x_0 + \tau\eta - y, \xi)| \right.$$

$$\left. + |\frac{d^k}{d\tau^k} \left(\vec{n}(y) \nabla_y K(x_0 + \tau\eta - y, \xi)\right) | \right) dS(y) d\xi \leq C_2^k k! |\eta|^k$$

for all $\eta \in I\!R^3$ with $|\eta| \leq R/2$, $\tau \in (-1, 1)$ and $k \in I\!N$. Now it follows from 3.45 and 3.48 and the previous estimate that there exists a constant $C_3 \in (0, \infty)$ with

$$(k!)^{-1} |\frac{d^k}{d\tau^k} \left(f(iB)\mathbf{g}\right)(x_0 + \tau\eta)| \leq (C_3|\eta|)^k$$

for all $\eta \in I\!R^3$ with $|\eta| \leq R/2$, $\tau \in (-1, 1)$ and $k \in I\!N$, which yields the analycity of $f(iB)\mathbf{g}$.

Next this analycity yields by 3.41 and the assumptions that G has nonempty interior and Ω is connected that

$$f(iB)\mathbf{g} = 0 \text{ for all } f \in C_0^\infty(I\!R \setminus \{0\}). \tag{3.49}$$

Choose a sequence $f_n \in C_0^\infty(I\!R \setminus \{0\}), n \in I\!N$ with $|f_n(\lambda)| \leq 1$ and $f_n(\lambda) \overset{n\to\infty}{\longrightarrow} 1$ for $\lambda \in I\!R \setminus \{0\}$ and $f_n(0) \overset{n\to\infty}{\longrightarrow} 0$.
By the spectral-theorem 3.49 implies

$$0 = \langle f_n(iB)\mathbf{g}, \mathbf{g}\rangle_X \overset{n\to\infty}{\longrightarrow} \langle (1 - P_{ker B})\mathbf{g}, \mathbf{g}\rangle_X$$

and hence $\mathbf{g} = P_{ker B}\mathbf{g} \in \ker B$. Together with 3.37 this yields $\mathbf{g} \in \mathcal{N}$, which completes the proof.

□

Theorem 2 *Suppose* $\mathbf{w} \in X$.
Then $T(t)\mathbf{w} \overset{t\to\infty}{\longrightarrow} 0$ *in* X *weakly, if and only if* $\mathbf{w} \in X^0 = \mathcal{N}^\perp$.

Proof:
First suppose that $T(t)\mathbf{w} \overset{t\to\infty}{\longrightarrow} 0$ in X weakly. From lemma 4 i) it follows
$< \mathbf{w}, \mathbf{a} >_X = < T(t)\mathbf{w}, \mathbf{a} >_X \overset{t\to\infty}{\longrightarrow} 0$, for all $\mathbf{a} \in \mathcal{N}$, i.e. $\mathbf{w} \in X^0$.

To prove the converse assume that $\mathbf{w} \in X^0$. Then it follows from lemma 1 i) that $\omega_0(\mathbf{w}) \neq \emptyset$. Hence it suffices to show that it can contain at most the zero-element. Since $\mathbf{w} \in X^0$, lemma 4 yields $\omega_0(\mathbf{w}) \subset X^0 = \mathcal{N}^\perp$. By the previos theorem it follows $\omega_0(\mathbf{w}) \in \mathcal{N} \cap \mathcal{N}^\perp = \{0\}$.

□

Let P be the orthogonal-projector on $\mathcal{N}$ in X.

Corollary 1 *Let* $\mathbf{w} \in X$ *be arbitrary. Then* $T(t)\mathbf{w} \overset{t\to\infty}{\longrightarrow} P\mathbf{w}$ *in* X *weakly.*

Proof:
Define $\mathbf{a} \overset{\text{def}}{=} P\mathbf{w} \in \mathcal{N}$ and $\mathbf{u}^{(1)}(t) \overset{\text{def}}{=} T(t)(1-P)\mathbf{w}$. Since $(1-P)\mathbf{w} \in \mathcal{N}^{\perp} = X^0$, theorem 2 yields

$$\mathbf{u}^{(1)}(t) \overset{t\to\infty}{\longrightarrow} 0 \text{ in } X \text{ weakly} . \tag{3.50}$$

