Picard-Einstein Metrics and Class Fields Connected with Apollonius Cycle

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with Appendices by
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Abstract

We define Picard-Einstein metrics on complex algebraic surfaces as Kähler-Einstein metrics with negative constant sectional curvature pushed down from the unit ball via Picard modular groups allowing degenerations along cycles. We demonstrate how the tool of orbital heights, especially the Proportionality Theorem presented in [H98], works for detecting such orbital cycles on the projective plane. The simplest cycle we found on this way is supported by a quadric and three tangent lines (Apollonius configuration). We give a complete proof for the fact that it belongs to the congruence subgroup of level $1+i$ of the full Picard modular group of Gauß numbers together with precise octahedral- symmetric interpretation as moduli space of an explicit Shimura family of curves of genus 3. Proofs are based only on the Proportionality Theorem and classification results for hermitian lattices and algebraic surfaces.

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Contents

1 Introduction

The main purpose of this article is to show that the world of complex algebraic surfaces is Picard-Einstein with a universal degeneration lifted finitely from a quadric and three tangents on the complex projective plane. The three tangent points are "points at infinity" (cusp points) from the non-euclidean metric viewpoint. I found this projective complexified *Apollonius configuration* in connection with Fuchsian systems of partial differential equations in Sakurai-Yoshida [S-Y] ("mysterious phenomenon", p. 1490; Figure 2, p. 1492). One calls a hermitian metric on a smooth complex surface $\mathring{\mathrm{X}}$ Picard-Einstein (in a wide sense), if it is Kähler-Einstein with negative constant sectional curvature. If, moreover, $\mathring{\mathrm{X}}$ is a Zariski open part of an algebraic surface X, then one says that X is Picard-Einstein (with Picard-Einstein metric) *degenerating* (at most) *along* $\mathrm{X} \setminus \mathring{\mathrm{X}}$. The Bergman metric on the two-dimensional complex unit ball $\mathbb{B}$ is Picard-Einstein, see [BHH], Appendix B, for a short approach. For a ball lattice $\Gamma \subset Aut_{hol}\,\mathbb{B}$ the (quasiprojective) quotient surface $X = X_\Gamma = \mathbb{B}/\Gamma$ (also any compactification $\hat{X}$ of X) is Picard-Einstein degenerating along the branch locus of the canonical quotient map $p = p_\Gamma : \mathbb{B} \longrightarrow \mathbb{B}/\Gamma$ (and along the compactification cycle). The Picard-Einstein property lifts to each finite cover $\hat{Y}$ of $\hat{X}$ degenerating (at most) along the preimages of branch loci of p_Γ and $\hat{Y} \longrightarrow \hat{X}$. We call $\hat{Y}$ *Picard-Einstein*, if it is finitely lifted (that means via finite covering) from a ball quotient surface $\mathbb{B}/\Gamma$ such that the Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ of $\mathbb{B}/\Gamma$ is the complex projective plane $\mathbb{P}^2$. If one finds a ball lattice with this property, then each complex projective surface is Picard-Einstein in the narrow sense because each such surface is a finite covering of $\mathbb{P}^2$, e.g. via general projections.

The first proof for the fact that $\mathbb{P}^2$ is Picard-Einstein (degenerating along six lines) can be found in [H86]. There we used the Picard modular group of Eisenstein numbers. The main result of this paper is to show that $\mathbb{P}^2$ is Picard-Einstein degenerating along the Apollonius configuration described above, see theorem 5.1. The corresponding group $\Gamma(1+i)$ is the congruence sublattice of $\Gamma := \mathbb{SU}(diag(1,1,-1), \mathfrak{O})$, $\mathfrak{O} = \mathbb{Z} + \mathbb{Z}i$, $i = \sqrt{-1}$, belonging to the ideal $\mathfrak{O}(1+i)$. This is a Picard modular group of Gauß numbers.

There are some papers which came already near to this result. First I have to mention Matsumoto's article [Mat]. There is proved that $\mathbb{P}^1 \times \mathbb{P}^1$ is the compactified ball quotient surface by a subgroup of $\Gamma(1+i)$ of index 2 but with $\Gamma = \mathbb{SU}(\left(\begin{smallmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right), \mathfrak{O})$. His proof is based on Mostow-Deligne's theorem [D-M] conversing multivalued solutions (hypergeometric functions) of a special Picard-Fuchs system by a (in [D-M] explicitly unknown) monodromy group acting on the ball. In the recent monograph of Yoshida [Y97] it appears in terms of admissible sequences, see Ch. VI, Table 1, case $d = 4$, $2+2+2+2$ (∞, ∞, ∞).

Already in [Ho83] we classified precisely the surface $\widehat{\mathbb{B}/\Gamma}$. The proof is reproduced in [H98], additionally with explicit description of the branch locus of p_Γ. The ramification locus (on $\mathbb{B}$) has been found before by Shvartsman [Sv1], [Sv2] via classification of some hermitian $\mathfrak{O}$-lattices. He calculated the Euler number of $\widehat{\mathbb{B}/\Gamma}$. The rationality of this surface has been proved before by Shimura [Sm64] after his celebrated general interpretations of arithmetic quotient varieties in [Sm63], which are called now "Shimura varieties". Since Shvartsmann's classification of Γ-elliptic points is not avalaible in publications, we fill that gap in sections 6, 7 classifying precisely the indefinite unimodular $rank-2$ sublattices of the Gauß lattice $\Lambda = \mathfrak{O}^3$ endowed with our diagonal hermitian metric of signature $(2,1)$. Very useful is Hashimoto's paper [Has] for this purpose.

The most natural way for finding a configuration (reduced cycle Z) on an orbiface (two-dimensional orbifold), which could be the degenerate locus of a Picard-Einstein metrics has been described in [H98]. Beside of quotient singularities we allow also cusp singularities on the surface. The irreducible components of the configuration (points and irreducible curves) are endowed with natural numbers or ∞ (weights) in an admissible manner. Then one gets an *orbital cycle*. The surface X together with the orbital cycle $\mathbf{Z}$ is called an *orbital surface*. The orbital surface germs around points are irreducible components of the orbital cycle are called *orbital points* or orbital curves, respectively. Points or curves with weight ∞ are called *cusp points* or *cusp curves*, respectively. They form a subcycle Z_∞ of Z whose support is denoted by X_∞. The finitely weighted points are *quotient points*. For details we refer to [H98], where we corresponded rational numbers to our orbital objects called *orbital hights*. The orbital surface hights (global hights H) generalize volumes of Γ- fundamental domains on $\mathbb{B}$ of arbitrary ball lattices Γ. The orbital curve hights (local hights h) do the same for the complex unit disc $\mathbb{D}$ and $\mathbb{D}$-lattice groups. Euler form and signature form define on this way two different orbital hights H_e, H_τ and h_e, h_τ called *Euler* or *signature hights*, respectively. A *finite uniformization* $Y \longrightarrow \mathbf{X}$ of an orbital surface

$\mathbf{X} = (X, \mathbf{Z})$ is a finite Galois covering $Y \longrightarrow X$ such that Y is smooth (outside cusp points) and the weights of the components of $\mathbf{Z}$ coincide with corresponding ramification indices. A *ball uniformization* of $\mathbf{X}$ is a (locally finite) infinite Galois covering (quotient map by a ball lattice) $\mathbb{B} \longrightarrow X_f := X \setminus X_\infty$ again with wights equal to corresponding ramification indices. We announce the following

Theorem 1.1 *For an orbital surface $\mathbf{X} = (X, \mathbf{Z})$ the following conditions are equivalent:*

(i) $\mathbf{X}$ has a ball uniformization

(ii) The proportionality conditions

(Prop 2) $H_e(\mathbf{X}) = 3H_\tau(X) > 0$

(Prop 1) $h_e(\mathbf{C}) = 2h_\tau(\mathbf{C}) < 0$ for all orbital curves $\mathbf{C} \subset \mathbf{Z}$

are satisfied, and there exists a finite uniformization Y of $\mathbf{X}$, which is of general type.

The direction (i) $\Rightarrow$ (ii) has been proved in [H98], see Proportionality Theorem IV.9.2. Notice that our h_τ is 3 times h_τ of [H98]. The other direction follows from the degree homogenity of the global heights and a well-known theorem of R.Kobayashi-Miyaoka-Yau applied to Y. Namely, it is easy to see that the (Prop 2)-condition lifts to the logarithmic Chern number condition $\bar{c}_1^2 = 3\bar{c}_2$ for Y.

□

In section 3 we use the explicit orbital hight machine for detecting suitable wights for points and curves on the Apollonius configuration on $\mathbb{P}^2$ such that the corresponding orbital surface satisfies the proportionality conditions. This has been done for demonstrating and understanding a general approach to detect Picard-Einstein metrics on surfaces. Any orbital configuration (X, Z) defines a system $Dioph(X, Z)$ of diophantine equations. It comes out from a system of a quadratic and some linear equations with rational coefficients closely related with (Prop 2) or (Prop 1), respectively, for which we have to determine inverse of natural numbers as solutions (the inverse of the weights we look for). There are at most finitely many solutions, see [H98], IV.10.

In the next section we transform the detected weights to seven properties (i),...,(vii) of a uniformizing ball lattice Γ' we look for using the Proportionality Theorem via the system $Dioph(X, Z)$ again, this time in converse direction: We know the weights but the data (Chern numbers, selfintersections) of X, Z are unknown. With the eight postulated properties we are able to determine these data and to classify surface and curves to get $\widehat{\mathbb{B}/\Gamma'} = \mathbb{P}^2$ and the Apollonius configuration back. In the sections 5,6,7 we prove that the congruence lattice $\Gamma(1+i)$ has all the eight properties.

In section 5 we prove that the structure of the factor group $\Gamma/\Gamma(2)$ is isomorphic to the binary octahedron group $2\mathbb{O}$. An essential point is to decide which of two possible unitary codes in $\mathbb{F}_2^8$ is defined by the intermediate factor group $\Gamma(1+i)/\Gamma(2)$. This is done by a non-elementary tool of algebraic topology (Armstrong's Theorem, see Theorem 8.2). Its application is well-prepared by the sections before. Knowing the code we find an intermediate ball lattice $\Gamma(2) \subset \Gamma_2 \subset \Gamma(1+i)$ with quotient surface $\mathbb{P}^1 \times \mathbb{P}^1$ and factor group Γ_2/Γ isomorphic to the binary dieder group $2S_3$ of order 12. Together with the appendix we prove that $\mathbb{P}^1 \times \mathbb{P}^1$ is the moduli space of the obviously $2S_3$-symmetric family of (double distinguished) curves $C_b : Y^3 = (X-1)(X+1)(X-b_1)^2(X-b_2)^2(X-b_3)^2$. The projective plane appears as moduli space of the (distinguished) curves via the map $C_b \mapsto \mathbb{P}b = (b_1 : b_2 : b_3)$.

In order to connect the family with octahedral-symmetric Picard modular forms it is important to know the surface $\widehat{\mathbb{B}/\Gamma(2)}$ because van Geemen [vGm] found a structure result for the ring of $\Gamma(2)$-modular forms in terms of theta constants and left open the problem of precise surface classification. For theta constants of Matsusaka's Γ_2'-level we refer also to [Mat]. Until now we know and announce that $\widehat{\mathbb{B}/\Gamma(2)}$ is a smooth rational surface with six cusp points. The curve part of the corresponding orbital cycle contains precisely ten smooth rational curves of wight 2 and selfintersection -1 on the blowing up model of the six cusp points. Nearby should be also congruence subgroups whose quotient surfaces are models of $E \times E$, E elliptic curve with complex $\mathbb{Q}(i)$-multiplication or of the Kummer surface $E \times E/\langle\iota\rangle$, ι the involution sending P to $-P$ on the abelian surface $E \times E$. Both surfaces together with ball quotient presentations are important. The first one should recognize Hirzebruch's abelian covers of $E \times E$ defined in [Hir] as Picard modular surfaces as it was done for Eisenstein numbers in [Ho86]. The second one could join Hilbert's 12-th problem for our special Shimura surface(s) with the 3-dimensional congruent number problem, see Narumiya-Shiga [N-S].

Next we turn our attention to a conjecture of Kobayashi [Kob] about complements of hypersurfaces in $\mathbb{P}^n$ to be (Kobayashi-) hyperbolic, if the degree of the sum of hypersurfaces is high enough. We refer to Dethloff- Schumacher-Wong [DSM] and the references given there for more details, restrict ourselves to $n = 2$ and ask for curve configurations Z on $\mathbb{P}^2$ such that $\mathbb{P}^2 \setminus supp(Z)$ is Picard-Einstein degenerated precisely along Z. This is a much stronger problem. For the complements of three quadrics or of a quadric and three lines the hyperbolicity is proved in general, but not individual. The orbital hight machine should be used to present more Picard-Einstein models. Is degree 5, as for the Apollonius configuration, the minimal possible degree ? In the mean time N.Vladov writes a detecting algorithm on MAPLE producing first experimental results: The declaration of all the three tangent points as cusp points leads only to our series of wights of cycle components such that the corresponding orbital surface satisfies the proportionality conditions. But there are also weight solutions for the cases that only 2, 1 or no of these three points are declared to be of cusp type (the other of quotient type), see Theorem 11.1. It seems to be quite possible that a good part of the 27 cases of the PDM (Picard-Deligne-Mostow) list of wighted lines on $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$, see e.g. [BHH], p. 201, can be lifted from the Apollonius configuration.

We finish the introduction with the following problem: Consider the class $\mathfrak{F}$ of all smooth compact complex algebraic surfaces finitely covering $\mathbb{P}^2$ with branch locus on the Appolonius configuration. Finite curve coverings of $\mathbb{P}^1$ branched over three points only are characterized as curves defined over number fields by a famous Theorem of Belyi [Bel] (for proof see also [Se89]). Find a similar characterization for the class $\mathfrak{F}$ of surfaces ! Belyi's curves are also characterized as compactified quotients curves by subgroups of $\mathbb{S}l_2(\mathbb{Z})$ acting on the upper half plane $\mathbb{H}$, see Shabat, Voevodsky [S-V], also [Se89], app. of 5.4. Which of the surfaces of $\mathfrak{F}$ are ball quotients by (a group commensurable with) a Picard modular group of Gauß numbers ?

2 The basic orbital surface: Plane with Apollonius cycle

We consider an orbital surface

$$\widehat{\mathbf{X}} = (\hat{X}; \widehat{\mathbf{C}}_0 + \widehat{\mathbf{C}}_1 + \widehat{\mathbf{C}}_2 + \widehat{\mathbf{C}}_3 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3) \tag{1}$$

with smooth compact complex algebraic surface $\hat{X}$ supporting the orbital cycle

$$\mathbf{Z}(\widehat{\mathbf{X}}) = \widehat{\mathbf{C}}_0 + \widehat{\mathbf{C}}_1 + \widehat{\mathbf{C}}_2 + \widehat{\mathbf{C}}_3 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3, \tag{2}$$

which consists of four orbital curves $\widehat{\mathbf{C}}_j$, $j = 0, 1, 2, 3$, on $\widehat{\mathbf{X}}$ with weight 4, three (finite) orbital abelian points $\mathbf{P}_j$, $j = 1, 2, 3$, of type $\mathbb{C}^2/Z_4 \times Z_4$ where $Z_4 \times Z_4 \subset \mathbb{G}l_2(\mathbb{C})$ denotes the abelian group generated by 2 opposite reflections of order 4, and $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ are precisely the orbital points at infinity. For the surface $\hat{X}$ and the reduced cycle

$$Z(\hat{\mathbf{X}}) = \hat{C}_0 + \hat{C}_1 + \hat{C}_2 + \hat{C}_3 + P_1 + P_2 + P_3 + K_1 + K_2 + K_3 \tag{3}$$

we claim the following conditions:

(i) The surface $\hat{X}$ is the projective plane $\mathbb{P}^2$

(ii) a) $\hat{C}_0$ is a quadric on $\mathbb{P}^2$;

b) $\hat{C}_1, \hat{C}_2, \hat{C}_3$ are projective lines on $\mathbb{P}^2$;

c) P_1, P_2, P_3 are the three different intersection points of these lines;

d) $\hat{C}_j$ is the tangent line of $\hat{C}_0$ at K_j, $j = 1, 2, 3$;

e) The *configuration divisor* $\hat{C}_0 + \hat{C}_1 + \hat{C}_2 + \hat{C}_3$ is *symmetric*. This means that there is an effective action of the symmetric group S_3 on $\mathbb{P}^2$ preserving $\hat{C}_0 + \hat{C}_1 + \hat{C}_2 + \hat{C}_3$.

Definition 2.1 *If these conditions are satisfied we call $\hat{C}_0 + \hat{C}_1 + \hat{C}_2 + \hat{C}_3$ a plane Apollonius configuration or Apollonius configuration on $\mathbb{P}^2$, the cyle $Z(\hat{\mathbf{X}})$ a reduced plane Apollonius cycle and each effective cycle with this support a plane Apollonius cycle.*

The properties a),b),c),d) mean that the Apollonius configuration on $\mathbb{P}^2$ consists of a plane quadric and three different tangent lines of it. We will see below that e) is automatically satisfied with a unique S_3-action. The following graphic describes the corresponding configuration together with three additional lines L_j joining P_j and K_j. For the rest of this section we work on $\hat{X} = \mathbb{P}^2$ and omit the hats over C_j. Moreover, we assume that all quadrics are non-degenerate, if the opposite is not stated.

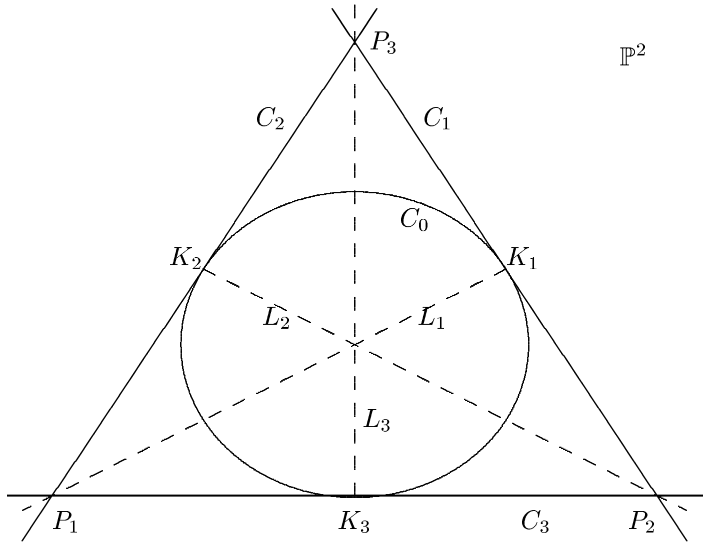

Figure 1.

Remark 2.2 *The three quadruples $\{P_j, K_1, K_2, K_3\}$, $j = 1, 2, 3$, are in general position. This means that each subtriple spans $\mathbb{P}^2$. Namely, the different points K_1, K_2, K_3 cannot ly on one line L because $(L \cdot C_0) = 2$. For the same reason, for example, the (tangent) line through P_1, K_2 cannot contain K_1 or K_3. By symmetry the argument is complete.*

Especially, we can choose the S_3-symmetric

Normalized Model 2.3

$$C_0 : (X + Y - Z)^2 - 4XY = X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ = 0;$$

$$\begin{array}{rclrclrcl}
C_1 : X & = & 0 & C_2 : Y & = & 0 & C_3 : Z & = & 0\,, \\
P_1 & = & (1:0:0) & P_2 & = & (0:1:0) & P_3 & = & (0:0:1)\,; \\
K_1 & = & (0:1:1) & K_2 & = & (1:0:1) & K_3 & = & (1:1:0)\,; \\
L_1 : Y & = & Z & L_2 : X & = & Z & L_3 : X & = & Y\,.
\end{array}$$

For finding the (unique) quadratic equation we refer to the end of this section (Lemma 2.10).

Proposition 2.4 *Up to $\mathbb{PG}l_3$-equivalence the Apollonius configuration is unique-ly determined. All Apollonius configurations are S_3-symmetric.*

Proof. Let $D_0 \subset \mathbb{P}^2$ be another quadric and D_1, D_2, D_3 three tangents touching D_0 in M_1, M_2, M_3, respectively. The intersection point of D_i, D_j is denoted by Q_k for $\{i, j, k\} = \{1, 2, 3\}$. Let π be the correspondence $P_1 \mapsto Q_1$, $K_j \mapsto M_j$, $j = 1, 2, 3$. By the main theorem of (elementary) projective geometry, this map extends uniquely to a projective transformation $\Pi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$, because the points P_1, K_1, K_2, K_3 (and their images) are in general position by the above remark. Π sends the C_0-tangents C_2, C_3 (through P_1 and K_2, K_3, respectively) to the D_0-tangents D_2, D_3. A quadric is uniquely determined by two given tangent lines and a point on it different from the touching points of the two tangents. Namely, the algebraic family of all plane quadrics is 5-dimensional. Going through three given points and two given tangent lines at two of them yield five linear conditions for the five (affine) parameters for the quadrics. Via projective transformation this can be also checked now more explicitly by example: Take $D_2 = X$-axis, $D_3 = Y$-axis in $\mathbb{C}^2 \subset \mathbb{P}^2$, $P_1 = (0,0)$, $K_2 = (1,0)$, $K_3 = (0,1)$. It is an easy calculation to see that the only quadric with tangents D_2, D_3 at K_2 or K_3, respectively, going through $K_1 := (2,1)$ is the circle $(X-1)^2 + (Y-1)^2 = 1$ with center $(1,1)$. Turning back to the general situation we see that Π sends the quadric C_0 to the quadric D_0. But then the tangent line C_1 at $K_1 \in C_0$ is sent to the tangent D_1 at $M_1 \in D_0$.

