Periodic and Solitary Traveling Wave Solutions for the Generalized Kadomtsev-Petviashvili Equation, II

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Abstract. As a continuation of our previous work, we improve here some results on convergence of periodic KP traveling waves to solitary ones as period goes to infinity. In addition, we present some qualitative properties of such waves, as well as nonexistence results, in the case of general nonlonearities. We suggest here an approach which does not use any scaling argument.

Key words: generalized Kadomtsev-Petviashvili equation, traveling waves, variational methods

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1 Introduction

Kadomtsev-Petviashvili equations, both original and generalized, appear in the theory of weakly nonlinear dispersive waves [7]. They read

$$\left.\begin{array}{c} u_t + u_{\xi\xi\xi} + f(u)_\xi + \varepsilon v_y = 0 \\ v_\xi = u_y \end{array}\right\} \tag{1}$$

or, eliminating v,

$$(u_t + u_{\xi\xi\xi} + f(u)_\xi)_\xi + \varepsilon u_{yy} = 0 \quad . \tag{2}$$

More precisely, these are KP-I equations if $\varepsilon = -1$, and KP-II equations if $\varepsilon = +1$. The original KP equations correspond to the case $f(u) = \frac{1}{2}u^2$, form a completely integrable Hamiltonian system, and were studied extensively by means of algebro-geometrical methods (see, e.g., [8]). There is also a number of papers dealing with more general equations (1) or (2), mainly in the case of power nonlinearity: [1, 3, 4, 5, 6, 10, 17, 21, 24], to mention a few. In particular, solitary traveling waves were studied [1, 4, 5, 6, 10, 17, 24]. Here we consider mainly the case of KP-I equations. Remark that KP-II equations do not posess solitary traveling waves at all (see [5] for the case of power nonlinearity and Section 4 below).

The present paper is a direct continuation of our previous work [17]. It concerns the existence of ground traveling waves, both periodic and solitary, and the limit behavior of periodic waves, as period goes to infinity. Corresponding equations for traveling waves read

$$\left.\begin{array}{c} -cu_x + u_{xxx} + f(u)_x + \varepsilon v_y = 0 \\ v_x = u_y \end{array}\right\} \tag{3}$$

and

$$(-cu_x + u_{xxx} + f(u)_x)_x + \varepsilon u_{yy} = 0 \quad , \tag{4}$$

respectively. Here $x = \xi - ct$, $c > 0$ is the wave speed. In [17], among other results we have proved that k-periodic in x ground waves converge to a solitary ground wave in a very strong sense (Theorem 5 of that paper). Unfortunately, that result does not cover the case of original KP equation, while includes the case $f(u) = u^3$. And the first aim of the present paper is to extend the results of Theorems 4 and 5, [17], in order to include nonlinearities like $f(u) = |u|^{p-1}$, $2 < p < 6$. This will be done in Section 2. Our second goal is to discuss, in Section 3, some qualitative properties of KP traveling waves. Here we follow very closely the paper [6] and point out only the main differences. Finally, in the Section 4, we discuss nonexistence of traveling waves, both solitary and periodic, for general nonlinearities.

All the assumptions we impose here are satisfied for the nonlinearities $f(u) = c|u|^{p-1}$ and $f(u) = c|u|^{p-2}u + \sum_{i=1}^{k} c_i|u|^{p_i-2}u$, with $c, c_i > 0$, $2 < p < 6$, $2 < p_i < p$. Unfortunately, the Assumptions (N) and (N1) below are not satisfied for a very interesting nonlinearity $f(u) = u^2 - u^3$.

2 Ground Waves

Denote by $F(u) = \int_0^u f(t)dt$ the primitive function of f. We make the following assumptions:

1^0 $f \in C(\mathbb{R})$, $f(0) = 0$;

2^0 $|f(u) \leq C(1 + |u|^{p-1})$, $2 < p < 6$, *and* $f(u) = o(|u|)$ *as* $u \to 0$;

3^0 *there exists* $\varphi \in C_0^\infty(\mathbb{R}^2)$ *such that*

$$\frac{1}{\lambda^2}\int_{\mathbb{R}^2} F(\lambda\varphi_x) \quad \to +\infty$$

as $\lambda \to +\infty$;

4^0 *there exists* $\mu > 2$ *such that* $\mu F(u) \leq uf(u)$ *for all* $u \in \mathbb{R}$.

