Effective solutions of Clebsch and C. Neumann systems

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Abstract

We solve in Riemann theta functions the classical Clebsch system and its particular case – the C. Neumann system. Going from a new Lax representation (with rational parameter) we work out the solutions. Separately, using relations between theta functions, we check that the corresponding expressions in theta functions satisfy Clebsch and Neumann systems.

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1 Introduction

The equations of motion of a rigid body in an ideal fluid are [3, 6, 5]

$$\dot{x} = x \times \frac{\partial H}{\partial p}\,, \qquad \dot{p} = x \times \frac{\partial H}{\partial x} + p \times \frac{\partial H}{\partial p} \tag{1}$$

where H is certain quadratic form in x and p. A nontrivial integrable case of equations (1) is the Clebsch case which is characterized with

$$H = \tfrac{1}{2}\sum_{j=1}^{3}(a_j p_j^2 + c_j x_j^2) \tag{2}$$

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and

$$\frac{c_2 - c_3}{a_1} + \frac{c_3 - c_1}{a_2} + \frac{c_1 - c_2}{a_3} = 0\,. \tag{3}$$

The Clebsch case (2), (3) can be easely reduced to the system with the Hamiltonian

$$H = \tfrac{1}{2}\sum_{j=1}^{3}(p_j^2 + a_j x_j^2) = h_1\ , \tag{4}$$

and then (3) is automatically satisfied [5]. So, the Clebsch system takes the form

$$\begin{array}{rclcl} \dot{x}_1 & = & p_3x_2 - p_2x_3 & & \dot{p}_1 = (a_3 - a_2)x_3x_2 \\ \dot{x}_2 & = & p_1x_3 - p_3x_1 & & \dot{p}_2 = (a_1 - a_3)x_1x_3 \\ \dot{x}_3 & = & p_2x_1 - p_1x_2 & & \dot{p}_3 = (a_2 - a_1)x_1x_2 \end{array} \tag{5}$$

where a_1, a_2, a_3 are different and nonzero constants. In vector notations the system reads

$$\begin{array}{rclll} \dot{x} & = & x \times p & x = (x_1, x_2, x_3)\ , & p = (p_1, p_2, p_3) \\ \dot{p} & = & x \times\, Ax & A = \ \mathrm{diag}\,(a_1\ ,\ a_2\ ,\ a_3)\ . & \end{array} \tag{6}$$

The additional first integrals are

$$\begin{aligned} &h_2 = a_1p_1^2 + a_2p_2^2 + a_3p_3^2 - a_2a_3x_1^2 - a_3a_1x_2^2 - a_1a_2x_3^2 \\ &h_3 = x_1p_1 + x_2p_2 + x_3p_3 = \langle\, x\ ,\ p\,\rangle \\ &h_4 = x_1^2 + x_2^2 + x_3^2\ . \end{aligned}$$

Theorem 1 *The Clebsch system* (5) *is equivalent to the Lax equation*

$$\dot{L} = [L, M] = LM - ML\,, \tag{7}$$

$$L = \begin{pmatrix} 1/a_1 & 0 & 0 & 0 \\ 0 & 1/a_2 & 0 & 0 \\ 0 & 0 & 1/a_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & -p_3/\sqrt{a_1a_2} & p_2/\sqrt{a_3a_1} & ix_1/\sqrt{a_1} \\ p_3/\sqrt{a_1a_2} & 0 & -p_1/\sqrt{a_2a_3} & ix_2/\sqrt{a_2} \\ -p_2/\sqrt{a_3a_1} & p_1/\sqrt{a_2a_3} & 0 & ix_3/\sqrt{a_3} \\ -ix_1/\sqrt{a_1} & -ix_2/\sqrt{a_2} & -ix_3/\sqrt{a_3} & 0 \end{pmatrix},$$

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \lambda + i \begin{pmatrix} 0 & 0 & 0 & -\sqrt{a_1}x_1 \\ 0 & 0 & 0 & -\sqrt{a_2}x_2 \\ 0 & 0 & 0 & -\sqrt{a_3}x_3 \\ \sqrt{a_1}x_1 & \sqrt{a_2}x_2 & \sqrt{a_3}x_3 & 0 \end{pmatrix},$$

satisfied for every $\lambda \in \mathbb{C}$.

The proof is straightforward.

A Lax representation for the Clebsch system, quadratic in the spectral parameter λ, is found in [15]. Another Lax pair, with parameter λ varying on an elliptic curve, is found in [1].

The kind of the Lax pair (7) affords to connect in a simple way the Clebsch system and Manakov's geodesic flow on $SO(4)$ [11].

$$(Z + \alpha h)^{\cdot} = [Z + \alpha h, \nu(Z) + \beta h], \quad Z = (z_{ij}) \in so(4),$$

$$\alpha = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad \beta = \operatorname{diag}(\beta_1, \beta_2, \beta_3, \beta_4), \quad \nu(Z) = (\nu_{ij} z_{ij}), \quad \nu_{ij} = \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j}.$$

This connection is well known and an isomorphism between the two cases is established in [2] with the help of above mentioned Lax pair with the spectral parameter varying on an elliptic curve.

In our case, the Clebsch system can be presented as a geodesic flow on $SO(4)$ with the specific metric

$$\nu_{12} = \nu_{13} = \nu_{23} = 0, \quad \nu_{j4} = -a_j, \quad j = 1, 2, 3.$$

The explicite change of variables is linear:

$$\left(z_1\,,\ z_2\,,\ z_3\,,\ z_4\,,\ z_5\,,\ z_6 \right) \ = \ \left(\frac{p_1}{\sqrt{a_2 a_3}}\,,\ \frac{p_2}{\sqrt{a_1 a_3}}\,,\ \frac{p_3}{\sqrt{a_1 a_2}}\,,\ -\frac{i x_1}{\sqrt{a_1}}\,,\ -\frac{i x_2}{\sqrt{a_2}}\,,\ -\frac{i x_3}{\sqrt{a_3}} \right).$$

The aim of this article is to solve the system (5).

Given a Lax pair (L, M), there exists an algorithm called algebro–geometric integration, leading to an integration of the system (5), that is to representation of $x_1, x_2, x_3, p_1, p_2, p_3$ as certain functions depending of the time t. These functions are always theta functions. Especially convenient and simple formulae are found by Dubrovin [5].

Let us sketch the algorithm of the algebro–geometric integration. Using the Lax equation (7), one constructs the algebraic curve:

$$\text{(8)} \qquad C: \ \det\big(L(\lambda) - \mu\big) = \mu^4 + I_1\mu^3\lambda + I_2\mu^2\lambda^2 + I_3\mu\lambda^3 + I_4\mu^2 + I_5\mu\lambda + I_6\lambda^2 + I_7 = 0,$$

where

$$\begin{aligned}
I_1 &= -1/a_1 - 1/a_2 - 1/a_3 \\
I_2 &= (a_1 + a_2 + a_3)(a_1 a_2 a_3)^{-1} \\
I_3 &= -(a_1 a_2 a_3)^{-1} \\
I_4 &= -I_3 h_2 \\
I_5 &= I_3 h_1 - I_2 h_4 \\
I_6 &= I_3 h_4 \\
I_7 &= I_3 h_3^2\,.
\end{aligned}$$

are first integrals of the system (5). Therefore, the curve C does not depend on t.

Adding four infinite points

$$\infty_j = \left(\lambda = \infty\,,\ \frac{\mu}{\lambda} = \frac{1}{a_j}\right), \quad j = 1, 2, 3, \quad \infty_4 = \left(\lambda = \infty\,,\ \frac{\mu}{\lambda} = 0\right),$$

we compactify the curve C to a (closed) Riemann surface Γ of genus 3.

Next, we fix a canonical basis in the one–dimensional homologies $H_1(\Gamma, \mathbb{Z})$. This basis defines the matrix of the periods B of Γ. B is a symmetric (3×3) matrix with negatively defined real part: $B^t = B$ and $\operatorname{Re} B < 0$.

The lattice of the periods $\Lambda = \{2\pi i N + BM \mid N, M \in \mathbb{Z}^3\}$ has rank 6. The Jacobi variety (Jacobian of Γ)

$$J(\Gamma) = \mathbb{C}^3/\Lambda$$

is a six–dimensional real torus.

Let us define the Riemann theta functions of genus 3. Recall that the Riemann theta function of genus g with characteristics $m, n \in \mathbb{R}^g$ is given by its Fourier series

$$\theta_B[\,{}^m_n\,](z) = \theta[\,{}^m_n\,](z) = \sum_{M\in\mathbb{Z}^g} \exp\left\langle \frac{B}{2}\left(M + \frac{m}{4}\right) + z + \pi i n\,,\, M + \frac{m}{2}\right\rangle , \tag{9}$$

$z \in \mathbb{C}^g$, B is a symmetric $(g \times g)$ matrix, $\operatorname{Re} B < 0$ [7]. If m and n are integers [1], they are called semi-periods.

Finally, we write down the solutions of (5) as functions Θ_3, which are ratios of genus 3 theta functions with arguments $z = z(t) \in \mathbb{C}^3$, linearly depending on the time t – it is said that the flow of (5) is linearized on the Jacobian $J(\Gamma)$ of Γ. The functions Θ_3 depend on certain constants which are Abelian integrals on Γ.

In this paper, the term "effective" has been used in the following sense. We consider that effective mathematical operations are summation, subtraction, multiplication, division and infinite summation when the sum is an entire function. The solutions of certain dynamical system are considered effective, when the general solution of the system is expressed via effectively computing functions.

For example, theta functions are effectively computed if there are no restrictions for the argument $z \in \mathbb{C}^g$ and the matrix B.

The obtained in [5] theta functions Θ_3 are not effective. One of the reasons is the symmetry

$$L(-\lambda) = -L^t(\lambda)\,, \qquad M(-\lambda) = -M^t(\lambda)$$

of the Lax matrices L and M. This symmetry means that there exists an holomorphic involution σ, acting on the Riemann surface Γ:

$$\sigma : \Gamma \to \Gamma\ , \qquad \sigma(\lambda, \mu) = (-\lambda, -\mu)\ , \qquad \sigma^2 = \text{identity}\ .$$

The involution σ has 4 fixed points: $\infty_1, \infty_2, \infty_3, \infty_4$. Hence, by the Riemann–Hurwitz formula, the factor–surface $\Gamma_1 = \Gamma/\sigma$ has genus 1, i.e. is an elliptic curve[2]. The Jacobian $J(\Gamma_1)$ of Γ_1 is a two–dimensional real torus, defined by a rank two sublattice Λ_2 of the lattice Λ. The factor

$$\operatorname{Prym}(\Gamma, \sigma) \overset{def}{=} J(\Gamma)/J(\Gamma_1)$$

is a two–dimensional complex variety, called Prym variety of Γ relative to σ [7].

The symmetry σ makes the three–dimensional theta functions factor in two–dimensional and one–dimensional theta functions [7]. Then, the solutions Θ_3 of (5) are reduced to expressions $\Theta_{2,1}$, including two–dimensional theta functions with arguments on $\operatorname{Prym}(\Gamma, \sigma)$ and one–dimensional theta functions with arguments on $J(\Gamma_1)$.

On the other hand, the elliptic theta functions on $J(\Gamma_1)$ can be reduced to two–dimensional theta functions exactly on $\operatorname{Prym}(\Gamma, \sigma)$. This can be done using the Fay formula [7, p.219]. As a result we get formulae only in terms of theta functions on $\operatorname{Prym}(\Gamma, \sigma)$.

In section 2, using two Fay formulae we get rid of the noneffectivity in Θ_2. The main result of this paper is the following theorem, giving effective solutions of the Clebsch system (5).

[1] Usually the notations $m/2$ and $n/2$ instead of m and n are used. We prefer the integer notations.

