A Non-linear Parabolic System Modeling Chemotaxis with Sensitivity Functions

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2nd August, 1999

Abstract

We study a model for chemotaxis on a general Lipschitz domain where the chemotatic response is specified by sensitivity functions. After finding a Lyapunov function for the system, we demonstrate existence of global solutions for different classes of sensitivity functions and show convergence to possibly non-trivial steady states.

1 Introduction

In this paper, we are going to study a mathematical model for chemotaxis. This phenomenon is the oriented motion of organisms sensitive to a concentration gradient of a chemical substance and appears in various biological processes. In particular, chemotaxis is known to be responsible for aggregation processes in the life cycle of certain unicellular organisms. We shall be concerned with the aggregation phase in the life cycle of the cellular slime molds *Dictyostelium discoideum* U moving towards regions of high concentration of the chemical substance cAMP (*cyclic adenosine monophosphate*) V, which they themselves produce. In times of shortage of food-supply, *Dictyostelium discoideum* cells appear that spontaneously secrete cAMP. Neighbouring cells now also start exuding the chemical in response to its increasing concentration. At the same time, the released cAMP is slowly destroyed by an enzyme. The amoebae form migrating multi-cellular slugs and move in streams towards the centre of highest concentration of cAMP. After a while, they come to a rest and erect themselves to a fruiting body, which aids the distribution of germinating spore cells.

Mathematical models for this phenomenon have been developed during recent years in order to predict aggregation patterns of cells as well as to test different biological hypotheses trying to explain chemotaxis. These models consist of two coupled non-linear partial differential equations of reaction-diffusion type based on the Keller-Segel [11] model and have proved to be of great mathematical interest in their own right. As a prototype, we state the following system for $U = U(t,x)$ and $V = V(t,x)$:

$$\begin{aligned} U_t &= \Delta U - \nabla(\chi U \nabla V) \\ V_t &= \alpha \Delta V - \beta V + \delta U \end{aligned} \tag{1}$$

on $(0,T) \times \Omega \subset I\!R \times I\!R^n$ with homogeneous Neumann boundary conditions

$$\nu \cdot \nabla U = \nu \cdot \nabla V = 0 \quad \text{on} \quad (0,T) \times \partial\Omega \tag{2}$$

(ν is the unit outer normal at points of $\partial\Omega$.) and initial conditions $U(0,x) = U_0(x)$ and $V(0,x) = V_0(x)$ for all $x \in \Omega$. χ, α, β and δ are usually positive constants.

At first, mathematicians studied a simplified version of (1) mostly on a two-dimensional smooth domain Ω proving existence of solutions and regularity properties, concentrating in particular on the possibility of blow-up at finite time. (See for example, Jäger and Luckhaus [10], Nagai [16], Mizutani and Nagai [14], Herrero and Velazquez [8], [9].) In 1996, Gajewski and Zacharias [6] and Nagai, Senba and Yoshida [17] first studied the fully instationary system (1) with (2).

We are going to study the fully instationary system (1) where the chemotactic coefficient χ and the δ in the production term for V are non-constant but of the form

$$\tilde{\chi}(V) := \chi S'(V) \quad \text{and} \quad \tilde{\delta}(V) := \delta S'(V). \tag{3}$$

The function S is called *sensitivity function* and specifies the ability of the amoebae U to sense the V-concentration. ($S'(V)$ denotes its first derivative with respect to V.) From the biological point of view, this model is more realistic as it has been observed that the cells do not react linearly to the V-concentration and in biological literature (see Murray [15], Schaaf [20]) , one finds the following forms of the sensitivity function:

$$S_1(V) = \frac{V}{1+cV}, \quad S_2(V) = \frac{V^2}{1+cV^2}, \quad S_3(V) = \log(V+c), \tag{4}$$

with constants $c \geq 1$. The above references suggest a chemotactic coefficient χ as in (3). In addition, the coefficient δ in the production term for V will in reality depend on the V-concentration, too. Rather than treating a more general production term, we confine ourselves to a case in which there exists a Lyapunov function for the system. Therefore, we assume the same V-dependence of χ and δ. (See Post [19] for a more detailed discussion.) We can now write our system as follows:

$$\begin{aligned} U_t &= \Delta U - \chi \nabla(U \nabla S(V)) \\ V_t &= \alpha \Delta V - \beta V + \delta U S'(V) \end{aligned} \tag{5}$$

in $(0,T)\times\Omega$, with Neumann boundary conditions

$$\nu\cdot\nabla U=\nu\cdot\nabla V=0 \quad \text{on } (0,T)\times\partial\Omega, \tag{6}$$

and initial values $U(0,x) = U_0(x)$, $V(0,x) = V_0(x)$. The system corresponds with general mathematical models for chemotaxis stated (but never studied) in literature.[†]

Although the sensitivity functions introduce additional non-linearities into the system, we will see that, at some stages, they facilitate proofs of higher regularity of the solutions.

We will treat a general bounded Lipschitz domain Ω, so that techniques for smooth domains cannot be applied. In the major part of the paper, we will have to confine ourselves to a two-dimensional setting, but, at some stages, we will obtain results for higher dimensional domains.

Formulating our results, we will always state the properties required of the sensitivity function S. The most general class of functions treated in this work will be the set

$$\mathcal{S} \;=\; \left\{S\in C^1(I\!R,I\!R): 0\le S(V),\ 0\le S'(V)\le C' \ \text{ for all } V\ge 0\right\}.$$

The main achievement of our work lies in the existence theorem (Theorem 5.1) of global solutions of system (5) with (6) on a general two-dimensional Lipschitz domain for different natural classes of sensitivity functions, which enables us to study the asymptotic solution behaviour. Moreover, we are able to prove convergence of the trajectories of the solutions to trivial *and* non-trivial steady states under varying conditions on the data of the system. The non-trivial stationary states seem to be of principle interest to us as their inhomogeneous distribution of the species can be viewed as a starting position for the erection of a fruiting body.

In this paper, we will principally concentrate on the cases of bounded sensitivity functions and the logarithmic sensitivity function $S(V) = \log(V + c)$. Note that equations (1) are the special case of the *direct measurement*, where $S(V) = S_0(V) = V$ in (5). See Post [19] for an enhancement of the results by Gajewski and Zacharias [6] and Nagai, Senba and Yoshida [17] for this (less realistic) case.

A main tool for nearly all our results is the function

$$F(U,V)=\int_\Omega\left\{U\log U-\chi US(V)+\frac{\chi}{2\delta}(\alpha|\nabla V|^2+\beta V^2)\right\}dx$$

which is shown to be a Lyapunov function for system (5), (6) for a general $S\in\mathcal{S}$ and in a possibly higher dimensional domain $\Omega\subset I\!R^n$. Note that, for the first time, we can obtain results for the fully instationary chemotaxis-system (with or without sensitivity function) on higher dimensional domains.

2 Preliminaries

We are going to work in a bounded domain $\Omega\subset I\!R^n$ with a Lipschitz continuous boundary $\partial\Omega$. We denote by $C^k(\bar\Omega)$, for k a non-negative integer, the usual spaces of k-times continuously

[†]See for example Childress and Percus [3] or Stevens [21].

differentiable functions on Ω and write $C(\bar\Omega) := C^0(\bar\Omega)$ and $C^\infty(\bar\Omega)$ for the space of infinitely often differentiable functions. By $L^p(\Omega)$ and $W^{1,p}(\Omega)$ with $1 \le p \le \infty$ we denote the Lebesgue spaces and Sobolev spaces of functions on Ω, where $H^1(\Omega) := W^{1,2}(\Omega)$. For $1 \le p \le \infty$, we write p' for its conjugate exponent satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. (The value of $\frac{1}{\infty}$ is defined to be 0.)

For a general Banach space X, $\|\cdot\|_X$ denotes its norm and X^* its dual, and the dual pairing between $f \in X^*$ and $g \in X$ will be denoted by $\langle f, g\rangle$. Furthermore, we write $L^p(0,T;X)$ (for $T > 0$ and $1 \le p \le \infty$) for the Banach space of all (equivalence classes of) Bochner measurable functions $f : (0,T) \to X$ such that $\|f(\cdot)\|_X \in L^p(0,T)$. Correspondingly, we denote by $C([0,T];X)$ the Banach space of continuous functions on $[0,T]$ with values in X.

In Section 6.2 we will use the Orlicz space $L^\Phi(\Omega)$ with $\Phi(s) = (1+s)\log(1+s) - s$ as Young function and some of its properties. See Kufner, John and Fučík [12] or Post [19], Appendix B, for details on this space.

In the estimates of this work, we are going to write C for possibly different constants whose exact form has no importance.

Definition 2.1 *A pair of functions (U,V) with*

$$\begin{aligned} U &\in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega)), \quad U_t \in L^2(0,T;(H^1(\Omega))^*) \\ V &\in C([0,T];H^1(\Omega)), \quad V_t \in L^2(0,T;L^2(\Omega)) \end{aligned}$$

is called a (weak) solution of (5), (6) if the following identities hold

$$\begin{aligned} \int_0^T \langle U_t, H\rangle ds &+ \int_0^T \int_\Omega (\nabla U - \chi U \nabla S(V))\nabla H \; dx\; ds = 0 \\ \int_0^T \int_\Omega V_t H dx ds &+ \int_0^T \int_\Omega (\alpha \nabla V \nabla H + \beta V H - \delta U S'(V) H) \; dx \; ds = 0 \end{aligned}$$

for all $H \in L^2(0,T;H^1(\Omega))$.

The various results of this work will be applicable to different classes of sensitivity functions. The most general class of functions considered will be the set

$$\mathcal{S} = \left\{ S \in C^1(I\!R, I\!R) : 0 \le S(V), \; 0 \le S'(V) \le C' \; \text{ for all } V \ge 0 \right\}.$$

At some stages, we will have to require additionally that $S \in \mathcal{S}$ is twice continuously differentiable and that

$$|S''(V)| \le C'' \tag{7}$$

for all $V \ge 0$. (Note that the functions S_1, S_2 and S_3 proposed in (4) all belong to our class $\mathcal{S}$ and fulfill (7).)

For the proof of global existence of solutions to system (5) with (6) we will confine ourselves in this paper to the cases of bounded sensitivity functions and the logarithmic sensitivity. For the logarithmic sensitivity function we will need the following result which is proven in Post [19] (Corollary B.2):

Corollary 2.2 *If U belongs to the Orlicz space $L^\Phi(\Omega)$, then there exists for every $\kappa > 0$ a constant $C = C(\kappa, \|U \log U\|_{L^1(\Omega)})$, such that*

$$\|U\|^2_{L^2(\Omega)} \;\le\; \kappa \, \|\nabla \sqrt{U}\|^2_{L^2(\Omega)} \;+\; C.$$

3 Local Existence and Uniqueness of Solutions

Theorem 3.1 *Let the sensitivity function S belong to the set $\mathcal{S}$. For positive initial values $U_0 \in L^2(\Omega)$ and $V_0 \in L^\infty(\Omega) \cap H^1(\Omega)$, there exist a $T > 0$ and a corresponding (weak) solution (U, V) of system (5), (6) with initial values $U(0,x) = U_0(x)$, $V(0,x) = V_0(x)$ for all $x \in \Omega$ satisfying*

$$\begin{aligned} &U \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)), \quad U_t \in L^2(0,T; (H^1(\Omega))^*), \\ &V \in C([0,T]; H^1(\Omega)) \cap L^\infty(0,T; L^\infty(\Omega)), \quad V_t \in L^2(0,T; L^2(\Omega)). \end{aligned}$$

U and V are both positive in the L^2-sense.

Furthermore, we can show that V belongs to $L^q(t,T; W^{1,p}(\Omega))$ for some $p > 2$ and $q > \dfrac{2p}{p-2}$ and that $U \in L^\infty(t,T, L^\infty(\Omega))$ for any $0 < t \le T$. (The system has a smoothing effect on the initial values.)

Assuming additionally that $S \in C^2(\mathbb{R}, \mathbb{R})$ with $|S''(V)| \le C''$ for all $V \ge 0$, and if $U_0 \in L^\infty(\Omega)$ and $V_0 \in W^{1,p}(\Omega)$ for some $p > 2$, we obtain $U \in L^\infty(0,T; L^\infty(\Omega))$, and we can show uniqueness of the solution.

Remark: The proof of the theorem follows the ideas of a proof of local existence for the system without sensitivity function by Gajewski and Zacharias [6] and can be found in detail in Post [19].

4 A Lyapunov Function for the System

As we will obtain results on $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ possibly bigger than 2, we refer to a theorem of existence by Amann [1], Theorem 14.7, for smooth domains. (See Post [19], Appendix A for more details.)

Theorem 4.1 *If $S \in C^1(\mathbb{R}, \mathbb{R})$, then*

$$F(U(t), V(t)) = \int_\Omega \left\{ U(t) \log U(t) - \chi U(t) S(V(t)) + \frac{\chi}{2\delta}(\alpha |\nabla V(t)|^2 + \beta (V(t))^2) \right\} dx \tag{8}$$

is a Lyapunov function for the system (5), (6).

Proof: We formally differentiate F with respect to t:

$$\begin{aligned} \frac{d}{dt} F(U(t), V(t)) \;&=\; \int_\Omega U_t(t)[\log U(t) - \chi S(V(t))] dx \;+\; \int_\Omega U_t(t) dx \\ &\quad + \chi \int_\Omega \left[\frac{\alpha}{2\delta} \frac{d}{dt} |\nabla V(t)|^2 + \frac{\beta}{\delta} V(t) V_t(t) - U(t) S'(V(t)) V_t(t) \right] dx \end{aligned}$$

Testing the first equation in (5) with $[\log U - \chi S(V)]$ and 1, respectively, gives

$$\begin{aligned}\int_\Omega U_t[\log U - \chi S(V)]dx &= -\int_\Omega (\nabla U - U\chi S(V))\nabla(\log U - \chi S(V))dx \\ &= -\int_\Omega U(t)|\nabla[\log U(t) - \chi S(V(t))]|^2 dx\end{aligned}$$

and

$$\int_\Omega U_t(t)dx = 0.$$

By the absolute continuity of the mapping $t \mapsto \|\nabla V(t)\|^2_{L^2(\Omega)}$, using the $V-$equation, we can write

$$\int_\Omega \frac{\alpha}{2\delta}\frac{d}{dt}|\nabla V(t)|^2 dx = \int_\Omega \left\{-\frac{\beta}{\delta}V(t)V_t(t) + U(t)S'(V(t))V_t(t) - \frac{1}{\delta}V_t^2\right\} dx,$$

so that we obtain

$$\frac{d}{dt}F(U(t),V(t)) = -\int_\Omega U(t)|\nabla[\log U(t) - \chi S(V(t))]|^2 dx \ - \ \frac{\chi}{\delta}\int_\Omega (V_t(t))^2 dx \ \le \ 0,$$

i.e., F decreases along solution trajectories. □

Remark: One can show that a solution to system (5), (6) ceases to exist in finite time if the Lyapunov function F becomes $-\infty$ for a finite t_0 as we can find a $p > 1$ (but close to 1) such that the L^p-norm of the function U will explode at this point of the time scale. (See Post [19], Proposition 3.2.) Since we will want to study at a later stage the long time behaviour of a solution of system (5), (6) we are interested in finding conditions ensuring the boundedness of F. In the following, let S belong to the class $\mathcal{S}$.