Since $\underline{\mathbf{a}}_1 = 0$ on $G = \Omega \setminus \Omega_0$, assumption 2.17 yields

$$F(t, \mathbf{u}^{(1)}(t) + \mathbf{a}) = F(t, \mathbf{u}^{(1)}(t)) \text{ for all } t \geq 0. \tag{3.51}$$

Since $\mathbf{u}^{(1)}$ soves 1.1-1.3 and $\mathbf{a} \in \mathcal{N} \subset \ker B$, it follows with 3.51 that $\mathbf{u}^{(1)} + \mathbf{a}$ also solves 1.1-1.3 in the sense of 2.26 with initial-condition $\mathbf{u}^{(1)}(0) + \mathbf{a} = \mathbf{w}$. Hence $T(t)\mathbf{w} = \mathbf{u}^{(1)}(t) + \mathbf{a}$ and the assertion follows from 3.50.

$\square$

4 Strong L^q-convergence of solutions

The aim of the following considerations is find sufficient conditions for strong convergence. Assume that in addition $\mathbf{S}(t, x, \mathbf{y})$ is independent of t, i.e. $\mathbf{S}(t, x, \mathbf{y}) = \mathbf{S}_0(x, \mathbf{y})$ and

$$(\mathbf{S}_0(x, \mathbf{y}) - \mathbf{S}_0(x, \tilde{\mathbf{y}}))\,(\mathbf{y} - \tilde{\mathbf{y}}) \geq 0 \tag{4.52}$$

for all $t \geq 0, \mathbf{y} \in I\!R^M$ and $x \in G$ with some function $\mathbf{S}_0 : \Omega \times I\!R^M \to I\!R^M$.
The main purpose of this assumption is to ensure that $T(t)\mathbf{w} \in D(B)$, $\partial_t(T(t)\mathbf{w}) \in L^2(\Omega)$ and $BT(\cdot)\mathbf{w} \in L^\infty((0,\infty), X)$, i. e. $||BT(t)\mathbf{w}||_X$ is bounded as $t \to \infty$ if $\mathbf{w} \in D(B)$ as shown in the following lemma. (For examle in the linear case $\mathbf{S}(t, x, y) = \sigma(t, x)y$ the condition that $\mathbf{S}$ is independent of t can be replaced by the weaker assumption

$$\partial_t\sigma \in L^\infty((0,\infty) \times G) \text{ and } |\partial_t\sigma(t,x)| \leq C_1\sigma(t,x)$$

for all $t \geq 0$ and $x \in G$ with some constant C_1 independent of t, x.)

Lemma 5 *For all* $\mathbf{w} \in D(B)$ *one has*

$$T(\cdot)\mathbf{w}) \in W^{1,\infty}((0,\infty), X) \cap L^\infty((0,\infty), D(B)) \tag{4.53}$$

Proof:
It follows from the assumption that there is a nonlinear operator $F_0 : X \to X$ with $F(t, \mathbf{w}) = F_0(\mathbf{w})$ and

$$| < F_0(\mathbf{w}) - F_0(\tilde{\mathbf{w}}), \mathbf{w} - \tilde{\mathbf{w}} >_X \leq 0 \text{ for all } \mathbf{w}, \tilde{\mathbf{w}} \in X$$

Suppose $\mathbf{w} \in D(B)$ and set $\mathbf{u}(t) \stackrel{\text{def}}{=} T(t)\mathbf{w}$. Let $\tau > 0$. Then in analogy to 2.28 one has

$$\frac{d}{dt}||\mathbf{u}(t+\tau) - \mathbf{u}(t)||_X^2$$

$$= 2 < F(t+\tau, \mathbf{u}(t+\tau)) - F(t, \mathbf{u}(t)), \mathbf{u}(t+\tau) - \mathbf{u}(t) >_X$$

$$= 2 < F_0(\mathbf{u}(t+\tau)) - F_0(\mathbf{u}(t)), \mathbf{u}(t+\tau) - \mathbf{u}(t) >_X \leq 0$$

Hence

$$|\tau|^{-2}||\mathbf{u}(t+\tau) - \mathbf{u}(t)||_X^2 \leq |\tau|^{-2}||\mathbf{u}(\tau) - \mathbf{w}||_X^2 \tag{4.54}$$

Now, it follws from 2.27 that

$$\mathbf{u}(\tau) - \mathbf{w} = (\exp(\tau B) - 1)\mathbf{w} - \int_0^\tau \exp((\tau - s)B)F(s, \mathbf{u}(s))ds,$$