[2] This is the curve where the parameter λ from the Lax pair in [1] vary

Theorem 2 *Let*

- Π *is an arbitrary symmetric* (2×2) *matrix with negatively defined real part*
- $r\neq 0$ *and* a_0 *are arbitrary complex constants*
- $\nu=(\nu_1,\nu_2)$ *and* $z_0=(z_{01},z_{02})$ *are arbitrary vectors in* $\mathbb{C}^2$.

Define

- *theta functions* $\theta[\begin{smallmatrix} m_1 & m_2\\ n_1 & n_2\end{smallmatrix}](u)=\theta[\begin{smallmatrix} m_1 & m_2\\ n_1 & n_2\end{smallmatrix}](u,\Pi)$, $m_1,m_2,n_1,n_2\in\mathbb{Z},\ u\in\mathbb{C}^2$
- *the vector* $U=(U_1,U_2)=$

$$=\left(\theta[\begin{smallmatrix} 0&0\\1&0\end{smallmatrix}](\nu)\frac{\partial\theta[\begin{smallmatrix} 1&0\\1&0\end{smallmatrix}](\nu)}{\partial\nu_2}-\theta[\begin{smallmatrix} 1&0\\1&0\end{smallmatrix}](\nu)\frac{\partial\theta[\begin{smallmatrix} 0&0\\1&0\end{smallmatrix}](\nu)}{\partial\nu_2}\ ,\quad \theta[\begin{smallmatrix} 1&0\\1&0\end{smallmatrix}](\nu)\frac{\partial\theta[\begin{smallmatrix} 0&0\\1&0\end{smallmatrix}](\nu)}{\partial\nu_1}-\theta[\begin{smallmatrix} 0&0\\1&0\end{smallmatrix}](\nu)\frac{\partial\theta[\begin{smallmatrix} 1&0\\1&0\end{smallmatrix}](\nu)}{\partial\nu_1}\right)$$

- *the differentiation* $\partial_U=U_1\frac{\partial}{\partial\nu_1}+U_2\frac{\partial}{\partial\nu_2}:=\ '$
- *the constants* $h=\Big(\ln\theta[\begin{smallmatrix} 0&0\\1&0\end{smallmatrix}](\nu)-\ln\theta[\begin{smallmatrix} 1&0\\0&0\end{smallmatrix}](\nu)\theta(\nu)\Big)'$ *and* $K=\Big(\ln\theta(\nu)\,\theta[\begin{smallmatrix} 1&0\\0&0\end{smallmatrix}](\nu)\Big)'$
- *the vector* $z=z_0+tU$

Then the general solution of the Clebsch system (5) *is given by*

$$\begin{aligned}
x_1 &= \frac{r}{W}\Big\{\theta[\begin{smallmatrix} 0&0\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 0&0\\1&1\end{smallmatrix}](z)+\theta[\begin{smallmatrix} 1&0\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 1&0\\1&1\end{smallmatrix}](z)\Big\}\\
x_2 &= \frac{ir}{W}\Big\{\theta[\begin{smallmatrix} 1&1\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 1&1\\1&1\end{smallmatrix}](z)+\theta[\begin{smallmatrix} 0&1\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 0&1\\1&1\end{smallmatrix}](z)\Big\}\\
x_3 &= \frac{-r}{W}\Big\{\theta[\begin{smallmatrix} 0&1\\1&0\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 0&1\\1&0\end{smallmatrix}](z)+\theta[\begin{smallmatrix} 1&1\\1&0\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 1&1\\1&0\end{smallmatrix}](z)\Big\}\\
p_1 &= \frac{-i}{W}\Big\{\partial_U\theta[\begin{smallmatrix} 0&0\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 0&0\\1&1\end{smallmatrix}](z)+\partial_U\theta[\begin{smallmatrix} 1&0\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 1&0\\1&1\end{smallmatrix}](z)\Big\}+hx_1\\
p_2 &= \frac{1}{W}\Big\{\partial_U\theta[\begin{smallmatrix} 1&1\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 1&1\\1&1\end{smallmatrix}](z)+\partial_U\theta[\begin{smallmatrix} 0&1\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 0&1\\1&1\end{smallmatrix}](z)\Big\}+hx_2\\
p_3 &= \frac{i}{W}\Big\{\partial_U\theta[\begin{smallmatrix} 0&1\\1&0\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 0&1\\1&0\end{smallmatrix}](z)+\partial_U\theta[\begin{smallmatrix} 1&1\\1&0\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 1&1\\1&0\end{smallmatrix}](z)\Big\}+hx_3\\
W &= \theta[\begin{smallmatrix} 0&0\\1&0\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 0&0\\1&0\end{smallmatrix}](z)+\theta[\begin{smallmatrix} 1&0\\1&0\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix} 1&0\\1&0\end{smallmatrix}](z)\\
a_1 &= \frac{\theta''[\begin{smallmatrix} 0&0\\1&1\end{smallmatrix}](\nu)-K\,\theta'[\begin{smallmatrix} 0&0\\1&1\end{smallmatrix}](\nu)}{\theta[\begin{smallmatrix} 0&0\\1&1\end{smallmatrix}](\nu)}+a_0\\
a_2 &= \frac{\theta''[\begin{smallmatrix} 1&1\\1&1\end{smallmatrix}](\nu)-K\,\theta'[\begin{smallmatrix} 1&1\\1&1\end{smallmatrix}](\nu)}{\theta[\begin{smallmatrix} 1&1\\1&1\end{smallmatrix}](\nu)}+a_0\\
a_3 &= \frac{\theta''[\begin{smallmatrix} 0&1\\1&0\end{smallmatrix}](\nu)-K\,\theta'[\begin{smallmatrix} 0&1\\1&0\end{smallmatrix}](\nu)}{\theta[\begin{smallmatrix} 0&1\\1&0\end{smallmatrix}](\nu)}+a_0\ .
\end{aligned}\tag{10}$$

Following the above described procedure for algebro–geometric integration, in section 2 we prove Theorem 2. In section 3 we prove Theorem 2 again using only relations between theta functions.

So, the solutions of the Clebsch system (5) are parameterized by the points of 9–dimensional open in $\mathbb{C}^9$ subset

$$\Omega = \left\{ \Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{pmatrix}, r \,,\, a_0 \,, \nu = (\nu_1, \nu_2) \,,\, z_0 = (z_{01}, z_{02}) \mid \operatorname{Re} \Pi < 0 \,,\, r \neq 0 \right\} .$$

The constants $\Pi_{11}, \Pi_{12}, \Pi_{22}, r, a_0, \nu_1, \nu_2, z_{01}, z_{02}$ are called algebro–geometric coordinates of the system (5). The two–dimensional torus, defined by fixing Π, r, a_0, ν is parameterized by the points $z_0 \in \mathrm{Prym}(\Gamma, \sigma)$. The vector $U = U(\nu, \Pi)$ is the winding vector on this invariant under the flow (5) torus.

Non-effective solutions for the Clebsch system are found by Kötter [10], see also [1]. In the Kötter's formulae, the vector ν is fixed and a_1, a_2, a_3 are parameters of the problem. As it is seen from Theorem 2, expressing ν via $a_1 - a_0$, $a_2 - a_0$ and $a_3 - a_0$ is a transcendent problem. Probably, this is the reason that in [10] and [1] the winding vector U and p_1, p_2, p_3 are not explicitly computed.

A requirement for effectivity of the solutions is the presenting the solutions as fast converging series. This is always true for the theta functions due to the condition $\operatorname{Re} B < 0$ (or $\operatorname{Re} \Pi < 0$).

Remark. The choice of the semi-periods of the theta functions in Theorem 2 is not canonical. It depends on the choice of the basis in $H_1(\Gamma, \mathbb{Z})$. It turns out that there exist 15 types of similar formulae. Every type corresponds to a nonzero period, equal to the difference between the first and the second group of theta periods in the formulae for $x_1, x_2, \ldots, p_3$. In Theorem 2, this difference is the semi-period $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ mod(2). Adding the permutations between $x_1 W, x_2 W, x_3 W$ and W one obtains $4!\, 15 = 360$ relations between theta functions, solving (5). In [10], Kötter uses one of these variants.

The special case of the Clebsch system, when the integral $\langle p, x \rangle$ vanishes, is called C. Neumann system and describes the motion of a particle on the sphere in the field of quadratic potential [14, 13]. It turns out that $\langle p, x \rangle = 0$ iff

$$\nu \in S = \left\{ \nu \;:\; \frac{\partial \theta\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right](\nu)}{\partial \nu_1} \frac{\partial \theta\left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right](\nu)}{\partial \nu_2} = \frac{\partial \theta\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right](\nu)}{\partial \nu_2} \frac{\partial \theta\left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right](\nu)}{\partial \nu_1} \right\} .$$

In addition, the C. Neumann system is invariant under the changes

$$U \to e^k U \;, \quad p_i \to e^k p_i \;, \quad a_i \to e^{-2k} a_i \;, \quad i = 1, 2, 3$$

for all constants k. Then, instead of ν to be a point on the one–dimensional variety S, we can fix $\nu = 0$ and $k \in \mathbb{C}$.

The constants $\Pi_{11}, \Pi_{12}, \Pi_{22}, k, a_0, z_{01}, z_{02}$ are algebro–geometric coordinates of the C. Neumann system.

Theorem 3 *The general solution of the C. Neumann system is*

$$
\begin{aligned}
x_1 &= \frac{\theta[\begin{smallmatrix}0\,0\\1\,1\end{smallmatrix}]\,\theta[\begin{smallmatrix}0\,0\\1\,1\end{smallmatrix}](z)}{\theta[\begin{smallmatrix}0\,0\\1\,0\end{smallmatrix}]\,\theta[\begin{smallmatrix}0\,0\\1\,0\end{smallmatrix}](z)} & p_1 &= -i\,\frac{\partial_U\theta[\begin{smallmatrix}1\,0\\1\,1\end{smallmatrix}]\,\theta[\begin{smallmatrix}1\,0\\1\,1\end{smallmatrix}](z)}{\theta[\begin{smallmatrix}0\,0\\1\,0\end{smallmatrix}]\,\theta[\begin{smallmatrix}0\,0\\1\,0\end{smallmatrix}](z)} & a_1 &= \frac{\partial_U^2\theta[\begin{smallmatrix}0\,0\\1\,1\end{smallmatrix}]}{\theta[\begin{smallmatrix}0\,0\\1\,1\end{smallmatrix}]} - a_0\\
x_2 &= i\,\frac{\theta[\begin{smallmatrix}1\,1\\1\,1\end{smallmatrix}]\,\theta[\begin{smallmatrix}1\,1\\1\,1\end{smallmatrix}](z)}{\theta[\begin{smallmatrix}0\,0\\1\,0\end{smallmatrix}]\,\theta[\begin{smallmatrix}0\,0\\1\,0\end{smallmatrix}](z)} & p_2 &= \frac{\partial_U\theta[\begin{smallmatrix}0\,1\\1\,1\end{smallmatrix}]\,\theta[\begin{smallmatrix}0\,1\\1\,1\end{smallmatrix}](z)}{\theta[\begin{smallmatrix}0\,0\\1\,0\end{smallmatrix}]\,\theta[\begin{smallmatrix}0\,0\\1\,0\end{smallmatrix}](z)} & a_2 &= \frac{\partial_U^2\theta[\begin{smallmatrix}1\,1\\1\,1\end{smallmatrix}]}{\theta[\begin{smallmatrix}1\,1\\1\,1\end{smallmatrix}]} - a_0\\
x_3 &= -\frac{\theta[\begin{smallmatrix}0\,1\\1\,0\end{smallmatrix}]\,\theta[\begin{smallmatrix}0\,1\\1\,0\end{smallmatrix}](z)}{\theta[\begin{smallmatrix}0\,0\\1\,0\end{smallmatrix}]\,\theta[\begin{smallmatrix}0\,0\\1\,0\end{smallmatrix}](z)} & p_3 &= i\,\frac{\partial_U\theta[\begin{smallmatrix}1\,1\\1\,0\end{smallmatrix}]\,\theta[\begin{smallmatrix}1\,1\\1\,0\end{smallmatrix}](z)}{\theta[\begin{smallmatrix}0\,0\\1\,0\end{smallmatrix}]\,\theta[\begin{smallmatrix}0\,0\\1\,0\end{smallmatrix}](z)} & a_3 &= \frac{\partial_U^2\theta[\begin{smallmatrix}0\,1\\1\,0\end{smallmatrix}]}{\theta[\begin{smallmatrix}0\,1\\1\,0\end{smallmatrix}]} - a_0
\end{aligned}
\tag{11}
$$

Here $z = z_0 + tU$, $\theta[\begin{smallmatrix}a\,b\\c\,d\end{smallmatrix}] \stackrel{def}{=} \theta[\begin{smallmatrix}a\,b\\c\,d\end{smallmatrix}](0,0)$, $U = k\left(\frac{\partial}{\partial\nu_2}\theta[\begin{smallmatrix}1\,0\\1\,0\end{smallmatrix}],\ -\frac{\partial}{\partial\nu_1}\theta[\begin{smallmatrix}1\,0\\1\,0\end{smallmatrix}]\right)$, *$k$ is an arbitrary constant and* $r = 1$.

