Pade's Approximation of Sampled Data Controllers

Zenith E. Vivas[(1)], and José J. Ferrer[(2)].

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Abstract

In this work we consider a sampled data control system, consisting of a continuous time plant and a controller, which is designed as: A uniform sampler plus a discrete time controller plus a zero order hold synchronizer. So, after fixing a sampling period $h > 0$, and in order to take into account the intersampling behaviour of the plant, we apply the lifting technique developed in [1]. Next, we consider certain output feedback conditions to construct a biparametric input output operator for the lifted system. Finally, after considering a representation of the family of non-linear and time-varying controllers, we come to our main objective of approaching any infinite-dimensional stabilizing controller by a discrete time but finite dimensional stabilizing one, all that by exhibiting an algorithm based on the theory of Pade's approximating polynomials.

Key words: hybrid stability, inner and outer functional, lifting technique, Pade's approximating polynomials.

(1) Institut fűr Mathematik der Humboldt Universität, Berlin, Germany.
(2) Universidad Simón Bolívar, Departamento de Sistemas y Control de Procesos, Caracas, Venezuela.

1 Introduction

The sampled data control systems have experienced great attention from the scientific community during the last few years. In fact, the main reason for that consists in the increased use of digital computers as control elements in industry. But, although it is true that digital controllers attain satisfactory results, it is also true that they can not avoid the presence of hidden oscillations to satisfy a certain performance criterion, such as disturbance rejection, at all. So we propose a way to design robust discrete controllers that do not only have a knowledge considering the intersampling behavior of the plant, but also achieve better results than the discrete controllers implemented in industry.

In Section 2, we introduce the setup for sampled data systems to be considered, some preliminaries, basic definitions and notation. Moreover, after fixing a sampling period $h > 0$, and in order to take into account the intersampling behavior of the plant, we consider a discrete representation for the continuous system by using the *lifting* technique [1], which sectionates every continuous time response of the system and converts it into a sequence of pieces of signals defined over every $[kh, (k+1)h)$ with $k \in \mathbb{Z}$, and hence, at every stage k, the lifted signals lie in an infinite-dimensional vector space. In Section 3, we consider certain output feedback conditions to construct a biparametric input-output operator for the lifted system. In Section 4 we introduce a representation of the family of non-linear and time-varying controllers, and next develop an algorithm to approximate sampled data controllers by Pade's polynomials. In Section 5, we deal with a numerical example, by considering the model of a stable stirred tank and next introducing some exogenous signals; So, applying the developed technology, we achieve some satisfactory stability results. In Section 6, we present some conclusions and remarks.

2 Basic Definitions and System Representation

Let us consider the sampled data system shown in Figure 1, where $\sum_{\mathtt{P}}$ represents a continuous linear time-invariant system, called the *Plant*; $\sum_{\mathtt{C}}$ represents a discrete linear constant system, called the *Controller*. The blocks marked by S and H represent a uniform *Sampler* and a *Zero Order Hold* (zoh),

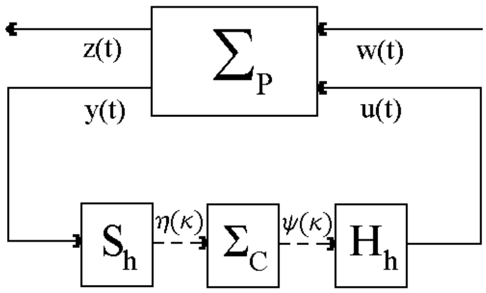

Figure 1: Sampled Data System

respectively, both of them operate synchronously every $h > 0$ units of time. Let us suppose that the plant $\sum_{\mathtt{P}}$ is represented by a *state space model* with dynamic equations:

$$\left\{ \begin{array}{rcllll} \frac{d}{dt}x(t) & := & \mathtt{A}\; x(t) & + \mathtt{B}_1\; w(t) & + \mathtt{B}_2\; u(t) & \\ z(t) & := & \mathtt{C}_1\; x(t) & + \mathtt{D}_{11}\; w(t) & + \mathtt{D}_{12}\; u(t) & \\ y(t) & := & \mathtt{C}_2\; x(t) & + \mathtt{D}_{21}\; w(t) & + \mathtt{D}_{22}\; u(t) & , \end{array} \right. \tag{1}$$

where the signals $w(t), z(t) \in \mathcal{L}_2\,[0, \infty)$ represent the exogenous input (or generalized disturbance vector) and the controlled output, respectively. The vector $x(t) \in \Re^n$, at time $t \geq 0$, represents the state of the system, and $u(t) \in \Re^{m_u}$, $y(t) \in \Re^{m_y}$ represent the input vector (or control law) and the measured signal at time $t \geq 0$, respectively. Here $\mathtt{A}$, $\mathtt{B}_1$, $\mathtt{B}_2$, $\mathtt{C}_1$,$\mathtt{C}_2$, $\mathtt{D}_{11}$, $\mathtt{D}_{12}$, $\mathtt{D}_{21}$ and $\mathtt{D}_{22}$ are constant real matrices of appropriate dimensions.