Since $\mathbf{w} \in D(B)$ one has $\tau^{-1}(\exp(\tau B) - 1)\mathbf{w} \stackrel{\tau\to 0}{\longrightarrow} B\mathbf{w}$ in$X$ strongly and hence

$$\overline{\lim_{\tau\to 0}}||\tau^{-1}(\mathbf{u}(\tau) - \mathbf{w})||_X \leq ||B\mathbf{w}||_X + \overline{\lim_{\tau\to 0}}\left(\tau^{-1}\int_0^\tau ||F(s, \mathbf{u}(s))||_X ds\right) \tag{4.55}$$

$$\leq ||B\mathbf{w}||_X + K_1\overline{\lim_{\tau\to 0}}\left(\tau^{-1}\int_0^\tau ||\mathbf{u}(s)||_X ds\right) = ||B\mathbf{w}||_X + K_1||\mathbf{w}||_X.$$

Now, 4.54 and 4.55 yield $\mathbf{u} \in W^{1,\infty}((0,\infty), X)$ and

$$||\partial_t \mathbf{u}||_{L^\infty((0,\infty),X)} \leq ||B\mathbf{w}||_X + K_1||\mathbf{w}||_X.$$

Next, 2.26 yields for all $\mathbf{c} \in D(B)$

$$| < \mathbf{u}(t), B\mathbf{c} >_X | \leq (||\partial_t \mathbf{u}(t)||_X + ||F(t, \mathbf{u}(t))||)\,||\mathbf{c}||_X.$$

$$\leq (||B\mathbf{w}||_X + K_1||\mathbf{w}||_X + ||F(t, \mathbf{u}(t))||)\,||\mathbf{c}||_X.$$

Finally, this implies $\mathbf{u}(t) \in D(B^*) = D(B)$ and

$$||B\mathbf{u}(t)||_X \leq ||B\mathbf{w}||_X + K_1||\mathbf{w}||_X + ||S(t, \mathbf{u}(t))||_X$$

$$\leq C_1(||B\mathbf{w}||_X + ||\mathbf{w}||_X)$$

with some $C_1 \in (0, \infty)$ independent of $\mathbf{w}$.

□

Lemma 6 *i) Let $K \subset \Omega_0$ be a bounded open set with $\overline{K} \subset \Omega_0$.*
Then $\mathbf{w} \in H^1(K)$ and

$$||\mathbf{w}||_{H^1(K)} \leq C_K ||\mathbf{w}||_{D(B)} \text{ for all } \mathbf{w} \in X^0 \cap D(B).$$

with some constant $C_K \in (0, \infty)$ depending only on K.
ii) Suppose in addition $E^{(2)} = 1$ on all of Ω.
Let $U \subset \Omega$ be a bounded open set with $\overline{U} \subset \Omega$.
Then $\mathbf{F} \in H^1(U)$ and

$$||\mathbf{F}||_{H^1(U)} \leq C_U ||\mathbf{w}||_{D(B)} \text{ for all } \mathbf{w} = (\mathbf{E}, \mathbf{F}) \in X^0 \cap D(B).$$

with some constant $C_U \in (0, \infty)$ depending only on U.

Proof:
i) Let $K \subset \Omega_0$ be a bounded open set with $\overline{K} \subset \Omega_0$. Choose $\chi \in C_0^\infty(\Omega_0)$ with $\chi = 1$ on K. Suppose $\mathbf{w} \in X^0 \cap D(B^2)$. Then lemma 3 ii) yields $\mathbf{w} \in H^2_{loc}(\Omega_0)$ and

$$\sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 |\nabla \mathbf{w}_k|^2 dx = \sum_{k=1}^{M+N} \int_{\Omega_0} \text{div } (\chi^2 \nabla \mathbf{w}_k) \overline{\mathbf{w}}_k dx$$

$$\leq C_{K,1} \sum_{k=1}^{M+N} \int_{\Omega_0} |\chi \nabla \mathbf{w}_k| \quad |\mathbf{w}_k| dx + \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 \Delta \mathbf{w}_k \overline{\mathbf{w}}_k dx$$

$$\leq C_{K,2} ||\mathbf{w}||_X^2 + 1/3 \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 |\nabla \mathbf{w}_k|^2 dx + \langle \chi^2 (B^2 \mathbf{w}), \mathbf{w} \rangle_X$$