Of course, there are 360 possible variants of solutions. Two of them are used in [13, 1], where non-effective (according to accepted in this paper definition) are only the vectors U.

Next, well known change of variables (see (72), section 4) brings the flow of C. Neumann system into the geodesic flow on the ellipsoid [12, 13]. In [16], Weierstrass deals with one of the 360 variants of the relations between theta functions, solving the equation of the ellipsoid $x_1^2/a_1 + x_2^2/a_2 + x_3^2/a_3 = 1$.

In section 4, we answer the question when a geodesic on the three–axial ellipsoid is closed; the answer requires a check whether some theta constant is a rational number.

2 Algebro–geometric integration of the Clebsch system

In this section we realize in details the sketched in the introduction scheme for algebro–geometric integration of the Clebsch system (5). As a result we obtain the formulae for x_1, x_2, x_3 from the Theorem 2. We comment algebro–geometric coordinates for the Clebsch system by the end of the section.

2.1 Lax pair for the Clebsch system

Straightforward calculations show that the Clebsch system is equivalent to the Lax equation $\dot{L} = [L, M] = LM - ML$, where the matrices $L(\lambda)$ and $M(\lambda)$ are defined by (7).

2.2 Riemann surface Γ, connected to the Lax equation

Using the matrix $L(\lambda)$, we construct the flat algebraic curve of degree four

$$C:\ \det\big(L(\lambda) - \mu\big) = \mu^4 + I_1\mu^3\lambda + I_2\mu^2\lambda^2 + I_3\mu\lambda^3 + I_4\mu^2 + I_5\mu\lambda + I_6\lambda^2 + I_7 = 0\,.$$

The coefficients $I_1, \ldots, I_7$ of C are polynomials of $x_1, x_2, x_3, p_1, p_2, p_3, a_1, a_2$ and a_3 and are first integrals of the Clebsch system. If the curve C is non–singular, its genus is equal to $\frac{1}{2}(4-1)(4-2) = 3$.

The curve C is compactified by adding four infinite points

$$\infty_k = \big(\lambda = \infty\,,\, \lambda/\mu = a_k\big), \quad k = 1,2,3, \quad \infty_4 = \big(\lambda = \infty\,,\, \lambda\mu = -I_3/I_6\big)\,.$$

The Riemann surface $\Gamma = C \cup \infty_1 \cup \infty_2 \cup \infty_3 \cup \infty_4$ has genus 3.

Next, we construct all necessary for the integration of (5) and standard for the theory of Riemann surfaces [7, 8] objects.

Let $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ be a basis of cycles in the one–dimensional homologies $H_1(\Gamma, \mathbb{Z})$ with intersection indices

$$\alpha_j \circ \alpha_k = \beta_j \circ \beta_k = 0\,, \qquad \alpha_j \circ \beta_k = -\beta_k \circ \alpha_j = \delta_{jk}\,, \qquad j,k = 1,2,3,$$

δ_{jk} is the Kronecker symbol.

Fix the basis $\omega_1, \omega_2, \omega_3$ of holomorphic on Γ differentials, normed by

$$\int_{\alpha_j} \omega_k = 2\pi i\, \delta_{jk} \quad j,k = 1,2,3, \quad i = \sqrt{-1}\,.$$

Denote by B the matrix of the periods for the Riemann surface Γ :

$$B = (B_{jk})_{j,k=1}^{3}\,, \qquad B_{jk} = \int_{\beta_j} \omega_k\,.$$

Let

$$\Lambda = \left\{\, 2\pi i N + BN \mid N, M \in \mathbb{Z}^3 \,\right\}$$

be the lattice of the periods of Γ. Λ has rank 6 in $\mathbb{C}^3 \cong \mathbb{R}^6$. The Jacobian of Γ

$$J(\Gamma) = \mathbb{C}^3/\Lambda$$

is a six dimensional real torus. Let

$$\mathrm{Div}(\Gamma) = \Big\{ D = \sum_{k=1}^{N} n_k P_k \mid n_k \in Z, P_k \in \Gamma \Big\}$$

be the group of divisors on Γ; $\deg D = \sum_{k=1}^{N} n_k$ is the degree of the divisor D and $\mathrm{Div}_p(\Gamma)$ are all divisors of degree p ($p \in \mathbb{Z}$). Then, the Jacobian $J(\Gamma)$ is isomorphic to $\mathrm{Div}_0(\Gamma)$ modulo the principal divisors, i.e. the divisors of the meromorphic functions $f : \Gamma \to \mathbb{CP}^1$. This isomorphism is realized via Abel's map

$$\begin{aligned} \mathcal{A}\ :\ \mathrm{Div}_0(\Gamma) \quad &\to \quad J(\Gamma)\ , \\ D = \sum_{k=1}^{N} (P_k - Q_k) \quad &\mapsto \quad \mathcal{A}(D) = \sum_{k=1}^{N} \int_{Q_k}^{P_k} (\omega_1, \omega_2, \omega_3)\,. \end{aligned} \tag{12}$$

Remark. We shall write D instead of $\mathcal{A}(D)$ for the Abel's map when this is not misleading. The same notations we shall also use for the multi-valued because of different possible integration paths between the points P_k and Q_k Abel's map (12) $\mathcal{A} : \mathrm{Div}_0(\Gamma) \to \mathbb{C}^3$.

Let $\theta_B(z)$, $z \in \mathbb{C}^3$, be the theta function on Γ and $\hat{\triangle}$ be the Riemann theta divisor [8]. The Riemann theta divisor $\hat{\triangle}$ possess the fundamental property that for every two points $P_1, P_2 \in \Gamma$,

$$\theta_B(P_1 + P_2 - \hat{\triangle}) = 0\,.$$

Let Ω_{PQ}, $P,Q\in\Gamma$, be the abelian differential of third kind with simple poles in P and Q, residues $(+1)$ in P, (-1) in Q and zero α–periods.

Finally, define the prime–form $E(P,Q)$, $P,Q\in\Gamma$ on Riemann surface Γ, via

$$E^2(P,Q)=\frac{\big(f(P)-f(Q)\big)^2}{df(P)\,df(Q)}\exp\int_{-Q+f^{-1}(f(Q))}^{-P+f^{-1}(f(P))}\Omega_{PQ}\ , \tag{13}$$

where $f:\Gamma\to\mathbb{CP}^1$ is an arbitrary meromorphic function [7].

2.3 Dubrovin's formulae for the matrix M

When the diagonal elements M_{kk} and M_{jj} of the matrix M in the Lax equation are different, then there exist simple formulae in theta functions for the entries M_{kj} and M_{jk} of M [5]. In the Lax pair (7), $M_{44}\neq M_{11}=M_{22}=M_{33}$ and $M_{k4}=-M_{4k}$, $k=1,2,3$. Then, for $k=1,2,3$,

$$\frac{M_{4k}M_{k4}}{a_k}=x_k^2=\frac{\theta_B(\infty_k-\infty_4+\hat z_0+t\hat U)\,\theta_B(\infty_4-\infty_k+\hat z_0+t\hat U)}{a_k\,\theta_B^2(\hat z_0+t\hat U)\,E^2(\infty_k,\infty_4)\,d\lambda^{-1}(\infty_k)\,d\lambda^{-1}(\infty_4)}\ ; \tag{14}$$

the integration paths γ_k from ∞_4 to ∞_k are fixed and same for $\pm(\infty_k-\infty_4)=\pm\mathcal{A}(\infty_k-\infty_4)$ and $E(\infty_k,\infty_4)$; $\hat U\in\mathbb{C}^3$ is the vector of the β–periods of the Abelian differential $\hat\Omega$ with a singularity $(-d\lambda)$+(holomorphic part) in ∞_4 and zero α–periods. The vector $\hat U$ reads

$$\hat U=(\hat U_1,\hat U_2,\hat U_3)=\frac{1}{d\lambda^{-1}(\infty_4)}\big(\,\omega_1(\infty_4)\,,\,\omega_2(\infty_4)\,,\,\omega_3(\infty_4)\,\big)\ . \tag{15}$$

Now, we intend to reduce the formulae (14) to the formulae in Theorem 2.

2.4 Symmetries on the Riemann surface Γ

The Clebsch system is invariant under the involution

$$\sigma:(x_1,x_2,x_3,p_1,p_2,p_3,t)\ \mapsto\ -(x_1,x_2,x_3,p_1,p_2,p_3,t)\ .$$

Denote by σ and σ^* the symmetries resulted from σ on the objects on the Riemann surface Γ. For the Lax matrices L and M the corresponding to σ symmetry is

$$L^t(\lambda)=-L(-\lambda)\,,\quad M^t(\lambda)=-M(-\lambda)\ .$$

The σ–action on Γ is:

$$\sigma:\Gamma\to\Gamma\,,\quad \sigma(\lambda,\mu)=(-\lambda,-\mu)\,,\quad \sigma(\infty_k)=\infty_k\,,\quad k=1,2,3,4\,. \tag{16}$$

The basis of cycles in $H_1(\Gamma,Z)$ can be chosen such, that [7]

$$\sigma(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)=(-\alpha_3,-\alpha_2,-\alpha_1,-\beta_3,-\beta_2,-\beta_1).$$

Then,

$$\sigma^*\omega_1=-\omega_3,\quad \sigma^*\omega_2=-\omega_2,\quad \sigma^*\omega_3=-\omega_1,\qquad \big(\,\sigma^*\omega_k(P)=\omega_k(\sigma P)\,\big)\ ;$$

the Abel map $\mathcal{A}$ possess symmetry

$$\sigma^* \int_\gamma (\omega_1, \omega_2, \omega_3) = \int_{\sigma\gamma} (\omega_1, \omega_2, \omega_3) = -\int_\gamma (\omega_3, \omega_2, \omega_1)\,, \tag{17}$$

where $\sigma\gamma$ is the σ–image of the integration path γ. For $\sigma^* : \mathbb{C}^3 \to \mathbb{C}^3$, the symmetry is

$$\sigma^*(z_1, z_2, z_3) = -(z_3, z_2, z_1), \qquad (z_1, z_2, z_3) \in \mathbb{C}^3.$$

When the integral[3] $I_7 = \langle p, x\rangle^2 (a_1 a_2 a_3)^{-1}$ does not vanish, then the holomorphic involution $\sigma : \Gamma \to \Gamma$ has exactly four fixed points $\infty_1, \infty_2, \infty_3, \infty_4$. Therefore, the factor Γ/σ is a well defined closed Riemann surface. According to the Riemann–Hurwitz formula, the genus of Γ/σ is equal to $-2 + 1 + \frac{4}{2} = 1$.