Next suppose that the controller $\sum_{\mathtt{C}}$ is given by:

$$\left\{ \begin{array}{rcl} \xi(k+1) & = & \mathtt{A_C}\,\xi(k) + \mathtt{B_C}\,\eta(k) \\ \psi(k) & = & \mathtt{C_C}\,\xi(k) + \mathtt{D_C}\,\eta(k)\ , \end{array} \right. \tag{2}$$

where at time $t = kh$, the vector $\xi(k)$ represents the state of $\sum_{\mathtt{C}}$; $\eta(k), \psi(k)$ represent the input and output signals of $\sum_{\mathtt{C}}$, respectively, and $\mathtt{A_C}$, $\mathtt{B_C}$, $\mathtt{C_C}$, $\mathtt{D_C}$ are constant real matrices of appropriate dimensions.

One of the main problems in designing a discrete controller that not only stabilizes the plant but also satisfies certain performance criteria, such as disturbance rejection, is to avoid the presence of hidden oscillations. Therefore, it is important to consider the intersampling behavior of the closed loop system during the design of such a controller. So, one technique that allows us to do that is the so-called *Lifting* (developed in [1]), which gives us a discrete representation of the continuous plant.

Definition 1 *Let $h > 0$ denote the intersampling size or sampling period. Let us define the lifting operator W by:*

$$W : \mathcal{L}_2\,[0,\infty) \to \ell_{\mathcal{L}_2[0,h)} \quad : \varphi(t) \to \varphi_k(\theta) \quad := \varphi(kh+\theta) \qquad \text{where} \quad t = kh+\theta \tag{3}$$

The images of the operator W will be denoted by $\overline{(.)}$.

We note that, as a consequence of applying this technique, the discrete model of the lifted plant $\sum_{\overline{\mathtt{P}}}$ (represented by the dynamic equations 1), becomes:

$$\left\{\begin{array}{rcll} x((k+1)h) & := & e^{\mathtt{A}h}x(kh)+ & \int\limits_0^h e^{\mathtt{A}(h-\theta)}\mathtt{B}_1 w(kh+\theta)d\theta + \phi(h)\mathtt{B}_2 u(kh) \\ z(kh+\vartheta) & := & \mathtt{C}_1 e^{\mathtt{A}\vartheta}x(kh)+ & \int\limits_0^\vartheta \mathtt{C}_1 e^{\mathtt{A}(\vartheta-\theta)}\mathtt{B}_1 w(kh+\theta)d\theta \\ & & + & [\mathtt{C}_1\ \phi(\vartheta)\ \mathtt{B}_2 + \mathtt{D}_{12}]\ \ u(kh) \end{array}\right. \tag{4}$$

where $\phi(\vartheta) := \int\limits_0^\vartheta e^{\mathtt{A}\,\zeta}\ d\zeta\ ,\ \vartheta \in [0,h]$

Evidently, the above equations can be restated in the following concise way:

$$\left\{\begin{array}{rcllll} x((k+1)h) & := & \overline{\mathtt{A}}\ x(kh)+ & \overline{\mathtt{B}}_1\ *\ w_k+ & \phi(h)\ \mathtt{B}_2\ u(kh) & \\ z(kh+\vartheta) & := & \overline{\mathtt{C}}_1\ x(kh)+ & \overline{\mathtt{D}}_{11}\ * w_k+ & \overline{\mathtt{D}}_{12}\ u(kh) & , \end{array}\right. \tag{5}$$

where $w_k(\vartheta) := w(kh+\vartheta)$; $z_k(\vartheta) := z(kh+\vartheta)$ with $\vartheta \in [0,h]$, and

$$\begin{array}{lcl} \overline{\mathtt{A}} & : & \Re^n \to \Re^n : (\overline{\mathtt{A}}\ x)(k) \ := e^{\mathtt{A}h}x(kh) \\ \overline{\mathtt{C}}_1 & : & \Re^n \to \mathcal{L}_2^{d1}\,[0,h] : \ (\overline{\mathtt{C}}_1\ x)(\vartheta) := \mathtt{C}_1 e^{\mathtt{A}\,\vartheta}x(\vartheta) \\ \overline{\mathtt{D}}_{12} & : & \Re^{m1} \to \mathcal{L}_2^{m1}\,[0,h] : \ (\overline{\mathtt{D}}_{12}\ u)(\vartheta) := [\mathtt{C}_1\ \phi(\vartheta)\ \mathtt{B}_2 + \mathtt{D}_{12}]\ \ u(kh) \end{array}$$