$$\leq C_{K,3} ||\mathbf{w}||_X^2 + 1/3 \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 |\nabla \mathbf{w}_k|^2 dx + \langle \chi^2 (B \mathbf{w}), B\mathbf{w} \rangle_X$$

$$\leq C_{K,4} \left(||B\mathbf{w}||_X^2 + ||\mathbf{w}||_X^2 \right) + 2/3 \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 |\nabla \mathbf{w}_k|^2 dx$$

by assumption 2.16. Hence

$$||\mathbf{w}||_{H^1(K)}^2 \leq ||\mathbf{w}||_X^2 + \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 |\nabla \mathbf{w}_k|^2 dx \leq 3C_{K,4} \left(||B\mathbf{w}||_X^2 + ||\mathbf{w}||_X^2 \right)$$

By lemma 2 the estimate holds for all $\mathbf{w} \in X^0 \cap D(B)$.

To prove ii) consider first $\mathbf{f} \in D(A^*) \cap (\text{ ker } A^*)^\perp$ with $A^*\mathbf{f} \in D(A)$.
Since $(\text{ker } A^*)^\perp = \overline{\text{ran} A}$ lemma 3 i) yields $\Delta \mathbf{f} = -AA^*\mathbf{f}$. From a similar cut-off argument as in the proof of the first part it follows that

$$||\mathbf{f}||_{H^1(U)}^2 \leq C_{U,4} \left(||A^*\mathbf{f}||_{L^2}^2 + ||\mathbf{f}||_{L^2}^2 \right) \tag{4.56}$$

Since the set of all $\mathbf{f} \in D(A^*) \cap (\ \ker\ A^*)^\perp$ with $A^*\mathbf{f} \in D(A)$ is dense in $D(A^*) \cap (\ \ker\ A^*)^\perp$, 4.56 holds for all $\mathbf{f} \in D(A^*) \cap (\ \ker\ A^*)^\perp$.

Now let $(\mathbf{E}, \mathbf{F}) \in X^0 \cap D(B)$.

Since $(0, \mathbf{g}) \in \mathcal{N}$ it follows from the assumption $E^{(2)} = 1$ on Ω that

$$\langle \mathbf{F}, \mathbf{g} \rangle_{L^2} = \langle (\mathbf{E}, \mathbf{F}), (0, \mathbf{g}) \rangle_X = 0 \text{ for all } \mathbf{g} \in (\ker A)^*,$$

in particular $\mathbf{F} \in D(A^*) \cap (\ \ker\ A^*)^\perp$. Finally, the assertion follows from 4.56.

$\square$

Remark 2 *As described in remark 1 the H^1_{loc}-regularty of $\underline{\mathbf{w}}_1$ for $\mathbf{w} \in X^0 \cap D(B)$ does generally not hold on the set $G = \Omega \setminus \Omega_0$ even if $E^{(j)} = 1$ on Ω.*

Lemma 7 *Suppose $E^{(2)} = 1$ on Ω.*
Then $(\mathbf{E}(t), \mathbf{F}(t)) \stackrel{\text{def}}{=} T(t)\mathbf{w}$ obeys

$$\Big(||\mathbf{E}(t)||_{L^r(K)} + ||\mathbf{F}(t)||_{L^r(U)} \Big) \stackrel{t\to\infty}{\longrightarrow} 0.$$

for all compact sets $K \subset \Omega_0$ and $U \subset \Omega$, $\mathbf{w} = (\mathbf{E}_0, \mathbf{F}_0) \in X^0 \cap D(B)$ and $r \in [2, 6)$.

Proof: Lemma 4 and lemma 5 yield

$$\mathbf{u} = (\mathbf{E}, \mathbf{F}) \stackrel{\text{def}}{=} T(\cdot)\mathbf{w} \in L^\infty((0, \infty), D(B) \cap X^0) \tag{4.57}$$

Hence, it follows from lemma 6 and Sobolev's imbedding theorem that

$$\{\mathbf{E}(t) : t \geq 0\} \text{ is precompact in } L^r(K) \text{ and } \{\mathbf{F}(t) : t \geq 0\} \text{ is precompact in } L^r(U).$$

Therefore, theorem 2 yields

$$||\mathbf{E}(t)||_{L^r(K)} + ||\mathbf{F}(t)||_{L^r(U)} \stackrel{t\to\infty}{\longrightarrow} 0.$$

$\square$

In the next lemma the strong L^r_{loc}-convergence of $\underline{\mathbf{u}}_1$ on the set G is proved, which in general does not follow from lemma 6, see remark 2.