Lemma 4.2 *We have for the solutions of (5), (6):*

$$\|U(t)\|_{L^1(\Omega)} = \|U_0\|_{L^1(\Omega)}$$

and there exists a constant $C > 0$, depending only on $\|U_0\|_{L^1(\Omega)}$ and $\|V_0\|_{L^1(\Omega)}$ such that

$$\|V(t)\|_{L^1(\Omega)} \le C.$$

Proof: Since U and V are positive, we obtain the assertions by testing both equations with $H \equiv 1$, respectively:

$$\frac{d}{dt}\int_\Omega U(t)dx = 0 \implies \|U(t)\|_{L^1(\Omega)} = \|U_0\|_{L^1(\Omega)} \tag{9}$$

and from

$$\frac{d}{dt}\int_\Omega V(t)dx = -\beta\int_\Omega V dx \ + \ \delta\int_\Omega US'(V)dx$$

we get

$$
\begin{aligned}
\|V(t)\|_{L^1(\Omega)} &= \|V_0\|_{L^1(\Omega)} e^{-\beta t} + e^{-\beta t}\int_0^t e^{\beta s}\delta \int_\Omega U(s)S'(V(s))dxds \\
&\overset{(9)}{\leq} \|V_0\|_{L^1(\Omega)} + C'\|U_0\|_{L^1(\Omega)}\frac{\delta}{\beta}(1-e^{-\beta t}) \\
&\leq \|V_0\|_{L^1(\Omega)} + \frac{C'\delta}{\beta}\|U_0\|_{L^1(\Omega)} \;\leq\; C
\end{aligned}
$$

by the formula of variation of the constant. □

In the next lemma we give a technical condition with which we can ensure boundedness of the Lyapunov function F and all its terms.

Lemma 4.3 *If there exists a positive constant c such that the inequality*

$$
\int_\Omega e^{aS(V)}dx \;\leq\; \exp\left(\frac{\chi\alpha}{2\delta\|U_0\|_{L^1(\Omega)}}\;\|\nabla V\|^2_{L^2(\Omega)} + \frac{\chi\beta}{2\delta\|U_0\|_{L^1(\Omega)}}\|V\|^2_{L^2(\Omega)} + c\right) \tag{10}
$$

holds with $a=\chi$, then the Lyapunov function F is bounded from below and there exists a $C>0$, independent of t, such that

$$
|F(U(t),V(t))| \leq C \quad \text{for all} \quad t \geq 0. \tag{11}
$$

If we can choose $a>\chi$ in (10), then we have additionally

$$
(\;0 \leq\;) \int_\Omega U(t)S(V(t))dx \;\leq\; C \text{ for all } t \geq 0 \tag{12}
$$

and the boundedness of all terms in F follows.

Proof: Suppose inequality (10) holds for an a, which will be determined later on. For a fixed $t \in (0,T)$, we set $\psi(x) := \frac{M}{\mu}e^{aS(V(x,t))}$, where $M := \|U(t)\|_{L^1(\Omega)} = \|U_0\|_{L^1(\Omega)}$ and $\mu := \int_\Omega e^{aS(V(x,t))}dx$. As $-\log(x)$ is a convex function, we know from Jensen's inequality that

$$
0 \;=\; -\log\int_\Omega \frac{\psi}{U}\frac{U}{M}dx \;\leq\; \int_\Omega\left(-\log\frac{\psi}{U}\right)\frac{U}{M}dx \;=\; \frac{1}{M}\int_\Omega U\log\frac{U}{\psi}dx,
$$

so that

$$
\begin{aligned}
0 &\leq \int_\Omega U(\log U - \log\psi)dx \;=\; \int_\Omega U(\log U - \log M + \log\mu - aS(V))dx \\
&= \int_\Omega U\log U dx + M(\log\mu - \log M) - a\int_\Omega US(V)dx \\
&\overset{(10)}{\leq} \int_\Omega U\log U dx - M\log M - a\int_\Omega US(V)dx \\
&\quad + M\log c \;+\; \frac{\chi\alpha}{2\delta}\|\nabla V\|^2_{L^2(\Omega)} \;+\; \frac{\chi\beta}{2\delta}\|V\|^2_{L^2(\Omega)} \;+\; Mc.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(a-\chi)\int_\Omega U(t)S(V(t))dx \;&\le\; F(U(t),V(t)) + M(\log\frac{c}{M}+c)\\
&\le\; F(U_0,V_0) + M(\log\frac{c}{M}+c) \;\le\; \tilde{C}
\end{aligned}
$$

with a positive constant $\tilde{C}$ and we obtain

$$
-M(\log\frac{c}{M}+c) \;\le\; F(U(t),V(t)) \;\le\; \tilde{C}
$$

by choosing $a=\chi$. Furthermore, it is clear, that if $a>\chi$, estimate (12) also holds:

$$
0 \;\le\; \int_\Omega U(t)S(V(t))dx \;\le\; \frac{\tilde{C}}{a-\chi}
$$

and it follows that

$$
\int_\Omega \left\{ U(t)\log U(t) + \frac{\chi}{2\delta}(\alpha|\nabla V(t)|^2 + \beta(V(t))^2) \right\} dx \;\le\; C + \chi\int_\Omega U(t)S(V(t))dx \;\le\; C,
$$

so that all terms in F are bounded. (Remember that $U\log U \ge -\frac{1}{e}$.) □

Corollary 4.4 Since

$$
\frac{\chi}{\delta}\int_\Omega V_t^2 dx \;+\; \int_\Omega U|\nabla(\log U - \chi S(V))|^2 dx \;=\; -\frac{d}{dt}F(U,V),
$$

it follows from Lemma 4.3, that

$$
\frac{\chi}{\delta}\int_0^t \|V_t(s)\|^2_{L^2(\Omega)}ds \;+\; \int_0^t\int_\Omega U|\nabla(\log U - \chi S(V))|^2 dxds \;\le\; F(U_0,V_0) - F(U,V) \;\le\; C
$$

whenever the Lyapunov function is bounded from below, where the constant C does not depend on t.

If additionally (12) holds, i.e., if all terms in the Lyapunov function are bounded, then $\|V(t)\|_{H^1(\Omega)}$ and $\|U(t)\log U(t)\|_{L^1(\Omega)}$ are bounded independently of t.

Remark: In general, we cannot exclude the possibility that particular terms in F become unbounded even if the whole Lyapunov function is bounded. If we can bound the term $US(V)$, however, boundedness of all terms in F follows.

Proposition 4.5 *If S is bounded, then the Lyapunov function $F(U(t),V(t))$ and all its terms are bounded independently of $t>0$.*

Proof: Here, inequality (10) holds trivially for every $a>\chi$:

$$
\int_\Omega e^{aS(V)}dx \le e^{aC}|\Omega| \le C(|\Omega|) \le \exp\left(\frac{\chi\alpha}{2\delta M}\;\|\nabla V\|^2_{L^2(\Omega)} + \frac{\chi\beta}{2\delta M}\|V\|^2_{L^2(\Omega)} + c\right)
$$

and the assertion follows from Lemma 4.3. □

Proposition 4.6 *For $S(V) = S_3(V) = \log(V+c)$, all terms in the Lyapunov function F are bounded for all $\chi < \infty$ if $n = 2$. If $n > 2$, then (11) holds for $\chi \le \frac{2n}{n-2}$. If $\chi < \frac{2n}{n-2}$, then (12) is also true.*

Proof: Since $S_3 \in \mathcal{S}$, we again only have to show inequality (10). For $1 \le a \le \dfrac{2n}{n-2}$ (If $n = 2, a < \infty$.), we have with an arbitrary $\varepsilon > 0$

$$\begin{aligned}
\int_\Omega e^{aS(V)}dx &= \int_\Omega (V+c)^a dx \;=\; C\left(\|V\|^a_{L^a(\Omega)} + 1\right) \;\le\; C\left(\|V\|^{a\theta}_{H^1(\Omega)}\|V\|^{a(1-\theta)}_{L^1(\Omega)} + 1\right)\\
&\le C\left[\left(\|\nabla V\|^{a\theta}_{L^2(\Omega)} + \|V\|^{a\theta}_{L^1(\Omega)}\right)\|V\|^{a(1-\theta)}_{L^1(\Omega)} + 1\right]\\
&= C\left(\|\nabla V\|^{a\theta}_{L^2(\Omega)}\|V\|^{a(1-\theta)}_{L^1(\Omega)} + \|V\|^a_{L^1(\Omega)} + 1\right) \;\le\; C(\|V_0\|_{L^1(\Omega)})\left(\|\nabla V\|^{a\theta}_{L^2(\Omega)} + 1\right)\\
&\le \varepsilon\|\nabla V\|^{2a\theta}_{L^2(\Omega)} + C,
\end{aligned}$$

by the Gagliardo-Nirenberg Inequality with $\theta = \dfrac{(1-\frac{1}{a})n}{1+\frac{n}{2}}$ [†] and Young's Inequality. For the last exponent we calculate

$$2a\theta \;=\; \frac{(a-1)2n}{1+\frac{n}{2}} \;<\; 4(a-1).$$

Since we can choose $\varepsilon > 0$ so small that

$$\varepsilon x^{4(a-1)} \le \exp\left(\frac{\chi\alpha}{2\delta\|U_0\|_{L^1(\Omega)}}x^2\right)$$

for all $x \in [0,+\infty)$, we get

$$\int_\Omega e^{aS(V)}dx \;\le\; \exp\left(\frac{\chi\alpha}{2\delta\|U_0\|_{L^1(\Omega)}}\|\nabla V\|^2_{L^2(\Omega)}\right) + C \;\le\; \exp\left(\frac{\chi\alpha}{2\delta\|U_0\|_{L^1(\Omega)}}\|\nabla V\|^2_{L^2(\Omega)} + C\right),$$

i.e., (10) holds and the assertions follow. □

5 Global Existence of Solutions

In this section, we will show that the solution (U,V) of system (5), (6) in a two-dimensional domain $\Omega \subset I\!R^2$ is global if all terms appearing in the Lyapunov function are bounded.

Unfortunately, we are not able to show this result for a general $S \in \mathcal{S}$. Nor can the arguments be carried over to the three-dimensional case due to dimension-dependent estimates (essentially Galiardo-Nirenberg's Inequality).

Theorem 5.1 *The solution (U,V) of (5), (6) on $\Omega \subset I\!R^2$ with initial conditions $U_0 \in L^\infty(\Omega)$, $V_0 \in W^{1,p}(\Omega)$ for some $p > 2$, where the sensitivity function $S \in \mathcal{S}$ is either bounded or $S(V) = \log(V+c)$, $c \ge 1$, is global in time.*

[†] Note that we need here the condition for a.

In the case of a bounded sensitivity function, we obtain for every $1 \le p < \infty$ a constant C such that

$$\|U\|_{L^\infty(0,\infty;L^p(\Omega))} \ + \ \|V\|_{L^\infty(0,\infty;L^\infty(\Omega))} \ \le \ C.$$

If $S(V) = \log(V+c)$, then there exist constants $C > 0$ and $r > 2$, independent of T, such that

$$\|U\|_{L^\infty(0,T;L^2(\Omega))} + \|V\|_{L^\infty(0,T;L^\infty(\Omega))} \ \le \ Ce^{CT^r} \tag{13}$$

and

$$\|U\|_{L^\infty(0,T;L^p(\Omega))} \ \le \ C\exp(pCe^{CT^r}) \tag{14}$$

for all $T > 0$ and every $2 < p < \infty$. If $\chi \le 1$, the exponential time-dependence for V in (13) can be improved to a polynomial dependence.

5.1 Bounded Sensitivity Functions

Proposition 5.2 *Let Ω be a Lipschitz domain in $I\!R^2$. If $S \in \mathcal{S}$ is bounded, then there exists a positive constant K, independent of $T > 0$, such that for every $1 \le p < \infty$ we have for the solution U of the first equation in (5) with homogeneous Neumann boundary conditions*

$$\|U\|_{L^\infty(0,T;L^p(\Omega))} \le K^p.$$

Proof: Let $e^{\chi S(V)} \le C_e$. (Note that $C_e \ge 1$.)

As we want to demonstrate the exact dependence on p of the bound for U in the space $L^\infty(0,T;L^p(\Omega))$, we are going to distinguish scrupulously between the different constants in the estimates of this proof.

W.l.o.g., we take $\chi = 1$ throughout the proof. (We can define $\tilde{S}(V) := \chi S(V)$ and $\tilde{\delta} := \dfrac{\delta}{\chi}$.) We will show by induction for all $p = 2^k$ with $k \in I\!N \cup \{0\}$ that

$$\|U(t)\|_{L^p(\Omega)} \le K^p \text{ for all } t \in [0,T]. \tag{15}$$

For $k = 0$, i.e., $p = 1$, this is true if

$$\|U(t)\|_{L^1(\Omega)} = \|U_0\|_{L^1(\Omega)} \le K.$$

Now let $p = 2^k \ge 2$ and suppose that (15) holds for $\hat{p} = 2^{k-1} = \dfrac{p}{2}$.