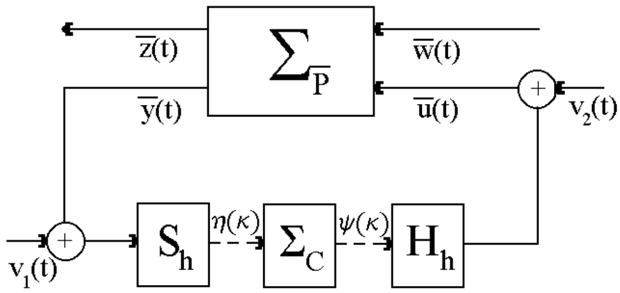

Figure 2: Sampled Data System after Lifting

but $\overline{\mathtt{B}}_1, \overline{\mathtt{D}}_{11}$ are the convolution operators:

$$\begin{aligned}
\overline{\mathtt{B}}_1 \ * w_k(h) \ : & \quad \mathcal{L}_2^{d1}[0,h] \rightarrow \Re^n : \overline{\mathtt{B}}_1(w) \ := \int_0^h e^{\mathtt{A}(h-\zeta)}\mathtt{B}_1 w(\zeta) d\zeta \\
\overline{\mathtt{D}}_{11} * w_k(\vartheta): & \quad \mathcal{L}_2^{d1}[0,h] \rightarrow \mathcal{L}_2^{d1}[0,h] : (\overline{\mathtt{D}}_{11}w)(\vartheta) := \int_0^{\vartheta} \mathtt{C}_1 e^{\mathtt{A}(\vartheta-\zeta)}\mathtt{B}_1 w(\zeta) d\zeta \quad .
\end{aligned}$$

After applying the *lifting* technique, the new plant looks different as shown in Figure 2. But, even though it resembles the standard discrete time state variable model, there is a big difference: Now the lifted model is an infinite-dimensional and time-varying one.

Note that for the lifted representation, the *Hilbert* spaces as well as $\mathcal{L}_2[0,h]$ and ℓ_2 for continuous and discrete signals, respectively, will be used. The signals v_1 and v_2 in Figure 2 were introduced in order to make the stability analysis and in order to take into account implementation errors.

Now, the resulting lifted model will be mathematically treated as a non-linear and time-varying system.

3 Input-Output Representation of the System $\sum_{\overline{\mathtt{P}}}$

In this section, we establish some general assumptions that will help us to preserve certain important conditions of the original plant, and further we give some basic definitions on stability of the system.
First, and in order to relax the notation, we will denote the input-output operator of the discrete time system $\sum_{\mathtt{C}}$ by C (i.e., $\mathrm{C}(z) := T_{\psi\eta}(z) : \psi(k) := T_{\psi\eta}(z)\eta(k)$). So, it is said that the (discrete time) controller $\sum_{\mathtt{C}}$ stabilizes the system $\sum_{\mathtt{P}}$ *hybridly* if and only if the (continuous) controller HCS stabilizes $\sum_{\mathtt{P}}$. Next, the sampling period h is said to be *non-pathological* with respect to a square matrix $\mathtt{A}$ if and only if, for any integer k and any eigenvalue μ of $\mathtt{A}$, $\mu + j\frac{2\pi k}{h}$ is not an eigenvalue of $\mathtt{A}$; Moreover, the sampled data system is said to be *uniformly exponentially* stable (UES) if there exist non-negative constants α and β such that $\|x(t)\| \leq \alpha e^{-\beta\, t}\ \forall t \geq t_0$ (it can be shown that in the case of *non-pathological* intersampling periods, *hybrid* stability and *uniform exponential* stability are equivalent). The system $\sum_{\mathtt{P}}$ is said to be *stabilizable* (or *detectable*) if there exists a matrix $\mathtt{M}$ (or $\mathtt{N}$) of appropriate dimensions such that $\mathtt{A}+\mathtt{BM}$ (or $\mathtt{A}+\mathtt{NC}$) is stable (i.e.,, all its eigenvalues are in the open left plane).