Remark that if $F(u) > 0$ for all $u \neq 0$, then 3^0 follows from 4^0.

Let $Q_k = (-k/2, k/2) \times \mathbb{R}$, $0 < k \leq \infty$. We set

$$D_{x,k}^{-1}u(x, y) = \int_{-k/2}^{x} u(s, y)ds \ , \quad k \in (0, \infty] \ . \tag{5}$$

We shall write simply D_x^{-1} in the case $k = \infty$. Define the Hilbert space X_k as the completion of $\{\varphi_x : \varphi \in C_k^\infty\}$, where C_k^∞ is the space of smooth functions on $\mathbb{R}^2$ which are k-periodic in x and have finite support in y, with respect to the norm $\|u\|_k = (u, u)_k^{1/2}$,

$$(u, v)_k = \int_{Q_k} u_x v_x + D_{x,k}^{-1}u_y \cdot D_{x,k}^{-1}v_y + cuv \ .$$

Similarly, $X = X_\infty$ is the completion of $\{\varphi_x : \varphi \in C_0^\infty(\mathbb{R}^2)\}$ with respect to the norm $\|u\| = \|u\|_\infty = (u,u)^{1/2} = (u,u)_\infty^{1/2}$. The operator $D_{x,k}^{-1}$ is well-defined on the space X_k, $k \in (0,\infty]$.

For k-periodic traveling waves, $k \in (0,\infty)$, equation (4) may be written in the form [17]

$$(-u_{xx} + D_{x,k}^{-2} u_{yy} + cu - f(u))_x = 0 \ . \tag{6}$$

Solitary waves are solutions of the same equation (6), with $k = \infty$. The action functional associated with (6) reads [17]

$$J_k(u) = \frac{1}{2}\|u\|_k^2 - \int_{Q_k} F(u) \ ; \tag{7}$$

J_k is of the class C^1 on X_k. We consider weak solutions of (6), i.e. critical points of J_k in X_k.

Now let us consider the so-called Nehari functional

$$I_k(u) = \langle J_k'(u), u\rangle = \|u\|_k^2 - \int_{Q_k} uf(u) \ , \tag{8}$$

and the Nehari manifold

$$S_k = \{u \in X_k : I_k(u) = 0, u \neq 0\} \ .$$

All traveling wave solutions lie in the corresponding Nehari manifold and we will find ground waves, i.e. solutions with minimal action among all nontrivial solutions, solving the following minimization problem

$$m_k = \inf\{J_k(u) : u \in S_k\} \ . \tag{9}$$

Remark that

$$J_k(u) = \int_{Q_k} \frac{1}{2}uf(u) - F(u) \ , \quad u \in S_k \ . \tag{10}$$

In what follows we will omit the subscript k if $k = \infty$ and write simply $J, I, \ldots$

Throughout this section, in addition to Assumptions 1^0–4^0, we impose the following one

(N) *For any* $u \in L^2(\mathbb{R}^2)$ *such that*

$$\int_{\mathbb{R}^2} uf(u) > 0 \ ,$$

the function of t

$$t^{-1} \int_{\mathbb{R}^2} uf(tu)$$

is strictly increasing on $(0, +\infty)$.

In the proof of Theorem 1, [17], we have introduced the Mountain Pass Values c_k for J_k and proved that they are uniformly bounded from below and above by positive constants. More precisely,

$$c_k = \inf_{\gamma \in \Gamma_k)} \max_{t \in [0,1]} J_k(\gamma(t))\} \ ,$$

where

$$\Gamma_k = \{\gamma \in C([0,1], X_k) : \gamma(0) = 0, J_k(\gamma(1)) < 0 \ .$$

Here we have defined Γ_k in a slightly different way than in [17], but it does not effect on the value of c_k. Consider also another minimax value

$$c_k' = \inf_{v \in X_k^+} \sup_{t>0} J_k(tv) \ ,$$

where

$$X_k^+ = \{v \in X_k : \int_{Q_k} F(v) > 0\} \ .$$

Due to Assumption 4^0, $X_k^+ \neq \emptyset$.