Testing the first equation of (5), (6) with $p\left(\dfrac{U}{e^{S(V)}}\right)^{p-1}$ gives

$$p\int_\Omega U_t \left(\frac{U}{e^{S(V)}}\right)^{p-1} dx + p\int_\Omega e^{S(V)} \nabla\left(\frac{U}{e^{S(V)}}\right) \nabla\left(\frac{U}{e^{S(V)}}\right)^{p-1} dx = 0. \tag{16}$$

We calculate for the two terms separately:

$$p\int_\Omega U_t \left(\frac{U}{e^{S(V)}}\right)^{p-1} dx = \frac{d}{dt}\int_\Omega \left(\frac{U}{e^{S(V)}}\right)^p e^{S(V)} dx + (p-1)\int_\Omega \left(\frac{U}{e^{S(V)}}\right)^p e^{S(V)} S'(V) V_t dx$$

and

$$
\begin{aligned}
p\int_\Omega e^{S(V)}\nabla\left(\frac{U}{e^{S(V)}}\right)\nabla\left(\frac{U}{e^{S(V)}}\right)^{p-1}dx &= p\int_\Omega e^{S(V)}\left|\nabla\left(\frac{U}{e^{S(V)}}\right)\right|^2(p-1)\left(\frac{U}{e^{S(V)}}\right)^{p-2}dx \\
&= \frac{4(p-1)}{p}\int_\Omega e^{S(V)}|\nabla W|^2dx
\end{aligned}
$$

with $W := \left(\frac{U}{e^{S(V)}}\right)^{\frac{p}{2}}$. Inserting this into (16) gives

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega W^2e^{S(V)}dx \quad + \quad & \frac{4(p-1)}{p}\int_\Omega e^{S(V)}|\nabla W|^2dx \;=\; -(p-1)\int_\Omega W^2S'(V)e^{S(V)}V_tdx \\
\leq\;& (p-1)C'C_e\int_\Omega |W^2V_t|dx \;\leq\; pC'C_e\|W\|^2_{L^4(\Omega)}\|V_t\|_{L^2(\Omega)} \\
\leq\;& pC_1C_e\|W\|_{H^1(\Omega)}\|W\|_{L^2(\Omega)}\|V_t\|_{L^2(\Omega)} \\
\leq\;& pC_1C_e(\|W\|_{L^2(\Omega)}+\|\nabla W\|_{L^2(\Omega)})\|W\|_{L^2(\Omega)}\|V_t\|_{L^2(\Omega)} \\
\leq\;& pC_1C_e\|W\|^2_{L^2(\Omega)}\|V_t\|_{L^2(\Omega)}+\frac{3p-4}{p}\|\nabla W\|^2_{L^2(\Omega)} \\
&+\frac{p^2p}{3p-4}(C_1C_e)^2\|W\|^2_{L^2(\Omega)}\|V_t\|^2_{L^2(\Omega)} \\
\leq\;& pC_1C_e\|W\|^2_{L^2(\Omega)}\|V_t\|_{L^2(\Omega)}+\frac{3p-4}{p}\|\nabla W\|^2_{L^2(\Omega)} \\
&+(C_1C_ep)^2\|W\|^2_{L^2(\Omega)}\|V_t\|^2_{L^2(\Omega)},
\end{aligned}
$$

applying Hölder's, Gagliardo-Nirenberg's and Young's Inequality. Since $1\leq e^{S(V)}$, we obtain

$$
\frac{d}{dt}\int_\Omega W^2e^{S(V)}dx+\|\nabla W\|^2_{L^2(\Omega)}\leq \|W\|^2_{L^2(\Omega)}(C_1C_ep\|V_t\|_{L^2(\Omega)}+(C_1C_ep)^2\|V_t\|^2_{L^2(\Omega)}). \tag{17}
$$

By Gagliardo-Nirenberg's, Poincaré's and Young's Inequality, we have, using the inductive assumption in the last step:

$$
\begin{aligned}
\|W\|^2_{L^2(\Omega)} &\leq C_2\left(\|\nabla W\|^2_{L^2(\Omega)}+\|W\|^2_{L^1(\Omega)}\right)^{\frac{1}{2}}\|W\|_{L^1(\Omega)} \\
&\leq C_2\left(\|\nabla W\|_{L^2(\Omega)}\|W\|_{L^1(\Omega)}+\|W\|^2_{L^1(\Omega)}\right) \\
&\leq \|\nabla W\|^2_{L^2(\Omega)}+C_3\|W\|^2_{L^1(\Omega)} \;\leq\; \|\nabla W\|^2_{L^2(\Omega)}+C_3\|U\|^p_{L^{\frac{p}{2}}(\Omega)} \\
&\leq \|\nabla W\|^2_{L^2(\Omega)}+C_3K^{\frac{p^2}{2}}
\end{aligned} \tag{18}
$$

On the other hand, since $\left(1-cp\|V_t\|_{L^2(\Omega)}\right)^2\geq 0$,

$$
C_1C_ep\|V_t\|_{L^2(\Omega)} \;\leq\; \frac{1}{2}+\frac{(C_1C_ep)^2}{2}\|V_t\|^2_{L^2(\Omega)}. \tag{19}
$$

Estimating the left hand side of (17) by (18) and using (19) on the right hand side, one obtains

$$
\frac{d}{dt}\int_\Omega W^2e^{S(V)}dx \;\leq\; \|W\|^2_{L^2(\Omega)}\left(\frac{3}{2}(C_1C_ep)^2\|V_t\|^2_{L^2(\Omega)}-\frac{1}{2}\right)+C_3K^{\frac{p^2}{2}}
$$

$$
\begin{aligned}
&= \frac{3}{2}(C_1C_ep)^2\|V_t\|^2_{L^2(\Omega)}\int_\Omega W^2dx - \frac{1}{2}\int_\Omega W^2dx + C_3K^{\frac{p^2}{2}} \\
&\le \left(\frac{3}{2}(C_1C_ep)^2\|V_t\|^2_{L^2(\Omega)} - \frac{1}{2C_e}\right)\int_\Omega W^2e^{S(V)}dx + C_3K^{\frac{p^2}{2}}
\end{aligned}
$$

since $1 \le e^{S(V)} \le C_e$. Putting $\varphi(s) := \left(\frac{3}{2}(C_1C_ep)^2\|V_t\|^2_{L^2(\Omega)} - \frac{1}{2C_e}\right)$, we can apply a variant of Gronwall's Lemma: With $g(t) := \int_\Omega W^2(t,x)e^{S(V(t,x))}dx$, the last inequality can be written as

$$
g'(t) \;\le\; \varphi(t)g(t) + C_3K^{\frac{p^2}{2}}. \tag{20}
$$

Defining now $h(t) := g(t)\ e^{-\int_0^t\varphi(s)ds}$, we calculate

$$
h'(t) \;=\; g'(t)\ e^{-\int_0^t\varphi(s)ds} \;-\; g(t)\ \varphi(t)\ e^{-\int_0^t\varphi(s)ds} \;\overset{(20)}{\le}\; C_3K^{\frac{p^2}{2}}\ e^{-\int_0^t\varphi(s)ds}.
$$

Integrating this last inequality, we obtain

$$
h(t) \;\le\; h(0) + C_3K^{\frac{p^2}{2}}\int_0^t e^{-\int_0^s\varphi(\tau)d\tau}ds \;=\; g(0) + C_3K^{\frac{p^2}{2}}\int_0^t e^{-\int_0^s\varphi(\tau)d\tau}ds,
$$

so that

$$
g(t) \;=\; e^{\int_0^t\varphi(s)ds}h(t) \;\le\; e^{\int_0^t\varphi(s)ds}g(0) \;+\; C_3K^{\frac{p^2}{2}}e^{\int_0^t\varphi(s)ds}\int_0^t e^{-\int_0^s\varphi(\tau)d\tau}ds,
$$

i.e.,

$$
\int_\Omega W^2(t)e^{S(V(t))}dx \;\le\; e^{\int_0^t\varphi(s)ds}\int_\Omega W_0^2e^{S(V_0)}dx \;+\; e^{\int_0^t\varphi(s)ds}C_3K^{\frac{p^2}{2}}\int_0^t e^{-\int_0^\tau\varphi(s)ds}d\tau.
$$

From the boundedness of the Lyapunov function F we know (see Conclusion 4.4) that V_t is bounded in $L^2(0,T;L^2(\Omega))$ independently of $T>0$, so that

$$
\int_0^t\varphi(s)ds \le C_4(pC_e)^2 - \frac{t}{2C_e} \le C_4(pC_e)^2.
$$

On the other hand,

$$
\int_0^t e^{-\int_0^\tau\varphi(s)ds}d\tau \le \int_0^t e^{\frac{\tau}{2C_e}}d\tau = 2C_e\left(e^{\frac{t}{2C_e}} - 1\right).
$$

Since

$$
\int_\Omega W_0^2e^{S(V_0)}dx \;=\; \int_\Omega \frac{U_0^p}{(e^{S(V_0)})^{(p-1)}}dx \;\le\; \int_\Omega U_0^pdx \;\le\; |\Omega|\ \|U_0\|^p_{L^\infty(\Omega)},
$$

we finally obtain

$$
\begin{aligned}
\int_\Omega W^2(t)e^{S(V(t))}dx &\le e^{C_4(pC_e)^2}|\Omega|\ \|U_0\|^p_{L^\infty(\Omega)} + e^{C_4(pC_e)^2}C_3K^{\frac{p^2}{2}}2C_e\left(1 - e^{-\frac{t}{2C_e}}\right) \\
&\le C_5e^{C_4(pC_e)^2} + C_6e^{C_4(pC_e)^2}C_eK^{\frac{p^2}{2}}.
\end{aligned}
$$

It follows that if $K \geq \|U_0\|_{L^1(\Omega)}$ is chosen sufficiently big, that is, if

$$C_5 e^{C_4 C_e^2} C_e \;\leq\; \frac{1}{2}K \implies C_5 e^{C_4 (pC_e)^2} C_e^{(p-1)} \;\leq\; C_5^{p^2} e^{C_4 (pC_e)^2} C_e^{p^2} \;\leq\; \frac{1}{2^{p^2}} K^{p^2} \;\leq\; \frac{1}{2} K^{p^2},$$

so that

$$C_5 e^{C_4 (pC_e)^2} \;\leq\; \frac{1}{2} C_e^{-(p-1)} K^{p^2}$$

and if

$$C_6 e^{C_4 C_e^2} C_e \;\leq\; \frac{1}{2}K^{\frac{1}{2}} \implies C_6 e^{C_4 (pC_e)^2} C_e^{p} \;\leq\; C_6^{p^2} e^{C_4 (pC_e)^2} C_e^{p^2} \;\leq\; \frac{1}{2^{p^2}} K^{\frac{p^2}{2}} \;\leq\; \frac{1}{2} K^{\frac{p^2}{2}},$$

so that

$$C_6 e^{C_4 (pC_e)^2} C_e K^{\frac{p^2}{2}} \;\leq\; \frac{1}{2} C_e^{-(p-1)} K^{p^2},$$

we obtain

$$\begin{aligned} \|U(t)\|^p_{L^p(\Omega)} \;&\leq\; C_e^{p-1} \int_\Omega W^2 e^{S(V)} dx \leq C_e^{p-1} \left(C_5 e^{C_4 (pC_e)^2} + C_6 e^{C_4 (pC_e)^2} C_e K^{\frac{p^2}{2}} \right) \\ &\leq\; \frac{1}{2} K^{p^2} + \frac{1}{2} K^{p^2} = K^{p^2}. \end{aligned}$$

Extracting the p-th root gives (15). □

Conclusion 5.3 Since $S'(V) \leq C'$, the right hand side $\delta U S'(V)$ of the equation for V is in $L^\infty(0,T;L^p(\Omega))$ for a $p > 2$ and it follows by standard arguments that V is bounded in $L^\infty(0,T;L^\infty(\Omega))$ independently of $T > 0$.

From the a-priori-estimates obtained in Corollary 4.4, we know that the estimate for V in $L^\infty(0,T;H^1(\Omega))$ neither depends on $T > 0$, so that we have proven the same regularity for the solutions $U(t)$ and $V(t)$, for almost all $t \in (0,\infty)$, as we had for the initial values. Therefore, we can, step by step, extend the interval of existence to $(0,\infty)$, and thus obtain global solutions. Furthermore, the norm estimates are valid on the whole positive real line, and Theorem 5.1 is proven for bounded sensitivity functions. Note that by the smoothing effect of the system, we have automatically global uniqueness of the extended solution.

5.2 The Logarithmic Sensitivity Function

The first step to prove Theorem 5.1 for the logarithmic sensitivity function will consist in finding (time-dependent) bounds for the $L^2(0,T;L^2(\Omega))$-norm of $US'(V)$.

In order to do so, we will first prove three results for more general sensitivity functions fulfilling certain convexity and concavity conditions, respectively. These results will be applied to the logarithmic function in Proposition 5.7.

Lemma 5.4 *Let $S \in \mathcal{S}$ and suppose that $\chi < 1$.*

If all the terms in the Lyapunov function $F(U,V)$ for system (5) are bounded and if the sensitivity function is twice continuously differentiable and fulfills the condition $S''(V)+(S'(V))^2 \le 0$ for all $V \ge 0$, then there exists a positive constant C, independent of $T>0$, such that

$$\|\nabla\sqrt{U}\|_{L^2(0,T;L^2(\Omega))} \le C(1+T)^{\frac{1}{2}}. \tag{21}$$

Proof: Obviously, the integral

$$\int_0^T \int_\Omega U|\nabla(\log U - \chi S(V))|^2 dxds. \tag{22}$$

is positive for all $T>0$.

Using that $\nabla(\log U) = \dfrac{\nabla U}{U}$ and $\nabla\sqrt{U} = \dfrac{\nabla U}{2\sqrt{U}}$, we can rewrite the integral in the following way:

$$\begin{aligned}
\int_0^T \int_\Omega U|\nabla(\log U \; - \; \chi S(V))|^2 dxds \; &= \; 4\int_0^T \int_\Omega |\nabla\sqrt{U}|^2 dxds - 2\chi\int_0^T \int_\Omega \nabla U \nabla S(V) dxds \\
&\quad + \chi^2 \int_0^T \int_\Omega U|\nabla S(V)|^2 dxds \\
&= \chi^2 \int_0^T \int_\Omega [U|\nabla V|^2 (S'(V))^2 - \nabla U\nabla S(V)]dxds - \; \chi(2-\chi)\int_0^T \int_\Omega \nabla U\nabla S(V)dxds \\
&\quad + 4(2-\chi)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)} ds \; - \; 4(1-\chi)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)} ds
\end{aligned} \tag{23}$$

We know that $\int_\Omega U_t \; dx = 0$ and by the first equation of (5) tested with $\log U$, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega U\log U dx \;&=\; \int_\Omega U_t \log U dx \; + \; \int_\Omega U_t \; dx \; = \; \int_\Omega U_t \log U dx \\
&= \; \chi\int_\Omega U\nabla S(V)\nabla \log U dx - \int_\Omega \nabla U\nabla \log U dx \\
&= \; \chi\int_\Omega \nabla U\nabla S(V) dx - 4\int_\Omega |\nabla\sqrt{U}|^2 dx,
\end{aligned}$$

so that

$$\begin{aligned}
-\chi(2-\chi)\int_0^T \int_\Omega \nabla U\nabla S(V)dxds \; &+ \; 4(2-\chi)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)} ds \\
&= -(2-\chi)\left(\int_\Omega U(t)\log U(t)dx - \int_\Omega U_0 \log U_0 dx\right).
\end{aligned} \tag{24}$$