Let $\pi : \Gamma \to \Gamma/\sigma$ be the natural projection. The basis in $H_1(\Gamma/\sigma, \mathbb{Z})$ is spanned by $\pi\alpha_1$ and $\pi\beta_1$; a basic differential in $H^1(\Gamma/\sigma, \mathbb{C})$ is $\pi_*(\omega_1 - \omega_3)$. Next, we define

$$\pi^* : J(\Gamma/\sigma) \to J(\Gamma)\ , \qquad \pi^*(D) = \pi^{-1}(D)\,, \qquad D \in \mathrm{Div}_0(\Gamma/\sigma)$$

and

$$\pi^* : \mathbb{C} \to \mathbb{C}^3\ , \qquad \pi^*(z) = (z, 0, -z)\ .$$

Let τ be the (1×1) matrix of the periods of Γ/σ. The constant τ and the matrix B of the periods of Γ are connected by [7]

$$B = \begin{pmatrix} \frac{1}{2}(\tau + \Pi_{11}) & \Pi_{12} & \frac{1}{2}(\tau - \Pi_{11}) \\ \Pi_{21} & 2\Pi_{22} & \Pi_{12} \\ \frac{1}{2}(\tau - \Pi_{11}) & \Pi_{21} & \frac{1}{2}(\tau + \Pi_{11}) \end{pmatrix}, \qquad \Pi_{12} = \Pi_{21}. \tag{18}$$

The matrix Π will be discussed in the next paragraph.

Finally, denote by $\triangle$ the Riemann theta divisor for the factor–surface Γ/σ.

2.5 Prym variety

The matrix $\Pi = (\Pi_{jk})_{j,k=1}^2$ from (18) is called Prym matrix for the pair (Γ, σ). The two–dimensional complex variety

$$\mathrm{Prym}\,(\Gamma, \sigma) = \mathbb{C}^2/\Lambda_4\ , \qquad \Lambda_4 = \left\{\, 2\pi i N + \Pi M \ \mid N\,, M \in \mathbb{Z}^2 \,\right\} \tag{19}$$

is called Prym variety for the pair (Γ, σ). Λ_4 is a rank 4 lattice, Π is a symmetric matrix with negatively defined real part. Actually, $\mathrm{Prym}\,(\Gamma, \sigma)$ is a four dimensional real torus. The map

$$p_* : \mathrm{Prym}\,(\Gamma, \sigma) \to J(\Gamma)\,, \quad (z_1, z_2) \mapsto (z_1, z_2, z_1) \tag{20}$$

linearly embed $\mathrm{Prym}\,(\Gamma, \sigma)$ as a codimension one subvariety into the Jacobian $J(\Gamma)$. Then, $p^* : J(\Gamma) \to \mathrm{Prym}\,(\Gamma, \sigma)$ has the form $p^*(z_1, z_2, z_3) = (z_1 + z_3\,,\, 2z_2)$. In terms of Abel's map, for $P, Q \in \Gamma$,

$$2\mathcal{A}(P - Q) = \pi^*\pi_*\mathcal{A}(P - Q)\ +\ p_* p^* \mathcal{A}(P - Q). \tag{21}$$

[3]see (8)

The role of a Riemann theta divisor on Prym (Γ, σ) plays the divisor of degree two

$$D = \hat{\triangle} - \pi^* \triangle \, , \qquad 2D = \infty_1 + \infty_2 + \infty_3 + \infty_4 \, .$$

As a final result, we can consider the Jacobian $J(\Gamma/\sigma)$ and Prym (Γ, σ) as linearly embedded in the Jacobian $J(\Gamma)$ submanifolds:

$$\begin{aligned} J(\Gamma/\sigma) &= \Big\{ (z, 0, -z) \bmod \Lambda \mid z \in \mathbb{C} \Big\} \, , \\ \mathrm{Prym}\,(\Gamma, \sigma) &= \Big\{ (z_1, z_2, z_1) \bmod \Lambda \mid (z_1, z_2) \in \mathbb{C}^2 \Big\} \, . \end{aligned} \tag{22}$$

The following identities between theta functions on $J(\Gamma)$, $J(\Gamma/\sigma)$ and Prym (Γ, σ) hold:

$$\theta_B(z_1, z_2, z_3) = \sum_{p=0,1} \theta_{2\tau}\big[\begin{smallmatrix} p \\ 0 \end{smallmatrix}\big](z_1 - z_3)\, \theta_{2\Pi}\big[\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\big](z_1 + z_3 \, , \, 2z_2) \, , \tag{23}$$

$$\theta_{2\Pi}\big[\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\big] \left(\int_D^{P_1+P_2} (\omega_1 + \omega_3 \, , \, 2\omega_2) \right) = c(P_1, P_2)\, \theta_{2\tau}\big[\begin{smallmatrix} p \\ 0 \end{smallmatrix}\big] \left(\int_D^{P_1+P_2} (\omega_1 - \omega_3) \right) \, , \tag{24}$$

where

$$c(P_1, P_2) = \frac{\theta_B(P_1 + P_2 - D)}{\theta_\tau(0)\, \theta_\tau\big(\pi_*(P_1 + P_2 - D)\big)} \quad . \tag{25}$$

The identities (23) and (24) are true for all $z_k \in \mathbb{C}$, $p = 0$ or 1 and $P_1, P_2 \in \Gamma$, see [7][4].

2.6 Linearization of the flow (5) on the Prym variety

Now, we can reduce the Dubrovin formulae (14), taking into account the involution $\sigma : \Gamma \to \Gamma$. The necessary and sufficient conditions for the existence of the symmetries $L^t(\lambda) = -L(-\lambda)$ and $M^t(\lambda) = -M(-\lambda)$ are the existence of σ and the requirement

$$\hat{z}_0 \in \mathrm{Prym}\,(\Gamma, \sigma), \qquad \text{i.e.} \qquad \hat{z}_0 = (z_{01}, z_{02}, z_{01}) \tag{26}$$

in (14), see [4].

It follows from (15) and (17) that $\hat{U} \in \mathrm{Prym}\,(\Gamma, \sigma)$. In order to define the formulae (14), we choose the integration paths γ_k such that

$$\gamma_1 - \sigma\gamma_1 = \alpha_2 \, , \quad \gamma_2 - \sigma\gamma_2 = \alpha_2 + \beta_2 \, , \quad \gamma_3 - \sigma\gamma_3 = \beta_2 \, .$$

Using the symmetry $\sigma^*(\omega_1, \omega_2, \omega_3) = -(\omega_3, \omega_2, \omega_1)$, it is easy to calculate that

$$\begin{aligned} \int_{\infty_4}^{\infty_1} (\omega_1 + \omega_3 \, , \, 2\omega_2) &= \tfrac{1}{2} \int_{\alpha_2} (\omega_1 + \omega_3 \, , \, 2\omega_2) = (0, \pi i) \\ \int_{\infty_4}^{\infty_2} (\omega_1 + \omega_3 \, , \, 2\omega_2) &= \tfrac{1}{2} \int_{\alpha_2 + \beta_2} (\omega_1 + \omega_3 \, , \, 2\omega_2) = (0, \pi i + \Pi_{22}) \\ \int_{\infty_4}^{\infty_3} (\omega_1 + \omega_3 \, , \, 2\omega_2) &= \tfrac{1}{2} \int_{\beta_2} (\omega_1 + \omega_3 \, , \, 2\omega_2) = (0, \Pi_{22}) \, . \end{aligned} \tag{27}$$

Now we reduce the thetas θ_B from (14) to theta functions $\theta_{2\Pi}$ with semi-periods.

[4] eq. (109), p. 95 and eq. (112), p. 96

Lemma 4 *Let*

$$u \stackrel{def}{=} \int_D^{Q_4+\infty_4} p^*\omega \;=\; \int_D^{Q_4+\infty_4} (\,\omega_1+\omega_3\,,\,2\omega_2\,) \;=\; p^*(Q_4+\infty_4-D), \tag{28}$$

where $Q_4, \sigma Q_4$ and ∞_4 are the zeroes of three–sheeted function $\mu : \Gamma \to \mathbb{CP}^1$, defined by (8). Then for $k = 1, 2, 3$,

$$\frac{\theta_B(\infty_k - \infty_4 + \hat{z}_0 + t\hat{U})\,\theta_B(\infty_4 - \infty_k + \hat{z}_0 + t\hat{U})}{\theta_B^2(\hat{z}_0 + t\hat{U})} \;=\; \frac{c(Q_4,\infty_4)\,c(\sigma Q_4,\infty_4)}{c(Q_4,\infty_k)\,c(\sigma Q_4,\infty_k)}\,F_k\,,$$

$$\begin{aligned} F_1 &= \frac{\pm 1}{w}\sum_{p=0,1}\theta_{2\Pi}\big[\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\big](u)\,\theta_{2\Pi}\big[\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\big](z)\,,\\ F_2 &= \frac{\pm i}{w}\sum_{p=0,1}\theta_{2\Pi}\big[\begin{smallmatrix} p & 1\\ 0 & 1\end{smallmatrix}\big](u)\,\theta_{2\Pi}\big[\begin{smallmatrix} p & 1\\ 0 & 1\end{smallmatrix}\big](z)\,,\\ F_3 &= \frac{\pm 1}{w}\sum_{p=0,1}\theta_{2\Pi}\big[\begin{smallmatrix} p & 1\\ 0 & 0\end{smallmatrix}\big](u)\,\theta_{2\Pi}\big[\begin{smallmatrix} p & 1\\ 0 & 0\end{smallmatrix}\big](z)\,,\\ w &= \sum_{p=0,1}\theta_{2\Pi}\big[\begin{smallmatrix} p & 0\\ 0 & 0\end{smallmatrix}\big](u)\,\theta_{2\Pi}\big[\begin{smallmatrix} p & 0\\ 0 & 0\end{smallmatrix}\big](z)\,, \qquad z = p^*(\hat{z}_0 + t\hat{U}). \end{aligned} \tag{29}$$

Proof. The function μ has divisor

$$\mathrm{Div}(\mu) = Q_4 + \sigma Q_4 + \infty_4 - \infty_1 - \infty_2 - \infty_3\ .$$

Therefore, on $J(\Gamma/\sigma)$,

$$\pi_*(Q_4+\infty_4) = \tfrac{1}{2}\,\pi_*(Q_4+\sigma Q_4 + 2\infty_4) = \tfrac{1}{2}\,\pi_*\sum_{k=1}^{4}\infty_k = \pi_* D\ .$$

Similarly, $\pi_*(\sigma Q_4 + \infty_4) = \pi_* D$. Applying (23) and (24) for $P_1 = Q_4$ or σQ_4 and for $P_2 = \infty_4$, we get

$$\begin{aligned} &\theta_B(\pm\infty_k \mp \infty_4 + \hat{z}_0 + t\hat{U})\\ &\stackrel{(23)}{=} \sum_{p=0,1}\theta_{2\tau}\big[\begin{smallmatrix} p\\ 0\end{smallmatrix}\big](\infty_k - \infty_4)\,\theta_{2\Pi}\big[\begin{smallmatrix} p & 0\\ 0 & 0\end{smallmatrix}\big](\pm\infty_k \mp \infty_4 + \hat{z}_0 + t\hat{U})\\ &= \sum_{p=0,1}\theta_{2\tau}\big[\begin{smallmatrix} p\\ 0\end{smallmatrix}\big](\infty_k + Q_4 - D)\,\theta_{2\Pi}\big[\begin{smallmatrix} p & 0\\ 0 & 0\end{smallmatrix}\big](\pm\infty_k \mp \infty_4 + z)\\ &\stackrel{(24)}{=} c^{-1}\big(\infty_k, \sigma^{\frac{1}{2}\mp\frac{1}{2}}Q_4\big)\sum_{p=0,1}\theta_{2\Pi}\big[\begin{smallmatrix} p & 0\\ 0 & 0\end{smallmatrix}\big](\pm\infty_k \mp \infty_4 + u)\,\theta_{2\Pi}\big[\begin{smallmatrix} p & 0\\ 0 & 0\end{smallmatrix}\big](\pm\infty_k \mp \infty_4 + z). \end{aligned} \tag{30}$$

Now, we convert the arguments $\pm\infty_k \mp \infty_4 = \pm\int_{\infty_4}^{\infty_k}(\omega_1+\omega_3, 2\omega_2)$ of theta functions into semi-periods of $\theta_{2\Pi}$, using (27), (28), (30) and the formula

$$\begin{aligned} \theta_{2\Pi}\big[\begin{smallmatrix} \xi_1 & \xi_2\\ \eta_1 & \eta_2\end{smallmatrix}\big](z_1,z_2) \;&=\; \theta_{2\Pi}\left(\,(z_1,z_2) + \pi i(\eta_1,\eta_2) + \tfrac{1}{2}(\xi_1,\xi_2)\Pi\,\right)\\ &\times \exp\left\langle\,\tfrac{1}{16}(\xi_1\,,\,\xi_2)\ ,\ (\xi_1,\xi_2)\Pi + \pi i(\eta_1,\eta_2) + 2(z_1,z_2)\,\right\rangle\ . \end{aligned}$$

The denominator W is obtained after putting $k = 4$ in (30). In order to get (29), we put $k = 1, 2, 3$ in (30) and multiply $(+)$ and $(-)$ variants (the expressions for F_k are squared). This proves Lemma 4. $\square$

The expressions F_1, F_2, F_3 and W correspond to the expressions for x_1, x_2, x_3 in Theorem 2. The next lemma simplifies the first formula in (29).