If a configuration is symmetric, then each projective transform of it is, by conjugation of the S_3-action. Since 2.3 is the symmetric we are through.

□

Corollary 2.5 *The action of the symmetric group S_3 on $\mathbb{P}^2$ preserving the configuration $C_0 + C_1 + C_2 + C_3$ is unique. It is determined by extending permutations of points $\pi : K_i \mapsto K_{\pi(i)}$, $P_i \mapsto P_{\pi(i)}$, $i = 1, 2, 3$, $\pi \in S_3$, to $\Pi \in Aut\,\mathbb{P}^2 = \mathbb{P}\mathbb{G}l_3(\mathbb{C})$. Especially for the normalized model 2.3 the group S_3 acts by permutation of canonical projective coordinates $(x : y : z)$ on $\mathbb{P}^2$.*

Proof. The general statement is a special case considered in the proof of Proposition 2.4 setting $D_0 = C_0$ and taking for D_1, D_2, D_3 an arbitrary permutation of C_1, C_2, C_3. For the normalized model the action is obvious.

□

Remark 2.6 *The lines L_1, L_2, L_3 defined in (1.4) have a common point.*

Proof. This can be checked now on any special model. The normalized model 2.3 yields $(1:1:1)$ as intersection point of the three lines.

□

Lemma 2.7 *Each projective representation $G \subset Aut\,\mathbb{P}^2$ of a finite group can be lifted to a linear representation $\tilde{G} \subset \mathbb{G}l_3(\mathbb{C})$. For given $d \in \mathbb{N}_+$ there is a unique central lift (group extension) $\tilde{G}_d \subset \mathbb{G}l_3(\mathbb{C})$ of G with the group $Z_{3d} \subset \mathbb{C}^*$ of 3d-th unit roots as (central) kernel. It consists of all lifts of elements of G with determinant in Z_d.*
Each finite lift $\tilde{G} \subset \mathbb{G}l_3(\mathbb{C})$ of G is a subgroup of $\tilde{G}_{3d}$ for a suitable $d \in \mathbb{N}_+$. The special lift $\tilde{G}_1 \subset \mathbb{S}l_3(\mathbb{C})$ has kernel Z_3 over G.

Proof. For each $g \in G$ we can find a lift $\tilde{g} \in \mathbb{G}l_3(\mathbb{C})$ because of the exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathbb{G}l_3(\mathbb{C}) \longrightarrow \mathbb{P}\mathbb{G}l_3(\mathbb{C}) \longrightarrow 1$$

of group homomorphism. The coset $\mathbb{C}^*\tilde{g}$ consists of all lifts of g. We can choose a special lift $\tilde{g}$ with determinant in Z_d. Then the subset $Z_{3d}\tilde{g}$ coincides with the set of all lifts of g with determinant in Z_d. By such choice $\tilde{g}$ for each $g \in G$ we obtain the group $\tilde{G}_d := \{Z_{3d}\tilde{g}; g \in G\} \subset \mathbb{G}l_3(\mathbb{C})$ together with exact sequences

$$1 \longrightarrow Z_{3d} \longrightarrow \tilde{G}_d \longrightarrow G \longrightarrow 1$$

$$1 \longrightarrow \mathbb{S}\tilde{G}_d \longrightarrow \tilde{G}_d \xrightarrow{det} Z_d \longrightarrow 1$$

with central kernels. Now it is clear that each finite representative lift of G to $\mathbb{G}l_3(\mathbb{C})$ is contained in one of the $\tilde{G}_d$ because the corresponding determinant group must be finite.

Write both exact sequences together in one diagram and complete it to a diagram with three exact rows and three exact columns. Then one gets an exact sequence

$$1 \longrightarrow Z_3 \longrightarrow \mathbb{S}\tilde{G}_d \longrightarrow G \longrightarrow 1.$$

Obviously, $\mathbb{S}\tilde{G}_d$ does not depend on d. It coincides with $\tilde{G}_1$. So the last sequence is nothing else but

$$1 \longrightarrow Z_3 \longrightarrow \tilde{G}_1 \longrightarrow G \longrightarrow 1. \qquad (4)$$

□

Denote for an arbitrary group H by $CF(H)$ the set of conjugation classes of finite subgroups of H. Obviously we get for all $n \in \mathbb{N}_+$ by $\mathbb{C}^*$- factorization a surjective map

$$CF(\mathbb{G}l_n(\mathbb{C})) \twoheadrightarrow CF(\mathbb{P}\mathbb{G}l_n(\mathbb{C})). \tag{5}$$

Let $CF_d(\mathbb{G}l_3(\mathbb{C}))$ be the subset of $CF(\mathbb{G}l_3(\mathbb{C}))$ consisting of all complete lifts $\tilde{G}_d$ with determinant d of finite subgroups G of $\mathbb{P}\mathbb{G}l_3(\mathbb{C})$. For $n = 3$ we get a bijective restriction of (5) to

$$CF_d(\mathbb{G}l_3(\mathbb{C})) \longleftrightarrow CF(\mathbb{P}\mathbb{G}l_3(\mathbb{C})). \tag{6}$$

For d $= 1$ one gets especially a bijection

$$CF(\mathbb{S}l_3(\mathbb{C})) = CF_1(\mathbb{G}l_3(\mathbb{C})) \longleftrightarrow CF(\mathbb{P}\mathbb{G}l_3(\mathbb{C})). \tag{7}$$

Let $P \subset \mathbb{G}l_3(\mathbb{C})$ be the subgroup of permutation matrices. We multiply the elements of P by their determinants to get a subgroup P_1 of $\mathbb{S}l_3(\mathbb{Z})$ isomorphic to S_3. We call it the *group of unimodular permutation matrices.*

Corollary 2.8 *Let $G \subset \mathbb{P}\mathbb{G}l_3(\mathbb{C})$ be a finite group isomorphic to S_3. Then G can be lifted uniquely to a subgroup of $\mathbb{S}l_3(\mathbb{C})$ conjugated to the group P_1 of unimodular permutation matrices. There exists a projective coordinate system on $\mathbb{P}^2$ such that G acts by permutation of coordinates.*

Proof. The second statement follows from the first because projective conjugations correspond to projective coordinate changes, and P_1 acts obviously by permutation of canonical coordinates. Take the lift $\tilde{G}_1 \subset \mathbb{S}l_3(\mathbb{C})$. It splits into $Z_3 \times \Sigma_3$ by (6) and Σ_3 projects isomorphically to $G \cong S_3$. Since Σ_3 is not abelian, the representation $\Sigma_3 \subset \mathbb{G}l_3(\mathbb{C})$ is not diagonizable that means it doesn't split into three characters. The only irreducible representations of S_3 are the characters 1, sgn (signature of permutations) and the faithful rank-2- representation δ realized as the dieder group of a regular triangle. Therefore there are, up to conjugation, only two faithful rank-3 representation, namely $1 + \delta$ and $\delta + sgn$, where only the latter has determinant $+1$. Therefore it is equivalent to the representation $P_1 \subset \mathbb{S}l_3(\mathbb{C})$. This means that the groups Σ_3 and P_1 are $\mathbb{G}l_3(\mathbb{C})$-conjugated, hence $\mathbb{S}l_3(\mathbb{C})$-conjugated. Since each $\mathbb{S}l_3$-lift of G must be contained in $\tilde{G}_1$ (via Lemma 2.7) we see that Σ_3 is the only possibility of isomorphic lifting.

□

Remark 2.9 *Let $\mathfrak{O}$ be an integral subdomain of the field $\mathbb{C}$ not containing primitive 3-rd unit roots. A finite group $G \subset \mathbb{P}\mathbb{G}l_3(\mathbb{C})$ has at most one unimodular lift $\tilde{G} \subset \mathbb{S}l_3(\mathfrak{O})$. If it exists, it must be isomorphic to G. Especially, S_3 has the unique representation $P_1 \subset \mathbb{S}l_3(\mathfrak{O})$.*

Proof. The first statement is true because

$$Z_3 = \left\{ \left(\begin{smallmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{smallmatrix} \right) \; ; \; \omega \, a \, 3-rd \, unit \, root \right\}$$

intersects $\mathbb{S}l_3(\mathfrak{O})$ trivially and because of the exact sequence (6). For the second one has only to lift the representation of S_3 permuting canonical coordinates and to apply the uniqueness statement of Corollary 2.8. □

Lemma 2.10 *For three projective lines C_1, C_2, C_3 on $\mathbb{P}^2$ intersecting each other in different points and for a given subgroup $\Sigma_3 \cong S_3$ of $\mathbb{P}\mathbb{G}l_3$ permuting them there is precisely one quadric C_0 with tangents C_1, C_2, C_3. For the canonical coordinate axes $X = 0$, $Y = 0$, $Z = 0$ of $\mathbb{P}^2$ the corresponding quadric (see 2.3, normalized model) has equation*

$$X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ = (X + Y - Z)^2 - 4XY = 0.$$

Proof. Assume that we find an Apollonius configuration $C_0 + C_1 + C_2 + C_3$ extending the given lines. Because of $\mathbb{P}\mathbb{G}l_3$-equivalence of such configurations (Proposition 2.4) we can assume that these lines are the coordinate axis

$$C_1 : X = 0 \quad , \quad C_2 : Y = 0 \quad , \quad C_3 : Z = 0$$

with intersection points

$$P_1 = (1:0:0) \quad , \quad P_2 = (0:1:0) \quad , \quad P_3 = (0:0:1).$$

Moreover, the action of S_3 on $\mathbb{P}^2$ can assumed to be the most natural one permuting coordinates (Corollary 2.5) with unique lift $S_3 \subset \mathbb{S}l_3(\mathbb{C})$ represented by permutation matrices (Corollary 2.8). Let

$$aX^2 + bY^2 + cZ^2 + 2dXY + 2eXZ + 2fYZ = 0.$$

be the homogeneous equation of an arbitrary plane quadric. All S_3-invariant quadrics must satisfy simultaneously three equations

$$\begin{aligned} aY^2 + bX^2 + cZ^2 + 2dXY + 2eYZ + 2fXZ &= 0 \\ aZ^2 + bY^2 + cX^2 + 2dYZ + 2eXZ + 2fXY &= 0 \\ aX^2 + bZ^2 + cY^2 + 2dXZ + 2eXY + 2fYZ &= 0, \end{aligned}$$

which have to be the same up to a factor. It follows that $a = b = c = 1$ (without loss of generality) and $d = e = f$. Therefore the S_3-invariant quadrics form a 1-parameter family

$$X^2 + Y^2 + Z^2 + 2dXY + 2dXZ + 2dYZ = 0.$$

The curve C_0 has contact of order 2 with $C_1 : X = 0$ at a point $K_1 = (0:1:t)$. Substituting the coordinates of K_1 in the quadratic equation, this means that the equation

$$0 = 1 + t^2 + 2dt = (t + d)^2 + (1 - d^2)$$

must have a unique solution for t. Therefore $d = -t = 1$ or $d = -t = -1$. By symmetry we conclude that there are precisely two S_3-invariant quadrics with tangent C_1, namely

$$X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ = (X + Y - Z)^2 - 4XY = 0$$

and

$$X^2 + Y^2 + Z^2 + 2XY + 2XZ + 2YZ = (X + Y + Z)^2 = 0.$$

But the latter one is degenerated. □

For later use we need the more general

Definition 2.11 *An Apollonius cycle on a smoooth compact complex algebraic surface Y is a cycle*

(8) $$\mathbf{Z} = v_0L_0 + v_1L_1 + v_2L_2 + v_3L_3 + P_1 + P_2 + P_3 + K_1 + K_2 + K_3,$$

where the v_i's are positive integers, $P_1, P_2, P_3, K_1, K_2, K_3$ are points on Y, and the L_i's are smooth complete algebraic curves on Y with the following intersection behaviour:

$$L_0 \cdot L_j = 2K_j \text{ for } j = 1, 2, 3; \; L_i \cdot L_j = P_k \text{ for } \{i, j, k\} = \{1, 2, 3\}.$$

The supporting reduced curve $L_0+L_1+L_3+L_3$ is called an Apollonius configuration on Y. The Apollonius cycle (configuration) is called symmetric, iff there is an algebraic S_3-action on Y, which preserves the cycle $\mathbf{Z}$ permuting effectively its components $\hat{C}_j$, P_j, K_j, $j = 1, 2, 3$, respectively.

Remark 2.12 *Obviously, $v_1 = v_2 = v_3$ holds in the symmetric case. If Y is the projective plane, then the Apollonius configuration is automatically of the (symmetric) plane Apollonius cycle consisting of a quadric and three tangent lines as defined in 2.1 .*

Namely, from $(L_j \cdot L_k) = 1$ for $0 < j < k \le 3$ and Bezout's theorem it follows that the L_j's are smooth curves of degree 1 on $\mathbb{P}^2$. Therefore L_1, L_2, L_3 are projective lines. Since $(L_0 \cdot L_j) = 2$ we conclude with the same argument that L_0 is a quadric touched by L_j at K_j. For symmetry we refer to Proposition 2.4.

3 Proportionality

Turn back now to the more precise notations of 2.1 not assuming in this section the symmetry condition (ii) e). We blow up each of the the points K_j twice such that the proper transforms of $\hat{C}_i$ for $i = 1, 2, 3$ on the resulting surface $\tilde{X}$ do not intersect the proper transform of $\hat{C}_0$. The exceptional divisor $E(\tilde{X} \longrightarrow \hat{X})$ on $\tilde{X}$ consists of three connected components. Each of them is a pair of transversally crossing smooth rational curves with selfintersection -1 or -2, respectively. Then we contract the three -2-curves to get a surface X' with three quotient singularities of type $\mathbb{C}^2/\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ lying on exceptional curves $E_1, E_2, E_3 \subset X'$. On this way we get an orbital birational morphism $\mathbf{X}' \longrightarrow \hat{\mathbf{X}}$ being isomorphic outside $X'_\infty = E_1 + E_2 + E_3$ and $\hat{X}_\infty = K_1 + K_2 + K_3$. The proper transforms of the $\hat{C}_j$ are denoted by C'_j, $j = 0, 1, 2, 3$, respectively. On this way we get a complete orbital surface

$$\mathbf{X}' = (X'; \mathbf{C}'_0 + \mathbf{C}'_1 + \mathbf{C}'_2 + \mathbf{C}'_3 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3)$$

called the canonical locally abelian model of $\hat{\mathbf{X}}$. The finite part supported by $X = X'_f = X' \setminus X'_\infty$ is the open orbital surface

$$\mathbf{X} = (X; \mathbf{C}_0 + \mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3)$$

with supporting non-compact curves $C_j = C'_{jf} = C'_j \setminus X'_\infty$. The orbital cycle $Z(\mathbf{X}')$ is described in the following picture

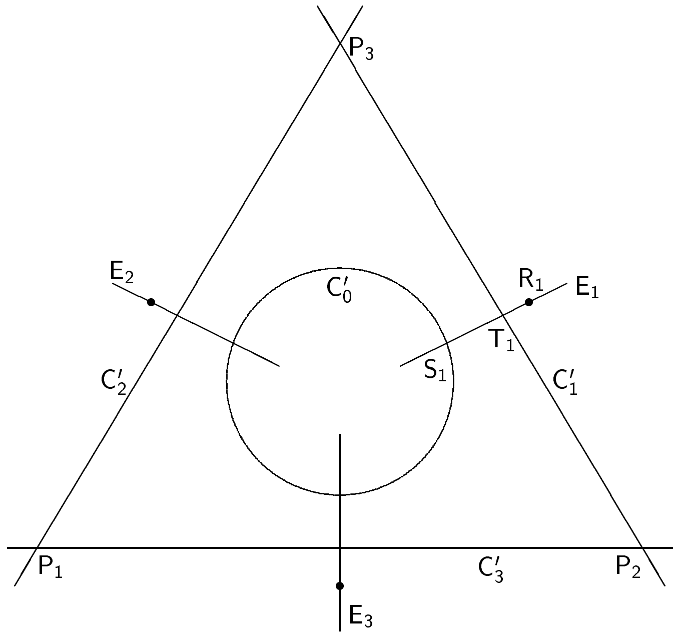

Figure 2. • singularity of type <2, 1>

The open orbital curves can be written as

$$\mathbf{C}_0 = 4C_0, \mathbf{C}_1 = (4C_1; \mathbf{P}_2 + \mathbf{P}_3), \mathbf{C}_2 = (4C_2; \mathbf{P}_1 + \mathbf{P}_3), \mathbf{C}_3 = (4C_3; \mathbf{P}_1 + \mathbf{P}_2)$$

The corresponding atomic graphs of the four orbital curves look like

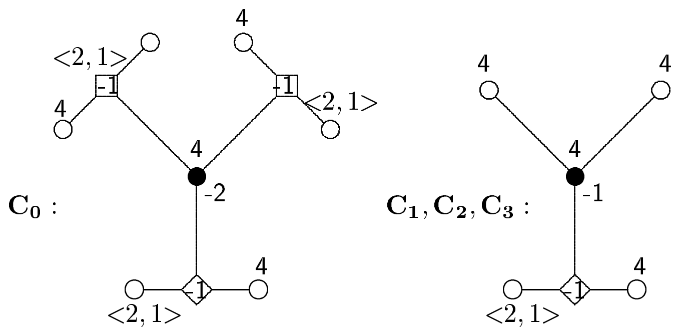

Figure 3.

and the molecular graph of the whole orbital cycle is

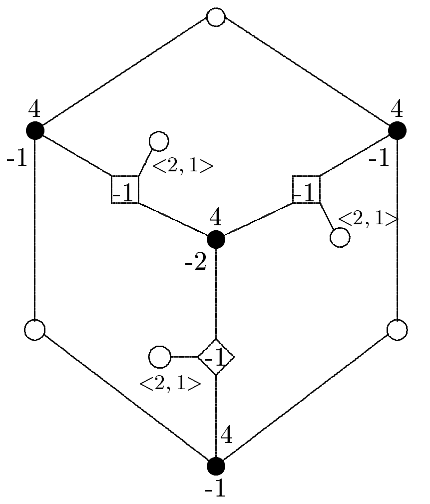

Figure 4.

In [H98], IV, Theorem 4.9.2, we proved that there are rather strong proportionality conditions for an orbital surface to be an orbital ball quotient. For this purpose we defined orbital heights for orbital curves and surfaces, which are rational numbers. First one has to draw the graph of an orbital curve $\hat{\mathbf{C}}$ on an arbitrary $\mathbb{B}$−orbital surface $\hat{\mathbf{X}}$ ($\mathbb{B}$−orbital means that only ball cusp singularities are allowed "at infinity"). On the open "finite" part $\mathbf{X}$ of $\hat{\mathbf{X}}$ at most quotient singularities are admitted. We restrict ourselves to abelian quotient singularities because in our example more complicated ones do not occur. Moreover we assume that the orbital curves are smooth for the same reason. The (atomic) graph of $\hat{\mathbf{C}}$ $= (v\hat{C}; \sum \mathbf{P}_i + \sum \mathbf{K}_j)$ looks star-like:

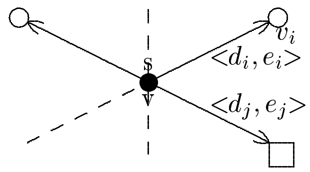

Figure 5.

The center represents the curve $\hat{C}$ weighted with $v \in \mathbb{N}_+$ and s is the selfintersection number (C'^2) on the minimal singularity resolution $\tilde{X} \longrightarrow X'$ of the canonical locally abelian resolution $X' \longrightarrow \hat{X}$, which replaces each cusp point K by an irreducible curve E_K (finite quotient of an elliptic curve) supporting (at most 4) cyclic surface singularities.