On the other hand, testing the second equation in (5) with $US'(V)$ gives

$$\begin{aligned}
\frac{1}{\alpha}\int_\Omega [V_t + \beta V - \delta US'(V)]US'(V)dx \;&=\; -\int_\Omega \nabla(US'(V))\nabla V dx \\
&= \; -\int_\Omega US''(V)|\nabla V|^2 dx - \int_\Omega \nabla U\nabla S(V)dx,
\end{aligned}$$

and hence

$$
\begin{aligned}
-\chi^2 \int_0^T \int_\Omega \nabla U \nabla S(V) dx ds &= \frac{\chi^2}{\alpha} \int_0^T \int_\Omega \left\{ V_t U S'(V) + \beta V U S'(V) - \delta (U S'(V))^2 \right\} dx ds \\
&\quad + \chi^2 \int_0^T \int_\Omega U |\nabla V|^2 S''(V) dx ds. \qquad (25)
\end{aligned}
$$

Inserting (24) and (25), identity (23) becomes

$$
\begin{aligned}
\int_0^T \int_\Omega U |\nabla (\log U &- \chi S(V))|^2 dx ds = \chi^2 \int_0^T \int_\Omega U |\nabla V|^2 [(S'(V))^2 + S''(V)] dx ds \\
&+ \frac{\chi^2}{\alpha} \int_0^T \int_\Omega \left\{ V_t U S'(V) + \beta V U S'(V) - \delta (U S'(V))^2 \right\} dx ds \\
&- (2-\chi) \left(\int_\Omega U(t) \log U(t) dx - \int_\Omega U_0 \log U_0 dx \right) \\
&- 4(1-\chi) \int_0^T \|\nabla \sqrt{U}\|^2_{L^2(\Omega)} ds. \qquad (26)
\end{aligned}
$$

Using now the positivity of the left hand side and the condition $S''(V) + (S'(V))^2 \le 0$, we obtain the estimate

$$
\begin{aligned}
4(1-\chi) \int_0^T \|\nabla \sqrt{U}\|^2_{L^2(\Omega)} ds &\le \frac{2\chi^4}{\delta \alpha^2} \|V_t\|^2_{L^2(0,T;L^2(\Omega))} + \frac{2\chi^4 \beta^2}{\delta \alpha^2} \|V\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + (2-\chi) \left(\|U(T) \log U(T)\|_{L^1(\Omega)} + \|U_0 \log U_0\|_{L^1(\Omega)} \right) \\
&\le C \Big(\|V_t\|^2_{L^2(0,T;L^2(\Omega))} + T \|V\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\quad + \|U \log U\|_{L^\infty(0,T;L^1(\Omega))} \Big) \\
&\le C(1+T), \qquad (27)
\end{aligned}
$$

where we used in the last step the a-priori-estimates from Corollary 4.4. Since $\chi < 1$, estimate (21) is proven. □

Lemma 5.5 *Let $S \in \mathcal{S}$ and suppose that $\chi > 1$.*

If all the terms in the Lyapunov function $F(U,V)$ are bounded and if the sensitivity function $S \in C^2(I\!R, I\!R)$ and fulfills the condition $S''(V) + (S'(V))^2 \ge 0$ for all $V \ge 0$, then there exists a constant C such that for all $T > 0$

$$
\|\nabla \sqrt{U}\|_{L^2(0,T;L^2(\Omega))} \le C(1+T)^{\frac{1}{2}}. \qquad (28)
$$

Proof: From the boundedness of the Lyapunov function (see Corollary 4.4) we know that the integral, whose positivity was used in the proof of Lemma 5.4, is also bounded from above:

$$
\int_0^T \int_\Omega U |\nabla (\log U - \chi S(V))|^2 dx ds \le C
$$

for all $T > 0$.

Using now identity (26), the assumed property $S''(V) + (S'(V))^2 \geq 0$ of the sensitivity function and the a-priori-estimates (See again Corollary 4.4.), we obtain

$$
\begin{aligned}
C &\geq -|\chi - 2| \left(\|U(t)\log U(t)\|_{L^1(\Omega)} + \|U_0 \log U_0\|_{L^1(\Omega)}\right) - C(\chi,\alpha,\beta,\delta)\|V_t\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad -2\delta\|US'(V)\|^2_{L^2(0,T;L^2(\Omega))} + 4(\chi-1)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)} ds \\
&\geq -C - 2\delta C'\|U\|^2_{L^2(0,T;L^2(\Omega))} + 4(\chi-1)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)} ds. \qquad (29)
\end{aligned}
$$

By Corollary 2.2 we infer that

$$
2\delta C'\|U\|^2_{L^2(0,T;L^2(\Omega))} \leq (\chi-1)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(0,T;L^2(\Omega))} ds + C(\delta,\chi,C')\, T,
$$

so that we have shown

$$
\|\nabla\sqrt{U}\|^2_{L^2(0,T;L^2(\Omega))} \leq C(1+T),
$$

which gives the claim. (Note that we used $\chi > 1$ in the last step.) □

Remarks:

1. The conditions for the sensitivity functions in Lemma 5.4 and Lemma 5.5 are equivalent to requiring that $e^{S(V)}$ be concave or convex, respectively, since

$$
\frac{d^2}{dV^2}\left(e^{S(V)}\right) = e^{S(V)}(S''(V) + (S'(V))^2).
$$

2. For our treatment of the logarithmic sensitivity function $S(V) = \log(V+c)$ we still need a result for the case $\chi = 1$. As in Lemma 5.4, we could use the positivity of the integral $\int_0^t \int_\Omega U|\nabla(\log U - \chi S(V))|^2 dx ds$ and equality (26).

However, we will now prove a more general result, which is applicable to sensitivity functions for which $e^{\chi S(V)}$ is concave.

Lemma 5.6 *Let $S \in \mathcal{S}$ be twice continuously differentiable. If all the terms in the Lyapunov function for system (5) are bounded and if the sensitivity function fulfills the condition $S''(V) + \chi(S'(V))^2 \leq 0$, then there exists a positive constant C, independent of T, such that*

$$
\|US'(V)\|_{L^2(0,T;L^2(\Omega))} \leq C(1+T)^{\frac{1}{2}}
$$

for all $T > 0$.

Proof: Testing the first equation with $S(V)$ gives

$$
\begin{aligned}
\int_\Omega U_t S(V)\, dx &= -\int_\Omega \nabla U \nabla S(V)\, dx + \chi\int_\Omega U|\nabla S(V)|^2 dx \\
&= -\int_\Omega S'(V)\nabla U\nabla V\, dx + \chi\int_\Omega U(S'(V))^2|\nabla V|^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_\Omega \nabla(US'(V))\nabla V dx \;+\; \int_\Omega U|\nabla V|^2[S''(V)+\chi(S'(V))^2]dx \\
&= \frac{1}{\alpha}\int_\Omega US'(V)V_t dx \;+\; \frac{\beta}{\alpha}\int_\Omega US'(V)V dx \;-\; \frac{\delta}{\alpha}\int_\Omega U^2(S'(V))^2 dx \\
&\quad +\int_\Omega U|\nabla V|^2[S''(V)+\chi(S'(V))^2]dx,
\end{aligned}
$$

where we used the second equation in the last step. Since

$$
\int_\Omega U_t S(V)\ dx = \frac{d}{dt}\int_\Omega US(V)dx \;-\; \int_\Omega US'(V)V_t\ dx,
$$

we now obtain by Hölder's and Young's Inequality

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega US(V)\ dx &= (1+\frac{1}{\alpha})\int_\Omega US'(V)V_t dx \;+\; \frac{\beta}{\alpha}\int_\Omega US'(V)V dx \\
&\quad -\frac{\delta}{\alpha}\int_\Omega U^2(S'(V))^2 dx \;+\int_\Omega U|\nabla V|^2[S''(V)+\chi(S'(V))^2]dx \\
&\leq C(\alpha,\delta)\ \|V_t\|^2_{L^2(\Omega)} + C(\alpha,\beta,\delta)\ \|V\|^2_{L^2(\Omega)} - \frac{\delta}{2\alpha}\ \|US'(V)\|^2_{L^2(\Omega)} \\
&\quad +\int_\Omega U|\nabla V|^2[S''(V)+\chi(S'(V))^2]dx.
\end{aligned}
$$

Using now the hypothesis on S, integration from 0 to T yields:

$$
\begin{aligned}
\int_\Omega U(t)S(V(t))dx &\leq \int_\Omega U_0 S(V_0)dx + C\Big(\ \|V_t\|^2_{L^2(0,T;L^2(\Omega))} + \|V\|^2_{L^2(0,T;L^2(\Omega))}\Big) \\
&\quad -\frac{\delta}{2\alpha}\int_0^T\int_\Omega U^2(s)(S'(V(s)))^2 dx.
\end{aligned}
$$

And with the estimates obtained with the Lyapunov function (See Corollary 4.4.), we find

$$
\begin{aligned}
\frac{\delta}{2\alpha}\|US'(V)\|^2_{L^2(0,T;L^2(\Omega))} &\leq \int_\Omega U_0S(V_0)dx + C\Big(\ \|V_t\|^2_{L^2(0,T;L^2(\Omega))} + \|V\|^2_{L^2(0,T;L^2(\Omega))}\Big) \\
&\leq C_0 + C\Big(\ \|V_t\|^2_{L^2(0,T;L^2(\Omega))} + T\|V\|^2_{L^\infty(0,T;L^2(\Omega))}\Big) \\
&\leq C(1+T),
\end{aligned}
$$

with C independent of T. □

Proposition 5.7 *Let $S(V)=\log(V+c)$. We can bound the solution U of the first equation in (5) in the space $L^2(0,T;L^2(\Omega))$ as follows*

$$
\|U\|_{L^2(0,T;L^2(\Omega))} \leq C(1+T)^{\frac{1}{2}},
$$

if $\chi \neq 1$.

For $\chi = 1$, we still have

$$
\|US'(V)\|_{L^2(0,T;L^2(\Omega))} \leq C(1+T)^{\frac{1}{2}}.
$$

The constant C in both estimates does not depend on the time $T>0$.

Proof: We know that all the terms in the Lyapunov function are bounded for all χ in the case of the logarithmic sensitivity function. (See Proposition 4.6.)

Also, $e^{S(V)}$ is obviously both concave and convex for $S(V) = \log(V + c)$ and we have

$$S''(V) + (S'(V))^2 = (\log(V + c))'' + ((\log(V + c))')^2 = -\frac{1}{(V + c)^2} + \frac{1}{(V + c)^2} = 0.$$

Hence, we can apply Lemma 5.5 if $\chi > 1$ and Lemma 5.4 if $\chi < 1$ in order to obtain

$$\|\nabla\sqrt{U}\|^2_{L^2(0,T;L^2(\Omega))} \leq C(1 + T).$$

Now, we can use Corollary 2.2 with $\kappa = 1$ to conclude that

$$\int_0^T \|U\|^2_{L^2(\Omega)} ds \ \leq \ \int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)} ds + CT \ \leq \ C(1 + T).$$

For $\chi = 1$, we apply Lemma 5.6, and the claim follows. □

Proposition 5.8 *Let S be the logarithmic sensitivity function. There exists a constant C, which is independent of the time $T > 0$, and an exponent $r > 2$ such that we have the following estimate for the solution (U, V) of system (5), (6):*

$$\|U\|_{L^\infty(0,T;L^2(\Omega))} \ + \ \|V\|_{L^\infty(0,T;L^\infty(\Omega))} \ \leq \ Ce^{CT^r}.$$

Proof:

(i) Using U itself as test function for the first equation in (5), we obtain

$$\begin{aligned}\frac{1}{2}\left(\|U(t)\|^2_{L^2(\Omega)} - \|U_0\|^2_{L^2(\Omega)}\right) \ &= \ \int_0^t \int_\Omega \frac{d}{ds}\left(\frac{1}{2}U^2(s)\right) dxds \ = \ \int_0^t \langle U_t(s), U(s)\rangle ds \\ &= \ \int_0^t \int_\Omega [\chi U \nabla S(V)\nabla U - |\nabla U|^2] dxds. \qquad (30)\end{aligned}$$

We apply Hölder's Inequality to the first term on the right hand side, where the coefficients fulfill

$$\frac{1}{r} + \frac{1}{p} = \frac{1}{2} \ .^\dagger$$

$$\chi\int_0^t \int_\Omega U\nabla S(V)\nabla U dxds \ \leq \ \chi C' \int_0^t \|U(s)\|_{L^r(\Omega)}\|\nabla V(s)\|_{L^p(\Omega)}\|\nabla U(s)\|_{L^2(\Omega)} \ ds \qquad (31)$$

By Gagliardo-Nirenberg's Inequality, we have

$$\begin{aligned}\|U\|_{L^r(\Omega)} \ &\leq \ C\|U\|^{\frac{2}{r}}_{L^2(\Omega)}\|U\|^{1-\frac{2}{r}}_{H^1(\Omega)} \ \leq \ C\|U\|^{\frac{2}{r}}_{L^2(\Omega)}\left(\|U\|^{1-\frac{2}{r}}_{L^2(\Omega)} + \|\nabla U\|^{1-\frac{2}{r}}_{L^2(\Omega)}\right) \\ &= \ C\left(\|U\|_{L^2(\Omega)} + \|U\|^{\frac{2}{r}}_{L^2(\Omega)}\|\nabla U\|^{1-\frac{2}{r}}_{L^2(\Omega)}\right),\end{aligned}$$

† Note that this condition is equivalent to $r = \dfrac{2p}{p-2} = \dfrac{2p'}{2-p'}$, so that $W^{1,p'}(\Omega) \hookrightarrow L^r(\Omega)$ and $L^{r'}(\Omega)$ is the greatest Lebesgue space, that is, the space with the smallest exponent, still contained in the dual of $W^{1,p'}(\Omega)$.

so that estimate (31) becomes

$$\begin{aligned}
\chi \int_0^t \int_\Omega U \nabla S(V) \nabla U dx ds \;\; &\le \;\; C \int_0^t \Big(\|U(s)\|_{L^2(\Omega)} \|\nabla V(s)\|_{L^p(\Omega)} \|\nabla U(s)\|_{L^2(\Omega)} \\
&\quad + \|U(s)\|_{L^2(\Omega)}^{\frac{2}{r}} \|\nabla V(s)\|_{L^p(\Omega)} \|\nabla U(s)\|_{L^2(\Omega)}^{2-\frac{2}{r}} \Big) ds \\
&\le \;\; C(\varepsilon_1, \varepsilon_2) \int_0^t \left(\|\nabla V(s)\|_{L^p}^2 + \|\nabla V(s)\|_{L^p}^r \right) \|U(s)\|_{L^2}^2 \; ds \\
&\quad + (\varepsilon_1 + \varepsilon_2) \int_0^t \|\nabla U(s)\|_{L^2(\Omega)}^2 ds, \qquad (32)
\end{aligned}$$

where we used twice Young's Inequality in the last step. Choosing $\varepsilon_1 + \varepsilon_2 \le 1$, we obtain from (30)

$$\begin{aligned}
\frac{1}{2} \left(\|U(t)\|_{L^2(\Omega)}^2 \;\; - \;\; \|U_0\|_{L^2(\Omega)}^2 \right) \;\; &\le \;\; \int_0^t (\varepsilon_1 + \varepsilon_2 - 1) \|\nabla U(s)\|_{L^2(\Omega)}^2 ds \\
&\quad + C \int_0^t \left(\|\nabla V(s)\|_{L^p(\Omega)}^2 + \|\nabla V(s)\|_{L^p(\Omega)}^r \right) \|U(s)\|_{L^2(\Omega)}^2 ds \\
&\le \;\; C \int_0^t \left(\|\nabla V(s)\|_{L^p(\Omega)}^2 + \|\nabla V(s)\|_{L^p(\Omega)}^r \right) \|U(s)\|_{L^2(\Omega)}^2 ds.
\end{aligned}$$