Lemma 1 *For every* $v \in X_k^+$ *there exists a unique* $t_k = t_k(v)$ *such that* $t_k v \in S_k$,

$$J_k(t_k v) = \max_{t>0} J_k(tv) \ ,$$

and $t_k(v)$ *depends continuously on* $v \in X_k^+$.

Proof. Assumption 4^0 implies that

$$\int_{Q_k} vf(v) > 0$$

for any $v \in X_k^+$. Therefore, due to Assumption **(N)**, the function

$$\frac{d}{dt} J_k(tv) = I_k(tv) = t^2(\|v\|_k^2 - t^{-1} \int_{Q_k} vf(tv))$$

vanishes at only one point $t_k = t_k(v) > 0$. Equation (10) and Assumption 4^0 imply that J_k is positive on S_k. Since $J_k(0) = 0$, we see that t_k is a point of maximum for $J_k(tv)$. Continuity of $t_k(v)$ is easy to verify. $\square$

Lemma 2 $c_k = c'_k = m_k$.

Proof. Since $uf(u)$ is subquadratic at 0 and the quadratic part of J_k is positive defined, we see that $I_k(v) > 0$ in a neighborhood of the origin, exept of 0. Hence, $I_k(\gamma(t)) > 0$, $\gamma \in \Gamma_k$, for small $t > 0$. Due to Assumption 4^0, for $v \in X_k^+$ we have

$$2J_k(v) = \|v\|_k^2 - 2\int_{Q_k} F(v) > \|v\|_k^2 - \mu \int_{Q_k} F(v) \geq$$

$$\geq \|v\|_k^2 - 2\int_{Q_k} vf(v) = I_k(v) \ .$$

Hence, $I_k(\gamma(1)) < 0$. Therefore, $\gamma(t)$ crosses S_k and this implies that $c_k \geq m_k$.

By Assumption 4^0, for any $v \in X_k^+$ we have $F(tv) \geq \alpha t^\mu$, $\alpha > 0$, if $t > 0$ is large enough. This implies that $J_k(tv) < 0$ for every $v \in X_k^+$ and sufficiently large $t > 0$. Hence, the half-axis $\{tv : t > 0\}$ generates in a natural way an element of Γ_k. This implies the inequality $c_k \leq c'_k$.

Now let $v \in S_k$. By the definition of I_k,

$$\sigma = \int_{Q_k} vf(v) > 0 \ ,$$

and **(N)** implies that

$$\frac{d}{dt} \int_{Q_k} F(tv) = f(t^{-1} \int_{Q_k} vf(tv) \geq$$

$$\geq t^{-1} \int_{Q_k} vf(tv) \geq \sigma > 0$$

provided $t \geq 1$. Hence, for $t > 0$ large enough

$$\int_{Q_k} F(tv) > 0 \; .$$

By definitions of c'_k and m_k, we see that $c'_k = m_k$. $\square$

Theorem 1 *Assume* 1^0–4^0 *and* **(N)** *to be fulfiled. Then, for any* $k \in (0, \infty)$, *there exists a minimizer* $u_k \in S_k$ *of (9) which is a critical point of* J_k. *Moreover,* $J_k(u_k) = m_k$ *is bounded from above and below by positive constants independent on* k.

Proof. In the proof of Theorem 1, [17], it is shown that there exists a Palais-Smale sequence $u_{k,n} \in X_k$ at the level c_k, i.e.

$$J'_k(u_{k,n}) \to 0 \; , \quad J_k(u_{k,n}) \to c_k$$

as $n \to \infty$. Moreover, $u_{k,n} \to u_k$ weakly in X_k and strongly in $L^p_{loc}(\mathbb{R}^2)$, where $u_k \in X_k$ is a nontrivial solution of (6). Therefore,

$$I_k(u_{k,n}) = \langle J'_k(u_{k,n}), u_{k,n} \rangle \to 0$$

and

$$J_k(u_{k,n}) - \frac{1}{2} I_k(u_{k,n}) = \int_{Q_k} (\frac{1}{2} u_{k,n} f(u_{k,n}) - F(u_{k,n})) \to c_k \; .$$

Due to 4^0, the integrand here is nonnegative and, since $u_{k,n} \to u_k$ in $L^2_{loc}(\mathbb{R}^2)$, we have

$$\int_{Q_k} (\frac{1}{2} u_k f(u_k) - F(u_k) \leq c_k \; .$$

However, u_k is a nontrivial solution, hence, $u_k \in S_k$. Therefore, we deduce from (10) that

$$J_k(u_k) = \int_{Q_k} (\frac{1}{2} u_k f(u_k) - F(u_k)) \geq m_k \; .$$

Now Lemma 2 implies that $J_k(u_k) = m_k$ and u_k is a ground wave solution.