The proper transform of $\hat{C}$ on X' or $\tilde{X}$ is denoted by C'. The arrows to small circles represent cyclic surface singularities P_i of type $\langle d_i, e_i\rangle$ of X' lying on C' and the circle itself represents the curve germ of weight v_i crossing C' at P_i. The abelian point $\mathbf{P}_i : \underset{v}{\circ}\xrightarrow{\langle d_i,e_i\rangle}\underset{v_i}{\circ}$ consist of the crossing curve germes of C, C_i with weights v, v_i, respectively. The small boxes represent cusp points lying on $\hat{C}$, and the arrow to the box represent the intersection point of E_K and C' on X' being a cyclic singularity of type $\langle d_j, e_j\rangle$ isomorphic, by definition, to the singularity of $\mathbb{C}^2/\langle\begin{pmatrix}\zeta & 0\\ 0 & \zeta^e\end{pmatrix}\rangle$, where ζ denotes a primitive d-th unit root. The weight t at the box is the selfintersection of (the proper transform of) E_K on $\tilde{X}$. We omit the arrow orientation and $\langle\ ,\ \rangle$, if $\langle d_i, e_i\rangle$ or $\langle d_j, e_j\rangle = \langle 1, 0\rangle$. This means that the corresponding intersection point is non-singular. The arrow orientation is also omitted, if the singularity of type $\langle d, e\rangle$ is symmetric. This means that its minimal resolution (linear tree of smooth rational curve with selfintersection numbers read off from the continued fraction of $\frac{d}{e}$) is symmetric. Examples are given in figure 3. For more details we refer to [H98]. There we defined (see IV, Definition 4.7.3 and restrict to our situation) the *Euler hight* of $\mathbf{C}$ by

(9)
$$h_e(\mathbf{C}) = e(C') - \sum(1 - \frac{1}{v_i d_i}) - \#C'_\infty,$$

and the signature (or selfintersection) hight

$$h_\tau(\mathbf{C}) = \frac{1}{v}[(C'^2) + \sum \frac{e_i}{d_i} + \sum \frac{e_j}{d_j}] \tag{10}$$

(which is $3\boldsymbol{\tau}_f(\hat{\mathbf{C}})$ in the notations of [H98]). The first sum runs over all abelian points $\mathbf{P}_i$ on $\mathbf{C}$ and the second sum in (10) over all arrows $\overset{\langle d_j,e_j\rangle}{\circ\!\longrightarrow\!\diamond}$ joining the center with a cusp box, see picture 5. Looking at the graphs 3 we can calculate

$$\begin{aligned} h_e(\mathbf{C}_0) &= e(C_0') - 3 = 2 - 3 = -1 \\ h_e(\mathbf{C}_k) &= e(C_k') - (1 - \frac{1}{4}) - (1 - \frac{1}{4}) - 1 = -\frac{1}{2},\ k = 1,2,3 \end{aligned} \tag{11}$$

$$\begin{aligned} h_\tau(\mathbf{C}_0) &= \frac{1}{4}[(C_0'^2) + 0 + 0] = \frac{1}{4}(-2) = -\frac{1}{2}, \\ h_\tau(\mathbf{C}_k) &= \frac{1}{4}[(C_k'^2) + 0 + 0] = \frac{1}{4}(-1) = -\frac{1}{4},\ k = 1,2,3. \end{aligned} \tag{12}$$

Until now we did not prove that the points K_1, K_2, K_3 are allowed to be considered as cusp points. For the corresponding orbital curves $\mathbf{E} = \mathbf{E}_K$, $K = K_i, i \in \{1,2,3\}$, on $\mathbf{X}'$ with formal (arbitrary) wight $w \in \mathbb{N}_+$ we have the graph

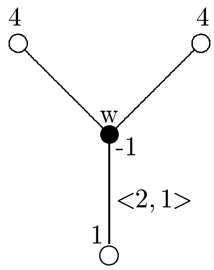

Figure 6.

$$(Prop\ \infty)$$

$$\begin{array}{lcl} h_e(\mathbf{E}) & = & 2 - (1-\frac{1}{4}) - (1-\frac{1}{4}) - (1-\frac{1}{2}) \quad = \quad 0, \\ h_\tau(\mathbf{E}) & = & \frac{1}{w}[-1 + \frac{0}{4} + \frac{0}{4} + \frac{1}{2}] \quad = \quad -\frac{1}{2}w \quad < 0. \end{array}$$

This is a cusp curve condition (see below) and really the graph appears in the graphical classification list of cusp points in [H98], III, Figure 3.5.3, type $(2,4,4)$. So we can change to the cusp graph of $\mathbf{K}$, which looks like

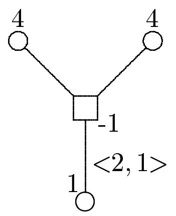

Figure 7.

Now we calculate the *hights* of $\mathbf{X}$ using Propsition 4.10.2 in [H98], chapter IV, as definition. The local contributions appear in [H98], IV, Table 10.2, the global ones in (4.10.2), (4.10.3) there. Since the open surface X is smooth and all quotient points on $\mathbf{X}$ are abelian the formulas for the *Euler height* and the *signature height* simplify to

$$\begin{aligned} H_e(\mathbf{X}) = e(X') - \sum(1 - \frac{1}{v_i})h_e(\mathbf{C}_i) - \sum h_e(\mathbf{P}_k) \\ - 2\#\{rational\, cusp points\, of\, \hat{X}\} \end{aligned} \tag{13}$$

$$H_\tau(\mathbf{X}) = \tau(X') - \frac{1}{3}\sum(v_i - \frac{1}{v_i})h_\tau(\mathbf{C}_i) - \sum h_\tau(\mathbf{P}_k) - \sum h_\tau(\mathbf{K}_m) \tag{14}$$

with

$$\begin{aligned} e(X') &= Euler\,number\,of\,X' \\ &= e(\tilde{X}) - \#\{components\,of\,E(\tilde{X} \longrightarrow X')\} = e(\tilde{X}) - 3, \\ e(\tilde{X}) &= \sum(-1)^i dim\,H^i(\tilde{X},\mathbb{C}) = Euler\,number\,of\,\tilde{X} \end{aligned}$$

and

$$\begin{aligned} \tau(X') &= \tau(\tilde{X}) + \#\{components\,of\,E(\tilde{X} \longrightarrow X')\} = \tau(\tilde{X}) + 3, \\ \tau(\tilde{X}) &= signature\,of\,\tilde{X} = signature\,of\,H_2(\tilde{X},\mathbb{R}). \end{aligned}$$

The sums in (13), (14) run over all orbital curves $\mathbf{C}_i$, all abelian points $\mathbf{P}_k$ on $\mathbf{X}$ and all cusp points $\mathbf{K}_m$ on $\hat{\mathbf{X}}$. The point contributions can be read off from the molecular graph of the orbital cycle $\mathbf{Z}(\hat{\mathbf{X}})$ connecting the graphs of orbital curves and points as demonstrated in our example in picture 4. Namely, for abelian points $\mathbf{P}$ and cusp points $\mathbf{K}$ we have

$$\begin{aligned} h_e(\mathbf{P}) &= 1 - \frac{1}{vd} - \frac{1}{v'd} + \frac{1}{v'vd}\ (\mathbf{P} : \underset{v}{\circ}\overset{\langle d,e\rangle}{\longrightarrow}\underset{v'}{\circ}\ in\,general) \\ &= 1 - \frac{1}{4} - \frac{1}{4} + \frac{1}{16} = \frac{9}{16}\ for\,our\,special\,\mathbf{P}'s, \\ 3h_\tau(\mathbf{P}) &= 3l_P + Tr(P) - \frac{e}{d} - \frac{e'}{d}\ (in\,general) \\ &= 3\cdot 0 + 0 - 0 - 0 = 0\ for\,our\,special\,\mathbf{P}'s, \\ 3h_\tau(\mathbf{K}) &= Tr(\mathbf{K}) + \sum_{j=1}^{4}(3l_j - \frac{e_j}{d_j})\ (in\,general) \\ &= -3 + 3\cdot 1 - \frac{1}{2} = -\frac{1}{2}\ for\,our\,special\,\mathbf{K}'s. \end{aligned} \tag{15}$$

Thereby l_P denotes the length of a resolution curve E_P (number of irreducible components of the linear tree E_P of rational curves) of the cyclic singularity P, $Tr(P)$ the trace of the intersection matrix of these components, $Tr(\mathbf{K})$ has the same meaning for the intersection matrix of $\tilde{E}_K$ being the preimage of $E_K \subset X'$ on $\tilde{X}$. The numbers l_j are the lengths of minimal resolutions of the cyclic surface singularities $Q_j \in X'$ of type $\langle d_j, e_j\rangle$ sitting on E_K.

Knowing $\hat{X} = \mathbb{P}^2$ we get $e(\hat{X}) = 3$, $\tau(\hat{X}) = 1$, hence

$$e(\tilde{X}) = 3 + 6 = 9 \quad , \quad \tau(\tilde{X}) = 1 - 6 = -5$$

and

$$e(X') = 9 - 3 = 6 \quad , \quad \tau(X') = -5 + 3 = -2. \tag{16}$$

Now we are able to calculte the heights of $\mathbf{X}$ explicitly substituting the local hights (11), (12), (15) and $e(X')$, $\tau(X')$ into (13), (14), respectively:

$$\begin{aligned} H_e(\mathbf{X}) &= 6 - (1 - \frac{1}{4})(-1) - 3(1 - \frac{1}{4})(-\frac{1}{2}) - 3\cdot\frac{9}{16} - 2\cdot 3 = \frac{3}{16}, \\ H_\tau(\mathbf{X}) &= -2 - \frac{1}{3}[(4 - \frac{1}{4})(-\frac{1}{2}) + 3(4 - \frac{1}{4})(-\frac{1}{4})] - 3\cdot 0 - 3\cdot(-\frac{1}{6}) = \frac{1}{16}. \end{aligned} \tag{17}$$

Summerizing ($Prop\ \infty$), (11) and (17) we proved in this section the following

Proposition 3.1 *The orbital surface $\hat{\mathbf{X}}$ of (1) with $\hat{X} = \mathbb{P}^2$ and orbital locus (2) supported by any Apollonius configuration 2.1, (ii) a),b),c),d) satisfies the proportionality conditions for ball quotient surfaces described in [H98] (IV.9, Theorem 4.9.2):*

$$\begin{array}{cr} h_e(\mathbf{E}_k) = 0\ ,\ h_\tau(\mathbf{E}_k) < 0\ ,\ k = 1,2,3; & (Prop\ \infty) \\ no\,condition\,here\,because\,all\,finite\,orbital\,points\,are\,abelian; & (Prop\ 0) \\ h_e(\mathbf{C}_i) = 2h_\tau(\mathbf{C}_i) < 0\ ,\ i = 0,1,2,3; & (Prop\ 1) \\ H_e(\mathbf{X}) = 3H_\tau(\mathbf{X}) > 0. & (Prop\ \ 2) \end{array}$$

□

Until now it is not generally known that the four proportionality conditions are sufficient for $\hat{\mathbf{X}}$ to be a ball quotient. In section 4 we prove it for our special plane orbital surface. This will be prepared in the next section translating precise hights and local conditions to geometric lattice conditions on the ball. For this purpose one has to read backwards the proof of the Proportionality Theorem 4.9.2 in [H98], well-prepared in the book parts before. In the section after we find an arithmetic ball lattice satisfying all these conditions.

4 Ball lattice conditions

We look for an arithmetic ball lattice $\Gamma \subset \mathbb{SU}((2,1),\mathbb{C}) \subset \mathbb{G}l_3(\mathbb{C})$ acting effectively on the complex two-ball

$$\mathbb{B} = \{(z_1, z_2) = (z_1 : z_2 : 1) \in \mathbb{P}^2; |z_1|^2 + |z_2|^2 < 1\} \subset \mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$$

via projective (fractional linear) transformations with postulated data described in 4.1 below (for special Γ' instead of general Γ). For the sake of simplicity we assume that all our ball lattices Γ are arithmetical (arithmetic defined subgrop of $\mathbb{SU}((2,1),\mathbb{C})$) and that they act effectively on $\mathbb{B}$.

Furthermore we use the following notions, see [H98], especially chapter IV, for more details. A *reflection* is an element $1 \neq \sigma \in \Gamma$ of finite order fixing a subdisc $\mathbb{D} = \mathbb{D}_\sigma$ of $\mathbb{B}$ pointwise. The disc $\mathbb{D}_\sigma$ is uniquely determined by σ. It is called a Γ-*reflection disc*, if such $\sigma \in \Gamma$ exists. If Γ is fixed we omit the prefix Γ-, also for further notations depending on Γ. For given subdisc $\mathbb{D}$ of $\mathbb{B}$ we call σ a $\mathbb{D}$-*reflection*, if $\mathbb{D} = \mathbb{D}_\sigma$ for a reflection σ. The group of $\mathbb{D}$-reflections in Γ is finite cyclic. Its order is called the Γ-*reflection order* at/of $\mathbb{D}$. It coincides, say by definition, with the ramification index of the natural locally finite quotient map $p : \mathbb{B} \longrightarrow \mathbb{B}/\Gamma$ along $\mathbb{D}$, and appears as wight of the orbital image curve $\mathbb{D}/\Gamma$ on the orbital quotient surface $\mathbb{B}/\Gamma$.

A Γ-cusp is a boundary point $\kappa \in \partial\mathbb{B}$ of $\mathbb{B}$ such that the unipotent elements of the isotropy group Γ_κ form a lattice in the unipotent radical of the parabolic group $\mathbf{P}_\kappa(\mathbb{R})$ of all elemets of $\mathbb{SU}((2,1),\mathbb{C})$ fixing κ. The set of all Γ-cusps is denoted by $\partial_\Gamma\mathbb{B}$. The quotient map p extends in a continuous manner to a unique surjective map $p^* : \mathbb{B}^* \twoheadrightarrow \widehat{\mathbb{B}/\Gamma}$ from $\mathbb{B}^* = \mathbb{B}^*(\Gamma) := \mathbb{B} \cup \partial_\Gamma\mathbb{B}$ onto the Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ of $\mathbb{B}/\Gamma$, which is a projective surface adding a finite number of normal points to $\mathbb{B}/\Gamma$.

An element $1 \neq \gamma \in \Gamma$ is called (honestly) *elliptic* if it has finite order and is not a reflection. It is equivalent to say that γ has precisely one fixed point Q on $\mathbb{B}$. In opposition we call $Q \in \mathbb{B}$ a Γ-*elliptic point*, if it is an *isolated fixed point* of Γ, which means that an elliptic element $\gamma \in \Gamma$ exists fixing Q.

Two subsets M, N of $\mathbb{B}^*$ are called Γ-*equivalent*, iff there is a $\gamma \in \Gamma$ such that $N = \gamma(M)$. Two points P, $Q \in \mathbb{B}^*$ are said to be Γ-*equivalent*, iff $\{P\}$ and $\{Q\}$ are. The Γ-equivalence classes of Γ-elliptic points, Γ-cusps or Γ-reflection discs are finite, see [H98].

We look for an arithmetic ball lattice Γ' satisfying seven special conditions. For the subdiscs $\mathbb{D}_i$ below we will use the following notation for the subgroup of all elements acting on $\mathbb{D}_i$:

$$\Gamma'_i := \{\gamma \in \Gamma';\, \gamma(\mathbb{D}_i) = \mathbb{D}_i\},\ i = 0, 1, 2, 3.$$

Postulates 4.1 *for the ball lattice* Γ'

(i) There are precisely three Γ'*-inequivalent* Γ'*-cusps* $\kappa_1, \kappa_2, \kappa_3 \in \partial\mathbb{B}$. *The corresponding cusp points* K_1, K_2, K_3 *on* $\hat{X} := \widehat{\mathbb{B}/\Gamma'}$ *are nonsingular.*

(ii) There is up to Γ'*-equivalence precisely one* Γ'*-reflection disc* $\mathbb{D}_0 \subset \mathbb{B}$ *with reflection order 4 such that* $\kappa_1, \kappa_2, \kappa_3 \in \partial\mathbb{D}_0$ *is a complete set of* Γ'_0*-inequivalent cusps for the quotient curve* $\mathbb{D}_0/\Gamma'_0$.

(iii) Up to Γ'*-equivalence there are precisely three* Γ'*-reflection discs* $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{D}_3$ *with reflection order 4 supporting at the boundary* $\partial\mathbb{D}_j$ *precisely one cusp up to* Γ'_j*-equivalence, namely* κ_j, $j = 1, 2, 3$, *respectively.*

(iv) Each Γ'*-reflection disc is* Γ'*-equivalent to one of the four discs above.*

(v) Up to Γ'*-equivalence there are precisely three* Γ'*-elliptic points* $O_1, O_2, O_3 \in \mathbb{B}$. *They coincide with the pairwise intersection points of* $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{D}_3$ *(for suitable choice of the three discs). The isotropy group* Γ'_{O_j}, $O_j := \mathbb{D}_k \cap \mathbb{D}_l$, $\{j, k, l\} = \{1, 2, 3\}$ *coincides with the abelian group of order 16 generated by the reflections of order 4 fixing the points of* $\mathbb{D}_k$ *or* $\mathbb{D}_l$, *respectively.*

(vi) The Euler-Bergmann volume of a Γ'*-fundamental domain is equal to* $\frac{3}{16}$.

(vii) There is a subgroup Σ_3 *of* $Aut_{hol}\mathbb{B}$ *isomorphic to* S_3 *normalizing* Γ', *which acts on* $\mathbb{D}_0$ *and permutes* $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{D}_3$.

We illustrate the situation in Picture 8 with a mixed 2- or 3-dimensional imagination (the latter around $\mathbb{D}_0$ with boundary points $\kappa_1, \kappa_2, \kappa_3$) of the real 4-dimensional unit ball $\mathbb{B}$.

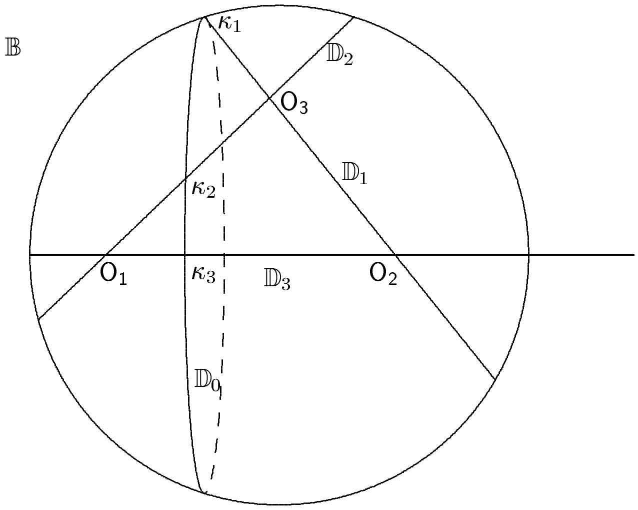

Figure 8. Representative Γ'-fixed point configuration on $\mathbb{B}$

Theorem 4.2 *Under the conditions (i) - (vii) it holds that $\hat{X} = \widehat{\mathbb{B}/\Gamma'}$ is the projective plane $\mathbb{P}^2$. The compactified branch locus of the quotient map*

$$p: \mathbb{B} \longrightarrow X = \mathbb{B}/\Gamma'$$

consists of a quadric $\hat{C}_0$ and three tangents $\hat{C}_j$, $j = 1, 2, 3$. These curves are the (compactified) images of the reflection discs $\mathbb{D}_0$ or $\mathbb{D}_j$, $j = 1, 2, 3$, respectively. There is up to $\mathbb{PG}l_3$-equivalence an - up to S_3-symmetry - unique projective coordinate system on $\mathbb{P}^2$ such that the projective lines $\hat{C}_j$, $j \neq 0$, are the coordinate axes and the quadric has the equation

$$\hat{C}_0 : (X + Y - Z)^2 - 4XY = X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ = 0. \tag{18}$$

In orbital surface terms we will prove mainly that

4.3 *The orbital ball quotient surface $\hat{\mathbf{X}} = \widehat{\mathbb{B}/\Gamma'}$ coincides, up to projective equivalence, with*

$$\hat{\mathbf{X}} = (\hat{X}; \hat{\mathbf{C}}_0 + \hat{\mathbf{C}}_1 + \hat{\mathbf{C}}_2 + \hat{\mathbf{C}}_3 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3)$$

described in section 2, (1), (2), (3) with properties 2.1 (i), (ii) a),b),c),d) (omitting the symmetry condition e) here).

The open curves $C_i = \hat{C}_i \setminus \{K_1, K_2, K_3\}$ are defined as images of the discs $\mathbb{D}_i$, $i = 0, 1, 2, 3$, the points P_j are the images of the elliptic points O_j, and the cusp points K_j are the images of the cusps κ_j with respect to the extended quotient map $p^* : \mathbb{B}^* \longrightarrow \widehat{\mathbb{B}/\Gamma'}$.

We use again the hight calculus for orbital surfaces developed in [H98] based on equivariant K-theory. The orbital heights of orbital ball quotient surfaces are links between differential geometric volumes of fundamental domains and algebraic-geometric invariants of surfaces or embedded curves. Mainly Euler heights h_e and signature heights h_τ are used. We dispose on the following strong

Theorem 4.4 *([H98], IV, Theorem 4.8.1, first part) For ball lattices $\Gamma \subset \mathbb{U}((2,1), \mathbb{C})$ with open orbital ball quotient $\mathbb{B}/\Gamma$ it holds that*

$$H_e(\mathbb{B}/\Gamma) = covol_{EB}(\Gamma) := vol_{EB}(\mathfrak{F}_\Gamma) := vol_{\gamma_2}(\mathfrak{F}_\Gamma).$$

□

Thereby $\mathfrak{F}_\Gamma$ denotes a Γ-fundamental domain on $\mathbb{B}$, and the volume is taken with respect to the $\mathbb{U}((2,1),\mathbb{C})$-invariant Euler-Bergmann (volume) form $\gamma_2 = \frac{1}{3}\gamma_1 \wedge \gamma_1$ on $\mathbb{B}$ with $\gamma_1 = -3\omega$ (Kähler-Einstein relation) the Ricci form and ω the Kähler form of the Bergmann metric on $\mathbb{B}$. For these details we refer to [BHH], Appendix B.