It follows now by Gronwall's Inequality and after taking the supremum over $t \in [0, T]$ on both sides of the inequality that

$$\|U\|_{L^\infty(0,T;L^2(\Omega))}^2 \;\le\; \|U_0\|_{L^2(\Omega)}^2 \exp\left(C \int_0^T \left[\|\nabla V(s)\|_{L^p(\Omega)}^2 + \|\nabla V(s)\|_{L^p(\Omega)}^r \right] ds \right). \qquad (33)$$

Thus, we need to control the right hand side of (33) to obtain a bound for U in the space $L^\infty(0,T;L^2(\Omega))$. In order to do so, we will use the following parabolic regularity result by Gröger [7]. The function $\tilde{V} := V - V_0$ solves the problem

$$\tilde{V}_t - \alpha \Delta \tilde{V} + \beta \tilde{V} = \alpha \Delta V_0 - \beta V_0 + \delta U S'(\tilde{V} + V_0) \quad \text{in} \;\; (0,T) \times \Omega \qquad (34)$$

with vanishing initial value $\tilde{V}_0 = 0$. Using the information that $V_0 \in W^{1,p}(\Omega)$ for a $p > 2$, we obtain from reference [7]

$$\begin{aligned}
\|\nabla \tilde{V}\|_{L^p(0,T;L^p(\Omega))} \;\; &\le \;\; C \|\alpha \Delta V_0 - \beta V_0 + \delta U S'(\tilde{V} + V_0)\|_{L^p(0,T;(W^{1,p'}(\Omega))^*)} \\
&\le \;\; C \left(\|V_0\|_{L^p(0,T;W^{1,p}(\Omega))} + \|US'(V)\|_{L^p(0,T;(W^{1,p'}(\Omega))^*)} \right) \\
&\le \;\; C \left(T^{\frac{1}{p}} \|V_0\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|US'(V)\|_{L^p(0,T;(W^{1,p'}(\Omega))^*)} \right) \\
&\le \;\; C \left(1 + T + \|US'(V)\|_{L^p(0,T;(W^{1,p'}(\Omega))^*)} \right).
\end{aligned}$$

In the terminology of Dore [4], this means that there is L^p-regularity on the interval $[0, T]$ for problem (34). He shows in [4], Theorem 4.2, that L^p-regularity for one p implies $L^{\tilde{p}}$-regularity for any $1 < \tilde{p} < \infty$, so that in particular

$$\int_0^T \|\nabla V(s)\|_{L^p(\Omega)}^2 ds \;\le\; \|\nabla \tilde{V}\|_{L^2(0,T;L^p(\Omega))}^2 + CT \;\le\; C \left(1 + T + \|US'(V)\|_{L^2(0,T;(W^{1,p'}(\Omega))^*)} \right)^2$$

and

$$\begin{aligned}
\int_0^T \|\nabla V(s)\|^r_{L^p(\Omega)} ds &\leq \|\nabla \tilde{V}\|^r_{L^r(0,T;L^p(\Omega))} + CT \\
&\leq C\left(1 + T + \|US'(V)\|_{L^r(0,T;(W^{1,p'}(\Omega))^*)}\right)^r
\end{aligned}$$

From the embedding of $L^{r'}(\Omega)$ into $(W^{1,p'}(\Omega))^*$, we thus obtain

$$\begin{aligned}
\int_0^T \left[\|\nabla V(s)\|^2_{L^p(\Omega)} + \|\nabla V(s)\|^r_{L^p(\Omega)}\right] ds &\leq C\left(1 + T + \|US'(V)\|_{L^2(0,T;L^{r'}(\Omega))}\right)^2 \\
&\quad + C\left(1 + T + \|US'(V)\|_{L^r(0,T;L^{r'}(\Omega))}\right)^r \\
&\leq C\left(1 + T + \|US'(V)\|_{L^2(0,T;L^2(\Omega))}\right)^2 \\
&\quad + C\left(1 + T + \|US'(V)\|_{L^r(0,T;L^{r'}(\Omega))}\right)^r
\end{aligned}$$

By interpolation, we find with $\dfrac{1}{r'} = 1 - \theta + \dfrac{\theta}{2} = 1 - \dfrac{\theta}{2} \quad \Leftrightarrow \quad \theta = 2\left(1 - \dfrac{1}{r'}\right) = \dfrac{2}{r}$

$$\begin{aligned}
\|US'(V)\|^r_{L^r(0,T;L^{r'}(\Omega))} &= \int_0^T \|U(s)S'(V(s))\|^r_{L^{r'}(\Omega)} ds \\
&\leq C \int_0^T \|U(s)S'(V(s))\|^{r(1-\theta)}_{L^1(\Omega)} \|U(s)S'(V(s))\|^{r\theta}_{L^2(\Omega)} ds \\
&\leq C(C')^{r(1-\theta)} \|U\|^{r(1-\theta)}_{L^\infty(0,T;L^1(\Omega))} \int_0^T \|U(s)S'(V(s))\|^{r\theta}_{L^2(\Omega)} ds \\
&= C\|U\|^{r-2}_{L^\infty(0,T;L^1(\Omega))} \|US'(V)\|^2_{L^2(0,T;L^2(\Omega))} \\
&\leq C\|US'(V)\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

Using the last two estimates, it follows from Proposition 5.7 that

$$\begin{aligned}
\int_0^T \left[\|\nabla V(s)\|^2_{L^p(\Omega)} + \|\nabla V(s)\|^r_{L^p(\Omega)}\right] ds &\leq C(1 + T^r + \|US'(V)\|^2_{L^2(0,T;L^2(\Omega))}) \\
&\overset{Prop. 5.7}{\leq} C(1 + T^r + (1 + T)) \leq C(1 + T^r),
\end{aligned}$$

so that inequality (33) becomes

$$\|U\|_{L^\infty(0,T;L^2(\Omega))} \leq Ce^{CT^r}.$$

(ii) We have shown in part *(i)* of this proof that the right hand side of the equation for V in (5) is bounded in the space $L^\infty(0,T;L^2(\Omega))$. We can therefore conclude that

$$\|V\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C\ \|U\|_{L^\infty(0,T;L^2(\Omega))} \leq Ce^{CT^r},$$

which completes the proof. □

Proposition 5.9 *The solution (U,V) of system (5), (6) with the logarithmic sensitivity function $S(V) = \log(V + c)$ is global.*

Proof: We can show that the pair $(U(T), V(T)) \in L^\infty(\Omega) \times W^{1,p}(\Omega)$ for a $p > 2$, so that we can extend the solution by the theorem of local existence up to a $T_1 > T$ and this extension is unique.

This procedure can be repeated and from existence on $[0, T_{n-1}]$ one infers existence on $[0, T_n]$ for a sequence $T < T_1 < ... < T_{n-1} < T_n$ for all $n \in I\!N$.

Suppose that $T_n \longrightarrow T_{max} < +\infty$ as $n \to +\infty$. From the estimates obtained in Proposition 5.8 we know that

$$\|U\|_{L^\infty(0,T_{max};L^2(\Omega))} \; + \; \|V\|_{L^\infty(0,T_{max};L^\infty(\Omega))} \; \leq \; Ce^{C(T_{max})^r},$$

so that T_{max} can be taken as a new starting point for an extension of the interval of existence (We have excluded explosion in $T_{max} < +\infty$.) and it follows that the solution (U, V) exists for all times. □

If $\chi \leq 1$, we can improve the T-dependence of the estimates with the following proposition.

Proposition 5.10 *Suppose $\chi \leq 1$. Defining $W := \dfrac{U}{e^{\chi S(V)}} = \dfrac{U}{(V+c)^\chi}$, we can rewrite system (5) in the following way*

$$\begin{aligned} (W(V+c)^\chi)_t &= \nabla((V+c)^\chi \nabla W) \\ V_t &= \alpha \Delta V - \beta V + \delta \frac{W}{(V+c)^{(1-\chi)}}. \end{aligned} \tag{35}$$

Now, there exist positive constants C and γ such that we have for the solution (W, V) of the system (35)

$$\|W\|_{L^\infty(0,T;L^2(\Omega))} + \|V\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C(1+T)^\gamma,$$

for every $T > 0$.

Proof: The equivalence between systems (5) and (35) with $W = \dfrac{U}{e^{\chi S(V)}} = \dfrac{U}{(V+c)^\chi}$ is obvious.

(i) We have for the solution W of the first equation in (35) with homogeneous Neumann boundary conditions

$$\|W\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla W\|^2_{L^2(0,T;L^2(\Omega))} \; \leq \; C(1+T)^\gamma,$$

where C and γ are independent of the time T.

We set $\psi := (V+c)^\chi$. We showed in Proposition 5.8 that V belongs to $L^\infty(0, T; L^\infty(\Omega))$ so that $\psi \in L^\infty(0, T; L^\infty(\Omega))$ follows. Moreover, because of $\chi \leq 1$,

$$\begin{aligned} \|\psi_t\|_{L^2(0,T;L^2(\Omega))} &= \chi \|(V+c)^{(\chi-1)} V_t\|_{L^2(0,T;L^2(\Omega))} \; = \; \chi \left\| \frac{V_t}{(V+c)^{(1-\chi)}} \right\|_{L^2(0,T;L^2(\Omega))} \\ &\leq \|V_t\|_{L^2(0,T;L^2(\Omega))} \; \leq \; C, \end{aligned} \tag{36}$$

by the a-priori-estimate obtained for V_t from the boundedness of the Lyapunov function (Corollary 4.4), where C does not depend on T.

Testing the first equation in (35) with W gives:

$$\langle (\psi W)_t, W\rangle \; + \; \int_\Omega \psi |\nabla W|^2 dx = 0,$$

which is because of

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \psi W^2 dx \;=\; \frac{1}{2}\int_\Omega \psi_t W^2 dx \;+\; \langle \psi W_t, W\rangle \;=\; \langle (\psi W)_t, W\rangle \;-\; \frac{1}{2}\int_\Omega \psi_t W^2 dx$$

equivalent to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \psi W^2 dx + \int_\Omega \psi |\nabla W|^2 dx = -\frac{1}{2}\int_\Omega \psi_t W^2 dx.$$

and by integration from 0 to t we obtain:

$$\frac{1}{2}\int_\Omega \psi W^2 dx\Big|_0^t + \int_0^t\int_\Omega \psi |\nabla W|^2 dx ds = -\frac{1}{2}\int_0^t\int_\Omega \psi_t W^2 dx ds,$$

so that (Note that $\psi \geq 1$.)

$$\begin{aligned}
\int_\Omega W^2(t)dx \;+\; 2\int_0^t\int_\Omega |\nabla W|^2 dx ds \;&\leq\; 2\int_\Omega (V_0+c)^\chi W^2(0)dx \;+\; \int_0^t\int_\Omega \psi_t W^2 dx ds \\
&\leq\; 2(\|V_0\|_{L^\infty(\Omega)}+c)^\chi \int_\Omega W^2(0)dx \;+\; \|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W^2\|_{L^2(0,t;L^2(\Omega))} \\
&=\; C_0\int_\Omega W^2(0)dx \;+\; \|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W\|^2_{L^4(0,t;L^4(\Omega))}. \qquad (37)
\end{aligned}$$

We will write for simplicity

$$Z_{0,t} := L^\infty(0,t;L^2(\Omega)) \cap L^2(0,t;H^1(\Omega))$$

and define the norm $\|w\|^2_{Z_{0,t}} := \left(\|w\|^2_{L^\infty(0,t;L^2(\Omega))} + \|\nabla w\|^2_{L^2(0,t;L^2(\Omega))}\right)$ for all $w \in Z_{0,t}$.

We know that

$$\|W\|^2_{L^2(0,T;L^2(\Omega))} \;=\; \left\|\frac{U}{e^{\chi S(V)}}\right\|^2_{L^2(0,T;L^2(\Omega))} \;\leq\; \|U\|^2_{L^2(0,T;L^2(\Omega))} \;\leq\; C(1+T),$$

for $\chi \neq 1$, and if $\chi = 1$,

$$\begin{aligned}
\|W\|^2_{L^2(0,T;L^2(\Omega))} \;&=\; \left\|\frac{U}{e^{S(V)}}\right\|^2_{L^2(0,T;L^2(\Omega))} \;=\; \left\|\frac{U}{(V+c)}\right\|^2_{L^2(0,T;L^2(\Omega))} \\
&=\; \|US'(V)\|^2_{L^2(0,T;L^2(\Omega))} \;\leq\; C(1+T),
\end{aligned}$$

by Proposition 5.7. Hence, we can apply the Gagliardo-Nirenberg Inequality to obtain

$$
\begin{aligned}
\|W\|^4_{L^4(0,t;L^4(\Omega))} &= \int_0^t \|W(s)\|^4_{L^4(\Omega)}ds \ \le\ C_1\int_0^t \|W(s)\|^2_{L^2(\Omega)}\|W(s)\|^2_{H^1(\Omega)}ds \\
&= C_1\int_0^t \|W(s)\|^2_{L^2(\Omega)}\left(\|W(s)\|^2_{L^2(\Omega)}+\|\nabla W(s)\|^2_{L^2(\Omega)}\right)ds \\
&\le C_1\|W\|^2_{L^\infty(0,t;L^2(\Omega))}\left(\|W\|^2_{L^2(0,t;L^2(\Omega))}+\|\nabla W\|^2_{L^2(0,t;L^2(\Omega))}\right) \\
&\le C_1\|W\|^2_{L^2(0,T;L^2(\Omega))}\|W\|^2_{L^\infty(0,t;L^2(\Omega))} \\
&\quad +\frac{C_1}{2}\left(\|W\|^4_{L^\infty(0,t;L^2(\Omega))}+\|\nabla W\|^4_{L^2(0,t;L^2(\Omega))}\right) \\
&\le C_2(1+T)\|W\|^2_{L^\infty(0,t;L^2(\Omega))}+\frac{C_1}{2}\|W\|^4_{Z_{0,t}} \\
&\le C_2^2(1+T)^2+\left(\frac{C_1}{2}+1\right)\|W\|^4_{Z_{0,t}}, \qquad (38)
\end{aligned}
$$

so that taking the supremum over $(0,t)$ in (37) yields

$$
\begin{aligned}
\|W\|^2_{Z_{0,t}} &\le C_0\|W(0)\|^2_{L^2(\Omega)}+\|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W\|^2_{L^4(0,t;L^4(\Omega))} \\
&\le C_0\|W(0)\|^2_{L^2(\Omega)}+\left(\frac{C_1}{2}+1\right)^{\frac{1}{2}}\|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W\|^2_{Z_{0,t}} \\
&\quad +C_2(1+T)\|\psi_t\|_{L^2(0,t;L^2(\Omega))} \\
&\le C_0\|W(0)\|^2_{L^2(\Omega)}+\tilde{C}\|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W\|^2_{Z_{0,t}}+C_2(1+T)\|\psi_t\|_{L^2(0,T;L^2(\Omega))} \\
&\le C_0\|W(0)\|^2_{L^2(\Omega)}+\tilde{C}\|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W\|^2_{Z_{0,t}}+\hat{C}(1+T)
\end{aligned}
$$

by (36). Now, let $t=t_1$ be so small that $\|\psi_t\|_{L^2(0,t_1;L^2(\Omega))}\le \frac{1}{2\tilde{C}}$. We then have

$$
\|W\|^2_{Z_{0,t_1}} \le 2(C_0\|W(0)\|^2_{L^2(\Omega)}+\hat{C}(1+T)). \qquad (39)
$$