The last statement of the theorem follows immediately from Lemma 2 and uniform estimates for c_k. $\square$

Remark 1 The Nehari variational principle suggested in [13] was used successfully in many papers (see, e.g., [2, 9, 14, 15, 16, 17, 23]). In all these papers, exept of [16], the geometry of Nehari manifold is simple enough: it is a bounded surface without boundary around the origin, like sphere. In the case we consider here the picture is different: S_k may looke like sphere if, e.g., $f(u) = |u|^{p-2}u$, and may be unbounded if, e.g., $f(u) = |u|^{p-1}$. Nevertheless, in any case S_k separates the origin and the domain of negative values of J_k, which is sufficient for our purpose. In [16] such a manifold is also unbounded in general, but there we have used different arguments.

Now we are going to study the behavior of u_k, as $k \to \infty$. Recall the definition of cut-off operators $P_k : X_k \to X$, [17]. Let $\chi_k \in C_0^\infty(\mathbb{R})$ be a nonnegative function such that $\chi_k(x) = 1$ for $x \in [-k/2, k/2]$, $\chi_k(x) = 0$ for $|x| \geq (k+1)/2$, and $|\chi_k'|, |\chi_k''| \leq C_0$, with some constant $C_0 > 0$. We set

$$P_k u(x, y) = [\chi_k(x) D_{x,k}^{-1} u(x, y)]_x \ .$$

Theorem 2 *Assume that* 1^0–4^0 *and* (N) *are satisfied. Let* $u_k \in X_k$ *be a sequence of ground wave solutions. Then there exist a nontrivial ground wave* $u \in X$ *and a sequence of vectors* $\zeta_k \in \mathbb{R}^2$ *such that, along a subsequence,* $P_k u_k(\cdot + \zeta_k) \to u$ *weakly in* X*. If in addition*

$$|f(u+v) - f(u)| \leq C(1 + |u|^{p-2} + |v|^{p-2})|v| \ , \quad v \in \mathbb{R}, \tag{11}$$

then, along the same subsequence,

$$\lim_{k\to\infty} \|u_k(\cdot + \zeta_k) - u\|_k = 0.$$

Proof. By Theorem 2, [17], there is a nontrivial solution $u \in X$ such that $P_k u_k(\cdot + \zeta_k) \to u$ weakly in X for some $\zeta_k \in \mathbb{R}^2$ (alonge a subsequence). Let us prove that u is a ground wave, i.e.

$$J(u) = \inf\{J(v) : v \in S\} = m \ .$$

First of all, for any $v \in S$ and any $\varepsilon > 0$, there exist k_ε and $v_k \in S_k$ such that

$$J_k(v_k) \leq J(v) + \varepsilon, \quad k \geq k_\varepsilon.$$

Indeed, since J and I are continuous, we can find $\varphi_k \in C_0^\infty(Q_k)$ such that $\eta_k = D_x\varphi_k \to v$ in X and, hence,

$$J(\eta_k) \to J(v), \quad I(\eta_k) \to I(v) = o.$$

Since $I(v) = 0$ and $v \neq 0$, we have

$$\int_{Q_k} vf(v) = \|v\|^2 > 0.$$

Hence,

$$\int_{Q_k} \eta_k f(\eta_k) > 0$$

for k large enough. Due to (N), there exists $\tau_k > 0$ such that $I(\tau_k\eta_k) = 0$ and $\tau_k \to 1$. Let v_k be a unique k-periodic function which coincides with $\tau_k\eta_k$ on Q_k. Then

$$J_k(v_k) = J(\tau_k\eta_k) \leq J(v) + \varepsilon$$

provided k is large enough.

In particular, we have

$$\limsup_{k\to\infty} m_k \leq m.$$

Now, exactly as in the proof of Theorem 5, [17], we see that

$$\liminf_{k\to\infty} m_k \geq J(u) \geq m.$$

Hence, $m = J(u)$ and u is a ground wave solution.