Lemma 5 *For all $P \in \Gamma$ and $k = 1, 2, 3$*

$$h(P) = \frac{c(P, \infty_4)\, c(\sigma P, \infty_4)}{c(P, \infty_k)\, c(\sigma P, \infty_k)} \exp \int_{\infty_1+\infty_2+\infty_3-\infty_k}^{P+\sigma P} \Omega_{\infty_k\infty_4} = \text{constant} = \pm 1\ . \tag{31}$$

Proof. Let $Q \in \Gamma$ be fixed. Then, the section $c(P, Q)$ has a simple zero at $P = \sigma Q$ [7]. Hence, the poles and zeroes of $h(P)$ cancel. Now, from (25), (31),

$$\theta_B(z + 2\pi i N + BM) = \theta_B(z) \exp\langle -\tfrac{1}{2} BM + z\,,\, M\rangle, \quad N, M \in \mathbb{Z}^2, \quad z \in \mathbb{C}^2$$

and [8]

$$\int_{\beta_j} \Omega_{PQ} = \mathcal{A}_j(P - Q), \qquad j = 1, 2, 3, \qquad \mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$$

follows, that $h(P)$ is a holomorphic function on Γ. This proves that $h(P)$ is a constant.

For $k = 1$, we put $P = \infty_2$ and $P = \infty_3$ in (31). Multiplication of the obtained expressions leads to a cancellation of the exponents. It remains to note, that $c(\infty_1, \infty_4) = c(\infty_2, \infty_3)$ and $c(\infty_1, \infty_3) = c(\infty_2, \infty_4)$ (see (25) and the identity $2D = \infty_1 + \infty_2 + \infty_3 + \infty_4$). Then, we get $h(\infty_2)h(\infty_3) = 1$, from where $h^2(P) = 1$ and $h(P) = \pm 1$.

The cases $k = 2, 3$ are similar. This proves Lemma 5. $\square$

It is seen from the kind of the infinite points ∞_k, that

$$\frac{\lambda(\infty_k)}{\mu(\infty_k)} = \frac{d\lambda(\infty_k)}{d\mu(\infty_k)} = a_k\,, \qquad k = 1, 2, 3\,, \qquad \lambda(\infty_4)\, \mu(\infty_4) = -\frac{I_3}{I_6}\,.$$

So,

$$\frac{d\mu(\infty_k)\, d\mu(\infty_4)}{\big(\mu(\infty_k) - \mu(\infty_4)\big)^2\, d\lambda^{-1}(\infty_k)\, d\lambda^{-1}(\infty_4)} = \frac{I_3}{I_6} a_k\ , \qquad k = 1, 2, 3. \tag{32}$$

From equations (14), (29), (31) for $P = Q_4$, (13) for $f = 1/\mu$, $P = \infty_4$, $Q = Q_4$ and from (32) we get the expressions for x_1, x_2, x_3 from Theorem 2 with multipliers $r = I_3/I_6$:

$$\begin{aligned}
x_k^2 &\overset{(14)}{=} \frac{\theta_B(\infty_k - \infty_4 + \hat{z}_0 + t\hat{U})\, \theta_B(\infty_4 - \infty_k + \hat{z}_0 + t\hat{U})}{a_k\, \theta_B^2(\hat{z}_0 + t\hat{U})\, E^2(\infty_k, \infty_4)\, d\lambda^{-1}(\infty_k)\, d\lambda^{-1}(\infty_4)} \\
&\overset{(29)}{=} F_k\ \frac{c(Q_4, \infty_4)\, c(\sigma Q_4, \infty_4)}{c(Q_4, \infty_k)\, c(\sigma Q_4, \infty_k)}\, \frac{1}{a_k\, E^2(\infty_k, \infty_4)\, d\lambda^{-1}(\infty_k)\, d\lambda^{-1}(\infty_4)} \\
&\overset{(31)}{=} \pm F_k \exp \int_{Q_4+\sigma Q_4}^{\infty_1+\infty_2+\infty_3-\infty_k} \Omega_{\infty_k\infty_4}\ \frac{1}{a_k\, E^2(\infty_k, \infty_4)\, d\lambda^{-1}(\infty_k)\, d\lambda^{-1}(\infty_4)} \\
&\overset{(13)}{=} \pm F_k \exp \int_{Q_4+\sigma Q_4}^{\infty_1+\infty_2+\infty_3-\infty_k} \Omega_{\infty_k\infty_4}\ \frac{\exp \int_{\infty_1+\infty_2+\infty_3-\infty_k}^{Q_4+\sigma Q_4} \Omega_{\infty_k\infty_4} d\mu(\infty_k)\, d\mu(\infty_4)}{a_k\, \big(\mu(\infty_k) - \mu(\infty_4)\big)^2\, d\lambda^{-1}(\infty_k)\, d\lambda^{-1}(\infty_4)} \\
&\overset{(32)}{=} \pm F_k \frac{I_3}{I_6}\ .
\end{aligned}$$

The following lemma calculates the vector U.

Lemma 6 *The vector $U = p^*\hat{U}$ satisfies the condition*

$$\left\langle U \ , \ \text{grad} \ln \frac{\theta_{2\Pi}[{}^{00}_{10}](u)}{\theta_{2\Pi}[{}^{10}_{10}](u)} \right\rangle \ = \ 0 \ . \tag{33}$$

Proof. Recall that $U = p\mathcal{A}(Q_4 + \infty_4 - D)$ and $D = \hat{\triangle} - \pi^*\triangle$. Then, for every $P \in \Gamma$,

$$\theta_B(P - \infty_4 + p_* u) = \theta_B(P + Q_4 - \hat{\Delta}) \equiv 0 \tag{34}$$

Take the Taylor expansion of $\theta_B(P - \infty_4 + p_*U)$ around the point ∞_4. Putting $P = \infty_4$ in (34), we get $\theta_B(U) = 0$. The second member in the expansion, when $P = \infty_4$, gives $\langle \hat{U} \, , \, \text{grad}\, \theta_B(p_*U) \rangle = 0$. The identities

$$\theta_B(p_*U) = \sum_{p=0,1} \theta_{2\tau}[{}^{p}_{0}](0)\, \theta_{2\Pi}[{}^{p\,0}_{0\,0}](u) = 0 \, , \qquad \theta_{2\Pi}[{}^{p\,0}_{0\,0}] = c(Q_4, \infty_4)\, \theta_{2\tau}[{}^{p}_{0}](0),$$

see (23) for $z = p_*U$ and (24) for $P_1 = Q_4, P_2 = \infty_4$, prove that

$$\left\langle U \ , \ \theta_{2\Pi}[{}^{00}_{10}](u)\, \text{grad}\, \theta_{2\Pi}[{}^{10}_{00}](u) - \theta_{2\Pi}[{}^{10}_{00}](u)\, \text{grad}\, \theta_{2\Pi}(u) \right\rangle = 0.$$

Then (33) is immediate. This finishes the proof of Lemma 6. □

In order to get more symmetrical formulae in Theorem 2, we replace every theta function $\theta[{}^{a\,b}_{0\,d}]$ with $\theta[{}^{a\,b}_{1\,d}]$. This can be done by the translations $z \mapsto z + (\pi i, 0)$ and $u \mapsto u + (\pi i, 0)$. Then, the formulae for x_1, x_2, x_3 from Theorem 2 are consequence of Lemma 6.

2.7 Algebro–geometric coordinates of the Clebsch system

The Clebsch system consists of six equations and depends on three constants a_1, a_2, a_3. Hence, the general solution of (5) has the form $\varphi = \varphi(c_1, \ldots, c_9, t)$, where φ is a nine–dimensional vector and $c_1, \ldots, c_9$ are constants with possible open relations between them.

As a first possibility, for $c_1, \ldots, c_9$ can be taken

$$a_1 \ , \ a_2 \ , \ a_3 \ , \ I_4 \ , \ I_5 \ , \ I_6 \ , \ J_7 \ , \ z_{01} \ , \ z_{02} \tag{35}$$

see (8) for $I_{4,5,6,7}$, a_1, a_2, a_3 and $z_0 = (z_{01}, z_{02})$ from Theorem 2. The first seven constants from (35) are the seven independent integrals for (5) with additional equations $\dot{a}_{1,2,3} = 0$. The point $z_0 = (z_{01}, z_{02})$ defines the initial position (at $t = 0$) of the phase trajectory on the Prym variety Prym (Γ, σ).

Second possible set of $c_1, \ldots, c_9$ is

$$I_1 \, , \ldots \, , I_7 \, , \, z_{01} \, , \, z_{02} \ .$$

The connection between I_1, I_2, I_3 and a_1, a_2, a_3 is a trivial one.

The third possible set is those of Theorem 2 :

$$\Pi_{11} \, , \, \Pi_{12} \, , \, \Pi_{22} \, , \, u_1 \, , \, u_2 \, , \, r \, , \, a_0 \, , \, z_{01} \, , \, z_{02} \, , \qquad \text{Re}\Pi < 0 \, , \, r \neq 0 \ . \tag{36}$$

We call the coordinates (36) algebro–geometric coordinates of the Clebsch system (5). The only question is whether $u = (u_1, u_2) = p^*\mathcal{A}(Q_4 + \infty_4 - D) \in \mathrm{Prym}\,(\Gamma, \sigma)$ is an arbitrary vector. The answer is positive. If the Riemann surface Γ with the involution $\sigma : \Gamma \to \Gamma$ of type (16) transforms in the way that the Prym matrix does not change, then the factor Γ/σ describes the Riemann surface of genus $g = 1$. The moduli space of the elliptic curve is one–dimensional and can be parameterized by the period τ. In our case, however, the full moduli (K, K'), $K' = \tau K$ of the elliptic curve Γ/σ , or, equivalently, the coefficients g_2 and g_3 of the curve

$$C_1 : \eta^2 = 4\xi^3 - g_2\xi - g_3$$

isomorphic to Γ/σ (see [9]), are two first integrals.