Step by step, we can partition $[0,T]$ into finitely many intervals $[t_{k-1},t_k]$ for $k=1,...,s$, where $t_0=0$ and $t_s=T$ such that

$$
\|\psi_t\|_{L^2(t_{k-1},t_k;L^2(\Omega))}\le\frac{1}{2\tilde{C}}
$$

and consequently

$$
\|W\|^2_{Z_{t_{k-1},t_k}}\le 2\left[2\int_\Omega (V(t_{k-1})+c)^\chi W^2(t_{k-1})dx+\hat{C}(1+T)\right] \qquad (40)
$$

for $k=1,...,s$. On the other hand, we can choose the partition of $[0,T]$ such that

$$
\|\psi_t\|_{L^2(t_{k-1},t_k;L^2(\Omega))}\ge\frac{1}{4\tilde{C}} \quad \text{for} \quad k=1,...,s-1,
$$

so that one has the following bound for the number s of intervals:

$$
(s-1)\left(\frac{1}{4\tilde{C}}\right)^2 \le \sum_{k=1}^{s-1}\|\psi_t\|^2_{L^2(t_{k-1},t_k;L^2(\Omega))}\le\sum_{k=1}^{s}\|\psi_t\|^2_{L^2(t_{k-1},t_k;L^2(\Omega))}=\|\psi_t\|^2_{L^2(0,T;L^2(\Omega))},
$$

i.e.,

$$s \leq 1 + (4\tilde{C})^2 \|\psi_t\|^2_{L^2(0,T;L^2(\Omega))} \ \leq \ C.$$

We will now show by induction that

$$\|W\|^2_{L^\infty(t_{k-1},t_k;L^2(\Omega))} \ \leq \ \|W\|^2_{Z_{t_{k-1},t_k}} \ \leq \ C(1+T)^{\left(\frac{3}{2}\right)^{(k-1)}}. \tag{41}$$

For $k=1$, the estimate is true since

$$\|W\|^2_{L^\infty(0,t_1;L^2(\Omega))} \leq \|W\|^2_{Z_{0,t_1}} \leq C(1+T)$$

and because of equation (39).

Let us assume that

$$\|W\|^2_{L^\infty(t_{k-2},t_{k-1};L^2(\Omega))} \ \leq \ \|W\|^2_{Z_{t_{k-2},t_{k-1}}} \ \leq \ C(1+T)^{\left(\frac{3}{2}\right)^{(k-2)}}.$$

As by standard results on parabolic equations,

$$\|V(t_{k-1})\|_{L^\infty(\Omega)} \ \leq \ C(k)\|W\|_{L^\infty(t_{k-2},t_{k-1};L^2(\Omega))} \ \leq \ C\|W\|_{L^\infty(t_{k-2},t_{k-1};L^2(\Omega))},$$

we calculate with inequality (40):

$$\begin{aligned}
\|W\|^2_{L^\infty(t_{k-1},t_k;L^2(\Omega))} &\leq \|W\|^2_{Z_{t_{k-1},t_k}} \\
&\leq 2\left[2(\|V(t_{k-1})\|_{L^\infty(\Omega)} + c)^\chi \|W(t_{k-1})\|^2_{L^2(\Omega)} + \hat{C}(1+T)\right] \\
&\leq 4\left(C(1+T)^{\frac{\chi}{2}\left(\frac{3}{2}\right)^{(k-2)}} + c^\chi\right) C(1+T)^{\left(\frac{3}{2}\right)^{(k-2)}} + 2\hat{C}(1+T) \\
&\leq C\left\{(1+T)^{\left(\frac{\chi}{2}+1\right)\left(\frac{3}{2}\right)^{(k-2)}} + (1+T)^{\left(\frac{3}{2}\right)^{(k-2)}} + (1+T)\right\} \\
&\leq C(1+T)^{\left(\frac{3}{2}\right)^{(k-1)}}
\end{aligned}$$

and (41) is proven.

Finally, we get the following time-independent estimate

$$\begin{aligned}
\|W\|^2_{Z_{0,T}} &\leq \sum_{k=1}^{s} \|W\|^2_{Z_{t_{k-1},t_k}} \leq \sum_{k=1}^{s} C(1+T)^{\left(\frac{3}{2}\right)^{(k-1)}} \\
&\leq \sum_{k=1}^{s} C(1+T)^{\left(\frac{3}{2}\right)^{(s-1)}} \leq sC(1+T)^{\left(\frac{3}{2}\right)^{(s-1)}}
\end{aligned}$$

and our claim follows with $\gamma = \dfrac{1}{2}\left(\dfrac{3}{2}\right)^{(s-1)}$.

(ii) It now follows for the solution V of the second equation of system (35) that

$$\|V\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C(1+T)^\gamma.$$

We have shown for the right hand side of the linear equation for V in (35) that

$$\|\delta W\|_{L^\infty(0,T;L^2(\Omega))} \leq \delta C(1+T)^\gamma$$

and it follows by standard arguments as in Ladyženskaja et al. [13] that $\|V\|_{L^\infty(0,T;L^\infty(\Omega))}$ depends Lipschitz continuously on $\|W\|_{L^\infty(0,T;L^2(\Omega))}$. $\square$

Remark: If we had homogeneous Dirichlet boundary conditions, we could obtain time-independent estimates for the solution (U,V) of system (5). (See Post [19].)

Conclusion 5.11 We have shown that

$$e^{\chi S(V)} = (V+c)^\chi \;\leq\; (\|V\|_{L^\infty(0,T;L^\infty(\Omega))} + c)^\chi \;\leq\; (Ce^{CT^r} + c)^\chi =: C_e.$$

To bound U in $L^\infty(0,T;L^p(\Omega))$ for $2 < p < \infty$, we can now apply the argument for bounded sensitivity functions: As in the proof of Proposition 5.2, we obtain

$$\int_\Omega W^2(t) e^{\chi S(V(t))} dx \;\leq\; C_1 e^{C_2(pC_e)^2} + C_3 e^{C_2(pC_e)^2} C_e K^{\frac{p^2}{2}},$$

where $K^{\frac{p}{2}}$ is the bound for $\|U\|_{L^\infty(0,T;L^{\frac{p}{2}}(\Omega))}$. To obtain $\|U(t)\|^p_{L^p(\Omega)} \leq K^{p^2}$, we require that $\|U_0\|_{L^1(\Omega)} \leq K$, $C_1 e^{C_2 C_e^2} C_e \leq \frac{1}{2}K$ and $C_3 e^{C_2 C_e^2} C_e \leq \frac{1}{2} K^{\frac{1}{2}}$, which means that K has to behave like $e^{C_3 C_e^2}$, i.e., $K = C\exp(Ce^{CT^r})$, and hence

$$\|U\|_{L^\infty(0,T;L^p(\Omega))} \leq C\exp(pCe^{CT^r})$$

for all $2 < p < \infty$, so that estimate (14) holds, too. We have thus proven Theorem 5.1 completely for the logarithmic sensitivity function.

6 Asymptotic Behaviour

6.1 Convergence to a Steady State

In this section, we are going to investigate the behaviour for $t \to +\infty$ of a weak solution $(U(t,x), V(t,x))$ of system (5), (6) in the sense of Definition 2.1. We will show for the class $\mathcal{S}$ of sensitivity functions convergence of a subsequence $(U(t_k), V(t_k))$ with $t_k \to +\infty$ to a possibly non-trivial steady state (U^*, V^*).

Theorem 6.1 *Let $\Omega \subset I\!R^2$ be a Lipschitz domain and $S \in \mathcal{S}$. If we have a global solution (U,V) of system (5), (6) for which all terms in the Lyapunov function $F(U,V)$ are bounded, then there exists a sequence $t_k \to +\infty$, a function V^* and a constant W^* such that*

$$V(t_k) \longrightarrow V^* \;\; \text{in } H^1(\Omega), \quad \frac{U(t_k)}{e^{\chi S(V(t_k))}} \longrightarrow W^* \;\; \text{in } L^p(\Omega) \;\; \text{for all} \;\; 1 \leq p < \infty,$$

$$U(t_k) \longrightarrow U^* := W^* e^{\chi S(V^*)} \quad in \ L^p(\Omega) \quad for \ all \quad 1 \le p < \infty$$

and

$$F(U(t_k), V(t_k)) \longrightarrow F(U^*, V^*) \quad as \quad t_k \to +\infty.$$

Furthermore, the limit function U^* has the form

$$U^* = \frac{\|U_0\|_{L^1(\Omega)} \ e^{\chi S(V^*)}}{\int_\Omega e^{\chi S(V^*)} dx}$$

and V^* solves the following boundary value problem

$$-\alpha \Delta V^* + \beta V^* \;=\; \delta U^* S'(V^*) \;=\; \delta \frac{\|U_0\|_{L^1(\Omega)} \ e^{\chi S(V^*)}}{\int_\Omega e^{\chi S(V^*)} dx} S'(V^*) \tag{42}$$

in Ω with $\nu \cdot \nabla V^* = 0$ on $\partial\Omega$.

Proof: Let us define $W(t,x) := \dfrac{U(t,x)}{e^{\chi S(V(t,x))}}$.

(i) There exists a sequence $t_k \to +\infty, \ t_k \in I\!R_+$, such that $V_t(t_k) \to 0$ in $L^2(\Omega)$, $V(t_k) \rightharpoonup V^*$ in $H^1(\Omega)$ and $W(t_k) \to W^*$ in $L^p(\Omega)$ for all $1 \le p < \infty$.

We have

$$C|\nabla\sqrt{W}|^2 \le 4e^{\chi S(V)}|\nabla\sqrt{W}|^2 = \frac{(\nabla U - \chi U \nabla S(V))^2}{U} = U|\nabla(\log U - \chi S(V))|^2. \tag{43}$$

Let us define the functional $I(t) := t(\|V_t(t)\|^2_{L^2(\Omega)} + \|\nabla\sqrt{W(t)}\|^2_{L^2(\Omega)})$. From (43) and Corollary 4.4, we have

$$\int_0^t (\|V_t(s)\|^2_{L^2(\Omega)} + \|\nabla\sqrt{W(s)}\|^2_{L^2(\Omega)}) ds \le C$$

for all $t \ge 0$, so that

$$\int_0^t I(s) ds \le Ct \ \text{ for all } t \ge 0.$$

Therefore, there exists a sequence $t_k \to +\infty$, such that $I(t_k) \le 2C$ for all t_k.[†] We therefore deduce that as $t_k \to +\infty$

$$\|V_t(t_k)\|^2_{L^2(\Omega)} + \|\nabla\sqrt{W(t_k)}\|^2_{L^2(\Omega)} \;\longrightarrow\; 0.$$

With the a-priori-estimates from the Lyapunov function (See Corollary 4.4 in Section 4.) we additionally know that $\|V(t_k)\|_{H^1(\Omega)} \le C$ for all t_k, so that there exists a V^* such that

[†]If there existed a $t_0 \ge 0$ with $I(t) > 2C$ for all $t \ge t_0$, then it would follow that

$$Ct \ge \int_0^t I(s) ds \ge \int_{t_0}^t I(s) ds > 2C(t - t_0),$$

i.e., $2t_0 > t$ for all $t \ge t_0$, which is impossible.

$V(t_k) \rightharpoonup V^*$ in $H^1(\Omega)$. (If necessary, here and later we pass to a subsequence of $\{V(t_k)\}$.) Also, we have

$$
\begin{aligned}
\|\sqrt{W(t_k)}\|^2_{H^1(\Omega)} &= \|\sqrt{W(t_k)}\|^2_{L^2(\Omega)} + \|\nabla\sqrt{W(t_k)}\|^2_{L^2} = \|W(t_k)\|_{L^1(\Omega)} + \|\nabla\sqrt{W(t_k)}\|^2_{L^2(\Omega)} \\
&\le \|U(t_k)\|_{L^1(\Omega)} + \|\nabla\sqrt{W(t_k)}\|^2_{L^2(\Omega)} = \|U_0\|_{L^1(\Omega)} + \|\nabla\sqrt{W(t_k)}\|^2_{L^2(\Omega)} \le C,
\end{aligned}
$$

and the existence of a W^* follows with $\sqrt{W(t_k)} \rightharpoonup \sqrt{W^*}$ in $H^1(\Omega)$.

Because of $\|\nabla\sqrt{W(t_k)}\|_{L^2(\Omega)} \to 0$, we even have strong convergence

$$
\sqrt{W(t_k)} \longrightarrow \sqrt{W^*} \text{ in } H^1(\Omega) \text{ as } t_k \to +\infty \quad \text{with } \nabla\sqrt{W^*} = 0,
$$

and by the compact embedding of $H^1(\Omega)$ into $L^p(\Omega)$,

$$
W(t_k) \longrightarrow W^* = \text{const. in } L^p(\Omega) \text{ for all } 1 \le p < \infty.
$$

(ii) We also have $e^{\chi S(V(t_k))} \longrightarrow e^{\chi S(V^*)}$ in $L^p(\Omega)$ as $t_k \to +\infty$ for all $1 \le p < \infty$.