From this theorem and condition (vi) for Γ' we get

$$covol_{EB}(\Gamma') = \frac{3}{16} \tag{19}$$

The signature form on $\mathbb{B}$ can be proportionally defined to be $\sigma = \frac{1}{3}(\gamma_1 \wedge \gamma_1 - 2\gamma_2)$. As for Euler hights we have

Theorem 4.5 *([H98], IV, Theorem 4.8.1, second part) For ball lattices $\Gamma \subset \mathbb{U}((2,1),\mathbb{C})$ it holds that*

$$H_\tau(\mathbb{B}/\Gamma) = \frac{1}{3}covol_{EB}(\Gamma) = covol_\sigma(\Gamma) = vol_\sigma(\mathfrak{F}_\Gamma).$$

□

This is the origin of the proportionality relation (*Prop* 2) for orbital ball quotient surfaces $\mathbb{B}/\Gamma$. From condition (vi) for Γ' we get now

$$H_\tau(\mathbb{B}/\Gamma') = \frac{1}{3}H_e(\mathbb{B}/\Gamma') = \frac{1}{16} \tag{20}$$

Now we change our attention to orbital curves comig from discs. Let $\mathbb{D} \subset \mathbb{B}$ be a (linearly embedded complete) disc whose image on $\mathbb{B}$ is an algebraic curve $\mathbb{D}/\Gamma$ on $\mathbb{B}/\Gamma$ (Γ-disc). For the finer object, the orbital curve $\mathbb{D}/\Gamma \subset \mathbb{B}/\Gamma$. Euler hight and covolume are connected by

Theorem 4.6 *([H98], IV.7, first part of (4.7.7))*

$$h_e(\mathbb{D}/\Gamma) = covol_{EP}(\Gamma_\mathbb{D}) := vol_{EP}(\mathfrak{F}_{\Gamma_\mathbb{D}}) := vol_\eta(\mathfrak{F}_{\Gamma_\mathbb{D}}),$$

□

where

$$\Gamma_\mathbb{D} := N_\Gamma(\mathbb{D})/Z_\Gamma(\mathbb{D}) \tag{21}$$

is the effectivized subgroup of all elements of Γ acting on $\mathbb{D}$,

$$N_\Gamma(\mathbb{D}) = \{\gamma \in \Gamma; \gamma(\mathbb{D}) = \mathbb{D}\}, Z_\Gamma(\mathbb{D}) = \{\gamma \in \Gamma; \gamma_\mathbb{D} = id_\mathbb{D}\}. \tag{22}$$

The volume of a $\Gamma_\mathbb{D}$-fundamental domain $\mathfrak{F}_{\Gamma_\mathbb{D}}$ is taken with respect to the $\mathbb{U}((1,1),\mathbb{C})$-invariant Euler-Poincaré form η on $\mathbb{D}$. This explicitly well-known volume form is normalized in such a way that the hight h_e of any compact quotient curve C of $\mathbb{D}$ by a torsion free $\mathbb{D}$-lattice N is nothing else but the Euler number $e(C) = 2-2g < 0$, g the genus of C. Assume for a moment that $N = N_G(\mathbb{D})$ for a torsion free cocompact ball lattice G and K is a canonical divisor of the smooth compact algebraic surface $\mathbb{B}/\Gamma$. By relative proportionality, see [BHH], appendix B.3.E, it holds that $3e(C) = -2(K\cdot C)$. Together with the adjunction formula $-(K\cdot C) = e(C) + (C^2)$ one gets $e(C) = 2(C^2)$. In order to calculate selfintersection numbers by means of volumes we define adequately the signature form on $\mathbb{D}$ to be $\frac{1}{2}\eta$. We proved also

Theorem 4.7 *([H98], IV.7, second part of (4.7.7))*

$$h_\tau(\mathbb{D}/\Gamma) = \frac{1}{2}covol_{EP}(\Gamma_\mathbb{D}) = \frac{1}{2}vol_\eta(\mathfrak{F}_{\Gamma_\mathbb{D}}).$$

□

This is the origin of proportionality condition (*Prop* 1) for orbital disc quotients on ball quotient surfaces. Especially we get

$$2h_\tau(\mathbb{D}/\Gamma'_i) = h_e(\mathbb{D}/\Gamma'_j)\ i = 0,1,2,3. \tag{23}$$

Now we check the admissibility of our cusp conditions, see ($Prop\,\infty$). There are precisely 3 cusp points K_1, K_2, K_3 on $\widehat{\mathbb{B}/\Gamma'}$ coming from $\kappa_1, \kappa_2, \kappa_3$ (condition (i)). The possible graphs of the corresponding orbital cusp points $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ are classified in [H98], III.3.5. We denote one of these points, say the first, by κ, K or $\mathbf{K}$, respectively. In general, each cusp point is the quotient of an elliptic singularity by a cyclic group G_κ of order 1, 2, 3, 4 or 6, see [H98], IV.4.5. Since two 4-reflection discs go through our special κ and there are no 2-reflection discs (condition (iv) and (ii), (iii) before), the group G_κ is cyclic of order 4, and the graph of $\mathbf{K}$ must look like

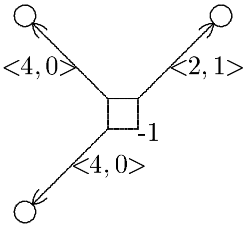

Figure 9.

(-1 in the box will be explained below, see (24). This means that K has a canonical smooth rational resolution curve E_κ supporting a surface singularity of cyclic quotient type $\langle 2,1\rangle$. In [H98] we call it the cusp curve corresponding to the center of the resolution graph 9 of κ. Remember that we have three of them: E_1, E_2, E_3 corresponding to $\kappa_1, \kappa_2, \kappa_3$, which are contracted to K_1, K_2, K_3, respectively, along the birational morphism $X' \longrightarrow \hat{X} = \widehat{\mathbb{B}/\Gamma'}$. Resolving the three singularities of type $\langle 2,1\rangle$ by rational -2-curves we get a birational morphism $\tilde{X} \longrightarrow X'$ with three connected exceptional curves $L_j + E_j$ on $\tilde{X}$ contracted to the nonsingular points K_j along $\tilde{X} \longrightarrow \hat{X}$ by the last part of condition (i). Omitting indices again, the smooth rational components L, E intersect each other transversally and $(L^2) = -2$. The contraction to a nonsingular point is only possible, if E has on $\tilde{X}$ selfintersection $(E^2)_{\tilde{X}} = -1$. So for all proper transforms of E_j on $\tilde{X}$ we get

$$(E_j^2)_{\tilde{X}} = -1,\ j = 1,2,3. \tag{24}$$

Proposition 4.8 *The compactified ball quotient surface $\hat{X} = \widehat{\mathbb{B}/\Gamma'}$ is smooth. Moreover, the closures $\hat{C}_i$ of $C_i := \mathbb{D}_i/\Gamma'$, $i = 0,1,2,3$, on $\hat{X}$ are smooth curves.*

Proof. The singularities of any Baily-Borel compactified ball quotient surface come from (honest) elliptic points and cusps. The cusp points K_j are nonsingular by (i). By condition (v) there are only three points $P_j \in \mathbb{B}/\Gamma'$ with elliptic preimages, namely the images of O_j, $j = 1,2,3$. Let O be one of them. The corresponding isotropy group Γ'_O is generated by reflections, see condition (iv) again. Therefore the points P_j are nonsingular (Chevalley criterion [Bou], V.5 Theorem 4); for our application, see [H98], I.1, Lemma 1.1.1 and IV.5, proof of Lemma IV.5.9). Now it is clear that $\hat{X}$ has to be smooth.

We denote by $\hat{C}$ be an arbitrary one of the curves $\hat{C}_i \subset \hat{X}$ and by C' its proper transform on X'. Assume that $\hat{C}$ goes through one of our cusp points K with canonical resolution curve E on X'. Its preimage on $\mathbb{B}$ is one of the Γ'-reflection discs $\mathbb{D} = \mathbb{D}_j$. It corresponds to one of the $\langle 4,0\rangle$ arrows in the cusp diagram 9. Looking down again to X' this diagram teaches us that C' intersects E locally transversal at (at most two) nonsingular surface points. By (ii) and (iii) E is intersected by precisely two of the reflection curves C'_j because the cusps κ_i are boundaries of precisely two of the corresponding reflection discs, see picture 8. So C' intersects E at one point only. Because of transversality this is a nonsingular point of C'. This point remains nonsingular on $\hat{C} \subset \hat{X}$ after contraction of E (or of $L + E$ starting from $\tilde{X}$). Locally around E_1, E_2, E_3 the intersection behaviour of these curves on X' with C'_j, $j = 0,1,2,3$, is described in Picture 2.

It remains to be proved that the non-compact curves $C_j \subset \mathbb{B}/\Gamma'$ are smooth. In [H98], IV.4, we proved that for Γ'-rational discs $\mathbb{D}$ on $\mathbb{B}$ the natural map $\mathbb{D}/\Gamma'_{\mathbb{D}} \longrightarrow \mathbb{D}/\Gamma'$ is the normalization (singularity resolution) of the latter curve on $\mathbb{B}/\Gamma'$. Our Γ'-reflection discs are arithmetic because Γ' is.

Curve singularities on $\mathbb{D}/\Gamma'$ come from (honest) Γ'-*cross points* Q on $\mathbb{D}$. Such a point Q is characterized by the property that through Q goes a Γ'-equivalent disc $\mathbb{D}'$ not being Γ'_Q-equivalent, see [H98], IV, Definition 4.4.5 and Proposition 4.4.6. Assume that Q is a Γ'-cross point of $\mathbb{D}$. Then it is the intersection point of two Γ'-reflection discs $\mathbb{D} = \mathbb{D}_\sigma$, $\mathbb{D}' = \mathbb{D}_\delta$ belonging to reflections σ, $\delta \in \Gamma'$, say. Then Q is an elliptic point because it is fixed also by the elliptic element $\sigma\delta$, which is not a reflection, because its representation on the tangent space $T_Q = T_Q(\mathbb{B})$ at $Q \in \mathbb{B}$ has two non-trivial eigenvalues, namely the non-trivial eigenvalue of σ and the non-trivial eigenvalue of δ. The only Γ'-elliptic points are the Γ'-orbits of O_1, O_2, O_3 by condition (iv). So we can assume without loss of generality that Q is one of these points, say $Q = O_3 = \mathbb{D}_1 \cap \mathbb{D}_2$, $\mathbb{D} = \mathbb{D}_1 = \mathbb{D}_\sigma$. The disc $\mathbb{D}'$ cannot coincide with $\mathbb{D}_2$ because the latter disc is not Γ'-equivalent with $\mathbb{D}_1$ by (iii). Therefore Q is the intersection point of three different reflection discs. But then the isotropy group Γ'_Q is not abelian because their elements produce at least three eigenlines in T_Q by the directions of the three reflection discs through Q. This contradicts to the second part of condition (iv). Hence, there is no Γ'-reflection disc $\mathbb{D}$ with Γ'-cross point; the image curves are smooth. This finishes the proof of the proposition.

□

It follows that the orbital quotient surface looks like

$$\widehat{\mathbb{B}/\Gamma'} = \hat{\mathbf{X}} = (\hat{X}; \hat{\mathbf{C}}_0 + \hat{\mathbf{C}}_1 + \hat{\mathbf{C}}_2 + \hat{\mathbf{C}}_3 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3)$$

we startet with in section 1 not knowing until now that $\hat{X} = \mathbb{P}^2$. Moreover, we have to prove the properties (i), (ii) a), ... ,d) before definition 2.1. Let us start with

c′) P_1, P_2, P_3 are the three different intersection points of the curves C_1, C_2, C_3.

This follows now immediately from (iv), because an intersection point of two of these reflection curves is necessarily an image point of Γ'-elliptic point. Up to Γ'-equivalence there are only three of them, namely O_1, O_2, O_3.

d′) $\hat{C}_j$ and $\hat{C}_0$ touch each other at K_j (with local intersection number 2), $j = 1, 2, 3$.

$\hat{C}_0$ goes through each of the cusp points K_j by (ii). The other reflection curve through K_j is $\hat{C}_j$ by (iii), see Figure 8. From the intersection graph 9 we deduced the intersection behaviour of the curves C'_0, C'_j, E_j locally around E_j, which is described in picture 2. Going back to $\tilde{X}$ we blow down first the -1-curve E_j. On the corresponding surface the proper transforms of C'_0 and C'_j intersect each other transversally. The proper transform of the -2-curve L becomes a smooth rational -1-curve denoted by L again supporting this intersection point. The intersection of the two C-curves with L are transversal, too. Now blow down the -1-curve L to K_j to see that the local situation of touching we look for is well-described in picture 1.

Now we relate Euler numbers e_i with selfintersections s'_i of C'_i on X' for $i = 0, 1, 2, 3$ using geometric height formulas (11), (12) for orbital curves $\mathbf{C}$ on open orbital surfaces:

$$h_e(\mathbf{C}) = e(C') - \sum(1 - \frac{1}{v_i d_i}) - \#C'_\infty \ ,$$

$$h_\tau(\mathbf{C}) = \frac{1}{v}[(C'^2) + \sum \frac{e_i}{d_i} + \sum \frac{e_j}{d_j}] \ .$$

The sums on the right-hand side can be read off from the atomic graph of the orbital curve $\mathbf{C}$ (or compact orbital curve $\mathbf{C}' = (vC'; \sum \mathbf{P}_i + \sum \mathbf{K}_m)$ which have been already described in Figure 3. Filling these contributions in the hight formulas we get together with (Prop 1), see 23,

$$\begin{aligned}
h_e(\mathbf{C}_0) &= e_0 - 3, \\
h_e(\mathbf{C}_j) &= e_j - (1 - \frac{1}{4}) - (1 - \frac{1}{4}) - 1 \ , \ j = 1, 2, 3; \\
\frac{1}{2} h_e(\mathbf{C}_0) &= \frac{1}{4}(s'_0 + 0 + 0), \\
\frac{1}{2} h_e(\mathbf{C}_j) &= \frac{1}{4}(s'_j + 0 + 0) \ , \ j = 1, 2, 3.
\end{aligned} \tag{25}$$

It follows that

$$s_0' = 2e_0 - 6 \quad , \quad s_j' = 2e_j - 5 \; , \; j = 1, 2, 3.$$

Blowing down the three rational -1-curves and the three rational -2-curves on $\tilde{X}$ to the cusp points K_1, K_2, K_3 we get on $\hat{X}$ the selfintersections $s_0 = s_0' + 6$, $s := s_j = s_j' + 2$ for the curves $\hat{C}_i$, $i = 0, 1, 2, 3$, because $\hat{C}_0$ goes through all three cusp points and each $\hat{C}_j$ only through one of them. It follows that

$$s_0 = 2e_0 \quad , \quad s = 2e - 3, \; e := e_j = e(\hat{C}_j), j > 0.$$

In a similar opposite use of hight formulas in comparision with their calculation in the previous section we can calculate now the Euler number and signature of $\hat{X}$ using (13) and (14):

$$H_e(\mathbf{X}) = e(X') - \sum (1 - \frac{1}{v_i}) h_e(\mathbf{C}_i) - \sum h_e(\mathbf{P}_j) - 2\#\{rational\, cusp\, points\}$$

$$H_\tau(\mathbf{X}) = \tau(X') - \frac{1}{3} \sum (v_i - \frac{1}{v_i}) h_\tau(\mathbf{C}_i) - \sum h_\tau(\mathbf{P}_j) - \sum h_\tau(\mathbf{K}_m)$$

The point contributions have been already substituted in 3, see (17). The left-hand sides are known from (20). So we get with the above substitutions ($h_\tau(\mathbf{C}_j) = s'/4 = (2e - 5)/4$ ...)

$$\frac{3}{16} = e(X') - (1 - \frac{1}{4})(e_0 - 3) - 3(1 - \frac{1}{4})(e - \frac{5}{2}) - 3 \cdot \frac{9}{16} - 2 \cdot 3$$

$$\frac{1}{16} = \tau(X') - \frac{1}{3}[(4 - \frac{1}{4})(2e_0 - 6)/4 + 3(4 - \frac{1}{4})(2e - 5)/4] - 3 \cdot 0 - 3 \cdot (-\frac{1}{6}).$$

Set $E := e(\hat{X}) = e(X') - 3$ and $S := \tau(\hat{X}) = \tau(X') + 3$. After substitution we obtain

$$\begin{array}{rcrcrcl} 8E & + & 6(3 - e_0) & + & 9(5 - 2e) & = & 39, \\ 16S & + & 10(3 - e_0) & + & 15(5 - 2e) & = & 41 \end{array} \tag{26}$$

Proposition 4.9 *Let Y be a smooth compact complex algebraic surface supporting a configuration $L_0 + L_1 + L_2 + L_3$ with smooth curves L_i, $i = 0, 1, 2, 3$ intersecting pairwise in at least one point. Assume that the invariants $E = e(Y)$, $S = \tau(Y)$, $e_0 = e(L_0)$ and $e = e(L_j)$, $j = 1, 2, 3$ satisfy the relations (26). Then $Y = \mathbb{P}^2$, and the curves $L_i's$, $i = 0, 1, 2, 3$, are rational.*

Proof. We need some basic facts of surface classification theory, which can be found in [BPV], for instance. Adding the first to the second equation of (26) we get the relation

$$64\chi + 22(3 - e_0) + 33(5 - 2e) = 119 \tag{27}$$

for the arithmetic genus $\chi = \chi(Y) = 4(E + S)$ of Y. The integers

$$3 - e_0 = 2g_0 + 1 \quad , \quad 5 - 2e = 4g + 1\,, \tag{28}$$

where g_0, g are the genera of L_0 or L_j, $j > 0$, respctively, are positive. From (27) we get $\chi < 0$ or

$$\chi(Y) = 1, g_0 = g(L_0) = 0, g = g(L_j) = 0. \tag{29}$$

We exclude the former case: Assume that $\chi < 0$. Then Y has negative Kodaira dimension. By surface classification theory Y must be a (blown up) ruled surface over a smooth compact curve B of genus q, say. The arithmetic genus of Y is equal to $\chi = 1 - q < 0$. The fibres of the fibration $Y \longrightarrow B$ are linear trees of rational curves. Since, by assumption, $L_1 + L_2 + L_3$ is a connected cycle it cannot belong to any finite union of fibres. Therefore one of the components covers B finitely. It follows that $g \geq q$. The identity (27) yields

$$64(1 - q) + 22(1 + 2g_0) + 33(1 + 4g) = 119,$$

hence $11g_0 + 33g = 16q$, which contradicts to $g \geq q > 1$.

We proved that the relations (29) must be satisfied. Altogether we solve(d) the simple linear system 26 of diophantine equations coming from the Proportionality Theorem. We get the surface invariants

$$\chi = 1,\ E = 3,\ S = 1,\ (K^2) = 9,\ (K^2)/E = 3,$$

where $(K^2) = 12\chi - E$ is the selfintersection index of a canonical divisor K on Y. We proved also that L_i, $i = 0, 1, 2, 3$, is rational by (29).

The extreme Chern quotient $(K^2)/E = 3$ with positive Euler number E is only possible for $Y = \mathbb{P}^2$ or for compact ball quotient surfaces $\mathbb{B}/\Gamma$ for torsion free ball lattices Γ by a theorem of Miyaoka-Yau, Kodaira-classification of surfaces and fine classification of rational surfaces, see [H98], V.2, Proposition 5.2.4, and the references given there. But $\mathbb{B}$, hence also $\mathbb{B}/\Gamma$, is hyperbolic in the sense of Kobayashi. Therefore it does not support any rational curve. The compact ball quotient case is excluded by the rationality of $L_i \subset Y$. Therefore Y must be the projective plane. $\square$

Corollary 4.10 *If Γ' satisfies the conditions (i),...,(vii), then $\hat{X} = \widehat{\mathbb{B}/\Gamma'}$ is the projective plane, $\mathbb{P}^2$, $\hat{C}_0$ is a quadric and $\hat{C}_1$, $\hat{C}_2$, $\hat{C}_3$ are tangent lines. In other words, $\hat{C}_0 + \hat{C}_1 + \hat{C}_2 + \hat{C}_3$ is a plane Apollonius configuration.*

Proof. We have only to summerize. $\hat{X}$ is a smooth surface by 4.8. Moereover, as Baily-Borel compactification $\hat{X}$ is projective, hence algebraic. We proved already that our four curves $\hat{C}_i$ are smooth, see Proposition 4.8. Together with c′), and (26) the assumptions of the proposition are satisfied. Therefore $hatX = \mathbb{P}^2$ and our curves are rational. More precisely, from Bezout's theorem and the intersection behaviour described in c′, d′ follows that the configuration is of Apollonius type.

$\square$

Now we finish the proof of 4.3 and Theorem 4.2. The projective lines $\hat{C}_j$ can be used as coordinate lines $X = 0, Y = 0, Z = 0$ of $\mathbb{P}^2$ such that the configuration divisor $\hat{C}_0 + \hat{C}_j + \hat{C}_1 + \hat{C}_1$ is S_3-invariant by Proposition 2.4 and Corollary 2.5 with the natural projective action of S_3 on $\mathbb{P}^2$ permuting coordinates. The uniqueness of the equation (18) of $\hat{C}_0$ comes from (the proof) of Lemma 2.10 verifying that this equation is the only S_3-symmetric possibility. Theorem 4.2 is proved.

The weights for the orbital cycle of a smooth orbital ball quotient surface come from reflection orders only, by definition. Therefore 4.3 follows now from these order postulates in 4.1 (ii),(iii) and from postulate (iv) forbidding other branch curves beside of C_i, $i = 0, 1, 2, 3$.