Indeed, Abel's map $\pi_*\mathcal{A}(\infty_k - \infty_4) = \int_{\infty_4}^{\infty_k}(\omega_1 - \omega_3)$ takes place in (30). After change $\lambda \mapsto c\lambda$ in (7), the Prym matrix and the modulus τ do not change, but (K, K') become (cK, cK') and $\pi_*\mathcal{A}$ becomes $c\,\pi_*\mathcal{A}$. The vector u is defined from

$$\frac{\theta_{2\Pi}[\begin{smallmatrix}1\,0\\0\,0\end{smallmatrix}](u)}{\theta_{2\Pi}(u)} = \frac{\theta_{2\tau}[\begin{smallmatrix}1\\0\end{smallmatrix}](0)}{\theta_{2\tau}(0)}\,, \qquad \frac{\theta_{2\Pi}[\begin{smallmatrix}1\,0\\0\,1\end{smallmatrix}](u)}{\theta_{2\Pi}[\begin{smallmatrix}0\,0\\0\,1\end{smallmatrix}](u)} = \frac{\theta_{2\tau}[\begin{smallmatrix}1\\0\end{smallmatrix}]\big(\mathcal{A}(\infty_1 - \infty_4)\big)}{\theta_{2\tau}\big(\mathcal{A}(\infty_1 - \infty_4)\big)}\,,$$

see (28) and (24). The right part of the first equality is invariant under the changes $\lambda \mapsto c\lambda$, while the right part of the second quality changes in the following way: $\mathcal{A}(\infty_1 - \infty_4) \mapsto c.\mathcal{A}(\infty_1 - \infty_4)$. So, we have another system of algebro–geometric coordinates:

$$\Pi_{11}\,,\ \Pi_{12}\,,\ \Pi_{22}\,,\ g_2\,,\ g_3\,,\ r\,,\ a_0\,,\ z_{01}\,,\ z_{02}\,.$$

3 Proof of Theorem 2

In this section we prove Theorem 2 using relations between two–dimensional theta functions

$$\theta[\begin{smallmatrix}m\\n\end{smallmatrix}](z) = \theta_{2\Pi}[\begin{smallmatrix}m_1\,m_2\\n_1\,n_2\end{smallmatrix}](z)\,,$$

where Π is a symmetric (2×2) matrix with complex entries Π_{jk} , $j, k = 1, 2$ and negative definite real part $\mathrm{Re}\,\Pi < 0$; m_j and n_j are integers, see definition (9).

3.1 Basic properties of the theta functions with semi-periods

The following identities are standard for theta functions [7]:

$$\begin{aligned} &(37) \qquad \theta[\begin{smallmatrix}m\\n\end{smallmatrix}](u + 2\pi i N + 2\Pi M) \\ &\qquad = \theta[\begin{smallmatrix}m\\n\end{smallmatrix}](u)\,\exp\left(-\langle \Pi M, M\rangle - \langle u, M\rangle + \pi i\langle m, N\rangle - \pi i\langle n, M\rangle\right),\end{aligned}$$

$$(38) \qquad \theta[\begin{smallmatrix}m\\n\end{smallmatrix}](u) = \theta(u + \pi i n + \Pi m)\exp\langle -\tfrac{1}{2}\Pi m + u + \pi i n\,,\ \tfrac{1}{2}m\rangle\,,$$

$$(39) \qquad \theta[\begin{smallmatrix}m+2M\\n+2N\end{smallmatrix}](u) = \theta[\begin{smallmatrix}m\\n\end{smallmatrix}](u)\,\exp\pi i\langle m, N\rangle\,,$$

satisfied for arbitrary $M, N \in \mathbb{Z}^g$ and $u \in \mathbb{C}^g$.

Using the fact that $\mathrm{Re}\,\Pi < 0$, we identify $\mathbb{C}^g$ and $\mathbb{R}^{2g}$ via

$$(40) \qquad u = [\begin{smallmatrix}m\\n\end{smallmatrix}] \stackrel{def}{=} 2\pi i n + 2\Pi m\,.$$

For every $u \in \mathbb{C}^2$, there exist unique vectors $m, n \in \mathbb{R}^2$ satisfying (40); m and n are called characteristics of the point $u \in \mathbb{C}^2$. We are interested in theta functions $\theta[{}^m_n](u)$ with semi-periods $m, n \in \mathbb{Z}$ and the points $u = [\,{}^p_q\,] \in \mathbb{C}^2$, $p, q \in \frac{1}{2}\mathbb{Z}$ (such p and q are called semi-periods of the point u).

Introduce the notations

$$\theta[{}^m_n] \stackrel{def}{=} \theta[{}^m_n](0)\,, \quad \theta(u) \stackrel{def}{=} \theta[{}^0_0](u)\,, \quad \theta \stackrel{def}{=} \theta(0)\,, \quad m,\, n \in \mathbb{Z}^2$$

for theta functions with semi-periods and for theta functions with argument $u = 0$.

If $\langle m, n\rangle \equiv 0 \bmod 2$, then (m, n) and the point $[\,{}^{m/2}_{n/2}\,] \in \mathbb{C}^2$ are called even semi-periods. If $\langle m, n\rangle \equiv 1 \bmod 2$, we say that m, n and the point $[\,{}^{m/2}_{n/2}\,] \in \mathbb{C}^2$ are odd semi-periods. The theta functions with odd (even) semi-periods are odd (even) functions [13]. Therefore, from (38) follows that

$$\theta[{}^m_n]([\,{}^{m'/2}_{n'/2}\,]) = 0 \tag{41}$$

if $\langle m + m', n + n'\rangle \equiv 1 \bmod 2$, $m, m', n, n' \in \mathbb{Z}$; also for $j = 1, 2$ and $z = (z_1, z_2) \in \mathbb{C}^2$,

$$\frac{\partial\theta[{}^m_n](u)}{\partial u_j} = \left[\frac{\partial\theta[{}^{m+m'}_{n+n'}](z)}{\partial z_j}\Big|_{z=0} - \frac{m'_j}{2}\theta[{}^{m+m'}_{n+n'}]\right] \exp\langle\, \tfrac{-1}{4}\Pi m\,,\, m\,\rangle\ \ i^{\langle\, -m'\,, n'+n\,\rangle} \tag{42}$$

where $u = (u_1, u_2) = \frac{1}{2}[\,{}^{m'_1}_{n'_1}\ {}^{m'_2}_{n'_2}\,]$, $z = (z_1, z_2)$, $m, n, m', n' \in \mathbb{Z}^2$.

Following Mumford [13], we introduce the linear spaces

$$\mathcal{R}^2(\ {}^m_n\ ) = \left\{ \begin{array}{l} \text{entire functions} f : \mathbb{C}^2 \to \mathbb{C}, \text{ satisfying } \ f(u + 2\pi i N + 2\Pi M) \\ = f(u)\ \exp\,\big[-\,\langle\, 2\Pi M, M\,\rangle\, -2\,\langle\, u, M\,\rangle\, + \pi i\,\langle\, m, N\,\rangle\, - \pi i\,\langle\, n, M\,\rangle\,\big] \\ \text{for all } M, N \in \mathbb{Z}^2 \text{ and } u \in \mathbb{C}^2 \end{array} \right.$$

It follows from (37), (38) and (42) that

- the function $\theta[{}^m_n](z)\,\theta[{}^{m'}_{n'}](z) \in \mathcal{R}^2(\ {}^{m+m'}_{n+n'}\ )$
- the function $\theta[{}^m_n](z)\,\partial_V\theta[{}^{m'}_{n'}](z) - \theta[{}^{m'}_{n'}](z)\,\partial_V\theta[{}^m_n](z)\ \in\ \mathcal{R}^2(\ {}^{m+m'}_{n+n'}\ )$

where ∂_V is a differentiation with respect to an arbitrary direction $V \in \mathbb{C}^2$.

It is convenient to carry in the argument $u \in \mathbb{C}^2$ in the characteristics of the theta functions

$$\theta[{}^m_n](u)\,\theta[{}^{m'}_{n'}](u) \;=\; \theta[{}^{m+\alpha}_{n+\beta}]\,\theta[{}^{m'+\alpha}_{n'+\beta}]\,\mathrm{e}^Q \tag{43}$$

$$\theta[{}^m_n](u)\,\partial_V\theta[{}^{m'}_{n'}](u) - \theta[{}^{m'}_{n'}](u)\,\partial_V\theta[{}^m_n](u) \;=\; \Big(\theta[{}^{m+\alpha}_{n+\beta}]\,\partial_V\theta[{}^{m'+\alpha}_{n'+\beta}] - \theta[{}^{m'+\alpha}_{n'+\beta}]\,\partial_V\theta[{}^{m+\alpha}_{n+\beta}]\Big)\ \mathrm{e}^Q, \tag{44}$$

where $u = \frac{1}{2}[\,{}^\alpha_\beta\,]$ and $Q = -\,\langle\,\Pi m\,,\, m\,\rangle\, -\,\langle\,\Pi m'\,,\, m'\,\rangle\, -\,\langle\,\pi i\,\alpha\,,\, \beta + m + m'\,\rangle\,.$

The relations (37) – (44) will be used for the proofs in the next paragraphs. The fact that the space $\mathcal{R}^2(\ {}^m_n\ )$ is four–dimensional [13], will be used repeatedly in the form of

Proposition 7 *Let $f_1(u), \dots, f_5(u)$ are five arbitrary functions from the space $\mathcal{R}^2\left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right)$, $m, n \in \frac{1}{2}\mathbb{Z}^2$. Then, there exist five constants $c_1, \dots, c_5$, not all of them equal to zero, and such that for all $u \in \mathbb{C}^2$*

$$c_1 f_1(u) + \cdots + c_5 f_5(u) = 0. \tag{45}$$

3.2 Proof of the identity $\dot{\mathbf{x}} = \mathbf{x} \times \mathbf{p}$

Let $x_1, x_2, x_3, p_1, p_2, p_3$ are the defined in Theorem 2 expressions in theta functions. We shall prove that these six functions satisfy first three equations of the Clebsch system. Start with the proof of the following identities between theta functions and their derivatives.

Proposition 8 *Let $\dot{}$ denotes a differentiations with respect to an arbitrary, but fixed direction in $\mathbb{C}^2$. Then for all z and ν, the following identities are true:*

$$\dot\theta[{}^{00}_{11}](z)\,\theta[{}^{00}_{10}](z) - \theta[{}^{00}_{11}](z)\,\dot\theta[{}^{00}_{10}](z) = +P_1\,\theta[{}^{11}_{11}](z)\,\theta[{}^{11}_{10}](z) + Q_1\,\theta[{}^{01}_{10}](z)\,\theta[{}^{01}_{11}](z) \tag{46}$$

$$\dot\theta[{}^{10}_{11}](z)\,\theta[{}^{10}_{10}](z) - \theta[{}^{10}_{11}](z)\,\dot\theta[{}^{10}_{10}](z) = +P_1\,\theta[{}^{01}_{11}](z)\,\theta[{}^{01}_{10}](z) + Q_1\,\theta[{}^{11}_{10}](z)\,\theta[{}^{11}_{11}](z) \tag{47}$$

$$\dot\theta[{}^{11}_{11}](\nu)\,\theta[{}^{11}_{10}](\nu) - \theta[{}^{11}_{11}](\nu)\,\dot\theta[{}^{11}_{10}](\nu) = +P_1\,\theta[{}^{00}_{11}](\nu)\,\theta[{}^{00}_{10}](\nu) + Q_1\,\theta[{}^{10}_{10}](\nu)\,\theta[{}^{10}_{11}](\nu) \tag{48}$$

$$\dot\theta[{}^{01}_{11}](\nu)\,\theta[{}^{01}_{10}](\nu) - \theta[{}^{01}_{11}](\nu)\,\dot\theta[{}^{01}_{10}](\nu) = +P_1\,\theta[{}^{10}_{11}](\nu)\,\theta[{}^{10}_{10}](\nu) + Q_1\,\theta[{}^{00}_{10}](\nu)\,\theta[{}^{00}_{11}](\nu) \tag{49}$$