The second part of the theorem follows from Theorem 3, [17], exactly as at the end of proof of Theorem 5, [17]. □

3 Qualitative Properties of Traveling Waves

Now we are going to study such properties of KP traveling waves as symmetry, regularity and decay. We start with the following

Lemma 3 *Under Assumptions* 1^0 *and* 2^0 *any traveling wave is continuous. Moreover, solitary (resp. periodic) wave tends to zero as* $(x, y) \to \infty$ *(resp.* $y \to \infty$*).*

Proof. For such a wave $u \in X_k$, we have

$$-cv_{xx} - v_{yy} + v_{xxxx} = f(u)_{xx} = g_x x, \tag{12}$$

Let

$$(\mathcal{F}_{k,x}h)(\xi) = \int_{-k/2}^{k/2} h(x)\exp(-i\xi x)dx$$

be the Fourier transform if $k = \infty$ (then we write simply $\mathcal{F}_x$), and the sequence of Fourier coefficients if $k < \infty$. In the last case $\xi \in (2\pi/k)\mathbb{Z}$. Now we get from (12)

$$\mathcal{F}_{k,x}\mathcal{F}_y u = p(\xi_1, \xi_2)(\mathcal{F}_{k,x}\mathcal{F}_y g), \tag{13}$$

where

$$p(\xi) = p(\xi_1, \xi_2) = \frac{\xi_1^2}{c\xi_1^2 + \xi_1^4 + \xi_2^2},$$

ξ_1 and ξ_2 are dual variables to x and y, respectively. If $k = \infty$, there is nothing to do. The proof of Theorem 1.1, [6], does not use any particular property of power nonlinearity, exept of its growth rate.

Now we explain how to cover the case of periodic waves. Recall the following Lizorkin theorem, [11]. Let $p(\xi)$, $\xi \in \mathbb{R}^n$, be of the class C^n for $|\xi_j| > 0$, $j = 1, \dots, n$. Assume that

$$|\xi_1^{k_1} \cdots \xi_n^{k_n} \frac{\partial^k p}{\partial\xi_1^{k_1} \cdots \partial\xi_n^{k_n}}| \le M,$$

with $k_j = 0$ or 1, $k = k_1 + \cdots k_n = 0, 1, \dots, n$. Then $p(\xi)$ is a Fourier multiplier on $L^r(\mathbb{R}^n)$, $1 < r < \infty$.

We rewrite now (13) as follows

$$\mathcal{F}_{k,x}u = \mathcal{F}_y^{-1}[p(\xi_1, \xi_2)\mathcal{F}_y\mathcal{F}_{k,x}g] = P(\xi_1)g,$$

where $P(\xi_1)$ is the operator $\mathcal{F}_y^{-1}p(\xi_1, \cdot)\mathcal{F}_y$ for any fixed ξ_1. It is easy to verify that $P(\xi_1) \in L(L^r(\mathbb{R}_y))$, the space of bounded linear operators in $L^r(\mathbb{R}_y)$. Moreover, due to the Lizorkin theorem, $p(\xi)$ is a multiplier in $L^r(\mathbb{R}^2)$. Hence, so is for $P(\xi_1)$ in the space $L^r(\mathbb{R}_x, L^r(\mathbb{R}_y)) = L^r(\mathbb{R}^2)$. It is not difficult to verify that $P(\xi_1)$ depends continuously on ξ_1 with respect to the norm in $L(L^r(\mathbb{R}_y))$ at any point $\xi_1 \neq 0$. Therefore, by Theorem 3.8 of Ch. 7, [22], we see that $P(\xi_1)$ is also a multiplier in the space

$L^r((-k/2,k/2),L^r(\mathbb{R}_y))_0 = L^r(Q_k)_0$ considered as the space of k-periodic in x functions. The subscript 0 means that for functions from this space $\mathcal{F}_{k,x}u$ vanishes at $\xi_1 = 0$. Since $p(0,\xi_2) = 0$, the corresponding multiplier vanishes on $\{u \in L^r(Q_k) : \mathcal{F}_{k,x}u = 0 \quad \text{if} \quad \xi_1 \neq 0\}$ and, hence, is a bounded operator on the entier space $L^r(Q_k)$. In fact, we need here an extension of that theorem for operator valued maltipliers which may be discontinuous at the point 0. However, in this case the proof presented in [22] works without any change.