$\square$

5 The Gauss congruence ball lattice

Let $\mathbb{Q}(i)$, $i=\sqrt{-1}$, be the field of Gauss numbers and $\mathfrak{O}=\mathbb{Z}[i]=\mathbb{Z}+\mathbb{Z}i$ the (maximal) order of Gauss integers in it. The center Z of the unitary group

$$\tilde{\Gamma} := \mathbb{U}((2,1),\mathfrak{O}) = \{g \in \mathbb{G}l_3(\mathfrak{O});\ {}^t\bar{g}\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&-1\end{smallmatrix}\right)g = \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&-1\end{smallmatrix}\right)\}$$

with Gauss integers as coefficients is generated by $\left(\begin{smallmatrix}i&0&0\\0&i&0\\0&0&i\end{smallmatrix}\right)$. The ineffective kernel of the action of $\tilde{\Gamma}$ on the ball $\mathbb{B}$ coincides with Z. We concentrate our attention to the *special Gauss ball lattice* $\Gamma := \mathbb{SU}((2,1),\mathfrak{O})$, which is an arithmetic ball lattice acting effectively on $\mathbb{B}$. It holds that $\tilde{\Gamma} = Z\cdot\Gamma$. The isomorphisms

$$\tilde{\Gamma}/Z \cong \Gamma \cong \mathbb{PU}((2,1),\mathfrak{O}) \cong \mathbb{PSU}((2,1),\mathfrak{O})$$

allow us to identify (sometimes, if we want) these groups. The most important role plays the congruence subgroup $\Gamma' := \Gamma(1+i)$ (*Gauss congruence ball lattice*) of the prime ideal of $\mathbb{Z}[i]$ generated by the prime divisor $1+i$ of 2 with residue field $\mathbb{F}_2$.

We want to prove the following

Theorem 5.1 *The arithmetic ball lattice Γ' satisfies all conditions (i),..,(vii) of 4.1. The Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma'}$ is equal to $\mathbb{P}^2$ with Apollonius configuration 1 supporting the orbital cycle of $\widehat{\mathbb{B}/\Gamma'}$.*

An essential role in the proof plays the theory of hermitian lattices, which is not so difficult in the case of $\mathfrak{O}-lattices$ with small ranks, because $\mathfrak{O}$ is an euclidean ring. The basic lattice is $\Lambda := \mathfrak{O}^3$ endowed with the indefinite unimodular hermitian form

$$\langle\ ,\ \rangle : \Lambda\times\Lambda \longrightarrow \mathfrak{O},\quad \langle\left(\begin{smallmatrix}a_1\\a_2\\a_3\end{smallmatrix}\right),\left(\begin{smallmatrix}b_1\\b_2\\b_3\end{smallmatrix}\right)\rangle = a_1\bar{b}_1 + a_2\bar{b}_2 - a_3\bar{b}_3.$$

We consider Γ as group of unimodular automorphisms of the hermitian $\mathfrak{O}$-lattice $\Lambda := \mathfrak{O}^3$. Then Γ' consists of all elements of Γ which restrict to an automorphism of the sublattice $\Lambda' := (1+i)\Lambda$. The factor group Γ/Γ' acts effectively on the residue space $\Lambda/\Lambda' \cong \mathbb{F}_2^3$. The hermitian structucture on Λ reduces to the canonical non-degenerate bilinear form on $\mathbb{F}_2^3$. Therefore Γ/Γ' appears as subgroup of the corresponding orthogonal group $\mathbb{O}(3,\mathbb{F}_2)$. This group consists of permutation matrices only, because the canonical basis vectors of $\mathbb{F}_2^3$ are the only ones with $(\mathbb{F}_2)$-norm 1 and norm 1 vectors in its orthogonal complement. Hence $\Gamma/\Gamma' \subseteq \mathbb{O}(3,\mathbb{F}_2)\cong S_3$.

We want to prove that the inclusion is the identity. It suffices to find two non-commuting elements in Γ/Γ'. Let $\mathfrak{a}\in\mathfrak{O}^3$ be a vector whose hermitian norm $\mathfrak{a}^2 := \langle\mathfrak{a},\mathfrak{a}\rangle$ is equal to ±1 or ±2. We define the reflection $R_\mathfrak{a} : \mathfrak{O}^3 \longrightarrow \mathfrak{O}^3$ by

$$R_\mathfrak{a} : \mathfrak{z} \mapsto \mathfrak{z} - \frac{2}{\mathfrak{a}^2}\langle\mathfrak{z},\mathfrak{a}\rangle\mathfrak{a}.$$

It sends $\mathfrak{a}$ to $-\mathfrak{a}$ and each vector of the orthogonal complement

$$\Lambda_\mathfrak{a} := \{\mathfrak{u}\in\mathfrak{O}^3;\ \mathfrak{u}\perp\mathfrak{a}\}$$

to itself. Therefore $id \neq R_\mathfrak{a}$ is an isometry of Λ. Its reduction $\bar{R}_\mathfrak{a}$ (modulo $1+i$) is the reflection isometry

$$r_{\bar{\mathfrak{a}}} : \mathbb{F}_2^3 \longrightarrow \mathbb{F}_2^3,\ \bar{\mathfrak{z}} \mapsto \bar{\mathfrak{z}} - (\bar{\mathfrak{z}},\bar{\mathfrak{a}})\bar{\mathfrak{a}},$$

where we overline by bar all kinds of reductions modulo $1+i$. This is a non-trivial isometry if and only if $\mathfrak{a}^2=\pm2$ and $\mathfrak{a}\not\equiv\mathfrak{o}$ *modulo* $1+i$.

The following examples yield two such reflections. Take

$$\mathfrak{a} = \begin{pmatrix}1+i\\1\\1\end{pmatrix},\ \mathfrak{b} = \begin{pmatrix}1\\i\\0\end{pmatrix}.$$

Both have norm 2. As reductions of the corresponding reflections we get $r_{(0,1,1)}$ or $r_{(1,1,0)}$ with matrix representations $\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right)$, respectively. Obviously, they generate $\mathbb{O}(3,\mathbb{F}_2)$.

Lemma 5.2 *We have an exact group sequence*

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \xrightarrow{red} S_3 \longrightarrow 1.$$

with a section $S_3 \longrightarrow \Gamma$ *sending* $S_3 \cong \mathbb{O}(3, \mathbb{F}_2)$ *to the stationary group* $\Gamma_{\mathbb{P}\mathfrak{c}} := \{\gamma \in \Gamma; \gamma(\mathfrak{c}) \in \mathfrak{O}\mathfrak{c}\}$ *for a vector* $\mathfrak{c} \in \mathfrak{O}^3$ *with negative norm* $\mathfrak{c}^2 = -3$.

Proof. The left-exact part comes from the definition of Γ' as kernel of the reduction homomorphism

$$\Gamma = \mathbb{SU}((2,1), \mathfrak{O}(1+i)) \xrightarrow{red} \mathbb{SU}((2,1), \mathfrak{O}/i\mathfrak{O}) \cong \mathbb{O}(3, \mathbb{F}_2) \quad .$$

The surjectivity of the reduction homomorphism has just been verified. The reflections $R_\mathfrak{a}$ and $R_\mathfrak{b}$ ($\mathfrak{a}, \mathfrak{b}$ as above), act trivially on the orthogonal complements $\Lambda_\mathfrak{a}$ or $\Lambda_\mathfrak{b}$, respectively, hence they fix $\mathfrak{c} = \left(\begin{smallmatrix} i \\ 1 \\ 2-i \end{smallmatrix}\right)$ generating the rank one lattice $\Lambda_\mathfrak{a} \cap \Lambda_\mathfrak{b}$. The norm 2 vectors $\mathfrak{a}$, $\mathfrak{b}$ have been choosen in such a way that their Gram matrix is

$$\begin{pmatrix} \mathfrak{a}^2 & \langle \mathfrak{a}, \mathfrak{b} \rangle \\ \langle \mathfrak{b}, \mathfrak{a} \rangle & \mathfrak{b}^2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

hence

$$\begin{aligned} R_\mathfrak{a} :& \quad \mathfrak{a} \mapsto -\mathfrak{a}, \ \mathfrak{b} \mapsto \mathfrak{b} - \mathfrak{a}, \ \mathfrak{c} \mapsto \mathfrak{c}; \\ R_\mathfrak{b} :& \quad \mathfrak{a} \mapsto \mathfrak{a} - \mathfrak{b}, \ \mathfrak{b} \mapsto -\mathfrak{b}, \ \mathfrak{c} \mapsto \mathfrak{c}. \end{aligned}$$

Looking at the corresponding matrix representation it is clear that the subgroup of $\Gamma_{\mathbb{P}\mathfrak{c}}$ generated by $-R_\mathfrak{a}$, $-R_\mathfrak{b}$ is isomorphic to S_3.

□

More generally we define *reflections* $\rho \in \mathbb{U}((2,1), \mathbb{C})$ as elements of finite order with precisely two different eigenvalues. The eigenspace $E(\rho)$ of the double eigenvalue of ρ is called the *reflection plane* of ρ. We call ρ a $\mathbb{B}$*-reflection*, iff $E(\rho)$ is an indefinite hermitian subspace of $\mathbb{C}^3$. In this case (only) $\mathbb{D}(\rho) := \mathbb{P}E(\rho) \cap \mathbb{B}$ is a complete (linear) subdisc of $\mathbb{B}$ called the *reflection disc* of ρ. The complete linear subdisc $\mathbb{D}$ of $\mathbb{B}$ is called a Γ*-reflection disc* iff there exists a $\mathbb{B}$-reflection $\rho \in \Gamma$ such that $\mathbb{D} = \mathbb{D}(\rho)$. Starting from $\mathbb{D}$ the $\mathbb{D}$*-reflection group* $Z_\Gamma(\mathbb{D})$ defined in (22) is finite and cyclic. Its order is called the *reflection order* of $\mathbb{D}$ w.r.t. Γ. The latter definitions apply to any ball lattice $\Gamma \subset \mathbb{U}((2,1), \mathbb{C})$.

Proof of Theorem 5.1(i). The second statement follows from the first by Theorem 4.2. So we have to check step by step the properties (i),...,(vii) of 4.1.

(i) By a result of Shvartsman [Sv1],[Sv2], the surface $\widehat{\mathbb{B}/\Gamma}$ has only one cusp point. We refer to [Zin] for the more general result, that the number of cusp points of Picard modular surfaces $\widehat{\mathbb{B}/\mathbb{U}((2,1), \mathfrak{O}_L)}$, L an arbitrary imaginary quadratic number field, coincides with the class number of L. It is also known that $\mathbb{B}^* = \mathbb{B} \cap \partial\mathbb{B}(L)$ setting $\partial\mathbb{B}(L) = \partial\mathbb{B} \cap \mathbb{P}^2(L)$ in this case.

With the above notations we get $\partial_\Gamma\mathbb{B} = \partial\mathbb{B}(\mathbb{Q}(i)) = \Gamma\kappa$ with $\kappa = \mathbb{P}\mathfrak{k}$, for each

$$\mathfrak{k} \in \Lambda_0 := \{\mathfrak{a} \in \Lambda; \mathfrak{a}^2 = 0\}$$

because $\partial_\mathbb{B}(\mathbb{Q}(i)) = \mathbb{P}\Lambda_0$. The set Λ_0 maps onto

$$\bar{V}_0 := \{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \} \subset \mathbb{F}_2^3$$

by reduction. The group $S_3 = \Gamma/\Gamma'$ acts effectively on $\bar{V}_0$ with bi-transitive restriction on the non-zero vectors. It follows that Γ acts bi-transitively on $\partial_\Gamma\mathbb{B}/\Gamma'$ completely represented by

$$k_1 = (0:1:1), \ k_2 = (1:0:1), \ k_3 = (1:1:0)$$

with ineffective kernel Γ'. Especially we get up to Γ'-equivalence precisely three cusps. This proves the first part of (i).

For the proof of the second part and later use we introduce the notations

$$X := \mathbb{B}/\Gamma' \subset \hat{X} := \widehat{\mathbb{B}/\Gamma'},\ Y := \mathbb{B}/\Gamma \subset \hat{Y} := \widehat{\mathbb{B}/\Gamma}.$$

We know that $Y = X/S_3 \subset \hat{Y} = \hat{X}/S_3$ considering $S_3 = \Gamma/\Gamma'$ now as subgroup of $Aut\,X = Aut\,\hat{X}$. If $z = \mathbb{P}\mathfrak{z}$, $\mathfrak{z} \in \mathbb{C}^3$, is a point of $\mathbb{B}^*$ we denote its image on X by Z. The quotient map of $\mathbb{B}$ onto $\mathbb{B}/\Gamma'$ is denoted by p'. These notations will be preserved also for the extensions of this projections to $\mathbb{B}^*$. Since the cusp points $K_i = p'(k_i)$ are S_3-equivalent, it suffices to show that an arbitrary one of them is non-singular. We move the ball inside of $\mathbb{P}^2$ such that $\infty := (0:0:1)$ becomes a $\mathbb{Q}(i)$-rational boundary point of the image ball $g\mathbb{B}$. For this purpose we choose $g \in \mathbb{G}l_3(\mathfrak{O})$ such that

$$^t\bar{g}\begin{pmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix} g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Such choice is possible. Namely the $\mathbb{Z}$-lattices $(\mathbb{Z}^3, \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right))$ and $(\mathbb{Z}^3, \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right))$ are isometric because they are unimodular, indefinite and have same rank, signature and type (defined by norms modulo 8). We refer to ([Se70], V.2). The isometry can be extended to isometries of hermitian $\mathfrak{O}$-lattices

$$(\mathfrak{O}^3, I) \cong (\mathfrak{O}^3, \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)) \cong (\mathfrak{O}^3, \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)) = \Lambda,\ I = \left(\begin{smallmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{smallmatrix}\right),$$

where the added first one is obvious. We get the Siegel domain

$$g\mathbb{B} = \mathbb{P}V_- \ :\ 2Im\,u - |v|^2 > 0,$$

$$V = (\mathbb{C}^3, I) = \mathbb{C} \otimes (\mathfrak{O}^3, I) = \mathbb{C} \otimes g\Lambda\ ,\ V_- = \{\mathfrak{x} \in V; \langle \mathfrak{x}, \mathfrak{x} \rangle_I < 0\}$$

On $g\mathbb{B}$ act $G := g\Gamma g^{-1} = \mathbb{SU}(I, \mathfrak{O})$ and its congruence subgroup $G' = G(1+i) = g\Gamma' g^{-1}$ with quotient group $G/G' = \Gamma/\Gamma' = S_3$. The stationary group of Γ at ∞ is generated by $\left(\begin{smallmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{smallmatrix}\right)$ and by its unipotent part

$$U_\infty(\mathfrak{O}) = \{\begin{pmatrix} 1 & i\bar{a} & \frac{i}{2}|a|^2 + r \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} =: [a, r];\ a \in \mathbb{C},\ r \in \mathbb{R}\} \cap \mathbb{S}l_3(\mathfrak{O}),$$

see [H98], IV.2, also for the next considerations. As torsion free nilpotent group of rank 3 each unipotent ball lattice has three generators. As generators of the unipotent congruence subgroup $U_\infty(\mathfrak{O})'$ one finds $[1+i, 1]$, $[1-i, 1]$ and $[0, 2]$. The covolume of $\mathbb{Z}(1+i)+\mathbb{Z}(1-i)$ in $\mathbb{C}$ and the covolume of $2\mathbb{Z}$ in $\mathbb{R}$ are both equal to 2. The selfintersection of the elliptic curve $T_\infty = T_\infty(G')$ in the cusp bundle $F_\infty = F_\infty(G')$ coincides with the characteristic number t of the unipotent lattice. This number can be calculated as -2 times the covolume volume quotient $\frac{2}{2}$, hence $(T_\infty^2) = t = -2$.

Now consider T_∞ as embedded curve in F_∞ . Endowed with trivial weight 1 it is an orbital curve $\mathbf{T}_\infty$. In order to get the canonical partial resolution of a cusp point $\mathbf{K}$ of $\hat{X}$ we look at the canonical abelization $\mathbf{X}' \longrightarrow \hat{\mathbf{X}}$ of the orbital surface $\hat{\mathbf{X}} = \widehat{\mathbb{B}/\Gamma}$. Following [H98], IV.5, the canonical orbital resolution $\mathbf{E} = \mathbf{E}_K$ of $\mathbf{K}$ concides with the orbital quotient curve $\mathbf{T}_\infty/Z_4$ with $Z_4 = \langle\sigma\rangle$ generated by the reflection $\sigma = \left(\begin{smallmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{smallmatrix}\right)$. From the classification of cusp points by resolution graphs in [H98], III.5, we know that $\mathbf{K}$ has to be of type (2,4,4), which means that $\mathbf{E}_K = (\mathbb{P}^1;\ \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3)$ with abelian points $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ of cyclic type $\langle 2, e_1\rangle$, $\langle 4, e_2\rangle$, $\langle 4, e_3\rangle$, respectively. We determine these types precisely together with the selfintersection (E^2) on the minimal resolution $\tilde{X}$ of X'. For this purpose we calculate the signature hights of our orbital curves, see (10). First we receive $h_\tau(\mathbf{T}_\infty) = (T_\infty^2) = -2$. Now we use the following

Proposition 5.3 *([H98], Theorems II.2.4, II.4.2). If $\mathbf{C} \longrightarrow \mathbf{D}$ is Galois-finite morphism of orbital curves and $h = h_e$ or $h = h_\tau$ denote Euler hights or signature hights, respectively, then it holds that*

$$h(\mathbf{C}) = [C : D]h(\mathbf{D}),\ [C : D] = deg(C \longrightarrow D).$$

$\square$

Applied to the Galois-covering $\mathbf{T}_\infty \longrightarrow \mathbf{E}$ of degree 4 we get

$$h_\tau(\mathbf{E}) = \frac{1}{4} \cdot h_\tau(\mathbf{T}_\infty) = \frac{1}{4} \cdot (-2) = -\frac{1}{2}.$$

The explicit formula (10) for signature hights yields

$$-\frac{1}{2} = (E^2) + \frac{e_1}{2} + \frac{e_2}{4} + \frac{e_3}{4},\ e_j \in \mathbb{N},$$

where the summands have to be smaller than 1. Since a Γ-reflection of order 4 belongs to the cusp group at least one abelian point on $\mathbf{E}$, say $\mathbf{P}_3$ has to be of type $\langle 4, 0\rangle$. The last identity reduces to

$$(E^2) = -\frac{1}{2} - \frac{e_1}{2} - \frac{e_2}{4} > -2,$$

hence $(E^2) = -1$ because the selfintersection must be negative (E is contractible to the cusp point K). Below we will see that there is no Γ'-reflection disc with Γ'-reflection order 2, see 7.10. Therefore $\mathbf{P}_1$ cannot be of type $\langle 2, 0\rangle$, hence $e_1 = 1$, $e_2 = 0$.

We proved that the graph of the orbital cusp point $\mathbf{K}$ is already drawn in figure 7. So E is a projective line supporting precisely one singular surface point $P = P_1$, $(E^2) = -1$, P of type $\langle 2, 1\rangle$ as it has been drawn already, for E_1 say, in figure 2. Therefore E contracts to the non-singular surface point K. The proof of property (i) is finished.

$\square$

6 Unimodular sublattices

Let $K = \mathbb{Q}(i)$ be the Gauß number field, $\mathfrak{O} = \mathbb{Z}[i]$ the ring of Gauß integers, V a finite dimensional K-vector space of dimension n with a hermitian metric $< , >$ with values in K. An $\mathfrak{O}$-module $\Lambda \subset V$, more precisely $(\Lambda, <,>_{|\Lambda})$, is called a sublattice of V, and a V-lattice, if moreover n coincides with the rank ($\mathfrak{O} - rank$) of Λ. A hermitian $\mathfrak{O}$-module Λ is a torsion free $\mathfrak{O}$-module of finite rank together with an hermitian form with values in K. It is a V-lattice in $V = K \otimes \Lambda$ endowed with the extended hermitian form. The dual lattice of Λ is the V-lattice

$$\Lambda^{\#} = \{\mathfrak{x} \in V = K \otimes \Lambda; < \mathfrak{x}, \mathfrak{l} > \in \mathfrak{O} \ for\, all\, \mathfrak{l} \in \Lambda\}.$$

Notice that $\Lambda \subseteq \Lambda^{\#}$ iff the hermitian form has (only) integral values on Λ. A hermitian $\mathfrak{O}$-lattice is called *unimodular* iff $\Lambda^{\#} = \Lambda$. This happens if and only if the hermitian form has integral values on Λ and the discriminant $d(\Lambda)$ is a unit (± 1). Two subsets M, N of a hermitian $\mathfrak{O}$-lattice are orthogonal, iff $< \mathfrak{m}, \mathfrak{n} >= 0$ for all $\mathfrak{m} \in M, \mathfrak{n} \in N$. We write $M \perp N$ in this case. The *orthogonal complement* of M in Λ is the sublattice

$$M^{\perp} = M^{\perp}_{\Lambda} = \{\mathfrak{l} \in \Lambda; \mathfrak{l} \perp M\}.$$

(We omit the index $_\Lambda$ if Λ is fixed and there is no danger of misunderstandings). Two sublattices M, N of Λ are called *orthogonal complementary* (in Λ), iff $M \cap N = O$, $M^{\perp} = N$ and $N^{\perp} = M$.