Lemma 8 *Suppose $\mathbf{w} = (\mathbf{E}_0, \mathbf{F}_0) \in X$, $R > 0$ and $r \in [1, 2)$.*
Then $(\mathbf{E}(t), \mathbf{F}(t)) \stackrel{\text{def}}{=} T(t)\mathbf{w}$ satisfies

$$||\mathbf{E}(t)||_{L^r(G \cap B_R)} \stackrel{t\to\infty}{\longrightarrow} 0.$$

Proof: Let $\mathbf{u}(t) = (\mathbf{E}(t), \mathbf{F}(t)) \stackrel{\text{def}}{=} T(t)\mathbf{w}$ with $\mathbf{w} \in D(B)$, and define $G^{(R)} \stackrel{\text{def}}{=} G \cap B_R$ and $M \stackrel{\text{def}}{=} ||\mathbf{u}||_{L^\infty((0,\infty),L^2(\Omega))}$.
Suppose $\delta > 0$. With $\gamma > 0$ as in 2.21 one has $G = \bigcup_{n \in I\!N} \{x \in G : \gamma(x) > 1/n\}$. Therefore there exists a subset $G_\delta^{(R)} \subset G^{(R)}$, such that

$$M|G^{(R)} \setminus G_\delta^{(R)}|^{(1/r-1/2)} \leq \delta/2, \tag{4.58}$$

and

$$\gamma(x) \geq c_\delta \text{ for all } x \in G_\delta^{(R)} \tag{4.59}$$

with some positive constant $c_\delta > 0$. In 4.58 $|G^{(R)} \setminus G_\delta^{(R)}|$ denotes the Lebesgue-measure of this set.
From 4.59 and lemma 1 one obtains

$$\mathbf{E} \in L^p((0,\infty), L^1_\gamma(G_\delta^{(R)})) \subset L^p((0,\infty), L^1(G_\delta^{(R)})). \tag{4.60}$$

Lemma 5 yields

$$\mathbf{E} \in W^{1,\infty}((0,\infty), L^2(\Omega)) \subset W^{1,\infty}((0,\infty), L^1(G_\delta^{(R)})). \tag{4.61}$$

By 4.60 and 4.61 the function $t \to ||\mathbf{E}(t)||^p_{L^1(G_\delta^{(R)})}$ is uniformly continuous and integrable over $(0,\infty)$ and hence

$$||\mathbf{E}(t)||_{L^1(G_\delta^{(R)})} \stackrel{t\to\infty}{\longrightarrow} 0.$$

Since $r \in (1,2)$, this yields

$$||\mathbf{E}(t)||_{L^r(G_\delta^{(R)})} \leq ||\mathbf{E}(t)||^\theta_{L^2(G_\delta^{(R)})} |||\mathbf{E}(t)||^{1-\theta}_{L^1(G_\delta^{(R)})} \leq M^\theta |||\mathbf{E}(t)||^{1-\theta}_{L^1(G_\delta^{(R)})} \stackrel{t\to\infty}{\longrightarrow} 0. \tag{4.62}$$

where $1/r = \theta/2 + 1 - \theta$. Next it follows from 4.58 that

$$||\mathbf{E}(t)||_{L^r(G^{(R)} \setminus G_\delta^{(R)})} \leq ||\mathbf{E}(t)||_{L^2(\Omega)} |G^{(R)} \setminus G_\delta^{(R)}|^{(1/r-1/2)} \tag{4.63}$$

$$\leq M|G^{(R)} \setminus G_\delta^{(R)}|^{(1/r-1-2)} \leq \delta/2.$$

Finally, the assertion follows from 4.62 and 4.63, since $\delta > 0$ is arbitrary.

□

Theorem 3 *Suppose $E^{(2)} = 1$ on Ω. Then it follows for all $q \in [1,2)$, $\mathbf{w} = (\mathbf{E}_0, \mathbf{F}_0) \in X^0$ and all compact $U \subset \Omega$ that*

$$\Big(||\mathbf{E}(t)||_{L^q(U)} + ||\mathbf{F}(t)||_{L^2(U)}\Big) \stackrel{t\to\infty}{\longrightarrow} 0.$$

where $(\mathbf{E}(t), \mathbf{F}(t)) \stackrel{\text{def}}{=} T(t)\mathbf{w}$.