Theorem 2 *[3] Considering a non-pathological intersampling size $h > 0$, also the sampled data system described by the dynamic equations 1, and its state vector $x := (x, \mathtt{H}u, \mathtt{H}\sigma\xi)'$, where σ is the forward-shift operator (i.e.,, $y = \sigma u \Leftrightarrow y(k) := u(k+1)$). If the pairs $(\mathtt{A}, \mathtt{B}_2)$ and $(\mathtt{C}_2, A)$ are stabilizable and detectable, respectively, then the digital controller $\sum_{\mathtt{C}}$ stabilizes $\sum_{\mathtt{P}}$ hybridly iff its state $\overline{x}$ converges UES.*

From now on, we will always assume that the continuous plant $\sum_{\mathtt{P}}$ satisfies the conditions of the above theorem; It is a well known fact that the lifted plant $\sum_{\overline{\mathtt{P}}}$ (which is the discrete representation of $\sum_{\mathtt{P}}$) will always be stabilizable internally. Further, in order to make the equations appearing next tractable, we can suppose without loss of generality that $\mathtt{D}_{11}$=$\mathtt{D}_{22}$=0 (by using certain standard loops transformations we can always return to the original model); And so we can observe the systems $\overline{u} \rightarrow \overline{y}$ and $\overline{w} \rightarrow \overline{z}$ involved in $\sum_{\overline{\mathtt{P}}}$ in detail. Now we put the emphasis on the input-output representation $T_{\overline{zw}}$ for the system $\overline{w} \rightarrow \overline{z}$ and, in the next section, we will treat the measured system $\overline{u} \rightarrow \overline{y}$ in detail. But before giving such a representation, it is convenient to recall a result that relates discrete and continuous frequential parameters.

Definition 3 *Let g be a real function over $[0,h]$, then its finite Laplace transform $L_h(g)$, is defined by:*

$$L_h(g)(s) := \int_0^h e^{-s\theta} g(\theta) d\theta \ . \tag{6}$$

So, the Z - transform can be related with the (standard unilateral) Laplace transform as follows.

Lemma 1 *[7] Let g be a given function that satisfies $\| \ g \ \|_{[kh,(k+1)h]} \leq e^{\beta\, k}$ for certain constants α, $\beta > 0$, with $\| \ . \ \|_{[kh,(k+1)h]}$ as the $\mathcal{L}_2$ - norm at the interval $[kh,(k+1)h]$. Then the Laplace transform of g exists for any complex s such that $Re(s) > \beta$, and*

$$L_h(Z(g)(s))|_{z=e^{sh}} := Z(L_h(g)(s))|_{z=e^{sh}} := e^{sh} L(g)(s) \ .$$

Now as a consequence of the above concept, we can represent the system $\sum_{\overline{\mathtt{P}}}$ as an input-output operator $T_{\overline{zw}}$, which becomes obvious in the followimg result.

Theorem 4 *Given a sampled data system represented by the equations 5 such that the pairs $(\overline{\mathtt{A}}, \overline{\mathtt{B}}_2)$ and $(\overline{\mathtt{C}}_2, \overline{\mathtt{A}})$ are stabilizable and detectable, respectively, then*

$$T_{\overline{zw}} : \ell_{\mathcal{L}_2[0,h]} \to \ell_{\mathcal{L}_2[0,h]} : \overline{w} \to \overline{z}$$

can be represented by:

$$T_{\overline{zw}}(s) := p(s) \mathtt{C}_{ext} (e^{-sh} I - \overline{\mathtt{A}})^{-1} \mathtt{B}_{ext} + \mathtt{D}_{ext} \ , \tag{7}$$

where

$$\begin{array}{ll} p(s) := \frac{s}{1-e^{-sh}}; & \mathtt{B}_{ext} := \overline{\mathtt{B}} + \widehat{\eta}(s); \\ \mathtt{C}_{ext} := \widehat{\mathtt{C}}(s) \ ; & \mathtt{D}_{ext} := \overline{\mathtt{D}} + \mathtt{C}_1 \ \widehat{\eta}(s) \ . \end{array}$$

$\widehat{(.)}$ denotes the composition of the operators $L_h \circ Z$ and $\eta\,(.)$ is the convolution kernel that appears at the integral equations (i.e.,, $\eta\,(\vartheta, \theta) := e^{\mathtt{A}(\vartheta - \theta)} \mathtt{B}_1$).

The proof is extended and explained in [6], where the signal $\overline{y}$ is assumed to be available for output feedback (i.e.,, $\overline{u} := K \overline{y}$; $K \in \Re^{m_u \times m_y}(s)$), and the stabilizability and detectability criteria are also assumed, in order to solve the integral equations as Volterra ones.

4 Discrete Time Control

In this section we consider a representation of the hybrid stabilizing controllers for $\sum_{\overline{\mathrm{P}}}$ and approximate them by using a technique based on *Pade's Approximating Polynomials.*

4.1 Representation of Controllers

Let $\mathcal{G}$ represent the class of (causal) distributions g of the form:

$$g(t) := \begin{cases} g_0(t) + \sum_{k=1}^{\infty} g_k \delta(t - t_k) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{8}$$

with $g_k(t) \in \mathcal{L}_2^{m_u \times m_y}[0, h)$, $k \in \{0, 1, ...\}$ and

$$\sum_{k=1}^{\infty} \| g_k(t) \|_2 < \infty \; ; \; 0 \leq t_1 \leq t_2 \leq ...$$

It must be clear that $\mathcal{G}$ represents a class of non-linear, time-varying, causal and infinite-dimensional systems that are bounded from $\mathcal{L}_2^{m_y}[0,\infty) \rightarrow \mathcal{L}_2^{m_u}[0, \infty)$.