$$\dot\theta[{}^{11}_{11}](z)\,\theta[{}^{00}_{10}](z) - \theta[{}^{11}_{11}](z)\,\dot\theta[{}^{00}_{10}](z) = +P_2\,\theta[{}^{01}_{10}](z)\,\theta[{}^{10}_{11}](z) + Q_2\,\theta[{}^{00}_{11}](z)\,\theta[{}^{11}_{10}](z) \tag{50}$$

$$\dot\theta[{}^{01}_{11}](z)\,\theta[{}^{10}_{10}](z) - \theta[{}^{01}_{11}](z)\,\dot\theta[{}^{10}_{10}](z) = +P_2\,\theta[{}^{11}_{10}](z)\,\theta[{}^{00}_{11}](z) + Q_2\,\theta[{}^{10}_{11}](z)\,\theta[{}^{01}_{10}](z) \tag{51}$$

$$\dot\theta[{}^{01}_{10}](\nu)\,\theta[{}^{10}_{11}](\nu) - \theta[{}^{01}_{10}](\nu)\,\dot\theta[{}^{10}_{11}](\nu) = -P_2\,\theta[{}^{11}_{11}](\nu)\,\theta[{}^{00}_{10}](\nu) - Q_2\,\theta[{}^{10}_{10}](\nu)\,\theta[{}^{01}_{11}](\nu) \tag{52}$$

$$\dot\theta[{}^{11}_{10}](\nu)\,\theta[{}^{00}_{11}](\nu) - \theta[{}^{11}_{10}](\nu)\,\dot\theta[{}^{00}_{11}](\nu) = -P_2\,\theta[{}^{01}_{11}](\nu)\,\theta[{}^{10}_{10}](\nu) - Q_2\,\theta[{}^{00}_{10}](\nu)\,\theta[{}^{11}_{11}](\nu) \tag{53}$$

$$\dot\theta[{}^{01}_{10}](z)\,\theta[{}^{00}_{10}](z) - \theta[{}^{01}_{10}](z)\,\dot\theta[{}^{00}_{10}](z) = +P_3\,\theta[{}^{00}_{11}](z)\,\theta[{}^{01}_{11}](z) + Q_3\,\theta[{}^{11}_{11}](z)\,\theta[{}^{10}_{11}](z) \tag{54}$$

$$\dot\theta[{}^{11}_{10}](z)\,\theta[{}^{10}_{10}](z) - \theta[{}^{11}_{10}](z)\,\dot\theta[{}^{10}_{10}](z) = +P_3\,\theta[{}^{10}_{11}](z)\,\theta[{}^{11}_{11}](z) + Q_3\,\theta[{}^{01}_{11}](z)\,\theta[{}^{00}_{11}](z) \tag{55}$$

$$\dot\theta[{}^{00}_{11}](\nu)\,\theta[{}^{01}_{11}](\nu) - \theta[{}^{00}_{11}](\nu)\,\dot\theta[{}^{01}_{11}](\nu) = -P_3\,\theta[{}^{01}_{10}](\nu)\,\theta[{}^{00}_{10}](\nu) - Q_3\,\theta[{}^{10}_{10}](\nu)\,\theta[{}^{11}_{10}](\nu) \tag{56}$$

$$\dot\theta[{}^{10}_{11}](\nu)\,\theta[{}^{11}_{11}](\nu) - \theta[{}^{10}_{11}](\nu)\,\dot\theta[{}^{11}_{11}](\nu) = -P_3\,\theta[{}^{11}_{10}](\nu)\,\theta[{}^{10}_{10}](\nu) - Q_3\,\theta[{}^{00}_{10}](\nu)\,\theta[{}^{01}_{10}](\nu) \tag{57}$$

where

$$P_1 = -\frac{\dot\theta[{}^{11}_{10}]\,\theta[{}^{11}_{11}]}{\theta[{}^{00}_{11}]\,\theta[{}^{00}_{10}]}, \qquad P_2 = \frac{\dot\theta[{}^{10}_{11}]\,\theta[{}^{01}_{10}]}{\theta[{}^{11}_{11}]\,\theta[{}^{00}_{10}]}, \qquad P_3 = \frac{\dot\theta[{}^{01}_{11}]\,\theta[{}^{00}_{11}]}{\theta[{}^{01}_{10}]\,\theta[{}^{00}_{10}]},$$

$$Q_1 = \frac{\dot\theta[{}^{01}_{11}]\,\theta[{}^{01}_{10}]}{\theta[{}^{00}_{11}]\,\theta[{}^{00}_{10}]}, \qquad Q_2 = -\frac{\dot\theta[{}^{11}_{10}]\,\theta[{}^{00}_{11}]}{\theta[{}^{11}_{11}]\,\theta[{}^{00}_{10}]}, \qquad Q_3 = -\frac{\dot\theta[{}^{10}_{11}]\,\theta[{}^{11}_{11}]}{\theta[{}^{01}_{10}]\,\theta[{}^{00}_{10}]} \tag{58}$$

are constants.

Proof: Begin with the proof of (46). Consider the functions
$f_1 = \dot\theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](z) - \theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](z)\,\dot\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](z)$, $f_2 = \theta[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](z)$, $f_3 = \theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](z)$,
$f_4 = \theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](z)$ and $f_5 = \theta(z)\,\theta[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}](z)$. According to Proposition 7, these functions are from $\mathcal{R}^2(\begin{smallmatrix}0&0\\0&1\end{smallmatrix})$ and are linearly dependent: $c_1 f_1(z) + \cdots + c_5 f_5(z) = 0$ for certain $(c_1, \ldots, c_5) \neq (0, \ldots, 0)$ and all $z \in \mathbb{C}^2$. Put $z = z_1 = [\begin{smallmatrix}1/2&0\\0&0\end{smallmatrix}]$. Then $f_1(z_1) = \cdots = f_4(z_1) = 0 \neq f_5(z_1)$, so that $c_5 = 0$. For $z = z_2 = 0$ one gets $f_4(z_2) \neq 0$ and $f_5(z_2) \neq 0$, so that $c_4 = 0$. Next, we put $z = z_3 = [\begin{smallmatrix}1/2&1/2\\0&0\end{smallmatrix}]$ and obtain $c_3 = -P_1 c_1$. Finally, for $z = z_4 = [\begin{smallmatrix}0&1/2\\0&0\end{smallmatrix}]$, we obtain $c_2 = -Q_1 c_1$. The identity (46) is proved.

The proofs of the rest identities from the proposition are similar. □

The next step is the proof of six new identities.

Proposition 9 *In the conditions of Proposition 7, the following identities are true*

$$\begin{aligned}
&\theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](\nu)\,\left\{\dot\theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](z) - \theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](z)\,\dot\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](z)\right\}\\
+\ &\theta[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](\nu)\,\left\{\dot\theta[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](z) - \theta[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}](z)\,\dot\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](z)\right\}\\
=\ &\theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](z)\,\left\{\dot\theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](\nu) - \theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](\nu)\,\dot\theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](\nu)\right\}\\
+\ &\theta[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](z)\,\left\{\dot\theta[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](\nu) - \theta[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}](\nu)\,\dot\theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](\nu)\right\}
\end{aligned}\tag{59}$$

$$\begin{aligned}
&\theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](\nu)\,\left\{\dot\theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](z) - \theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](z)\,\dot\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](z)\right\}\\
+\ &\theta[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](\nu)\,\left\{\dot\theta[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](z) - \theta[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}](z)\,\dot\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](z)\right\}\\
=\ &\theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}](z)\,\left\{\dot\theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}](\nu) - \theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](\nu)\,\dot\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](\nu)\right\}\\
+\ &\theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](z)\,\left\{\dot\theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](\nu) - \theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](\nu)\,\dot\theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](\nu)\right\}
\end{aligned}\tag{60}$$

$$\begin{aligned}
&\theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](\nu)\,\left\{\dot\theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](z) - \theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](z)\,\dot\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](z)\right\}\\
+\ &\theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](\nu)\,\left\{\dot\theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](z) - \theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](z)\,\dot\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](z)\right\}\\
=\ &\theta[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}](z)\,\left\{\dot\theta[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}](\nu) - \theta[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}](\nu)\,\dot\theta[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}](\nu)\right\}\\
+\ &\theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](z)\,\left\{\dot\theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](\nu) - \theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](\nu)\,\dot\theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](\nu)\right\}
\end{aligned}\tag{61}$$

$$\begin{aligned}
&\theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](\nu)\,\left\{\dot\theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](z) - \theta[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}](z)\,\dot\theta[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}](z)\right\}\\
+\ &\theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](\nu)\,\left\{\dot\theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](z) - \theta[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}](z)\,\dot\theta[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}](z)\right\}\\
=\ &\theta[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](z)\,\left\{\dot\theta[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](\nu) - \theta[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}](\nu)\,\dot\theta[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}](\nu)\right\}\\
+\ &\theta[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}](z)\,\theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](z)\,\left\{\dot\theta[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}](\nu)\,\theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](\nu) - \theta[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}](\nu)\,\dot\theta[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}](\nu)\right\}
\end{aligned}\tag{62}$$

$$
\begin{aligned}
& \theta\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right](\nu)\ \left\{\dot\theta\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right](z)\,\theta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right](z)-\theta\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right](z)\,\dot\theta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right](z)\right\}\\
+\ & \theta\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right](\nu)\ \left\{\dot\theta\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right](z)\,\theta\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right](z)-\theta\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right](z)\,\dot\theta\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right](z)\right\}\\
=\ & \theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](z)\,\theta\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](z)\ \left\{\dot\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](\nu)-\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\,\dot\theta\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](\nu)\right\}\\
+\ & \theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](z)\,\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](z)\ \left\{\dot\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)-\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](\nu)\,\dot\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)\right\}
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
& \theta\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right](\nu)\ \left\{\dot\theta\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right](z)\,\theta\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right](z)-\theta\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right](z)\,\dot\theta\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right](z)\right\}\\
+\ & \theta\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right](\nu)\ \left\{\dot\theta\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right](z)\,\theta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right](z)-\theta\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right](z)\,\dot\theta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right](z)\right\}\\
=\ & \theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](z)\,\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](z)\ \left\{\dot\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](\nu)-\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\,\dot\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](\nu)\right\}\\
+\ & \theta\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](z)\,\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](z)\ \left\{\dot\theta\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)-\theta\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](\nu)\,\dot\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)\right\}
\end{aligned}
\tag{64}
$$

Proof: The identity (59) is the sum of (46) multiplied by $\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right](\nu)$, (47) multiplied by $\theta\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)$, (48) multiplied by $-\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](z)\theta\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right](z)$ and (49) multiplied by $-\theta\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right](z)\theta\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](z)$.

The identities (60)–(64) can be proved in a similar manner. $\square$

Now, it is easy to prove

Proposition 10 *The equation $\dot x = x \times p$ is fulfilled (even when the differentiation is with respect to an arbitrary direction in $\mathbb{C}^2$).*

Proof. The sum of (59) and (60) is exactly $\dot x_1 = p_3 x_2 - p_2 x_3$. Next, summing up (61) and (62) we get $\dot x_2 = p_1 x_3 - p_3 x_1$. Finally, sum of (63) and (64) gives $\dot x_3 = p_2 x_1 - p_1 x_2$. This proves the proposition. $\square$

3.3 Proof of the identity $\dot{\mathbf{p}} = \mathbf{x} \times \mathbf{A}\mathbf{x}$

We shall check that the functions $p_j(t)$ and $x_j(t)$ defined in Theorem 2 satisfy the identity $\dot p = x \times Ax$.

We have two types of differentiations for the expressions of the kind $\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z)$ in the formulae in Theorem 2. Denote by $\dot{}$ the directional derivative with respect to U of the theta function with argument z:

$$
\begin{aligned}
\left\{\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z)\right\}^{\cdot} &\overset{def}{=} \theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\,\dot\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z) = \theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\,\partial_U\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z)\\
\left\{\partial_U\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z)\right\}^{\cdot} &\overset{def}{=} \partial_U\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\,\dot\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z) = \partial_U\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\,\partial_U\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z)
\end{aligned}
$$

According to the definition of $z = z_0 + tU$ from Theorem 2, the differentiation $\dot{}$ is the time t differentiation.