Proposition 6.1 *Let $(\Lambda, < \ , \ >)$ be a unimodular hermitian $\mathfrak{O}$-lattice, M and N orthogonal complementary sublattices of Λ, then $M^{\#}/M \cong N^{\#}/N$ as $\mathfrak{O}$-modules.*

Proof. Let p_K be the orthogonal projection of $V = K \otimes \Lambda$ along $U = K \otimes M$ onto $W = K \otimes N$ and p its restriction to Λ. Since $M = \Lambda \cap N^{\perp}$, by assumption, we get the left-exact sequence of $\mathfrak{O}$-modules

$$0 \longrightarrow M \longrightarrow \Lambda \xrightarrow{p} N^{\#} \longrightarrow 0.$$

This sequence is exact. For the proof take an element $\mathfrak{f} \in N^{\#}$. It defines a covector $\mathfrak{F}^* =< \mathfrak{f}, \, . \, > \in W^*$ with integral values on N. Its restriction to $N^{\#}$ is denoted by f^*. Since Λ is a unimodular hermitian lattice, the pull back $p^*(\mathfrak{f}^*) = \mathfrak{f}^* \circ p$ is equal to $\mathfrak{v}^* =< \mathfrak{v}, \, . \, >$ on Λ for a unique $\mathfrak{v} \in \Lambda$. For any $\mathfrak{n} \in N^{\#}$ we calculate

$$< p(\mathfrak{v}), \mathfrak{n} >=< \mathfrak{v}, \mathfrak{n} >= \mathfrak{v}^*(\mathfrak{n}) = \mathfrak{f}^*(p(\mathfrak{n})) =< \mathfrak{f}, p(\mathfrak{n}) >=< \mathfrak{f}, \mathfrak{n} >,$$

hence $\mathfrak{f} = p(\mathfrak{v})$. For the same reasons we dispose with obvious notations also on the exact sequence

$$0 \longrightarrow N \longrightarrow \Lambda \xrightarrow{q} M^{\#} \longrightarrow 0$$

of $\mathfrak{O}$-modules. From both exact sequences we get the $\mathfrak{O}$-module isomorphisms

$$M^{\#}/M \cong \Lambda/q^{-1}(M) = \Lambda/(N + M) \cong N^{\#}/N.$$

□

Corollary 6.2 *Under the conditions of the proposition, M is unimodular if and only if its Λ-orthogonal complement N is unimodular.*

Proof. The following conditions are equivalent: M is unimodular, $M^{\#}/M = \mathfrak{O}$, $N^{\#}/N = \mathfrak{O}$, N is unimodular.

□

For arbitrary hermitian $\mathfrak{O}$-lattices Λ and sublattices M we denote by $Aut\Lambda \subset End_{\mathfrak{O}}\Lambda$ the isometry group of Λ and by $Aut(\Lambda, M)$ its subgroup of isometries sending M to M.

Corollary 6.3 *Let Λ be a unimodular hermitian $\mathfrak{O}$-lattice, M a unimodular sublattice and N its orthogonal complement in Λ. Then $\Lambda = M \oplus N$ and*

$$Aut(\Lambda, M) = Aut(\Lambda, N).$$

Proof. M and N are orthogonal complementary in Λ because each $\mathfrak{m} \in \Lambda \cap N^{\perp}$ belongs to $M^{\#} = M$. By 6.2 N is unimodular, too. Now decompose $\mathfrak{l} \in \Lambda$ in $\mathfrak{l} = \mathfrak{m} + \mathfrak{n}$, $\mathfrak{m} \in K \otimes M$, $\mathfrak{n} \in K \otimes N$. Obviously, $\mathfrak{m} \in M^{\#} = M$ and $\mathfrak{n} \in N^{\#} = N$. This proves the first statement, the second follows immediately.

□

We need classification results for unimodular lattices.

Proposition 6.4 *(see Hashimoto [Has], Prop. 3.8). Let $(V, <\ ,\ >)$ be a hermitian space of dimension r over K of signature (p_+, p_-) which contains a unimodular V-lattice ($\mathfrak{O}$-sublattice of V of rank r).*

(i) If r is odd, then there is only one genus of unimodular V-lattices.

(ii) If r is even, then the set of unimodular V-lattices consists of at most two genera. The cardinality of this set is 2 if and only if $p_- \equiv r/2 \,\mathrm{modulo}\, 2$.

□

A genus consists, by definition, of all V-lattices which are locally $\mathbb{U}(V)$-isometric at all natural primes p. More precisely, two such lattices M, M' belong to the same genus iff for each natural prime p there exists

$$\gamma_p \in \mathbb{U}(V_p),\ V_p = V \otimes \mathbb{Q}_p = V \otimes K_p$$

endowed with the $<\ ,\ >$-extending form, sending $M_p = M \otimes \mathbb{Z}_p = M \otimes \mathfrak{O}_p$ to M'_p. The V-lattices M, M' belong to the same class if and only if $g(M) = M'$ for a suitable $g \in \mathbb{U}(V)$.

Proposition 6.5 *(see Hashimoto [Has], Theorem 3.9). If the hermitian metric on V is indefinit, then each genus of unimodular V-lattices consists of one class.*

□

Corollary 6.6 *There are precisely two isometry classes of indefinite unimodular hermitian $\mathfrak{O}$-lattices of rank 2; one is odd and the other even. They are represented by $(\mathfrak{O}^2, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right))$ or $(\mathfrak{O}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, respectively.*

Proof. Let Λ be an unimodular indefinite hermitian $\mathfrak{O}$-lattice of rank 2 and $V = \Lambda \otimes K$. Since the signature of $V = (V, <\ ,\ >)$ must be $(1, -1)$, we know from Proposition 6.4 (ii) that there exist precisely two unimodular genera of V-lattices. From Proposition 6.5 follows that there are precisely two classes of such lattices. The members of each class are isometric. The discriminant $d(\Lambda)$ is -1 because Λ is unimodular (and indefinite). Therefore the discriminants of V and $(K^2, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right))$ coincide. Dimensions, discriminants and signatures form a complete set of invariants of isometry classes of non-degenerated hermitian vector spaces over K by a basic theorem of Landherr [Lan]. Therefore V and K^2 are isometric. It follows that Λ is isometric to precisely one of the two standard lattices announced in the corollary.

□

Corollary 6.7 *All definite unimodular hermitian $\mathfrak{O}$-lattices Λ of rank 2 are isometric to the standard lattice $(\mathfrak{O}^2, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$.*

Proof. Since the discriminant of Λ (and $V = K \otimes \Lambda$) is $+1$ we can assume, by Landherr's theorem again, that $V = K^2$ endowed with standard metric. There is only one genus of definite unimodular K^2-lattices by Proposition 6.4 (ii). The class number of the genus can be read off as $h(4, 2) = 1$ in the table of [Has], p. 76, where 4 comes from 4-th unit root generating our cyclotomic field K, and 2 denotes the lattice rank.

□

We say that two hermitian $\mathfrak{O}$-lattices with integral values have the same *parity*, iff they are both odd or both even, respectively. Let $\tilde{\Gamma} = \tilde{\Gamma}(\Lambda)$ be the automorphism group of a fixed hermitian $\mathfrak{O}$-lattice Λ, and Γ' a subgroup of finite index. A Γ'-class of sublattices of Λ is a Γ'-orbit of one (arbitrary) sublattice of Λ. The normal subgroup of elements with determinant 1 of any subgroup G of the linear group of a finite dimensional vector space is denoted by $\mathbb{S}G$. Usually we set

$$\Gamma = \Gamma(\Lambda) := \mathbb{S}\tilde{\Gamma} = \mathbb{S}\tilde{\Gamma}(\Lambda). \tag{30}$$

Theorem 6.8 *Let Λ be an indefinite unimodular $\mathfrak{O}$-lattice of signature (p_+, p_-) of odd rank $r = p_+ + p_-$. With the above notations it holds that:*

(i) If $p_+ \geq 2$, then there exists precisely one $\tilde{\Gamma}$ -class containing a definite unimodular sublattice of rank 2.

(ii) If $p_- \geq 2$ or $(p_+, p_-) = (2,1)$, then there exist precisely two $\tilde{\Gamma}$-classes of indefinite unimodular rank-2 sublattices.

The parity and discriminant form under the conditions of (ii) a complete invariant system for $\tilde{\Gamma}$-classes of unimodular rank-2 sublattices of Λ.

Proof. Λ is isometric to $\mathfrak{O}^r$ with diagonal standard form $diag(1,..,1,-1,..,-1)$ corresponding to the given signature. This follows from the Propositions 6.4 (i) and 6.5 (uniqueness of genus and class) and Landherr's theorem. It is immediately clear that under the given conditions definite or odd indefinite unimodular rank-2 sublattices exist. But also even ones under the conditions of (ii), namely the lattice generated by $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ in the second case, respectively by

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \\ 0 \end{pmatrix}$ in the first case.

Let E, E' be two unimodular rank-2 sublattices of Λ of same parity and discriminant. By Corollaries 6.7 (definite case) and 6.6 (indefinite case) E and E' are isometric. With Corollary 6.3 we proved that E and $E^\perp$ are orthogonal complementary; the same is true for E' and $E'^\perp$. From Corrollary 6.2 follows that $E^\perp$ and $E'^\perp$ are unimodular. In the signature $(2,1)$ case the orthogonal complements have rank 1 and the same discriminant $+1$. Therefore they are isometric. In the other cases they are indefinite, have odd rank and the same signature by our assumptions. Applying Landherr's theorem, the Propositions 6.4 and 6.5 again, we see that $E^\perp$ and $E'^\perp$ are isometric again. We know also that

$$E \oplus E^\perp = \Lambda = E' \oplus E'^\perp,$$

see Corollary 6.3. The isometries from E onto E' and from $E^\perp$ onto $E'^\perp$ can be composed to an automorphism of Λ showing that E and E' belong to the same $\tilde{\Gamma}$ -class.

□

Let $\Lambda \cong \mathfrak{O}^r$ be an indefinite unimodular lattice of odd rank r as in the above theorem. For two unimodular rank-2 sublattices E, E' of Λ of same discriminant and parity we denote by $Isom(E, E')$ set of isometries of E onto E' and set

$$\Gamma'(E, E') = \{\gamma \in \Gamma';\ \gamma(E) = E'\}.$$

Corollary 6.9 *Under the conditions of the theorem the restriction maps*

$$\tilde{\Gamma}(E, E') \longrightarrow Isom(E, E')\ ,\ \Gamma(E, E') \longrightarrow Isom(E, E')$$

are surjective. The isometry class $Cl_\Lambda(E)$ of sublattices of Λ containing E, the $\tilde{\Gamma}$-class $\tilde{\Gamma} \cdot \{E\}$ and the Γ-class $\Gamma \cdot \{E\}$ coincide.

Proof. For the first map the surjectivity has already been proved. For the second we remark that $\tilde{\Gamma} = \mathfrak{O}^* \cdot \Gamma$ because r is odd. The last statement follows immediately.

□

Now come back to the Picard modular group $\Gamma = \mathbb{SU}((2,1), \mathfrak{O})$, the special automorphism group of the standard unimodular lattice $\Lambda = \mathfrak{O}^3$ of signature $(2,1)$, and its congruence subgroup $\Gamma(\pi)$.

Proposition 6.10 *There are precisely three Γ-classes of unimodular rank-2 sublattices E of Λ completely represented by lattices with Gram matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (definit), $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (indefinit, odd) or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (even), respectively. The Γ-class splits into three $\Gamma(\pi)$-classes if and only if E is not even. In the even case we have only one class $\Gamma(\pi) \cdot \{E\} = \Gamma \cdot \{E\}$.*

Proof. The first statement follows from Theorem 6.8 together with the most obvious realizations by the Λ-orthogonal complements $(0,0,1)^\perp, (1,0,0)^\perp$ or $(1,1,1)^\perp$, respectively. It follows that

6.11 *all non-even unimodular rank-2 sublattices of $\mathfrak{O}^3$ have an orthogonal basis.*

Remember that $\Gamma/\Gamma(\pi) \cong S_3 \cong \mathbb{O}(3, \mathbb{F}_2)$ and that there is a sectional subgroup in Γ also denoted by S_3, see Lemma 5.2. The lattice E defines a residue subplane $\mathbb{E} = E/\pi E \subset \Lambda/\pi\Lambda = \mathbb{F}_2^3$ endowed with the standard non-degenerate quadratic form defined also as residue form of the hermitian form on Λ. The residue maps $E \to \mathbb{E}$ are compatible with discriminants and parity. There are precisely two S_3-orbits of non-degenerate subplanes of $\mathbb{V} = \mathbb{F}_2^3$, namely

$$\{\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3\} = S_3 \cdot \{\mathbb{E}_1\}, \mathbb{E}_1 = (1,0,0)^\perp \ (in\, \mathbb{V}) \tag{31}$$

consisting of all odd planes and the even plane $\mathbb{E}_0 = (1,1,1)^\perp$. Let $E_i \subset \Lambda$ be three definite or three odd unimodular rank-2 lattices with residue planes $\mathbb{E}_i$, $i = 1,2,3$, respectively. They belong to different $\Gamma(\pi)$-classes because $\Gamma(\pi)$ acts trivially on $\mathbb{F}_2^3$. Now let, for instance, $\tau = (2,3) \in S_3 \subset \Gamma$ be the transposition sending $\mathbb{E}_2$ to $\mathbb{E}_3$ and $\mathbb{E}_1$ to itself. Set $E = E_1, \mathbb{E} = \mathbb{E}_1$, $E' = \tau(E), E^\perp = \mathfrak{O}\mathfrak{a}$ and $E'^\perp = \mathfrak{O}\mathfrak{a}'$. Then we have $\mathfrak{a} \equiv \mathfrak{a}' \, mod\, \pi$. Choose an orthogonal base $(\mathfrak{b}, \mathfrak{c})$ of E such that $\mathfrak{b}^2 = 1$, hence $\mathfrak{c}^2 = d(E) = \pm 1$. Then $\tau(\mathfrak{b}), \tau(\mathfrak{c})$ is an orthogonal basis of E'. If $\mathfrak{b} \equiv \tau(\mathfrak{b}) \, mod\, \pi$, and (consequently) $\mathfrak{c} \equiv \tau(\mathfrak{c}) \, mod\, \pi$ we set $\mathfrak{b}' = \tau(\mathfrak{b})$, $\mathfrak{c}' = \tau(\mathfrak{c})$. The correspondence $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \mapsto (\mathfrak{a}', \mathfrak{b}', \mathfrak{c}')$ extends to an automorphism γ of Λ which descends to the identical map on the residue space $\mathbb{V}$. Therefore γ belongs to $\tilde{\Gamma}(\pi)$ sending E to E'. A multiplicative modification by a power of $diag(i,i,i)$ yields an element of $\Gamma(\pi)$ with the same quality. Therefore E and E' belong to the same $\Gamma(\pi)$-class. We have to modify the proof, if $\tau(\mathfrak{b}) \equiv \mathfrak{c}$ and $\tau(\mathfrak{c}) \equiv \mathfrak{b} \, mod\, \pi$. Then we change to the orthogonal bases $(\mathfrak{b}', \mathfrak{c}') = (\mathfrak{c}, \mathfrak{b})$ or $(\pi\mathfrak{b} + \mathfrak{c}, \mathfrak{b} + \bar{\pi}\mathfrak{c})$ in the definite or indefinite case, respectively, preserving the Gram matrix. Now the same argument works. In the even case S_3 acts effectively on the set of bases of $\mathbb{E} = \mathbb{E}_0$. Forgetting the order of base vectors we have only 3, say $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$. Each of them has the symmetric Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Now we can repeat the same base transposition argument starting with any $\tau \in S_3$ of order 2. It has to preserve one of the basis, $\mathfrak{B}_1$ say, up to the order. The rest is clear.

□

At the end of this section we draw a representative plane picture of projective images of unimodular rank-2 lattices E_i representing $\mathbb{E}_i$, $i = 0,1,2,3$. We distinguish for $i = 1,2,3$ definite and indefinite representatives by upper index $+$ or $-$, respectively. By Proposition 6.10 we have a complete system of representatives $E_0, E_1^+, E_2^+, E_3^+, E_1^-, E_2^-, E_3^-$ of $\Gamma(\pi)$-classes. For the rest of this section we denote the subplanes $\mathbb{R} \otimes E_i^\pm$ of the canonical hermitian signature (2,1) space $\mathbb{C}^3$ by $E_i^\pm$ and the projective lines $\mathbb{P}E_i^\pm \subset \mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ by $L_i^\pm$. The orthogonal complements of $\mathfrak{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\mathfrak{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathfrak{c} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathfrak{e} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ in $\mathbb{C}^3$ yield the special representatives $E_1^- = \mathfrak{a}^\perp$, $E_2^- = \mathfrak{c}^\perp$, $E_3^+ = \mathfrak{a}^\perp$ and $E_0 = \mathfrak{e}^\perp$. Using projective coordinates $(x : y : z)$ the corresponding lines are described by linear equations:

$$L_0 : X - Y + Z = 0,\ L_1^- : Y = 0,\ L_2^- : X = 0,$$
$$L_3^+ = L_\infty : Z = 0 \, (the\, infinite\, line)$$

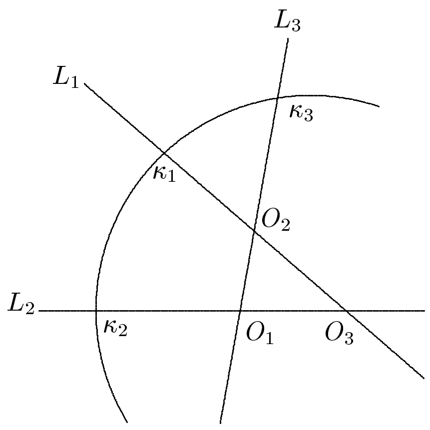

Figure10.

on the (real) projective plane where the (real) ball points ly inside of the (unit) circle, the Γ-cusps sit on the circle. All intersection points of the lines are real, hence all visible in the real picture. Restricting to $\mathbb{B}$ we forget L_3^+ and the marked points. Then we get for the remaining lines and points the dual unweighted graph

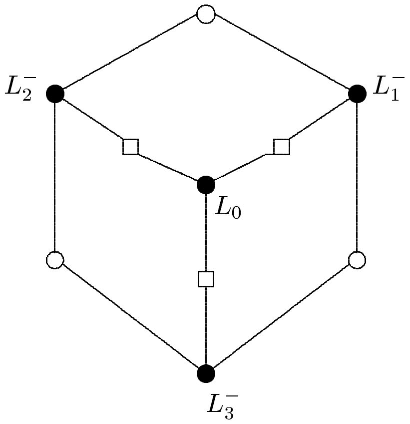

Figure11.

Applying $S_3 \subset \Gamma$ (the alternating subgroup $A_3 \subset S_3$ is sufficient) we get similar graphs including also $L_3^- = \mathbb{P}E_3^-$. Alltogether we get the

$\Gamma(\pi)$-graph of indefinite unimodular rank-2 lattices

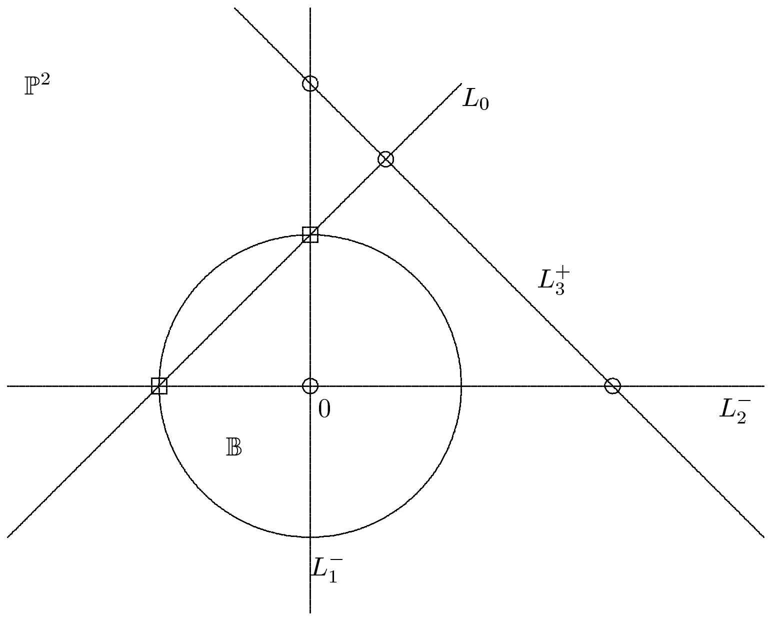

Figure12.

In the next section we determine positive wights as reflection orders, calculate (negative) heights as Euler-Poincare volumes of fundamental domains in discs $\mathbb{D}_i$, $i = 1, 2, 3$, cutten out as intersections of L_i^- with the ball $\mathbb{B}$.