Definition 5 *Let us denote by $\widehat{\mathcal{G}}$ the image under the composition of the operators L_h and Z (or generalized Z-transform) of all elements of $\mathcal{G}$, i.e.,,*

$$\widehat{\mathcal{G}} = \left\{ \widehat{g}(s, z) : \widehat{g}(s, z) = \sum_{k=0}^{\infty} L_h(g_k(\theta)) z^{-k} \right\} \tag{9}$$

Note that $\widehat{\mathcal{G}}$ can be immersed in a subspace of the Hardy space $\mathcal{H}_2(\mathcal{D})$, where $\mathcal{D}$ is the unitary complex disc, and thus, as a consequence of the canonical representation theorem for $\mathcal{H}_2$-operators, it is not hard to see that, for every $\widehat{g}(s, z) \in \widehat{\mathcal{G}}$, there exist inner functionals $f_k^o(s)$ and exterior functionals F_k over $\mathcal{L}_2^{m_y}[0, \infty)$ such that:

$$\widehat{g}(s, z) := \sum_{j=1}^{\infty} f_k^o \mathtt{B}_k(z) F_k(s) \; z^j \; , \tag{10}$$

where $\mathtt{B}_k(z)$ is the Blaschke product,$\forall k = 0, 1, 2, ...$

Proposition 1 *Given a controller* $\sum_{\mathtt{C}}$ *that stabilizes* $\sum_{\mathtt{P}}$ *hybridly, let us consider the operator* $HCS(.)$ *:*

$$HCS(t) : \mathcal{L}_2^{m_y}[0,\infty) \to \mathcal{L}_2^{m_u}[0,\infty) \ .$$

Then $\widehat{HCS} \in \widehat{\mathcal{G}}$ *.*

Proof: Recall that by definition

$$\left\|\widehat{HCS}\right\|_2 := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|\widehat{HCS}(e^{i\theta})\right|^2 d\theta \right\}^{\frac{1}{2}} \quad ,$$

therefore,

$$\left\|\widehat{HCS}\right\|_2 \leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{\infty} \left|L_h(HCS_k(e^{i\theta}))z^k\right|^2 d\theta \right\}^{\frac{1}{2}} \quad .$$

On the other hand, since $\sum_{\mathtt{C}}$ stabilizes $\sum_{\mathtt{P}}$ hybridly, $\widehat{HCS}$ is a bounded operator with respect to the $\mathcal{H}_2$-norm. So, after choosing a non pathological sampling period $h > 0$, the sampled data system is UES, and by using Lemma 1, we have that the sum in the right-hand side of the above inequality is finite.

Next, by observing that $HCS(t) := HCS_k(t) := HCS(kh + \vartheta)$, where $t := kh + \vartheta$ with $\vartheta \in [0, h)$, is a non-linear, time-varying operator that lies in an infinite-dimensional space. Furthermore, we can always find a cannonical representation for every bounded linear operator $\widehat{HCS} \in \mathcal{H}_2^{m_y \times m_u}(\mathcal{D})$, where $\mathcal{D}$ is the unitary complex disc:

$$L_h(HCS_k(e^{i\theta})) := f_k^o(s)\mathtt{B}_k(z)F_k(s) \ ,$$

where $f_k^o(s)$ is an inner functional, $F_k(s)$ is an exterior functional and $\mathtt{B}_k(z)$ is a Blaschke product. $\triangle$

4.2 Finite Approximation of a Controller

Let us observe that, for every bounded linear operator $\widehat{H\mathcal{C}S} \in \mathcal{H}_2^{m_u \times m_y}(\mathcal{D})$, where $\mathcal{D}$ is the unitary disc, there always exists a canonical representation, and due to the fact that $\mathcal{H}_2(\mathcal{D})$ is a reflexive space, we can extend $\widehat{H\mathcal{C}S}$ to an $f^o \in \mathcal{L}_2^{m_y}[0,\infty)$ in the way we show next:

Proposition 2 *Given the bounded linear operator*

$$H\mathcal{C}S : \mathcal{L}_2^{m_y}[0,\infty) \to \mathcal{L}_2^{m_u}[0,\infty) : y(t) \to u(t) .$$