Proof:
First consider $\mathbf{w} = (\mathbf{E}_0, \mathbf{F}_0) \in X^0 \cap D(B)$, $(\mathbf{E}(t), \mathbf{F}(t)) \stackrel{\text{def}}{=} \mathbf{u}(t) \stackrel{\text{def}}{=} T(t)\mathbf{w}$ and define $M \stackrel{\text{def}}{=} ||\mathbf{u}||_{L^\infty((0,\infty),L^2(\Omega))}$.
Suppose $\delta > 0$. Choose a compact set $K \subset U \cap \Omega_0$ with $M|(U \cap \Omega_0) \setminus K|^{(1/q-1/2)} \leq \delta$. Then Hölder's inequality yields

$$||\mathbf{E}(t)||_{L^q(U)} \leq ||\mathbf{E}(t)||_{L^q(U\cap G)} + ||\mathbf{E}(t)||_{L^q(K)} + ||\mathbf{E}(t)||_{L^2(U)}|(U \cap \Omega_0) \setminus K|^{(1/q-1/2)}$$

$$\leq ||\mathbf{E}(t)||_{L^q(U\cap G)} + ||\mathbf{E}(t)||_{L^q(K)} + \delta.$$

Now, lemma 7 and lemma 8 yield $\limsup_{t\to\infty} ||\mathbf{w}(t)||_{L^q(U)} \leq \delta$, which proves the assertion for $\mathbf{w} = (\mathbf{E}_0, \mathbf{F}_0) \in X^0 \cap D(B)$.
In order to prove the theorem for all $\mathbf{w} \in X^0$ note that from 4.52 and a similar estimate as in 2.28 one obtains

$$\frac{d}{dt}||T(t)\mathbf{w} - T(t)\tilde{\mathbf{w}}||_X^2$$

$$= 2 < F(t, T(t)\mathbf{w}) - F(t, T(t)\tilde{\mathbf{w}}), T(t)\mathbf{w} - T(t)\tilde{\mathbf{w}} >_X \leq 0$$

and therefore

$$||T(t)\mathbf{w} - T(t)\tilde{\mathbf{w}}||_X \leq ||\mathbf{w} - \tilde{\mathbf{w}}||_X.$$

With $||\underline{\mathbf{u}}_1||_{L^q(U)} + ||\underline{\mathbf{u}}_2||_{L^2(U)} \leq C_{q,U}||\mathbf{u}||_X$ for all $\mathbf{u} \in X$, the assertion follows for arbitrary $\mathbf{w} = (\mathbf{E}_0, \mathbf{F}_0) \in X^0$ from the density of $X^0 \cap D(B)$ in X^0 as shown in lemma 2.

$\square$

In the case of Maxwell's equations 1.4-1.6 the assumption $E^{(2)} = 1$ on Ω can be omitted using the compactness-result in [5] [9] and [12].

Under the general assumptions considered so far it cannot be expected that the assertion of the previons theorem holds for $q = 2$ or sets U which may overlap the boundary $\partial\Omega$. However, for the system corresponding to the scalar wave-equation the result can be improved in this direction. Consider

$$\partial_t^2 \varphi = \operatorname{div} (E\nabla\varphi) - S(x, \partial_t\varphi) \tag{4.64}$$

supplemented by the initial-boundary-onditions

$$\varphi = 0 \text{ on } (0, \infty) \times \partial\Omega \tag{4.65}$$

$$\varphi(0, x) = f_0(x) \text{ and } \partial_t\varphi(0, x) = f_1(x). \tag{4.66}$$

Here the nonlinear function $S : \Omega \times I\!R^3 \to I\!R^3$ obey the assumptions 2.15 - 2.21. According to 4.52 it is assumed that S is independent of t and monotone with respect to $y \in I\!R^3$.