7 Elements of finite order

Let K be a number field, $\mathfrak{O} = \mathfrak{O}_K$ its ring of integers, Γ a subgroup of $\mathbb{G}l_m(\mathfrak{O})$, $\mathfrak{a} \subset \mathfrak{O}$ an ideal and $\Gamma(\mathfrak{a}) \subseteq \Gamma$ the corresponding congruent subgroup defined as kernel of the natural group homomorphism

$$\Gamma \longrightarrow \mathbb{G}l_m(\mathfrak{O}) \longrightarrow \mathbb{G}l_m(\mathfrak{O}/\mathfrak{a}).$$

Lemma 7.1 *If $\gamma^n = 1$ for $\gamma \in \Gamma(\mathfrak{a})$, Then $\mathfrak{a}$ divides $\zeta - 1$ in $\mathfrak{O}_L$, $L = K(\zeta)$, where ζ is an arbitrary eigenvalue of γ (a suitable n-th unit root).*

Proof. For any ζ-eigenvector $\mathfrak{x} = \begin{pmatrix} x_1 \\ \cdots \\ x_m \end{pmatrix} \in K(\zeta)^m$ of g the (fractional) ideal

$$\mathfrak{I}(\mathfrak{x}) := \mathfrak{O}_L x_1 + ... + \mathfrak{O}_L x_m$$

and the ideal class $cl(\mathfrak{x}) := cl(\mathfrak{I}(\mathfrak{x}))$ in the ideal class group $Pic\mathfrak{O}_L$ of L are well-defined. Choose an integral ideal $\mathfrak{q} \in cl(\mathfrak{x})$ such that $(\mathfrak{q}, \mathfrak{a}) = 1$ in $\mathfrak{O}_L$. Since

$$\mathfrak{q} = (\lambda)\mathfrak{I}(\mathfrak{x}) = \mathfrak{I}(\lambda\mathfrak{x})$$

we can assume that $\mathfrak{q} = \mathfrak{I}(\mathfrak{x})$, hence $x_i \in \mathfrak{q} \subset \mathfrak{O}_L$. By assumption, $\mathfrak{a}$ divides (each coefficient of) $\gamma - E$, E the unit matrix of order m. This implies

$$\mathfrak{a} \,|\, (\gamma - E)\mathfrak{x} = (\zeta - 1)\mathfrak{x}, \ \mathfrak{a} \,|\, (\zeta - 1)\mathfrak{I}(\mathfrak{x}) = (\zeta - 1)\mathfrak{q}$$

and finally $\mathfrak{a} \,|\, (\zeta - 1)$ in $\mathfrak{O}_L$.

□

Corollary 7.2 *In the special case of the field $K = \mathbb{Q}(i)$ of Gauß numbers there are at most two possibilities for non-trivial ideals $\mathfrak{a} \subset \mathfrak{O}$ such that $\Gamma(\mathfrak{a})$ contains non-trivial elements of finite order, namely $\mathfrak{a} = (\pi) = (1 + i)$ or $\mathfrak{a} = (2)$. The only orders of such elements are 2 and 4. Elements of order 4 belong to $\Gamma(\pi) \setminus \Gamma(2)$. Especially, $\Gamma(\pi^3)$ is a torsion free group.*

Proof. Let $n = \prod p_j^{k_j}$, p_j natural primes, be the order of $1 \neq \gamma \in \Gamma(\mathfrak{a})$. Then γ has a primitive n-th unit root $\zeta = \zeta_n$ as eigenvalue. The degrees of the minimal polynom $\Phi_n(X)$ of ζ over $\mathbb{Q}$ and of the field extension $L = \mathbb{Q}(\zeta)/\mathbb{Q}$ coincide with the Euler product $\phi(n) = \prod(p_j - 1)p_j^{k_j - 1}$. Since $\Phi_n(X)$ divides $X^{n-1} + X^{n-2} + ... + X + 1$ in $\mathbb{Z}[X]$ and $\mathfrak{a} \,|\, (\zeta - 1)$ it follows that

$$N_{K/\mathbb{Q}}(\mathfrak{a})^{\phi(n)/2} = N_{L/\mathbb{Q}}(\mathfrak{a}) \,|\, N_{L/\mathbb{Q}}(1 - \zeta) = \Phi(1) \,|\, n.$$

This is not possible for an odd prime $p = n > 3$ because in this case the exponent $\phi(n)/2$ is greater than 1. Also $p = 3$ is excluded because $N(\mathfrak{a}) \,|\, 3$ can only realized by $\mathfrak{a} = \mathfrak{O}$ in the ring auf integral Gauß numbers. Therefore only elements γ of 2-power order $n = 2^k$ occur. The norm $1 < a = N_{K/\mathbb{Q}}(\mathfrak{a})$ has to satisfy $a^{2^{k-1}} \,|\, 2^k > 1$, hence $a = 2$, $2^{k-1} \leq k$ which is only possible for $k = 1$ or 2, $n = 2$ or 4. In both cases we have $\mathfrak{a} \,|\, 1 - \zeta = 2$ or $1 - i$, respectively.

□

We concentrate our further considerations to subgroups of $\tilde{\Gamma} = \mathbb{U}((2,1), \mathfrak{O})$, $\mathfrak{O} = \mathfrak{O}_K$, $K = \mathbb{Q}(i)$, especially to $\Gamma = \mathbb{S}\tilde{\Gamma}$, again. Elements of order 4 in Γ belong to the $\mathbb{G}l_3(K)$-conjugation classes of $diag(1, i, -i)$, $diag(-1, i, i)$ or $diag(-1, -i, -i)$, and elements of order two are conjugated to $diag(1, -1, -1)$. The conjugacy classes of the latter three types exhaust the set of all semisimple elements $1 \neq \sigma \in \Gamma$ with a double eigenvalue. This follows easily from the fact that the characteristic polynomials $\chi_\gamma(T)$ have to ly in $\mathfrak{O}[T]$. Semisimple elements in $\tilde{\Gamma}$ with precisely two eigenvalues are called Λ-*reflections*. The *reflection lattice* $E(\sigma) \subset \Lambda$ is defined to be the intersection of $\Lambda = \mathfrak{O}^3$ with the eigenspace of the double eigenvalue of σ. Obviously, it has $\mathfrak{O}$-rank 2.

Proposition 7.3 *For each Λ-reflection $\sigma \in \Gamma(\pi)$ is the reflection lattice $E(\sigma)$ unimodular.*

Proof. We can assume that the eigenvalues of σ are $1,-1,-1$. Otherwise σ has order 4 by Corollary 7.2 and we can take σ^2 because $E(\sigma)=E(\sigma^2)$. Let $\mathfrak{a}\in\Lambda$ and $(\mathfrak{b},\mathfrak{c})$ be an $\mathfrak{O}$-base of $E^\perp$ or E, respectively. On K^3 the reflection σ acts by the correspondence

$$\mathfrak{u}=x\mathfrak{a}+y\mathfrak{b}+z\mathfrak{c}\mapsto\sigma(\mathfrak{u})=x\mathfrak{a}-y\mathfrak{b}-z\mathfrak{c},\ x,y,z\in K.$$
$$-\sigma(\mathfrak{u})=\mathfrak{u}-2x\mathfrak{a}=\mathfrak{u}-2\frac{<\mathfrak{u},\mathfrak{a}>}{<\mathfrak{a},\mathfrak{a}>}\mathfrak{a}.$$

Since $\mathfrak{a}$ is primitive, we find vectors $\mathfrak{u}\in\mathfrak{O}^3$ such that $<\mathfrak{u},\mathfrak{a}>=1$. The image $\sigma(\mathfrak{u})$ has to be integral, therefore

$$\mathfrak{a}^2=<\mathfrak{a},\mathfrak{a}>\ \textit{divides}\ 2.$$

Assume that $\mathfrak{a}^2$ is not a unit, that means $\mathfrak{a}^2=\pm 2$. The $-\sigma$-images of the canonical basis vectors $\mathfrak{e}_i$, $i=1,2,3$, of $\mathfrak{O}^3$ are $\mathfrak{e}_i\pm\bar{a}_i\mathfrak{a}$, where a_i is the i- th coordinate of $\mathfrak{a}$. Now we see that σ does not induce the identical map on $\mathfrak{O}^3/\pi\mathfrak{O}^3$. Namely, $\mathfrak{e}_i\equiv\sigma(\mathfrak{e}_i)\,mod\,\pi$ implies $\bar{a}_i\mathfrak{a}\equiv 0\,mod\,\pi$, hence $|a_i|^2\equiv 0\,mod\,\pi$, which is not possible for $i=1,2,3$, because $\mathfrak{a}$ is primitive. So we proved that $\mathfrak{a}^2$ is a unit which means that $\mathfrak{O}\mathfrak{a}$ is unimodular. Corollary 6.2 implies the unimodularity of E.

□

For $\Gamma'\subseteq\tilde{\Gamma}$ and any pair of orthogonal complementary sublattices $\mathfrak{O}\mathfrak{a}\perp E$ of Λ we have a pair of restriction homomorphisms

$$Aut\,E\longleftarrow\Gamma'(E,E)\longrightarrow Aut\,\mathfrak{O}\mathfrak{a}$$

For unimodular E and $\Gamma'=\tilde{\Gamma}$ one gets a pair of cartesian projections

$$Aut\,E\longleftarrow\tilde{\Gamma}(E,E)=\tilde{\Gamma}(\mathfrak{O}\mathfrak{a},\mathfrak{O}\mathfrak{a})\cong Aut\,E\times Aut\,\mathfrak{O}\mathfrak{a}\longrightarrow Aut\,\mathfrak{O}\mathfrak{a}\cong\mathfrak{O}^*,$$

where the surjectivity on the left-hand side comes from Corollary 6.9 and the identity from Corollary 6.3. Restricting to Γ we get an exact sequence

$$\begin{array}{ccccc} 1\longrightarrow\mathbb{S}Aut\,E\longrightarrow & \Gamma(E,E)=\Gamma(\mathfrak{O}\mathfrak{a},\mathfrak{O}\mathfrak{a}) & \longrightarrow Aut\,\mathfrak{O}\mathfrak{a}\cong\mathfrak{O}^*\longrightarrow 1, \\ & \| & \\ & Aut\,E & \end{array}$$

where the vertical isomorphism sends ρ to $\rho\times det\rho$. It restricts via intersections with $\Gamma(\pi)$ to the obviously splitting exact sequence

$$\begin{array}{ccccc} 1\longrightarrow(\mathbb{S}Aut\,E)(\pi)\longrightarrow & \Gamma(\pi)(E,E)=\Gamma(\pi)(\mathfrak{O}\mathfrak{a},\mathfrak{O}\mathfrak{a}) & \longrightarrow Aut\,\mathfrak{O}\mathfrak{a}\cong\mathfrak{O}^*\longrightarrow 1, \\ & \| & \\ & (Aut\,E)(\pi) & \end{array}\tag{32}$$

Lemma 7.4 *Each maximal finite subgroup G of $(\mathbb{S}Aut\,E)(\pi)$ is cyclic of order 4.*

Proof. G has only elements of order dividing 4 by Corollary 7.2. Consider G as subgroup of $\mathbb{S}l_2(\mathbb{C})=\mathbb{S}l(\mathbb{R}\otimes E)$. It acts on the projective line $\mathbb{P}^1(\mathbb{C})$. If G is not abelian, then it must be a binary dieder group $2D_2$ (quaternion group) or $2D_4$ because of the orders of elements of G, see e.g. [Bri]. $2D_n$, D_n the dieder group of order $2n$, is represented by the subgroup of $\mathbb{S}l_2(\mathbb{C})$ generated by $\sigma=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ and $diag(\zeta_{2n},\zeta_{2n}^{-1})$. Therefore only $2D_2$ survives. There is a canonical $\mathbb{S}l_2(\mathfrak{O})$- representation with generators

$$\mathfrak{i}=diag(i,-i),\ \mathfrak{j}=\left(\begin{smallmatrix}0&i\\i&0\end{smallmatrix}\right)\ \mathfrak{k}=-\mathfrak{i}\cdot\mathfrak{j}=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right).$$

Let σ_1,σ_2,σ_3 be the corresponding elements of $G\subset\mathbb{S}l(E)$. The product $1\neq\tau=-i\cdot\sigma_1$ has order 2. We choose primitive (orthogonal) eigenvectors $\mathfrak{a},\mathfrak{b}\in E$ of τ. By Proposition 6.10 there exist an $\mathfrak{O}$-base of E with Gram matrix $diag(1,1)$, $diag(1,-1)$ or $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. We can now repeat the arguments of Proposition 7.3 to show that $\mathfrak{a}^2$ and $\mathfrak{b}^2$ have to be units. This kills already the third case of even lattice. The contradiction comes from the existence of such a reflection. Therefore $G\ncong 2D_2$ in the even case. In the odd cases we see that $\mathfrak{a},\mathfrak{b}$ is an $\mathfrak{O}$-base of E sent to $i\mathfrak{a}$ or $-i\mathfrak{b}$ by σ_1, respectively. We work with matrix representations with respect to this base,

$$M(\sigma_1)=\left(\begin{smallmatrix}i&0\\0&-i\end{smallmatrix}\right)\ \textit{and}\ M(\sigma_2)=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),$$

say. From $\sigma_1\sigma_2 = -\sigma_2\sigma_1$ follows that $a = b = 0$. Therefore we get the canonical representation. Descending to residue classes modulo π it is clear now that G acts not trivially on $E/\pi E$. Therefore G is not contained in the congruence subgroup $(\mathbb{S}Aut\,E)(\pi)$ in contradiction to our assumption. We proved that G is an abelian group. The abelian groups in $\mathbb{S}l_2(\mathbb{C})$ are cyclic. The central subgroup of $(\mathbb{S}Aut\,E)(\pi)$ generated by $-id$ is obviously not maximal.

□

Theorem 7.5 *(i) Each maximal finite subgroup T of $\Gamma(\pi)$ is isomorphic to $\mathfrak{O}^* \times \mathfrak{O}^*$.*

(ii) The set of all these groups coincides with the set of intersections

$$\Gamma(\pi)(E,E) \cap \Gamma(\pi)(E',E') = \Gamma(\pi)(\mathfrak{O}\mathfrak{c}) \cap \Gamma(\pi)(\mathfrak{O}\mathfrak{b}),$$

where $E = \Lambda_{\mathfrak{c}}$ and $E' = \Lambda_{\mathfrak{b}}$ are two different unimodular rank-2 sublattices of Λ with orthogonal vectors $\mathfrak{b}$, $\mathfrak{c}$ of hermitian norms ± 1.

(iii) Each element $\delta \in \Gamma(\pi)$ of order 2 is a square of a reflection $\rho \in \Gamma(\pi)$ of order 4.

(iv) Each element $\gamma \in \Gamma(\pi)$ of finite order is a reflection or a product of two reflections.

(v) A non-trivial element of finite order of $\Gamma(\pi)$ has order 2 if and only if it belongs to $\Gamma(\pi^2) = \Gamma(2)$. It has order 4 if and only if it belongs to $\Gamma(\pi) \setminus \Gamma(\pi^2)$.

Proof. First we prove that T is commutative. Assuming the opposite we find two non-commuting elements $\sigma, \tau \in T$. Changing, if necessary to $-\sigma$ or $-\tau$, we assume that σ and τ have the simple eigenvalue $+1$, see the preparations of 7.3. Let $\mathfrak{a}, \mathfrak{b} \in \Lambda$ be corresponding (primitive) eigenvectors of σ, τ, respectively. If σ has order 4, then σ^2 is a reflection. Therefore $\Lambda_{\mathfrak{a}}$ and for the same reason also $\Lambda_{\mathfrak{b}}$ is a reflection lattice which has to be unimodular by Proposition 7.3. If they coincide, then $\mathfrak{O}\mathfrak{a} = \mathfrak{O}\mathfrak{b}$. We can restrict the exact sequence (32) applied to $E = \Lambda_{\mathfrak{a}} = \Lambda_{\mathfrak{b}}$ to the finite subgroup $< \sigma, \tau >$ of $\Gamma(\pi)(\mathfrak{a})$ to get a splitting exact finite group sequence

$$1 \longrightarrow R \longrightarrow \mathfrak{O}^* \cdot < \sigma, \tau > \longrightarrow \mathfrak{O}^* \cong Aut\,\mathfrak{O}\mathfrak{a} \longrightarrow 1$$

By Lemma 7.4 the subgroup R is cyclic (of order $1, 2$ or 4). Therefore the middle group is abelian in contradiction to our assumption. It follows that $\Lambda_{\mathfrak{a}} \neq \Lambda_{\mathfrak{b}}$. The intersection of both is generated by a common (primitive) eigenvector $\mathfrak{e} \in \Lambda$ of σ and τ. There is an element ρ of order 2 in $< \sigma, \tau >$ such that $\rho(\mathfrak{e}) = \mathfrak{e}$. For instance, take $\rho = \sigma^2\tau^2$, if σ and τ have order 4. Then $\rho \neq id$ because otherwise $\sigma^2 = \sigma^-2 = \tau^2$, hence $\mathfrak{a} = \mathfrak{b}$ (up to a unit factor) by the above choice. With $E = \Lambda_{\mathfrak{e}}$ we found again an unimodular reflection lattice with $< \sigma, \tau > \subset \Gamma(\pi)(E,E)$. Repeating the above argument, we see that σ and τ commute. Therefore T is a (maximal) abelian subgroup of $\Gamma(\pi)$.

T has a simultaneous (primitive) eigenbase $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \Lambda$. We can choose via elements of order 2 one of these eigenvectors, $\mathfrak{c}$ say, such that $E = \Lambda_{\mathfrak{c}}$ is a unimodular reflection lattice, $T \subset \Gamma(\pi)(E,E)$. We get from (32) the splitting exact sequence

$$1 \longrightarrow S \longrightarrow T \longrightarrow \mathfrak{O}^* = Aut\,\mathfrak{O}\mathfrak{c}$$

of finite abelian groups, S a subgroup of $\mathfrak{O}^*$ by Lemma 7.4. Working with matrix representations with respect to $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ we see that at least the reflection group $< diag(i,i,-1) >$ is a subgroup of T. This cyclic group cannot exhaust T because it is not maximal and $Aut\,E$ has non-trivial elements $diag(\mu,\nu)$, $\mu, \nu \in \mathfrak{O}^*$, with respect to any $\mathfrak{O}$-base of E, which can be easily extended to elements of $\Gamma(\pi)$ via $\mathfrak{O}\mathfrak{c}$. So T must be a product of two cyclic groups, Take a subgroup $< diag(i,i,-1) > \times < \delta >$ of T with δ of order 2. then $E(\delta)$ is a unimodular reflection $lattice\Lambda'_{\mathfrak{b}}$. The eigenvectors $\mathfrak{b}'$ and $\mathfrak{c}$ of δ are orthogonal because they belong to the different eigenvalues $+1$ or -1 of δ, respectively. Therefore $diag(i,i,1)$ acts by i-multiplication on $\mathfrak{O}\mathfrak{b}' \subset E = \Lambda_{\mathfrak{c}}$, hence the vectors $\mathfrak{b}'$ and $\mathfrak{c}$ are common eigenvectors of T. By Corollary 6.2 we know that $\mathfrak{b}'^2, \mathfrak{c}^2 \in \{\pm 1\}$. So $\mathfrak{O}\mathfrak{b}' + \mathfrak{O}\mathfrak{c}$ and its Λ-orthogonal complement $\mathfrak{O}\mathfrak{a}'$ are unimodular, and T acts on $\mathfrak{O}\mathfrak{a}'$, too. So we can assume that $\mathfrak{a} = \mathfrak{a}'$, $\mathfrak{b} = \mathfrak{b}'$, $\mathfrak{c}$. These vectors form an $\mathfrak{O}$- base of Λ because their Gram matrix is diagonal with $\mathfrak{O}^*$- coefficients. Via $\mathfrak{O}$-linear extension to Λ of three independent $\mathfrak{O}^*$-actions on the base vectors we get with obvious notation and identification

$$T \subset \mathbb{S}diag(\mathfrak{O}^*, \mathfrak{O}^*, \mathfrak{O}^*) = \mathfrak{O}^{*2} \subset T,$$

because T is maximal, hence $T \cong \mathfrak{O}^* \times \mathfrak{O}^*$. We proved (i) and (ii). (iii) follows immediately because we can extend each subgroup $< \delta > \subset \Gamma(\pi)$ of order 2 into a maximal one T which allows us to work with diagonal representations with respect to an orthogonal basis of Λ: $\delta = diag(1,-1,-1) = diag(-1,i,i)^2$. (iv) Embed γ again into a maximal finite T as above. If γ is not a reflection, then it has diagonal representation $diag(1,i,-i) = diag(i,-1,i) \cdot diag(-i-i,-1)$. (v) It is now also clear that δ belongs to $\Gamma(2)$. For the elements of order 4 we refer to Corollary 7.2.

□

Notation. $E_{\mathbb{R}} := \mathbb{R} \otimes E$ for each $\mathfrak{O}$-lattice E.

Definition-Remark 7.6 *We call a Λ-reflection δ a $\mathbb{B}$- reflection, iff $L(\delta) := \mathbb{P}E_{\mathbb{R}}(\delta)$ intersects $\mathbb{B}$. We denote the corresponding (complete linear) subdisc $\mathbb{D}(\delta) = L(\delta) \cap \mathbb{B}$ or by $\mathbb{D}_{\mathfrak{a}}$, where $\mathfrak{a} \in \mathbb{C}^3$ is an arbitrary non-trivial vector orthogonal to E. δ is a $\mathbb{B}$-reflection if and only if $a^2 > 0$ or, equivalently, $E(\delta)$ is indefinite.*

Corollary 7.7 *Any three different projective lines*

$$L_j = \mathbb{P}E_{j\mathbb{R}} \subset \mathbb{P}^2 = \mathbb{P}(\Lambda_{\mathbb{R}})$$

of unimodular rank-2 sublattices of Λ have no common intersection point Q on $\mathbb{B}$.