There exists an operator $f^{\,o}(s)$ over the space $\mathcal{L}_2^{m_y}[0,\infty)$, and an exterior functional $F(s,z) \in \mathcal{H}_2^{m_y \times m_u}(\mathcal{D})$ such that a canonical representation for $\widehat{H\mathcal{C}S}(s,z)$ is given by:

$$\widehat{H\mathcal{C}S}(s,z) := F(s,z)\, f^{\,o}(s) . \tag{11}$$

Furthermore, $\widehat{H\mathcal{C}S}$ is extended to this $f^{\,o}(s)$ over $\mathcal{L}_2^{m2}[0,\infty)$ as:

$$\widehat{H\mathcal{C}S}(y) := \frac{1}{2\pi}\int_0^{2\pi} y(e^{i\theta})F(s,z)\,\overline{f^{\,o}}(e^{i\theta})d\theta . \tag{12}$$

Proof: As a consequence of the canonical representation theorem, for every $L_h(H\mathcal{C}S_k(s))$ there exists an inner functional $f_k^o(s)$ over $\mathcal{L}_2^{m_y}[0,h)$, an exterior functional $F_k(s)$, and a Blaschke product $\mathtt{B}_k(z)$ such that:

$$L_h(H\mathcal{C}S_k(s)) := F_k(s)\mathtt{B}_k(z)f_k^o(s) .$$

Therefore

$$\widehat{H\mathcal{C}S}(s,z) := \sum_{k=0}^{\infty} F_k(s)\mathtt{B}_k(z)f_k^{\,o}(s)z^k .$$

Next, find $F(s,z)$, the Z-transform of $F_k(s)\mathtt{B}_k(z)$, and take $f^{\,o}(s)$ such that the following diagram commutes:

$$\begin{array}{ccc} Y(s) & \xrightarrow{\widehat{f_k^o}(s)} & U(s) \\ L_h \uparrow & & \uparrow L_h \\ y(kh+\vartheta) & \xrightarrow[f_k^o(s)]{} & u(kh+\vartheta) \end{array}$$

Finally, the extension is obtained immediately by applying the Ries'z factorization theorem. $\triangle$

Theorem 6 *Let us consider a continuous controller HCS that stabilizes the lifted system $\sum_{\overline{\mathrm{P}}}$ hybridly. Then, there exists a sequence of rational functions,* $\left\{\frac{p_k^1(z)}{p_k^2(z)}\right\}$, *$p_k^1\ (z), p_k^2\ (z)$ polynomials in z, that does not only approximate HCS but also stabilizes the lifted system $\sum_{\overline{\mathrm{P}}}$ hybridly.*

Proof: As a consequence of Beurling's theorem, every controller $\widehat{HCS} \in \widehat{\mathcal{G}}$ can be represented by a sequence $\{f_k^o\}_{k=0}^{\infty}$ of inner and bounded operators:

$$\widehat{HCS}(s,z) := \sum_{k=0}^{\infty} f_k^o(s) F_k(s,z)\ .$$

From the above relation, it follows that

$$\widehat{HCS}(s,z) \in \oplus_{k=1}^{\infty} f_k^o \mathcal{H}_2\ .$$

Now, given $f \in \mathcal{L}_2^{m_y}[0,\infty)$, let us denote by $\mathcal{P}[f]$ the space generated by f and z. So, remembering Beurling's theorem, we have as a consequence that $\mathcal{P}[f] := f\mathcal{H}_2$. In this sense the approximation by polynomials comes just right because of the fact that they are dense in $\mathcal{H}_2$, and then we can use a technique based on *Pade's approximating polynomials* in order to construct the required sequence.
Now, by taking a natural number m sufficiently large and considering the *m-th ratio of Pade's approximating polynomials* [5] at each k, we generate a sequence of polynomial ratios with degree at most m that approximates the controller $\widehat{HCS}$, i.e.,,

$$\widehat{HCS}(s,z) \cong \frac{p_0^1}{p_0^2}(z) + \sum_{k=1}^{\infty} \frac{p_k^1}{p_k^2}(z)\ . \tag{13}$$

Moreover, as this approximation is uniform and under the assumptions UES, the approximating controller, which lies in a finite dimensional space, stabilizes the system hibridly. $\triangle$

4.3 Pade's Approximating Polynomials

In this subsection we adapt Pade's approximation algorithm [2] and apply it to any stabilizing controller. So, let us observe the following algorithm:

PART I.