For a domain $\Omega_1 \subset \Omega$ let $H^1(\Omega_1)$ be the usual first order Sobolev space and $\overset{0}{H^1}(\Omega_1)$ denotes the closure of $C_0^\infty(\Omega_1)$ in $H^1(\Omega_1)$.
Next, $D(\mathcal{A}) \subset \overset{0}{H^1}(\Omega)$ is defined as the set of all $f \in \overset{0}{H^1}(\Omega)$, such that

$$\mathcal{A}f \overset{\text{def}}{=} -\operatorname{div}(E\nabla f) \in L^2(\Omega).$$

It is well known that for $f_0 \in \overset{0}{H^1}(\Omega))$ and $f_1 \in L^2(\Omega))$ problem 4.64 - 4.66 admits a unique solution $\varphi \in C([0,\infty), \overset{0}{H^1}(\Omega))$ with $\partial_t\varphi \in C([0,\infty), L^2(\Omega))$. The usual energy-estimate yields

$$\partial_t\varphi \in L^\infty((0,\infty)L^2(\Omega)), \nabla\varphi \in L^\infty((0,\infty), L^2(\Omega)). \tag{4.67}$$

If in addition $f_1 \in \overset{0}{H^1}(\Omega))$ and $f_0 \in D(\mathcal{A})$ then $\varphi \in C([0,\infty), D(\mathcal{A}))$
and $\partial_t\varphi \in C([0,\infty), \overset{0}{H^1}(\Omega))$ with

$$\partial_t\nabla\varphi, \partial_t^2\varphi \in L^\infty((0,\infty)L^2(\Omega)),\ \operatorname{div}(E\nabla\varphi) = \mathcal{A}\varphi(\cdot) \in L^\infty((0,\infty), L^2(\Omega)). \tag{4.68}$$

In order to consider problem 4.64 - 4.66 is the setting of section 2 the following operators are introduced. Let $D(A) \overset{\text{def}}{=} \overset{0}{H^1}(\Omega, \mathbb{C})$, $A\varphi \overset{\text{def}}{=} \nabla\varphi$. $D(A^*)$ is the space of all vector-fields $\mathbf{a} \in L^2(\Omega, \mathbb{C}^3)$ with $A^*\mathbf{a} = -\operatorname{div}\mathbf{a} \in L^2(\Omega)$. Next, $D(B) \overset{\text{def}}{=} D(A) \times D(A^*)$ and

$$B(\mathbf{w}_1, .., \mathbf{w}_4) \overset{\text{def}}{=} (-A^*(\mathbf{w}_2, .., \mathbf{w}_4), EA\mathbf{w}_1) = (\operatorname{div}(\mathbf{w}_2, .., \mathbf{w}_4), E\nabla\mathbf{w}_1)$$

for $\mathbf{w} \in D(B)$.

Suppose $\varphi \in C([0,\infty), \overset{0}{H^1}(\Omega))$ is for $f_0 \in \overset{0}{H^1}(\Omega))$ and $f_1 \in L^2(\Omega))$ a solution of problem 4.64 - 4.66. Then $\mathbf{u} \overset{\text{def}}{=} (\partial_t\varphi, E\nabla\varphi) \in C([0,\infty), L^2(\Omega, \mathbb{R}^4))$ is a weak soltion of 2.26, i.e.

$$\frac{d}{dt}\langle \mathbf{u}(t), \mathbf{a}\rangle_X = -\langle \mathbf{u}(t), B\mathbf{a}\rangle_X + \langle F_0(\mathbf{u}(t)), \mathbf{a}\rangle_X \text{ for all } \mathbf{a} \in D(B)$$

where $F_0 : L^2(\Omega, \mathbb{R}^4) \to L^2(\Omega, \mathbb{R}^4)$ is defined by

$$F_0(\mathbf{u}) \overset{\text{def}}{=} -(S(\cdot, \mathbf{u}_1(\cdot)), 0)\,.$$

If $f_0 \in D(\mathcal{A})$ and $f_1 \in \overset{0}{H^1}(\Omega))$ then $\mathbf{u}(0) \in D(B)$ and hence by lemma 5
$\mathbf{u} \in L^\infty((0,\infty), D(B))$, whence again 4.68.