From the Trudinger-Moser Inequality (which can be shown to hold on a general Lipschitz domain by extension of the functions to a smooth set if necessary), we have an estimate of the form

$$
\|e^{\chi S(V)}\|_{L^p(\Omega)} \le C(\|V\|_{H^1(\Omega)}, p)
$$

for all $V \in H^1(\Omega)$ and all $1 \le p < \infty$. Since the sequence $\{V(t_k)\}$ is uniformly bounded in $H^1(\Omega)$, it therefore follows that $\{e^{\chi S(V(t_k))}\}$ is uniformly bounded in every $L^p(\Omega)$. We use

$$
\begin{aligned}
|e^{\chi S(V(t_k,x))} - e^{\chi S(V^*(x))}| &\le \chi S'(\tilde{V}(x)) e^{\chi S(\tilde{V}(x))} |V(t_k,x) - V^*(x)| \\
&\le \chi C' (e^{\chi S(V(t_k,x))} + e^{\chi S(V^*(x))}) |V(t_k,x) - V^*(x)|,
\end{aligned}
$$

where $\tilde{V}(x)$ is an intermediate value between $V(t_k,x)$ and $V^*(x)$, and we obtain

$$
\begin{aligned}
\|e^{\chi S(V(t_k))} - e^{\chi S(V^*)}\|_{L^p(\Omega)} &= \left(\int_\Omega |e^{\chi S(V(t_k,x))} - e^{\chi S(V^*(x))}|^p dx\right)^{\frac{1}{p}} \\
&\le \chi C' \left(\int_\Omega (e^{\chi S(V(t_k,x))} + e^{\chi S(V^*(x))})^p |V(t_k,x) - V^*(x)|^p dx\right)^{\frac{1}{p}} \\
&\le \chi C' C \left(\|e^{\chi S(V(t_k))}\|_{L^{2p}} + \|e^{\chi S(V^*)}\|_{L^{2p}}\right) \|V(t_k) - V^*\|_{L^{2p}(\Omega)} \\
&\le C \|V(t_k) - V^*\|_{L^{2p}(\Omega)} \longrightarrow 0.
\end{aligned}
$$

Thus, $e^{\chi S(V(t_k))} \longrightarrow e^{\chi S(V^*)}$ in $L^p(\Omega)$ for all $1 \le p < \infty$.

(iii) The remaining assertions in the theorem hold.

The convergence of the $\{U(t_k)\}$ follows from

$$\begin{aligned}
\|U(t_k) - W^* e^{\chi S(V^*)}\|^p_{L^p(\Omega)} &\leq C \int_\Omega \Big\{ |W(t_k,x) e^{\chi S(V(t_k,x))} - W(t_k,x) e^{\chi S(V^*(x))}|^p \\
&\quad + |W(t_k,x) e^{\chi S(V^*(x))} - W^*(x) e^{\chi S(V^*(x))}|^p \Big\}\, dx \\
&\leq C \int_\Omega \Big(W(t_k,x) [e^{\chi S(V(t_k,x))} - e^{\chi S(V^*(x))}] \Big)^p dx \\
&\quad + C \int_\Omega \Big([W(t_k,x) - W^*(x)] e^{\chi S(V^*(x))} \Big)^p dx \\
&\leq C\, \|W(t_k)\|^p_{L^{2p}(\Omega)} \|e^{\chi S(V(t_k))} - e^{\chi S(V^*)}\|^p_{L^{2p}(\Omega)} \\
&\quad + C\, \|W(t_k) - W^*\|^p_{L^{2p}(\Omega)} \|e^{\chi S(V^*)}\|^p_{L^{2p}(\Omega)} \\
&\leq C \Big(\|e^{\chi S(V(t_k))} - e^{\chi S(V^*)}\|^p_{L^{2p}(\Omega)} + \|W(t_k) - W^*\|^p_{L^{2p}(\Omega)} \Big) \\
&\longrightarrow 0
\end{aligned}$$

for all $1 \leq 2p < \infty$, so that $U(t_k) \longrightarrow U^*$ in every $L^p(\Omega)$.

Passing to the limit $t_k \to +\infty$ in the weak formulation of the V-equation,[†] one obtains the stationary equation (42). Since on one hand

$$\|U^*\|_{L^1(\Omega)} = \|W^* e^{\chi S(V^*)}\|_{L^1(\Omega)} = W^* \int_\Omega e^{\chi S(V^*(x))} dx$$

and on the other hand $\|U^*\|_{L^1(\Omega)} = \|U_0\|_{L^1(\Omega)}$, we obtain

$$W^* = \frac{\|U_0\|_{L^1(\Omega)}}{\int_\Omega e^{\chi S(V^*)} dx}, \qquad \text{i.e.,} \qquad U^* = \frac{\|U_0\|_{L^1(\Omega)}\ e^{\chi S(V^*)}}{\int_\Omega e^{\chi S(V^*)} dx}.$$

Finally, testing the difference of the equations for the $V(t_k)$ and V^* with $(V(t_k) - V^*)$ yields the strong convergence of the sequence $\{V(t_k)\}$ to V^* in $H^1(\Omega)$ and the values of F converge as well. □

Remark: We can generalize Theorem 6.1 to higher dimensions. Let us consider the case $n = 3$. By the continuous embedding of $H^1(\Omega)$ into $L^p(\Omega)$ for $1 \leq p \leq 6$, we obtain in the proof of the theorem strong convergence of the sequence $\{V(t_k)\}$ to V^* in $L^6(\Omega)$ and $W(t_k) \longrightarrow W^*$ in $L^3(\Omega)$.

If the sensitivity function is bounded, $e^{\chi S(V)}$ is globally Lipschitz continuous so that we obtain $e^{\chi S(V(t_k))} \longrightarrow e^{\chi S(V^*)}$ in $L^p(\Omega)$ for $1 \leq p \leq 6$ and we still get

$$\begin{aligned}
\|U(t_k) - W^* e^{\chi S(V^*)}\|^2_{L^2(\Omega)} &\leq C \int_\Omega \Big(W(t_k,x) [e^{\chi S(V(t_k,x))} - e^{\chi S(V^*(x))}] \Big)^2 dx \\
&\quad + C \int_\Omega \Big([W(t_k,x) - W^*(x)] e^{\chi S(V^*(x))} \Big)^2 dx
\end{aligned}$$

[†]Note that by continuity of $S'(V)$ we have pointwise convergence of $S'(V(t_k)) \to S'(V^*)$ and it follows by the Dominated Convergence Theorem that $S'(V(t_k)) \to S'(V^*)$ in every $L^p(\Omega)$.

$$
\begin{aligned}
&\le C\, \|W(t_k)\|^2_{L^3(\Omega)} \|e^{\chi S(V(t_k))} - e^{\chi S(V^*)}\|^2_{L^6(\Omega)} \\
&\quad + C\, \|W(t_k) - W^*\|^2_{L^3(\Omega)} \|e^{\chi S(V^*)}\|^2_{L^6(\Omega)} \\
&\le C \left(\|e^{\chi S(V(t_k))} - e^{\chi S(V^*)}\|^2_{L^6(\Omega)} + \|W(t_k) - W^*\|^2_{L^3(\Omega)} \right) \\
&\longrightarrow 0,
\end{aligned}
$$

i.e., we have $U(t_k) \longrightarrow U^*$ in $L^2(\Omega)$ as well as the other results of the theorem.

If $S(V) = S_3(V) = \log(V + c)$ (and $\chi < 6$), then $(V(t_k) + c)^\chi \longrightarrow (V^* + c)^\chi$ in the space $L^{\frac{6}{\chi}}(\Omega)$ and $W(t_k) \longrightarrow W^*$ in $L^3(\Omega)$, so that we have convergence of the the functions $U(t_k) = W(t_k)(V(t_k) + c)^\chi$ for $p \le \frac{6}{2+\chi}$ if $\chi \le 4$. The rest of the theorem follows, too.

6.2 Trivial and Non-trivial Steady States

In this section, we will, in analogy to results by Gajewski and Zacharias in [6] for the equations without sensitivity function, study examples of solutions tending to trivial and non-trivial steady states, respectively.

On one hand, we will find conditions on the data of the problem which ensure convergence of the solution to the trivial constant state $(1, C_V)$ where the constant C_V fulfills $\beta C_V = \delta S'(C_V)$. (W.l.o.g. we will assume here that $\|U_0\|_{L^1(\Omega)} = |\Omega|$, so that $C_U = \dfrac{1}{|\Omega|}\|U_0\|_{L^1(\Omega)} = 1$.)

On the other hand, we will give an example for the logarithmic sensitivity function $\log(V + c)$, $c \ge 1$, where the limit steady state (U^*, V^*) found in Section 6.1 is non-constant, i.e., different from the trivial constant solution $(1, C_V)$, provided the chemotactic coefficient as well as the production rate are large.

Proposition 6.2 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain. Let the sensitivity function $S \in \mathcal{S}$ be twice continuously differentiable and satisfy the conditions $-\chi S''(V) \le 1$ and $S''(V) \le -\gamma\chi(S'(V))^2$ for a $\gamma > 1 + \dfrac{\alpha}{4}$. If the chemotactic coefficient χ and the coefficient δ in the production term for V are sufficiently small, then we obtain convergence of the solution $(U(t), V(t))$ of system (5) with (6) to the trivial constant steady state $(1, C_V)$ with $\beta C_V = \delta S'(C_V)$ in $L^\Phi(\Omega) \times L^2(\Omega)$ as $t \to \infty$.*

Proof: Let us define the functional

$$
F_*(U(t), V(t)) = \int_\Omega \left\{ U(t) \log U(t) - \chi(U(t) - 1) S(V(t)) + \frac{1}{2}(V(t) - C_V)^2 \right\} dx. \tag{44}
$$

We will show that under (smallness) conditions on the parameters of the system there exists a constant $b > 0$, such that

$$
\frac{d}{dt} F_*(U(t), V(t)) \;\le\; -b F_*(U(t), V(t)).^\dagger \tag{45}
$$

†Note that as a consequence of this estimate, $F_*(U, V)$ is a second Lyapunov function for system (5) with (6).

If we differentiate $F_*(U(t),V(t))$ with respect to t, using the equations (5) with (6) and the relation $\frac{|\nabla U|^2}{U} = 4|\nabla\sqrt{U}|^2$ as well as $\beta C_V = \delta S'(C_V)$, we obtain

$$\begin{aligned}
\frac{d}{dt}F_*(U,V) &= \int_\Omega \{U_t(\log U - \chi S(V)) \;+\; \chi V_t(1-U)S'(V) \;+\; (V-C_V)V_t\}\,dx \\
&\overset{(5)}{=} -\int_\Omega \Big\{U|\nabla(\log U - \chi S(V))|^2 \;+\; \alpha\nabla V\nabla[\chi(1-U)S'(V)+V]\Big\}\,dx \\
&\quad -\int_\Omega (\beta V - \delta U S'(V))(\chi(1-U)S'(V) + (V-C_V))dx \\
&= -4\int_\Omega |\nabla\sqrt{U}|^2 dx \;+\; 2\chi\int_\Omega \nabla U\nabla S(V)dx - \chi^2\int_\Omega U|\nabla S(V)|^2 dx \\
&\quad +\chi\alpha\int_\Omega U|\nabla V|^2 S''(V)dx \;+\; \chi\alpha\int_\Omega \nabla U\nabla S(V)dx \\
&\quad -\alpha\int_\Omega |\nabla V|^2(\chi S''(V)+1)dx \;-\; \beta\int_\Omega (V-C_V)^2 dx \\
&\quad -\delta\int_\Omega (S'(C_V) - US'(V))(V-C_V)dx \\
&\quad -\chi\int_\Omega (\beta V - \beta C_V + \delta S'(C_V) - \delta U S'(V))(1-U)S'(V)dx \\
&= -4\int_\Omega |\nabla\sqrt{U}|^2 dx \;+\; \chi(2+\alpha)\int_\Omega \nabla U\nabla S(V)dx \\
&\quad +\chi\int_\Omega U|\nabla V|^2(\alpha S''(V) - \chi(S'(V))^2)dx \\
&\quad -\alpha\int_\Omega |\nabla V|^2(\chi S''(V)+1)dx \;-\; \beta\int_\Omega (V-C_V)^2 dx \\
&\quad -\delta\int_\Omega \left[(S'(C_V)-S'(V))(V-C_V) \;+\; S'(V)(1-U)(V-C_V)\right]dx \\
&\quad -\chi\int_\Omega \beta(V-C_V)(1-U)S'(V)dx \\
&\quad -\chi\delta\int_\Omega [S'(C_V) - S'(V) + S'(V)(1-U)](1-U)S'(V)dx,
\end{aligned}$$

We are going to use the conditions required of the sensitivity function $-\chi S''(V) \le 1$ and $S''(V) \le -\chi\gamma(S'(V))^2$. Moreover, we apply Young's Inequality to the second term on the right hand side where we write $\nabla U\nabla S(V)$ as $\frac{\nabla U}{2\sqrt{U}}2\sqrt{U}\nabla S(V)$ with an $\varepsilon_1 > 0$.

$$\begin{aligned}
\frac{d}{dt}F_*(U,V) &\le -4\int_\Omega |\nabla\sqrt{U}|^2 dx \;+\; \int_\Omega \left(\frac{\varepsilon_1}{2}|\nabla\sqrt{U}|^2 \;+\; \frac{2(2+\alpha)^2\chi^2 U|\nabla S(V)|^2}{\varepsilon_1}\right)dx \\
&\quad -\chi^2\int_\Omega U|\nabla S(V)|^2(1+\alpha\gamma)dx \;-\; \beta\int_\Omega (V-C_V)^2 dx \\
&\quad -\delta\int_\Omega \left[(S'(C_V)-S'(V))(V-C_V) \;+\; S'(V)(1-U)(V-C_V)\right]dx \\
&\quad -\chi\int_\Omega \beta(V-C_V)(1-U)S'(V)dx \\
&\quad -\chi\delta\int_\Omega [S'(C_V) - S'(V) + S'(V)(1-U)](1-U)S'(V)dx,
\end{aligned}$$

Note that with $\gamma > 1 + \frac{\alpha}{4}$, it is possible to choose $\varepsilon_1 < 8$ such that the coefficient $1 + \alpha\gamma - \frac{2(2+\alpha)^2}{\varepsilon_1} \geq 0$, and it follows that $4 - \frac{\varepsilon_1}{2} =: \varepsilon_2 > 0$. Using Young's Inequality three more times, we obtain

$$\begin{aligned} \frac{d}{dt}F_*(U,V) \;\; &\leq \;\; -\varepsilon_2 \int_\Omega |\nabla\sqrt{U}|^2 dx \; - \; \left(\beta - \delta C'' - \frac{\delta C'}{2} - \frac{\chi\beta C'}{2} - \frac{\chi\delta C' C''}{2}\right) \int_\Omega (V - C_V)^2 dx \\ &\quad + \; \left(\frac{\delta C'}{2} + \frac{\chi\beta C'}{2} + \frac{\chi\delta C'C''}{2}\right) \int_\Omega (U-1)^2 dx. \end{aligned} \tag{46}$$