- **Step1:** Given a stabilizing controller:

$$\widehat{HCS}_k(s,z) := \frac{p_{k0}^1(s,z)}{p_{k0}^2(s,z)} + \sum_{j=1}^{\infty} \frac{p_{kj}^1(s,z)}{p_{kj}^2(s,z)} \,. \tag{14}$$

- **Step2:** Introduce the following reparametrization $s = e^{-zh}$ to obtain:

$$\widehat{HCS}_k(z) := \frac{p_{k0}^1(z)}{p_{k0}^2(z)} + \sum_{j=1}^{\infty} \frac{p_{kj}^1(z)}{p_{kj}^2(z)} \,. \tag{15}$$

- **Step3:** Expand around $z = 1$ in Taylor's series, to obtain:

$$\widehat{HCS}_k(z) \cong \sum_{j=1}^{\infty} q_{kj} z^j \,. \tag{16}$$

- **Step4:** Fix an order $r > 0$ for the approximating controller.

PART II.
In order to build the controller of reduced order r, to approximate $\widehat{HCS}_k(z)$, let us consider the following two subroutines:
a) Reduction of the numerator $N_r(z)$.

- **Step1:** Establish the initial moments $\forall i = 1, ..., r-1$ as:

$$n_{ki} := q_{ki} + \sum_{j=1}^{i} p_{kj}^2 q_{kj} \,. \tag{17}$$

- **Step 2:** Establish the numerator as:

$$N_r(z) := \sum_{i=1}^{r-1} n_{ki} z^i \,. \tag{18}$$

b) Reduction of the denominator $\mathtt{D}_r(z)$.

- **Step1:** Take m_1 and m_2 to be the integral parts of $n/2$ and $(n-1)/2$, respectively.

- **Step2:** Express the system p_{kj}^2 as $D(z) := \mathtt{D}_e(z) + \mathtt{D}_o(z)$, where

$$\begin{aligned} \mathtt{D}_e(z) &:= \prod_{j=1}^{m_1} (1 + \tfrac{z^2}{\zeta_j^2}) \\ \mathtt{D}_o(z) &:= p_{k1}^2 z \prod_{j=1}^{m_2} (1 + \tfrac{z^2}{\xi_j^2}) \ . \end{aligned} \tag{19}$$

- **Step3:** Sort ζ_j^2 and ξ_j^2 such that:

$$0 \prec \zeta_1^2 \prec \xi_1^2 \prec \zeta_2^2 \prec \xi_2^2 \prec \ldots$$

- **Step4:** Take m_{1r} and m_{2r} as the integral parts of $\frac{r}{2}$ and $\frac{r-1}{2}$, respectively.

- **Step5:** Discard the factors greater than ζ_r^2 and ξ_j^2, let:

$$\mathtt{D}_r(z) := \mathtt{D}_{er}(z) + \mathtt{D}_{or}(z) \ , \tag{20}$$

  where

$$\begin{aligned} \mathtt{D}_{er}(z) &:= \prod_{j=1}^{m_{1r}} (1 + \tfrac{z^2}{\zeta_j^2}) \\ \mathtt{D}_{or}(z) &:= p_{k1}^2 z \prod_{j=1}^{m_{2r}} (1 + \tfrac{z^2}{\xi_j^2}) \end{aligned}$$

5 Numerical Example

Let us consider a stirred tank (presented in [4]) with two incoming liquid flows $F_1(t)$ and $F_2(t)$, with material concentration c_1 and c_2 respectively. Let us denote by $F(t)$ and $c(t)$ the flow and the concentration of liquid coming out of the tank. Let us consider:

$$\left\{ \begin{array}{l} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} := \begin{bmatrix} -0.01 & 0 \\ 0 & -0.02 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -0.25 & 0.75 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} := \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{array} \right. \tag{21}$$

as nominal values of system's parameters. Let us take, moreover, into account thatthe concentrations c_1 and c_2 are perturbed by two exogenous signals $w_0(t)$ and $w_1(t)$, like fluctuation in the volume of the tank of 20 percent of the initial volume V_0 and a white noise with intensity 1, respectively. And considering further that these two signals incide on the process through the matrix:

$$\begin{pmatrix} 0 & 0 \\ 0.015 & 0.005 \end{pmatrix} \tag{22}$$

By using a non-pathological intersampling size $h = 0.1$, and considering the following constant state feedback matrix of gains:

$$\begin{pmatrix} 0.8049 & 0.5634 \\ 0.5806 & 0.7927 \end{pmatrix} \tag{23}$$

we obtain that $T_{z_0 w_0}(s, z)$ and $T_{z_1 w_1}(s, z)$ can be approximated [6] by another operators, which lies in a finite dimensional space:

$$T_{z_i w_i}(z) := \frac{z}{z-1} \mathtt{C}_f \left(zI - \overline{A} \right)^{-1} \mathtt{B}_f + \mathtt{D}_f \ , \ i = 0, 1 \tag{24}$$