Next it is shown that

$$\nabla\varphi(t) \overset{t\to\infty}{\longrightarrow} 0 \text{ and } \partial_t\varphi(t) \overset{t\to\infty}{\longrightarrow} 0. \text{ in } L^2(\Omega) \text{ weakly.} \tag{4.69}$$

for all $f_0 \in \overset{0}{H^1}(\Omega))$ and $f_1 \in L^2(\Omega)$. For this purpose let $\mathbf{w} \overset{\text{def}}{=} (f_1, E\nabla f_0) \in L^2(\Omega, I\!R^4)$. Then $(\partial_t\varphi(t), E\nabla\varphi(t)) = \mathbf{u}(t) = T(t)\mathbf{w}$ solves 2.26. In order to apply theorem 2 it suffices to show

$$\mathbf{w} \in X^0 \tag{4.70}$$

Suppose $\mathbf{a} \in \mathcal{N}$. Then $\mathbf{a}_1 \in \overset{0}{H^1}(\Omega))$, with $\nabla \mathbf{a}_1 = 0$, which implies $\mathbf{a}_1 = 0$. Moreover, div $(\mathbf{a}_2, .., \mathbf{a}_4) = 0$ by the definition of A, B. Hence

$$\langle \mathbf{w}, \mathbf{a}\rangle_X = \int_\Omega [E^{-1}(\mathbf{w}_2, .., \mathbf{w}_4)](\mathbf{a}_2, .., \mathbf{a}_4)dx = \int_\Omega (\mathbf{a}_2, .., \mathbf{a}_4)\nabla f_0 dx = 0$$

since $f_0 \in \overset{0}{H^1}(\Omega))$. Thus, 4.70 and 4.69 are proved. In the following theorem local strong convergence in the energy-norm is shown.

Theorem 4 *For all $R \in (0, \infty)$, $f_0 \in \overset{0}{H^1}(\Omega))$ and $f_1 \in L^2(\Omega))$ one has*

$$\Big(||\nabla\varphi(t)||_{L^2(\Omega\cap B_R)} + ||\partial_t\varphi(t)||_{L^2(\Omega\cap B_R)}\Big) \overset{t\to\infty}{\longrightarrow} 0.$$

Proof:

By a density-argument it suffices to consider $f_0 \in D(\mathcal{A})$ and $f_1 \in \overset{0}{H^1}(\Omega)$.
Choose $\chi \in C_0^\infty(B_{2R})$ with $\chi(x) = 1$ on B_R and define $\Omega_R \overset{\text{def}}{=} \Omega \cap B_{2R}$ and $\varphi_R(t, x) \overset{\text{def}}{=} \chi(x)\varphi(t, x)$. It follows easily from 4.68 using Poincare's inequality that $\varphi_R \in L^\infty((0, \infty), \overset{0}{H^1}(\Omega \cap B_{2R}))$ and $\partial_t\varphi_R \in L^\infty((0, \infty), \overset{0}{H^1}(\Omega \cap B_{2R}))$.
Since $\Omega \cap B_{2R}$, is bounded, the imbedding $\overset{0}{H^1}(\Omega \cap B_{2R}) \hookrightarrow L^2(\Omega \cap B_{2R})$ is compact. Hence

$$\{\varphi(t) : t \geq 0\} \text{ is precompact in } L^2(\Omega \cap B_R) \tag{4.71}$$

$$\text{and } \{\partial_t\varphi(t) : t \geq 0\} \text{ is precompact in } L^2(\Omega \cap B_R). \tag{4.72}$$

for all $R \in (0, \infty)$. Next, one obtains by 2.15and the definition of $\mathcal{A}$ that

$$c_0||\nabla(\varphi(t_1) - \varphi(t_2))||^2_{L^2(B_R)} \leq \int_\Omega \chi E\nabla(\varphi(t_1) - \varphi(t_2))\nabla(\varphi(t_1) - \varphi(t_2))dx$$

$$= -\int_\Omega (\varphi(t_1) - \varphi(t_2)) \text{ div } (\chi E\nabla[\varphi(t_1) - \varphi(t_2)])\, dx$$

$$\leq ||\varphi(t_1) - \varphi(t_2)||_{L^2(B_{2R})} \Big(||\mathcal{A}(\varphi(t_1) - \varphi(t_2))||_{L^2(\Omega)}$$

$$+K_R||\nabla(\varphi(t_1) - \varphi(t_2))||_{L^2(\Omega)}\Big) \text{ for all } t_1, t_2 \geq 0.$$

which implies by 4.67, 4.68 and 4.71 also

$$\{\nabla\varphi(t) : t \geq 0\} \text{ is precompact in } L^2(\Omega \cap B_R) \tag{4.73}$$

Finally, the result follows from 4.69, 4.72 and 4.73.

□

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