Now to conclude we can use the same reiteration argument, as in [6]. $\square$

Remark 2 If f is of the class C^∞, then $u \in H^\infty(Q_k) = \cap H^s(Q_k)$.

We need also the following additional assumption:

(N1) *$f \in C^1(\mathbb{R})$ and, for any $v \in L^2(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f(v)v > 0$, we have*

$$\int_{\mathbb{R}^2} f(v)v < \int_{\mathbb{R}^2} f'(v)v^2$$

and $\int_{\mathbb{R}^2} f(tv)v > 0 \quad \forall t > 0$.

Calculating the derivative of $t^{-1}\int_{\mathbb{R}^2} f(tv)v$, we see that (N1) implies (N).

Let us introduce the functional

$$L_k(v) = \int_{Q_k} [\frac{1}{2}f(v)v - F(v)], \quad v \in X_k.$$

As we have seen, $L_k = J_k$ on S_k and $L_k(v) \geq 0$, $\forall v \in X_k$.

Lemma 4 *Under Assumption* (N1), *$L_k(tv)$ is a strictly increasing function of $t > 0$, provided $\int_{Q_k} f(v)v > 0$.*

Proof. It follows immediately from the following elementary identity

$$\frac{d}{dt}L_k(tv) = \frac{1}{2t}[\int_{Q_k} f'(tv)t^2v - \int_{Q_k} f(tv)tv]. \qquad \square$$

We need also the following dual characterization of ground traveling waves

Lemma 5 *Suppose Assumptions* 1^0–4^0 *and* (N1) *to be satisfied. For nonzero $u \in X_k$, $k \in (0,\infty]$, the following statements are equivalent:*

(i) *u is a ground wave,*

(ii) $I_k(u) = 0$ *and* $L_k(u) = m_k = \inf\{L_k(v) : v \in S_k\}$,

(iii) $I_k(u) = 0 = \sup\{I_k(v) : v \in X_k, L_k(v) = m_k\}$.

Proof. Implication $(i) \Rightarrow (ii)$ is proved in Section 2.

To prove $(ii) \Rightarrow (i)$ assume that $u \in X_k$ satisfies (ii). Since $J_k = L_k$ on S_k, there exists a Lagrange multiplier λ such that

$$\lambda I_k'(u) = J_k'(u).$$

Then

$$\lambda\langle I_k'(u), u\rangle = \langle J_k'(u), u\rangle = I_k(u) = 0.$$

On the other hand

$$\langle I_k'(u), u\rangle = 2\|u\|_k^2 - \int_{Q_k} f'(u)u^2 - \int_{Q_k} f(u)u =$$

$$= 2I_k(u) + \int_{Q_k} f(u)u - \int_{Q_k} f'(u)u^2 =$$

$$= \int_{Q_k} f(u)u - \int_{Q_k} f'(u)u^2.$$

However, $\int_{Q_k} f(v)v > 0$ on S_k and, due to (N1), $\langle I_k'(u), u\rangle < 0$. Therefore, $\lambda = 0$ and u is a ground wave.

Now let us prove $(ii) \Rightarrow (iii)$. For u as in (ii), $I_k(u) = 0$. Assume that there is $v \in X_k$ such that $L_k(v) = m_k$ and $I_k(v) < 0$. Then $\int Q_k f(v)v > 0$ and there exists $t_0 \in (0,1)$ such that $I_k(t_0 v) = 0$. By Lemma 4, $L_k(t_0 v) < L_k(v) = m_k$, which is impossible.

Finally, we prove $(iii) \Rightarrow (ii)$. Let $u \in X_k$ satisfies (iii). Then, $L_k(u) \geq m_k$. Assume that $L_k(u) > m_k$. Again we have $\int_{Q_k} f(u)u > 0$. By Lemma 4, there exists $t_0 \in (0,1)$ such that $L_k(t_0 u) = m_k$. However, $I_k(t_0 u) > 0$ and this contradicts (iii). $\square$.

Now we are ready to prove the symetry property for all kinds of ground waves we consider. As in [6], we use the approach suggested in [12] (see also [23]).

Theorem 3 *In addition to Assumptions* 1^0–4^0 *and* (N1)*, suppose that* $f \in C^2(\mathbb{R})$. *Than any ground wave* $u \in X_k$, $k \in (0, \infty]$, *is symmetric with respect to some line* $\Delta = \{(x, y) \in \mathbb{R}^2 : y = b\}$.