We also need the following estimate:

$$\|U - 1\|^2_{L^2(\Omega)} \;\leq\; \frac{1}{k}\|\nabla\sqrt{U}\|^2_{L^2(\Omega)} \tag{47}$$

with a positive constant k. In order to prove (47), we remind that, by the continuous Sobolev Embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$, there exists a $\bar{C} > 0$ such that

$$\|U\|^2_{L^2(\Omega)} \;\leq\; \bar{C}\left(\|U_x\|^2_{L^1(\Omega)} \;+\; \|U_y\|^2_{L^1(\Omega)}\right)$$

for all $U \in W^{1,1}(\Omega)$ with vanishing spatial mean value. Therefore, we can calculate for our function U

$$\begin{aligned} \|U - 1\|^2_{L^2(\Omega)} \;\; &\leq \;\; \bar{C}\left(\|U_x\|^2_{L^1(\Omega)} \;+\; \|U_y\|^2_{L^1(\Omega)}\right) \\ &= \;\; 4\bar{C}\left(\|\sqrt{U}(\sqrt{U})_x\|^2_{L^1(\Omega)} \;+\; \|\sqrt{U}(\sqrt{U})_y\|^2_{L^1(\Omega)}\right) \\ &\leq \;\; 4\bar{C}\|U\|_{L^1(\Omega)}\|\nabla\sqrt{U}\|^2_{L^2(\Omega)} \;\leq\; 4\bar{C}|\Omega| \; \|\nabla\sqrt{U}\|^2_{L^2(\Omega)}, \end{aligned}$$

which is (47) with $\frac{1}{k} = 4\bar{C}|\Omega|$. Using (47) in estimate (46), we finally obtain

$$\begin{aligned} \frac{d}{dt}F_*(U,V) \;\; &\leq \;\; - \; \left(\varepsilon_2 k - \frac{\delta C'}{2} - \frac{\chi\beta C'}{2} - \frac{\chi\delta C'C''}{2}\right) \int_\Omega (U-1)^2 dx \\ &\quad - \; \left(\beta - \delta C'' - \frac{\delta C'}{2} - \frac{\chi\beta C'}{2} - \frac{\chi\delta C'C''}{2}\right) \int_\Omega (V - C_V)^2 dx \end{aligned}$$

It is obvious that we can choose χ and δ so small that

$$\left(\varepsilon_2 k - \frac{\delta C'}{2} - \frac{\chi\beta C'}{2} - \frac{\chi\delta C'C''}{2}\right) \;=\; b_1 \;>0 \tag{48}$$

and

$$\left(\beta - \delta C'' - \frac{\delta C'}{2} - \frac{\chi\beta C'}{2} - \frac{\chi\delta C'C''}{2}\right) \;=\; b_2 \;>0. \tag{49}$$

Besides, because of $\int_\Omega U dx = |\Omega|$,

$$\begin{aligned} -\,\chi\int_\Omega (U-1)S(V)dx \;\; &= \;\; -\,\chi\int_\Omega (U-1)(S(V) - S(C_V))dx \\ &\leq \;\; \chi^2\int_\Omega (U-1)^2 dx \;+\; (C')^2\int_\Omega (V - C_V)^2 dx \\ &\leq \;\; \chi^2\int_\Omega (U-1)^2 dx \;+\; K\int_\Omega (V - C_V)^2 dx, \end{aligned} \tag{50}$$

with any $K \geq (C')^2$. For the $\int_\Omega (U-1)^2 dx$-term we need that

$$\int_\Omega U \log U dx \;\leq\; \int_\Omega (U-1)^2 dx, \tag{51}$$

which follows from $(U-1)^2 - U(\log U - 1) - 1 \;\geq\; 0$ for all $U \geq 0$. (See Post [19].) We go on calculating as follows

$$\begin{aligned}
\frac{d}{dt} F_*(U,V) \;&\leq\; -\frac{b_1}{2}\int_\Omega (U-1)^2 dx \;-\; \frac{b_2}{2\beta}\,\beta \int_\Omega (V-C_V)^2 dx \\
&\quad - \frac{b_1}{2\chi^2}\,\chi^2 \int_\Omega (U-1)^2 dx - \frac{b_2}{2K}\,K \int_\Omega (V-C_V)^2 dx \\
&\leq\; -\frac{b_1}{2}\int_\Omega U \log U dx \;-\; \frac{b_2}{2\beta}\,\beta \int_\Omega (V-C_V)^2 dx \\
&\quad - \frac{b_2}{2K}\left(\chi^2 \int_\Omega (U-1)^2 dx + K\int_\Omega (V-C_V)^2 dx\right) \\
&\leq\; \frac{b_2}{2K}\int_\Omega \left\{ -U\log U + \chi(U-1)S(V) - \beta(V-C_V)^2 \right\} dx \;\leq\; -\frac{b_2}{2K} F_*(U,V)
\end{aligned}$$

if K is chosen so large that $\dfrac{b_2}{K}$ is smaller than $\dfrac{b_1}{\chi^2}$, $\dfrac{b_2}{\beta}$ and b_1. Thus, we have proven estimate (45) under the smallness conditions for χ and δ (48) and (49) with $b = \dfrac{b_2}{2K}$. By Gronwall's Lemma, it now follows that

$$\begin{aligned}
\int_\Omega U \log U dx \;&-\; \chi \int_\Omega (U(t)-1) S(V(t)) dx \;+\; \frac{1}{2}\|V(t) - C_V\|^2_{L^2(\Omega)} \\
&=\; F_*(U(t),V(t)) \;\leq\; e^{-bt} F_*(U_0,V_0).
\end{aligned}$$

Similarly to (50), we can estimate

$$\begin{aligned}
-\chi\int_\Omega (U(t)-1)S(V)dx \;&=\; -\chi\int_\Omega (U(t)-1)(S(V) - S(C_V))dx \\
&\geq\; -\frac{1}{4}\,\|V(t) - C_V\|^2_{L^2(\Omega)} \;-\; \chi^2 (C')^2 \|U(t) - 1\|^2_{L^1(\Omega)} \\
&\geq\; -\frac{1}{4}\,\|V(t) - C_V\|^2_{L^2(\Omega)} \;-\; \chi^2 (C')^2 \|U(t)\log U(t)\|_{L^1(\Omega)},
\end{aligned}$$

where we applied again (51) in the last step, and we obtain under the additional smallness condition $\chi \;<\; \dfrac{1}{C'}$ the existence of a positive constant $\tilde{C}$ such that

$$\tilde{C}\|U(t)\log U(t)\|_{L^1(\Omega)} \;+\; \frac{1}{4}\|V(t) - C_V\|^2_{L^2(\Omega)} \;\leq\; e^{-bt} F_*(U_0,V_0).$$

Using here the estimate $\|U-1\|_{L^\Phi(\Omega)} \;\leq\; \|U\log U\|_{L^1(\Omega)}$ (See Post [19] for the proof.) finally gives

$$\tilde{C}\|U(t) - 1\|_{L^\Phi(\Omega)} \;+\; \frac{1}{4}\|V(t) - C_V\|^2_{L^2(\Omega)} \;\leq\; e^{-bt} F_*(U_0,V_0),$$

which yields the claimed convergence. □

Proposition 6.3 *Consider system (5), (6) with* $S(V) = \log(V+c)$, $c \geq 1$, *in the two-dimensional domain* $\Omega = \{(x_1, x_2) : 0 < x_1 < a,\ 0 < x_2 < b\}$. *Let*

$$U_0(x_1, x_2) \;=\; V_0(x_1, x_2) \;=\; 1 \;+\; \cos\frac{\pi x_1}{a} \tag{52}$$

for $(x_1, x_2) \in \Omega$.

There exist (sufficiently large) coefficients χ, δ *and* β, *such that the solution* (U, V) *of system (5), (6) will tend to a non-constant steady state* (U^*, V^*) *in the sense of Theorem 6.1.*

Proof: From Theorem 5.1 we know that the solution $(U(t), V(t))$ of system (5) is global for the initial values given in (52). Furthermore, a subsequence $(U(t_k), V(t_k))$ convergences by Theorem 6.1 to a steady state (U^*, V^*) satisfying

$$U^* = \frac{\|U_0\|_{L^1(\Omega)}\ e^{\chi S(V^*)}}{\int_\Omega e^{\chi S(V^*)} dx} \tag{53}$$

and

$$-\alpha \Delta V^* + \beta V^* = \delta U^* S'(V^*). \tag{54}$$

Since

$$\|U_0\|_{L^1(\Omega)} = \int_0^b \int_0^a \left(1 + \cos\frac{\pi x_1}{a}\right) dx_1 \;=\; ab + b \sin\frac{\pi x_1}{a}\bigg|_0^a \;=\; ab \;=\; |\Omega|,$$

we obtain by (53) and (54) for the trivial, i.e., constant steady state (C_U, C_V) that $C_U = 1$ and that C_V satisfies the relation $\beta C_V = \delta S'(C_V)$. It follows for the logarithmic sensitivity function that

$$C_V^2 + c\, C_V \;=\; \frac{C_V}{S'(C_V)} \;=\; \frac{\delta}{\beta}, \tag{55}$$

so that we calculate

$$C_V = -\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}}. \tag{56}$$

Hence, the value of the Lyapunov function

$$F(U(t), V(t)) = \int_\Omega \left\{ U(t) \log U(t) - \chi U(t) S(V(t)) + \frac{\chi}{2\delta}(\alpha |\nabla V(t)|^2 + \beta (V(t))^2) \right\} dx$$

at the point $(1, C_V)$ is

$$\begin{aligned}
F(1, C_V) \;&=\; \chi \int_\Omega \left[\frac{\beta}{2\delta} C_V^2 - \log(C_V + c)\right] dx \;\stackrel{(55)}{=}\; \chi ab \left[\frac{\beta}{2\delta}\left(\frac{\delta}{\beta} - c\, C_V\right) - \log(C_V + c)\right] \\
&=\; \chi ab \left[\frac{1}{2} - \frac{\beta c}{2\delta}\, C_V - \log(C_V + c)\right] \\
&\stackrel{(56)}{=}\; \chi ab \left[\frac{1}{2} - \frac{\beta c}{2\delta}\left(-\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}}\right) - \log\left(\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}}\right)\right] \\
&=\; \chi ab \left[\frac{1}{2} + \frac{\beta c^2}{4\delta} - \frac{\beta c}{2\delta}\sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}} - \log\left(\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}}\right)\right].
\end{aligned} \tag{57}$$

Now, we are going to determine the value of $F(U_0, V_0)$. Integral calculations, involving principally trigonometric relations, show that

$$\begin{aligned}\int_\Omega U_0 \log U_0 dx &= \int_0^b \int_0^a \left(1 + \cos\frac{\pi x_1}{a}\right) \log\left(1 + \cos\frac{\pi x_1}{a}\right) dx_1 dx_2 \\ &= \frac{ab}{\pi} \int_0^\pi (1 + \cos y) \log(1 + \cos y)\ dy \ = \ ab(1 - \log 2)\end{aligned}$$

and†

$$\begin{aligned}-\chi \int_\Omega U_0 S(V_0) dx &= -\chi \int_0^b \int_0^a \left(1 + \cos\frac{\pi x_1}{a}\right) \log\left(1 + c + \cos\frac{\pi x_1}{a}\right) dx_1 dx_2 \\ &= -\frac{\chi ab}{\pi} \int_0^\pi (1 + \cos y) \log(1 + c + \cos y) dy \\ &= -\chi ab \left(1 + c - \sqrt{c^2 + 2c} + \log\frac{1 + c + \sqrt{c^2 + 2c}}{2}\right).\end{aligned}$$

Finally

$$\begin{aligned}\frac{\chi}{2\delta} \int_\Omega \left(\alpha |\nabla V_0|^2 + \beta V_0^2\right) dx &= \frac{\chi ab}{2\delta\pi} \int_0^\pi \left(\alpha \sin^2 y + \beta\left(1 + 2\cos y + \cos^2 y\right)\right) dy \\ &= \frac{\chi ab}{4\delta}\left(\frac{\alpha\pi^2}{a^2} + 3\beta\right)\end{aligned}$$

Inserting the last three equalities into $F(U_0, V_0)$, we obtain

$$\begin{aligned}F(U_0, V_0) &= \int_\Omega \left\{U_0 \log U_0 - \chi U_0 S(V_0) + \frac{\chi}{2\delta}(\alpha|\nabla V_0|^2 + \beta V_0^2)\right\} dx \\ &= ab\chi\left(\frac{\alpha\pi^2}{4\delta a^2} + \frac{3\beta}{4\delta} - 1 - c + \sqrt{c^2 + 2c} - \log\frac{1 + c + \sqrt{c^2 + 2c}}{2} + \frac{1 - \log 2}{\chi}\right) \quad (58)\end{aligned}$$

We will now show, that for big values of χ, δ and β

$$F(U_0, V_0) \ < \ F(1, C_V). \tag{59}$$

By (57) and (58), inequality (59) is equivalent to

$$\begin{aligned}&\frac{1 - \log 2}{\chi} + \frac{\alpha\pi^2}{4\delta a^2} + \frac{3\beta}{4\delta} - \frac{\beta c^2}{4\delta} + \frac{\beta c}{2\delta}\sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}} + \log\left(\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}}\right) \\ &\qquad < \frac{3}{2} + c - \sqrt{c^2 + 2c} + \log\frac{1 + c + \sqrt{c^2 + 2c}}{2}.\end{aligned} \tag{60}$$

One can choose the parameters of the problem such that (60) is satisfied.

†One can show that $\int_0^\pi (1 + \cos y) \log(b + \cos y) dy = \pi\left[b - \sqrt{b^2 - 1} + \log\left(\frac{b + \sqrt{b^2 - 1}}{2}\right)\right]$ for $b \geq 1$.

Suppose for instance that $\delta = 3\beta$ and $c = 1$. Under these assumptions, (60) reduces to the condition

$$\frac{1-\log 2}{\chi} \quad + \quad \frac{\alpha\pi^2}{4\delta a^2} \; < \; \frac{7}{3} - \sqrt{3} + \log\frac{2+\sqrt{3}}{2} - \frac{1}{6}\sqrt{\frac{13}{4}} - \log\left(\frac{1}{2} + \sqrt{\frac{13}{4}}\right) \simeq 0.09,$$

so that (59) is fulfilled provided χ and δ are chosen sufficiently large.

Since we know that the values of the Lyapunov function F decrease along the evolution of the solution (U, V), this proves that the limit steady state (U^*, V^*) cannot be equal to the trivial stationary state $(1, C_V)$, so that we must have convergence of the solution to a non-constant stationary state. □

Remark: The example, which proves the existence of a non-trivial stationary solution, was chosen to be one-dimensional for simplicity. However, we know that by continuity we obtain the same situation in a neighbourhood of the initial values (52).

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