Denote by $'$ the directional derivative with respect to U of the theta functions with argument ν:

$$
\begin{aligned}
\left\{\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z)\right\}' &\overset{def}{=} \theta'\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z) = \partial_U\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z)\\
\left\{\partial_U\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z)\right\}' &\overset{def}{=} \theta''\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z) = \partial_U^2\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right](z)
\end{aligned}
$$

Recall that W denotes the common denominator of the functions $x_1, x_2, x_3, p_1, p_2, p_3$, see Theorem 2. Next proposition explains the choice of the vector U in Theorem 2.

Proposition 11 *The function $J = W'/W$ does not depend on the time t.*

Proof. From the definition of U it follows that

$$\partial_U \theta[{}^{00}_{10}](\nu) = \theta'[{}^{00}_{10}](\nu) = \theta[{}^{00}_{10}](\nu) \left\{ \frac{\partial\theta[{}^{10}_{10}](\nu)}{\partial\nu_2}\frac{\partial\theta[{}^{00}_{10}](\nu)}{\partial\nu_1} - \frac{\partial\theta[{}^{00}_{10}](\nu)}{\partial\nu_2}\frac{\partial\theta[{}^{10}_{10}](\nu)}{\partial\nu_1} \right\},$$

$$\partial_U \theta[{}^{10}_{10}](\nu) = \theta'[{}^{10}_{10}](\nu) = \theta[{}^{10}_{10}](\nu) \left\{ \frac{\partial\theta[{}^{00}_{10}](\nu)}{\partial\nu_2}\frac{\partial\theta[{}^{10}_{10}](\nu)}{\partial\nu_1} - \frac{\partial\theta[{}^{10}_{10}](\nu)}{\partial\nu_2}\frac{\partial\theta[{}^{00}_{10}](\nu)}{\partial\nu_1} \right\}.$$

Then,

$$\frac{\theta'[{}^{00}_{10}](\nu)}{\theta[{}^{00}_{10}](\nu)} = \frac{\theta'[{}^{10}_{10}](\nu)}{\theta[{}^{10}_{10}](\nu)} \overset{def}{=} J(\nu) \tag{65}$$

and

$$\frac{W'}{W} = \frac{\theta'[{}^{00}_{10}](\nu)\,\theta[{}^{00}_{10}](z) - \theta'[{}^{10}_{10}](\nu)\,\theta[{}^{10}_{10}](z)}{\theta[{}^{00}_{10}](\nu)\,\theta[{}^{00}_{10}](z) - \theta[{}^{10}_{10}](\nu)\,\theta[{}^{10}_{10}](z)} = J(\nu) = J$$

The proposition is proved. □

According to the definition of x_j, p_j and a_j in Theorem 2 and the differentiations $\cdot$ and $'$

$$p = -ix' - iJx + hx = -ix' - i\hat{J}x\,, \qquad i = \sqrt{-1}\,, \tag{66}$$

$$a_j = \frac{-ip'_j}{x_j}, \qquad j = 1, 2, 3\,, \qquad \text{i.e.} \qquad p' = iAx\ . \tag{67}$$

We proved that

$$\dot{x} = x \times p = -ix \times x'\ . \tag{68}$$

But the formulae for x_1, x_2 and x_3 are symmetric with respect to ν and z, so a symmetric identity to (68) is valid:

$$x' = -ix \times \dot{x}\ .$$

Now, we can calculate

$$\begin{aligned} \dot{p} &\overset{(66)}{=} -i(x' + \hat{J}x)^{\cdot} = -i(\dot{x}' + \hat{J}\dot{x}) \\ &\overset{(68)}{=} (x' \times x)' + (\hat{J}x' \times x) = x'' \times x + (\hat{J}x' \times x) = (x' + \hat{J}x)' \times x \\ &= ip' \times x \overset{(67)}{=} x \times Ax\ . \end{aligned}$$

The identity $\dot{p} = x \times Ax$ is proved.

3.4 Proof that $\mathbf{a_j}$ are constants

It remains to prove that a_1, a_2, a_3 do not depend on t.

Proposition 12 *The following identities hold*

$$\Big(\ln\big(\theta'\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)-\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\theta'\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)\big)\Big)'=\Big(\ln\big(\theta'\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](\nu)-\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](\nu)\theta'\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](\nu)\big)\Big)'$$

$$=\Big(\ln\big(\theta'\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right](\nu)-\theta\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right](\nu)\theta'\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right](\nu)\big)\Big)'=\Big(\ln\theta(\nu)\theta\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right](\nu)\Big)'\stackrel{def}{=}K,$$

see the definition of K in Theorem 2.

Proof: The functions $f_1(\nu)=\theta'\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)-\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\theta'\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)$, $f_2(\nu)=\theta(\nu)\theta\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right](\nu)$, $f_3(\nu)=\theta'\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right](\nu)-\theta\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right](\nu)\theta'\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right](\nu)$, $f_4(\nu)=\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)$, $f_5(\nu)=\theta\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](\nu)$ are from $\mathcal{R}^2_\Pi\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and according to Proposition 7 they are linearly dependent: $c_1f_1(\nu)+\cdots+c_5f_5(\nu)=0$, $\nu\in\mathbb{C}^2$. Putting $\nu=\left[\begin{smallmatrix}0&1/2\\1/2&0\end{smallmatrix}\right]$, we get $c_5=0$. Then, we put $\nu=\left[\begin{smallmatrix}0&0\\1/2&0\end{smallmatrix}\right]$ and get $c_4=0$. If $c_1=0$, then after putting $\nu=0$ we get $c_3=0$. But this gives also $c_2=0$, which is impossible since at least one $c_j\neq 0$. Therefore, we can consider $c_1=1$ and $f_1(\nu)=c_2f_2(\nu)+c_3f_3(\nu)$. According to the Proposition 7 we have $f_3(\nu)\equiv 0$ due to the special choice of the differentiation $'$. Then, $f_1(\nu)=c_2\theta(\nu)\theta\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right](\nu)$ and

$$K=\Big(f_1(\nu)\Big)'=\Big(\ln c_2\,\theta(\nu)\theta\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right](\nu)\Big)'=\Big(\ln\big(\theta'\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)-\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\theta'\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)\big)\Big)'.$$

The rest equalities are proved in a similar way. The proposition is proved. □

As a consequence from the Proposition 9, we obtain the identities

$$\frac{\theta''\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)-K\theta'\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)}{\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)}=\frac{\theta''\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)-K\theta'\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)}{\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)}=a_1-a_0 \tag{69}$$

$$\frac{\theta''\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](\nu)-K\theta'\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](\nu)}{\theta\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right](\nu)}=\frac{\theta''\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](\nu)-K\theta'\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](\nu)}{\theta\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right](\nu)}=a_2-a_0 \tag{70}$$

$$\frac{\theta''\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right](\nu)-K\theta'\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right](\nu)}{\theta\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right](\nu)}=\frac{\theta''\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right](\nu)-K\theta'\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right](\nu)}{\theta\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right](\nu)}=a_3-a_0\ , \tag{71}$$

see Theorem 2.

Finally, we check that $a_1=-ip_1'/x_1$. Indeed, according to (69)

$$\frac{-ip_1'}{x_1}=a_1-a_0=\frac{\left\{\theta''\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)-K\theta'\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\right\}\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](z)+\left\{\theta''\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)-K\theta'\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)\right\}\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](z)}{\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right](z)+\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](\nu)\,\theta\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right](z)}.$$

This coincides with the constant a_1, defined in Theorem 2.

Similarly, one checks that $a_2=-ip_2'/x_2$ and $a_3=-ip_3'/x_3$ coincide with the definitions in Theorem 2. Theta constants $a_j(\nu)$ do not depend on the time t, because K and a_0 do not depend on t. Therefore, we proved that A does not depend on t.

So, checking the validity of the identities $\dot{x}=x\times p$, $\dot{p}=x\times Ax$ and $\dot{A}=0$, we obtain another proof of Theorem 2.

4 Closed geodesics on the 3 - axial ellipsoid

Let us change the variables in the Neumann system as follows

$$\begin{aligned} \xi_1(t) &= \frac{x_1(t)}{\sqrt{a_1}} \qquad & \eta_1 &= \sqrt{a_1}\, p_1(t) \\ \xi_2(t) &= \frac{x_2(t)}{\sqrt{a_2}} \qquad & \eta_2 &= \sqrt{a_2}\, p_2(t) \\ \xi_3(t) &= \frac{x_3(t)}{\sqrt{a_3}} \qquad & \eta_3 &= \sqrt{a_3}\, p_3(t)\,. \end{aligned} \tag{72}$$

Then the vector $\xi(t) = \big(\xi_1(t), \xi_2(t), \xi_3(t)\big)$ lives on the ellipsoid

$$E \ : \ \frac{\xi_1^2}{a_1} + \frac{\xi_2^2}{a_2} + \frac{\xi_3^2}{a_3} = 1\,.$$

So, $\xi(t)$ and $\eta(t) = \big(\eta_1(t), \eta_2(t), \eta_3(t)\big)$ define the geodesic flow on the ellipsoid E up to a preparametrization of the time t. But we are interesting in the closed geodesics, so we don't need to know this preparametrization explicitly.

The ellipsoid E is a real one, that is $\xi_i \in \mathbb{R},\ i = 1, 2, 3,\ t \in \mathbb{R}$. It is easy to check that the matrix B has to be real. Using the periodicity laws for the theta functions we obtain that

$$\begin{aligned} & \text{a geodesic } \big(\xi(t), \eta(t)\big) \text{ on } E \text{ is closed} \\ \Longleftrightarrow\ & \xi(t) \text{ and } \eta(t) \text{ are periodic functions} \\ \Longleftrightarrow\ & \exists\, T > 0\,,\ T \in \mathbb{R} : \xi(t+T) = \xi(t),\ \eta(t+T) = \eta(t) \text{ for every } t \in \mathbb{R} \\ \Longleftrightarrow\ & \exists\, T > 0 \text{ and } \exists\, N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix},\ N_{1,2} \in \mathbb{Z} : TU = BN\,, \\ & U = (U_1, U_2) \text{ is the winding vector from Theorem 3} \\ \Longleftrightarrow\ & \text{the number} \end{aligned}$$

$$\mathcal{D} = \frac{B_{11}\,\frac{\partial}{\partial \nu_1}\theta\big[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\big](0, B) + B_{12}\,\frac{\partial}{\partial \nu_2}\theta\big[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\big](0, B)}{B_{21}\,\frac{\partial}{\partial \nu_1}\theta\big[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\big](0, B) + B_{22}\,\frac{\partial}{\partial \nu_2}\theta\big[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\big](0, B)}$$

is a rational one.

The number $\mathcal{D}$ depends only on the constants B_{11}, B_{12} and B_{22}, i.e $\mathcal{D}$ is a function of the matrix $B \in \mathcal{H}_2^{\mathbb{R}}$, where

$$\mathcal{H}_2^{\mathbb{R}} = \left\{ B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix},\ B_{js} \in \mathbb{R},\ B < 0 \right\}.$$

$\mathcal{D} = \mathcal{D}(B)$ is not a constant, so the closed geodesics are dense set in $\mathbb{R}^3$ (or the set of those B which correspond to the closed geodesics, is dense in $\mathcal{H}_2^{\mathbb{R}}$).

In the singular case

$$B = \begin{pmatrix} -\infty & c \\ c & -\infty \end{pmatrix},\quad c \in \mathbb{R},$$

E becomes a sphere and all geodesics are closed.

Unfortunately, the check whether the number $\mathcal{D}(B)$ is rational, is not an effective operation due to the infinite summation.

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