Proof. Choose b in such a way that

$$\int_{\Delta^+ \cap Q_k} [\frac{1}{2} f(v)v - F(v)] = \int_{\Delta^- \cap Q_k} [\frac{1}{2} f(v)v - F(v)] = \frac{m_k}{2},$$

where Δ^+ and Δ^- are corresponding upper and lower half-planes. Let $u^\pm$ be a symmetric (with respect to Δ) function such that $u^\pm = u$ on $\Delta^\pm$. Then $u^\pm \in X_k$ and

$$L_k(u^\pm) = L_k(u) = m_k.$$

By Lemma 5, $I_k(u^\pm) \le 0$. On the other hand,

$$I_k(u^+) + I_k(u^-) = 2I_k(u) = 0.$$

Using Lemma 5, we conclude that $u^\pm$ is a ground wave.

To conclude that $u^\pm = u$ and, hence, complete the proof it is sufficient to use the same unique continuation result, as in [6], and just here we need the assumption $f \in C^2(\mathbb{R})$ and Lemma 3. Remark that a periodic version (with $\Pi = \Delta^\pm$) of unique continuation Theorem A.1, [6], can be proved exactly as that theorem itself. $\square$

In addition, we formulate the following direct generalization of results of [6] for decay of solitary waves.

Theorem 4 *Suppose Assumptions* 1^0 *and* 2^0 *to be satisfied. Let* $u \in X_k$, $k \in (0, infty]$, *be a traveling wave. If* $k = \infty$, *then*

$$r^2 u \in L^\infty(\mathbb{R}^2), \qquad r^2 = x^2 + y^2.$$

If $0 < k < \infty$, *then*

$$y^2 u \in L^\infty(Q_k).$$

The proof is essentially the same as in [6]. In the case $k < \infty$ one needs only to use the partially periodic Fourier transform, as in Lemma3.

4 On Nonexistence of Traveling Waves

In this section we turn to general KP equations (3), with $\varepsilon = \pm 1$, and discuss the nonexistence problem. We use the same approach as in [5].However, the case of periodic waves is more involved (see the proof of Lemma 6). Here we consider traveling waves belonging to the space

$$Y_k = \{u \in X_k : u \in H^1(Q_k), u_{xx}, D_{x,k}^{-1} u_{yy} \in L^2(Q_k), f(u)u \in L^1(Q_k)\}$$

if $k < \infty$, and

$$Y = Y_\infty = \{u \in X : u \in H^1(\mathbb{R}^2), u_{xx}, D_{x,k}^{-1} u_{yy} \in L^2_{loc}(\mathbb{R}^2), f(u)u \in L^1(\mathbb{R}^2)\}.$$

First, we collect some useful identities.

Lemma 6 *Suppose that f satisfies Assumptions 1^0 and 2^0. Let $u \in Y_k$, $k \in (0, \infty]$, be a solution of equations (3). Then*

$$\int_{Q_k} [\frac{c}{2} u^2 + \frac{3}{2} u_x^2 + \varepsilon \frac{v^2}{2} - uf(u) + F(u)] = 0, \tag{14}$$

$$\int_{Q_k} [\frac{c}{2} u^2 + \frac{1}{2} u_x^2 + \varepsilon \frac{v^2}{2} + F(u)] = 0, \tag{15}$$

$$\int_{Q_k} [cu^2 + u_x^2 - \varepsilon v^2 - f(u)u] = 0. \tag{16}$$

Proof. First, we remark that, for any k, (16) is an extention to the case $\varepsilon = \pm 1$ of $I_k(u) = 0$ stated in Section 2. Therefore, we concentrate at (14) and (15) only.

In the case of solitary waves ($k = \infty$) the calculations carried out in the proof of Theorem 1.1, [5], work equally well for general nonlinearities. Therefore, we look at periodic waves ($k < \infty$).

Fixed $\kappa \in (0, 1)$, let $\varphi_T \in C_0^\infty(\mathbb{R})$ be a nonnegative function such that $\varphi_T = 1$ on $[-T/2, T/2]$, $\varphi_T(x) = 0$ if $|x| \geq (T + T^\kappa)/2$, and $\varphi^{(j)}(x) \leq C_j/|x|^j$, $j = 1, 2, \ldots$, if $T/2 \leq |x| \leq (T + T^\kappa)/2$ (the construction of such a function will be given later on).