where

$$T_{z_0 w_0}(z) := \frac{z}{z-1} \begin{pmatrix} 0.01 & 325 \\ 1 & 2.2\times10^4 \end{pmatrix} \begin{pmatrix} z-0.99 & -1 \\ -1 & z-0.99 \end{pmatrix}^{-1} \begin{pmatrix} 28.3 & -67.2 \\ -47 & -42.2 \end{pmatrix} + \begin{pmatrix} 1105 & 2525 \\ 7.6\times10^4 & 1.7\times10^5 \end{pmatrix}$$

and

$$T_{z_1 w_1}(z) := \frac{z}{z-1} \begin{pmatrix} 0.01 & 325 \\ 1 & 2.2\times10^4 \end{pmatrix} \begin{pmatrix} z-0.99 & -1 \\ -1 & z-0.99 \end{pmatrix}^{-1} \begin{pmatrix} 25.5 & -73.9 \\ -49 & -48.9 \end{pmatrix} + \begin{pmatrix} 173 & 353 \\ 1.1\times10^4 & 2.4\times10^4 \end{pmatrix}$$

But, since the plant is stabilizable and detectable, we can build a dynamic controller (linear state feedback + state observer) to replace the poles suitably and also reject a given perturbation signal. The general expression for this controller reads:

$$\widehat{HCS} := K'(sI - A_k - \mathtt{B}_k K' - L\mathtt{C}_k)^{-1} L \ . \tag{25}$$

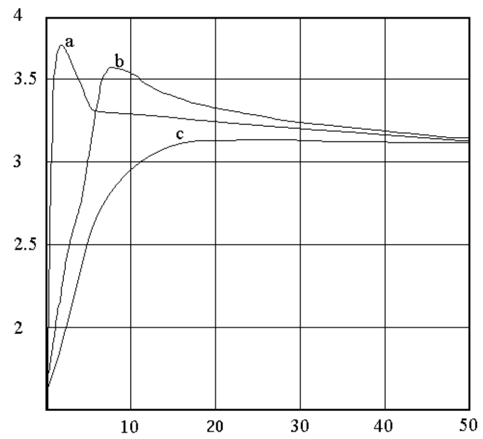

Figure 3: Natural Responses of the Plant

Hence, let us consider (in this case) the following continuous time controller:

$$\widehat{HCS}(s) := \begin{pmatrix} \frac{0.3333s+0.6333}{s^2+3.1352s+2.3755} & \frac{0.2405s+0.3405}{s^2+3.1352s+2.3755} \\ \\ \frac{-0.5522s-0.9539}{s^2+3.1352s+2.3755} & \frac{0.777s+0.957}{s^2+3.1352s+2.3755} \end{pmatrix} \tag{26}$$

This controller has an unknown discrete representation, $\widehat{HCS}(s,z)$, which lies in an infinite-dimensional space but it can be approximated by a sequence of discrete controllers which lie in a finite dimensional space. So, by using the procedure depicted above, and if we consider the full order model ($r = 2$), and the 2-nd Pade's approximating polynomial for the sampled data unknown controller, then we obtain:

$$\mathtt{C}_{.1,2}(z) := \begin{pmatrix} \frac{0.6333z^2-1.2666z+0.6333}{5.5107z^2-4.751z-0.7597} & \frac{0.3405z^2-0.681z+0.3405}{5.5107z^2-4.751z-0.7597} \\ \\ \frac{-0.9539z^2+1.9078z-0.9539}{5.5107z^2-4.751z-0.7597} & \frac{0.957z^2-1.914z+0.957}{5.5107z^2-4.751z-0.7597} \end{pmatrix}. \tag{27}$$

Next, in order to make some comparisons between this and previous methods, let us observe the natural responses of the plant (shown in Figure 3) if: a) the continuous controller (see eq. 26) is applied; b) this proposed controller (see eq. 27) is applied; and c) the continuous controller after discretizing it (by Tustin transformation) is applied.

6 Conclusions

In this work, we have shown that, given a suitable discrete representation for a continuous time system $\sum_{\mathrm{P}}$, the family of all sampled data stabilizing controllers is contained in the vector space: $\widehat{\mathcal{G}} = \left\{ \widehat{g}(s,z) := \sum_{k=0}^{\infty} L_h(g_k(\theta)) z^{-k} \right\}$, where L_h denotes the finite Laplace transform, and s, z are the continuous and discrete frequential parameters, respectively; These two parameters allowed us to model the closed loop system in the intersampling period. Moreover, since the stabilizing controllers lie in an infinite-dimensional space, we have shown a procedure (based on the theory of Pade's approximation) to build discrete and finite-dimensional controllers that approximate them to a given degree of accuracy, but which also stabilize the continuous time plant $\sum_{\mathrm{P}}$ hybridly.

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