Multiplying the first equation (3) by $x\varphi_T u$ and integrating over $\mathbb{R}^2$, we get, after a number of integrations by parts,

$$\frac{c}{2}\int \varphi_T u^2 - \int \varphi_T u f(u) + \int \varphi_T F(u) + \frac{3}{2}\int \varphi_T u_x^2 +$$
$$+\varepsilon\frac{1}{2}\int \varphi_T v^2 + \frac{1}{2}\int x\varphi_T' u^2 - \int x\varphi_T' u f(u) + \int x\varphi_T' F(u) +$$
$$2\int \varphi_T' u u_x + \int x\varphi_T'' u u_x + \frac{3}{2}\int x\varphi_T' u_x^2 + \varepsilon\frac{1}{2}\int x\varphi_T' v^2 = 0.$$

Dividing the last identity by T, we are going to pass to the limit as $T \to \infty$. First, we point out that here the integrals containing φ_T are taken over

$$Q_T \cup Q_T' \cup Q_T'' = Q_T \cup \{(T/2, (T+T^\kappa)/2) \times \mathbb{R}\} \cup \{(-(T+T^\kappa)/2, -T/2) \times \mathbb{R}\},$$

while that ones containing φ_T' and φ_T'' are over $Q_T' \cup Q_T''$. Moreover, $\varphi_T = 1$ on Q_T.

Now let $g \in L^1_{loc}(\mathbb{R}^2)$ be a function which is k-periodic in x. Then, it is easy to verify that

$$\lim_{T\to\infty} \frac{1}{T}\int_{Q_T} g = \int_{Q_k} g.$$

Next, due to the properties of φ_T, all the integrals over Q_T' can be estimated from above by $\int_{Q_T'} g$, with a nonnegative k-periodic in x function $g \in L^1_{loc}(\mathbb{R}^2)$. Now

$$\frac{1}{T}\int_{Q_T'} g \le \frac{T^\kappa + 1}{T}\int_{Q_k} g.$$

This justifies the passage to the limit and gives rise to (14).

Identity (15) can be proved exactly as (2.8), [5], with the only change: take there cut-off functions χ_j depending on y only.

Now we construct the function φ_T. Fix $\varepsilon > 0$ and let

$$g(x) = \begin{cases} 1 & \text{if } \; x \le T/2 \\ 1 - \log(x/T) & \text{if } \; T/2 < x \le (T+T^\kappa)/2 + \varepsilon \\ 0 & \text{if } \; x > (T+T^\kappa)/2 + \varepsilon \end{cases}.$$

We choose a nonnegative function $h \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} h \subset (0, \varepsilon)$ and $\int h = 1$, and set

$$\tilde{\varphi}_T(x) = \int h(x-t) g(t)\,dt, \qquad \varphi_T(x) = \tilde{\varphi}_T(|x|).$$

For this function it is easy to verify all the properties we need. □

Theorem 5 *Suppose that $f \in C(\mathbb{R})$ satisfies Assumption 4^0. Then there is no nontrivial traveling wave $u \in Y_k$, $k \in (0, \infty]$, provided $\varepsilon = +1$, or $\varepsilon = -1$ and $\mu \geq 6$.*

Proof. Adding (14), (15) and subtracting (16), we get

$$\int_{Q_k} u_x^2 = -2\varepsilon \int_{Q_k} v^2.$$

This rules out the case $\varepsilon = +1$. In the case $\varepsilon = -1$ (KP-I equations) the last identity together with (14) and (16), respectively, implies

$$\int_{Q_k} [\frac{c}{2}u^2 + \frac{5}{2}v^2 - f(u)u + F(u)] = 0$$

and

$$\int_{Q_k} [cu^2 + 3v^2 - f(u)u] = 0.$$

Eliminating v, we get

$$2c \int_{Q_k} u^2 = \int_{Q_k} [6F(u) - f(u)u].$$

If $\mu \geq 6$, we have

$$2c \int_{Q_k} u^2 \leq \int_{Q_k} [\mu F(u) - f(u)u] \leq 0.$$

Hence, $u = 0$ and we conclude. □

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