Proof. Let $Q = \mathbb{P}\mathfrak{q}, \mathfrak{o} \neq \mathfrak{q} \in \Lambda$, be a common intersection point. Then $\mathfrak{q}$ is a common eigenvector of reflections $\sigma_j \in \Gamma(\pi)$ with reflection lattice E_j. The group $G =< \sigma_1, \sigma_2, \sigma_3 >$ is a subgroup of

$$\Gamma(\pi)(\Lambda_{\mathfrak{q}}, \Lambda_{\mathfrak{q}}) = \Gamma(\pi)(\mathfrak{O}\mathfrak{q}, \mathfrak{O}\mathfrak{q}).$$

Since $\mathfrak{q}^2 < 0$ the lattice $\Lambda_{\mathfrak{q}}$ is positive definite. Therefore G is a finite group, see (32) or a sequence before. It follows from the theorem that G is abelian. Therefore the elements of G has three simultaneous eigenvectors, one of them is $\mathfrak{q}$. The projected finite abelian group $\mathbb{P}G$ has a faithful representation on the tangent space $T_Q(\mathbb{P}^2)$ with only two eigenlines there. But L_j is an eigenline of $\mathbb{P}\sigma_j$ for $j = 1,2,3$. This is a contradiction.

□

Proposition 7.8 *The surface $\mathbb{B}/\Gamma(\pi)$ is smooth. There are precisely three $\Gamma(\pi)$-orbits of $\Gamma(\pi)$-elliptic points on $\mathbb{B}$. Its union is the Γ-orbit of $O = (0:0:1) \in \mathbb{B}$ consisting of all $\mathfrak{q} \in \Lambda$ with $\mathfrak{q}^2 = -1$. Each subgroup $\Sigma \cong S_3$ of Γ acts transitively as permutation group on the three orbits via conjugation. The isotropy group $\Gamma(\pi)_Q$ of each $\Gamma(\pi)$- elliptic point Q is the product of two cyclic groups each generated by a reflection of order 4. $\mathfrak{O}\mathfrak{q}$ is the intersection of the corresponding reflection lattices E and E'. Both are unimodular, indefinite and odd.*

Proof. We have to show that each stationary group $\Gamma(\pi)_Q$ of a $\Gamma(\pi)$-elliptic points $Q \in \mathbb{B}$, is generated by $\mathbb{B}$- reflections of $\Gamma(\pi)_Q$, see [H98]. $\Gamma(\pi)_Q$ is a finite subgroup containing an element γ which is not a reflection, by the definition of elliptic points. This means that Q is an isolated fixed point of γ. Then γ has order 4 and $Q = \mathbb{P}\mathfrak{q}$ for a primitive eigenvector $\mathfrak{q} \in \Lambda$ of γ. Since γ has only simple eigenvalues $1, i, -i$, it has precisely three eigenlines in K^3. Extend γ to a maximal (abelian) subgroup T of $\Gamma(\pi)$. Then γ is a product of two reflections $\sigma, \tau \in T$ by the proof of (iv). The three eigenlines of T must be the same as those for γ. Therefore σ and τ belong to $\Gamma(\pi)_Q$ and both are $\mathbb{B}$-reflections. Moreover, $\mathfrak{q}$ is a common eigenvector of T, hence $\mathfrak{O}^* \times \mathfrak{O}^* \cong T \subset \Gamma(\pi)_Q$. The inclusion is the identical map because of the maximality of T.

T acts on the orthogonal complement $E = E(Q) = \Lambda_{\mathfrak{q}}$ of $\mathfrak{q} in \Lambda$. It is a Λ-reflection lattice because $E = E(\gamma^2)$ and γ^2 is a Λ-reflection because its order is 2. By Proposition 7.3 E is unimodular and also $\mathfrak{O}\mathfrak{q}$ is by Corollary 6.2. Therefore $\mathfrak{q}^2 = -1$, because $Q = \mathbb{P}\mathfrak{q}$ belongs to $\mathbb{B}$, hence $\mathfrak{q}^2$ must be negative. Thus E is a definite odd unimodular sublattice of Λ. The last conclusion is correct for each $\mathfrak{q} \in \Lambda$ with $\mathfrak{q}^2 = -1$. The set

$$S = \{\mathfrak{O}\mathfrak{q}; \mathfrak{q} \in \Lambda, \mathfrak{q}^2 = -1\}$$

and the set of all unimodular definite odd rank-2 sublattices E of Λ correspond bijectively to each other via orthogonality. The latter set is the Γ- orbit of one of its members by Proposition 6.10. Therefore the former set is the Γ-orbit of one element, hence $\mathbb{P}S = \Gamma \cdot O$. We get on this way also precisely one Γ-conjugacy class of stabilizer groups Γ_Q of $\Gamma(\pi)$-elliptic points Q. They split into three different $\Gamma(\pi)$-classes by the second part of Proposition 6.10. □

Corollary 7.9 *Each $\mathfrak{q} \in \Lambda$ with hermitian norm $\mathfrak{q}^2 = -1$ extends uniquely, up to $\mathfrak{O}^*$-factors and order of numeration, to an orthogonal basis $(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{q})$ of Λ. The both unique unimodular (indefinite odd) rank-2 sublattices of Λ with intersection $\mathfrak{O}\mathfrak{q}$ are the reflection planes*

$$E_1 = \Lambda_{\mathfrak{a}_1} = \mathfrak{O}\mathfrak{a}_2 + \mathfrak{O}\mathfrak{q}, \quad E_2 = \Lambda_{\mathfrak{a}_1} = \mathfrak{O}\mathfrak{a}_1 + \mathfrak{O}\mathfrak{q}, \tag{33}$$

Moreover, we set

$$E_3 := \Lambda_{\mathfrak{q}} = \mathfrak{O}\mathfrak{a}_1 + \mathfrak{O}\mathfrak{a}_2.$$

The set of residue planes of the lattices E_j, $j = 1, 2, 3$, coincides with the set of the three unimodular odd subplanes in $\mathbb{F}_2^3 = \Lambda/\pi\Lambda$, explicitly described in (31) (with possibly other numeration). Two -1-vectors $\mathfrak{q}, \mathfrak{q}'$ belong to the same $\Gamma(\pi)$-orbit if and only if $\mathfrak{q} \equiv \mathfrak{q}' \bmod \pi$.

Proof. Define E_1, E_2 as the unique pair of $\Gamma(\pi)$-reflection lattice containing $\mathfrak{q}$. Their existence has been proved above (as $E(\sigma), E(\tau)$). The uniqueness comes from Corollary 7.7. Choose an orthogonal base $\mathfrak{a}_1, \mathfrak{a}_2$ of the definite unimodular lattice $\Lambda_{\mathfrak{q}}$. Then $(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{q})$ is an orthogonal base of Λ. It holds that $\mathfrak{a}_1^2 = \mathfrak{a}_2^2 = 1$, $\mathfrak{O}\mathfrak{q} = \Lambda_{\mathfrak{a}_1} \cap \Lambda_{\mathfrak{a}_2}$ with unimodular $\Lambda_{\mathfrak{a}_1}, \Lambda_{\mathfrak{a}_2}$. These lattices must coincide with E_1, E_2 but also with the unimodular sum lattices in (33). It is clear that $E_j/\pi E_j$ is a unimodular odd subplane of $\mathbb{F}_2^3$. It is easy to see that if $E_j \equiv E_k \bmod \pi$, then $E_l/\pi E_l$ must be a line in $\mathbb{F}_2^3$ for $\{j, k, l\} = \{1, 2, 3\}$, which is not possible. $S_3 \subset \Gamma$ acts transitively on the set of unimodular odd subplanes $\{\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3\}$ permuting indices. Therefore Γ acts transitively on them, hence on the set of their orthogonal complements in $\mathbb{F}_2^3$ represented for instance by the -1-vectors $(1, 0, \pi), (0, 1, \pi), (0, 0, 1) \in \Lambda$. These vectors represent different $\Gamma(\pi)$-orbits and all of them by the proposition. The last statement follows immediately.

□

Proposition 7.10 *The irreducible components of the branch locus of the quotient map $\mathbb{B} \longrightarrow \mathbb{B}/\Gamma(\pi)$ are smooth. It consists of 4 curves*

$$C_0 = \mathbb{D}_0/\Gamma_0, \; C_1 = \mathbb{D}_1/\Gamma_1, C_2 = \mathbb{D}_2/\Gamma_2, \; C_3 = \mathbb{D}_3/\Gamma_3, \tag{34}$$

where $\mathbb{D}_j = \mathbb{B} \cap L_j$, $L_j = \mathbb{P}(E_j\mathbb{R})$, E_j an arbitrary unimodular indefinite rank-2 sublattice of Λ with (non-degenerate) residue subplane $\mathbb{E}_j$ of $\mathbb{F}_2^3$ with the notations of (31) and $\Gamma_j = \Gamma(\pi)(E_j, E_j)$, $j = 0, 1, 2, 3$, respectively. The ramification index is 4 for all four components. The action of $S_3 = \Gamma/\Gamma(\pi)$ on $\mathbb{B}/\Gamma(\pi)$ permutes the curves C_1, C_2, C_3 and restricts to an effective action on C_0. C_k intersects C_l in precisely one point P_m for any triple $\{k, l, m\} = \{1, 2, 3\}$. The intersection is transversal. The points P_1, P_2, P_3 are the images of all $\Gamma(\pi)$-elliptic points on $\mathbb{B}$. The $\mathbb{B}$-reflection discs $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3$ do not intersect $\mathbb{D}_0$.

Proof. The branch locus comes from reflection discs and elliptic points, see [H98], IV, Corollary 4.6.3. Each elliptic points Q is the intersection point of precisely two reflection discs, see 7.7, 7.8, say of $\mathbb{D} = \mathbb{P}E_{\mathbb{R}}$, $\mathbb{D}' = \mathbb{P}E'_{\mathbb{R}}$. The unimodular reflection lattices $E, E' \subset \Lambda$ project onto different residue subplanes $\mathbb{E}_1, \mathbb{E}_2$, say, of $\mathbb{F}_2^3$ by Corollary 7.9. With this observation the S_3-action on the branch components is clarified. Now let R be an arbitrary point of $\mathbb{D} = \mathbb{P}E_{\mathbb{R}}$. We prove that R cannot be a honest $\Gamma(\pi)$-cross point of $\mathbb{D}$. This means, by definition, that

$$\{\Gamma(\pi)R\}_{\mathbb{D}} = \Gamma(\pi)R \cap \mathbb{D} = \Gamma(\pi)(\mathbb{D}, \mathbb{D})R$$

with

$$\Gamma(\pi)(\mathbb{D}, \mathbb{D}) := \Gamma(\pi)(E; E).$$

We start with $\mathbb{D} = \mathbb{D}_1$. If R is not elliptic, then also $\gamma(R)$ is not for each $\gamma \in \Gamma(\pi)$. Therefore $\gamma(\mathbb{D})$ is the only $\Gamma(\pi)$-reflection disc throug $\gamma(R)$. Especially $\mathbb{D}$ is the only reflection disc through $\gamma(R)$, if $\gamma(R) \in \mathbb{D}$, hence $\gamma(\mathbb{D}) = \mathbb{D}$, $\gamma \in \Gamma(\pi)(\mathbb{D}, \mathbb{D})$. If $R = Q \in \mathbb{D}$ is elliptic, $\{Q\} = \mathbb{D}_1 \cap \mathbb{D}_2$ with the above notations, then $\gamma(\mathbb{D}_1)$ and $\gamma(\mathbb{D}_2)$ are the only reflection discs through $\gamma(Q)$ for $\gamma \in \Gamma(\pi)$. Observe that corresponding planes $\mathbb{E}_1$ and $\mathbb{E}_2$ are preserved and not transposed because γ acts trivially on $\mathbb{F}_2^3$. If $\gamma(Q) \in \mathbb{D} = \mathbb{D}_1$, then $\gamma(\mathbb{D}) \neq \mathbb{D}_2$, hence $\gamma(\mathbb{D}) = \mathbb{D}_1 = \mathbb{D}$, $\gamma \in \Gamma(\pi)(\mathbb{D}, \mathbb{D})$. From the abscence of honest $\Gamma(\pi)$-cross points on $\mathbb{D}$ it follows that the quotient curve $C = \mathbb{D}/\Gamma(\pi)(\mathbb{D}, \mathbb{D})$ is smooth, see [H98], IV, Proposition 4.4.6. For $\mathbb{D} = \mathbb{D}_0$ we know that each $R \in \mathbb{D}$ is not an elliptic point because elliptic points are intersections of two discs coming from odd lattices by Proposition 7.8, but E_0 is even and intersection points of three reflection discs do not exist on $\mathbb{B}$ by Corollary 7.7. Now the the same argument for non-elliptic points on $\mathbb{D}_1$ works.

$\mathbb{D}_0$ cannot intersect $\mathbb{D}_1$ because an intersection point $Q \in \mathbb{B}$ would be an elliptic point as intersection point of two reflection lines L_0, L_1. But then Q is also an intersection point of two lines coming from unimodular odd lattices, hence of three reflection lines, which is not possible by Corollary 7.7 again.

□

Proposition 7.11 *Each Γ-cusp $\kappa = \mathbb{P}\mathfrak{k}$, $\mathfrak{k} \in \Lambda$ a primitive isotropy vector, is the intersection of precisely two $\Gamma(\pi)$-reflection lines L_0, L_1. Both come from unimodular indefinite lattices E_0, E_1, where the first one is even and the other odd. Each unimodular even (hence indefinite) lattice E_0 contains isotropy vectors $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3$ representing all the possible non- trivial residue isotropy vectors*

$$\begin{pmatrix} 0\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\1\\0 \end{pmatrix} \in \mathbb{F}_2^3.$$

They can be choosed as A_3-orbit of $\mathfrak{k}_1$, where A_3 is the alternating subgroup of a group (isomorphic to and identified with) $S_3 \subset \Gamma(E_0, E_0)$ acting on E such that the corresponding reflection lines $L_1, L_2, L_3 \neq L_0$ through κ_1, κ_2 or κ_3, respectively, intersect each other pairwise on $\mathbb{B}$. These three elliptic intersection points represent the three $\Gamma(\pi)$-orbits of all $\Gamma(\pi)$-elliptic points.

Proof. If L_0, L_1 through κ exist they must be unique by Corollary 7.7. Since the Γ-cusps form only one Γ-orbit as already mentioned above, it suffices to check the situation at one cusp. This has been already done in the graphic 10, where $\kappa_1 = (0:1:1)$ appears as intersection of L_0 and L_1^-. The application of $\Gamma \supset S_3$ shows that the cusps κ_j of the residue class with corresponding index is the intersection of two reflection lines of type L_0 and L_j. Now we show that we can choose an S_3 in $\Gamma(E_0, E_0)$. Remember that we can find an $\mathfrak{O}$-base of $E_0 = \Lambda_{\mathfrak{c}}$ with Gram matrix $\left(\begin{smallmatrix} 0 & 1\\ 1 & 0 \end{smallmatrix}\right)$ and $\mathfrak{c}^2 = 1$. Using matrix representations with respect to this base it is easy to check that $\left(\begin{smallmatrix} 0 & i\\ i & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1\\ -1 & 0 \end{smallmatrix}\right)$ generate a subgroup of $\mathbb{S}Aut(E_0)$ isomorphic to S_3, where the first element generates A_3. Sending $\mathfrak{c}$ to $\mathfrak{c}$ it extends to a subgroup S_3 of $\Gamma(E_0, E_0)$ depending on the chosen base. The last statement can be checked by example. We refer to picture 10 again, with lines L_1, L_2 meeting L_0 in cusps κ_1 or κ_2, respectively. The corresponding isotropy vectors $\mathfrak{k}_1 = (0,1,1)$, $\mathfrak{k}_2 = (1,0,-1)$ have Gram matrix $\left(\begin{smallmatrix} 0 & 1\\ 1 & 0 \end{smallmatrix}\right)$. The above generator of the correspnding subgroup A_3 sends $\mathfrak{k}_1$ to $i\mathfrak{k}_2$, hence κ_1 to κ_2, L_1 to L_2, L_2 to a reflection line L_3 and the intersection point $O_3 = O \in \mathbb{B}$ of L_1, L_2 to an intersection point $O_1 \in \mathbb{B}$ of L_2 and L_3.

□

Remark 7.12 *Restricting to reflection discs $\mathbb{D}_j = L_j \cap \mathbb{B}$ we realized the situation described in Figure 8.*

Proof of Theorem 4.2. It remains to check the properties (ii),...,(vii) postulated in 4.1 For $\Gamma' = \Gamma(\pi)$.

(ii) Let $\kappa_1, \kappa_2, \kappa_3 \in \partial_K \mathbb{D}_0$ be an S_3-orbit for $S_3 \subset \Gamma_0 = \Gamma(E_0, E_0)$, see 7.11, and $\kappa \in \partial_K \mathbb{D}_0$ arbitrary. We have to show that $\kappa \in \Gamma_0' \kappa_j$. Assume, for instance, that $\kappa \equiv \kappa_1 \, mod\, \pi$. Then $\kappa = L_0 \cap L_1'$, hence $\kappa = \gamma\kappa_1$ for a suitable $\gamma \in \Gamma$ with $\kappa_1 = L_0 \cap L_1$. Since pairs of reflection lines through one point are unique, γ acts on L_0 and transferes L_1 to L_1'. Therefore γ or $\gamma \circ (2,3)$ sends L_j to L_j', $j = 2,3$. This property can be assumed now for our γ. Then $\gamma \equiv E \, mod\, \pi$ which means that γ belongs to Γ'.

(iii) Take $\mathbb{D}_1 = \mathbb{B} \cap L_1$ described in 7.11. Then $\kappa_1 = L_1 \cap L_0 \in \partial_K \mathbb{D}_1$ is fixed by $(2,3) \in S_3 \subset \Gamma(E_0, E_0)$. For arbitrary $\kappa \in \partial_K \mathbb{D}_0$ take $\gamma \in \Gamma$ such that $\kappa = \gamma\kappa_1$. By the same argument as above, γ acts on L_1 and sends L_0, L_2 to L_0' or L_2', respectively (if not take $\gamma \circ (2,3)$). It follows again that γ belongs to Γ_1'.

(iv) see Proposition 6.10.

(v) see Propositin 7.8.

(vi) The proof of the following theorem is completely published in [H98].

Theorem 7.13 *([H98], V, Theorem 5A.4.7). Let K be an imaginary quadratic number field with ring of integers $\mathfrak{O}_K$, discriminant $D = D_{K/\mathbb{Q}} \neq -3$, Dirichlet character $\chi(n) = (\frac{D}{n})$ (generalized quadratic residue, Jacobi symbol) and corresponding Dirichlet series $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$. Then for $\Gamma = \mathbb{SU}((2,1),\mathfrak{O}_K)$ with fundamental domain $\mathfrak{F}_\Gamma$ on $\mathbb{B}$ it holds that*

$$vol_{EB}(\Gamma) = \frac{3|D|^{5/2}}{32\pi^3} L(3,\chi).$$

It is now easy to calculate for $K = \mathbb{Q}(i)$ the $\mathfrak{F}_\Gamma$-volume $\frac{1}{32}$ in the case of Gauß numbers. This was first proved by Shvartsman [Sv1]. Since $\Gamma/\Gamma(\pi) \cong S_3$ it follows that $vol_{EB}(\Gamma(\pi)) = \frac{3}{16}$.

(vii) see Proposition 7.11.

Theorem 4.2 is proved.

$\square$

8 The binary octehedron group and moduli of curves of Shimura equation type

We look for the structure of factor groups of the group tower

$$\Gamma(\pi^4) \subset \Gamma(\pi^3) \subset \Gamma(\pi^2) \subset \Gamma(\pi) \subset \Gamma = \mathbb{SU}((2,1),\mathfrak{O}), \mathfrak{O} = \mathbb{Z} + \mathbb{Z}i.$$

Until now we know only the structure S_3 of the top factor group.

Theorem 8.1 *The congruence subgroup $\Gamma(\pi)$ is generated by reflections of order 4.*

We need

Theorem 8.2 *(Armstrong [Arm]). Let G be a group acting homeomorphically and proper discontineously on a local compact, linearly and simply-connected metric space X and S the normal subgroup of G generated by the elements of G fixing at least one point of X. Then the fundamental group $\pi_1(X/G)$ of the quotient space X/G is isomorphic to the factor group G/S.*

□

Proof of Theorem 8.1. We apply Armstrong's theorem to $X = \mathbb{B}$, $G = \Gamma(\pi)$. By 5.1 the Baily-Borel compactification $\mathbb{B}/\Gamma(\pi)$ is the projective plane. Therefore the fundamental group $\pi_1(\mathbb{B}/\Gamma(\pi)$ is trivial. According to 8.1 the group $\Gamma(\pi)$ is generated by all its elements of finite order. Since each element of finite order is a product of (at most two) reflections by Theorem 7.5 (iv), the group $\Gamma(\pi)$ is generated by reflections. The only orders of $\Gamma(\pi)$-reflections are 2 and 4, see Corollary 7.2. But each reflection of order 2 is a square of a reflection of order 4 by 7.5 (iii).

□

Let $Z_2 \cong (\mathbb{Z}/2\mathbb{Z}, +) = (\mathbb{F}_2, +)$ be the cyclic group of order 2. First we show that

(35) $$\Gamma(\pi^m)/\Gamma(\pi^{m+2}) \cong Z_2^s \; with \; 2^s = [\Gamma(\pi^m) : \Gamma(\pi^{m+2})] \; for \, m \geq 2.$$

Namely, each element of this factor group has order 2 because

$$(E + \pi^m A)^2 = E + 2\pi^m A + \pi^{2m} A^2 \equiv E \, mod \, \pi^{m+2},$$

$A \in Mat_2(\mathfrak{O})$, E the unit matrix. Two elements s, t of a group G whose non-trivial elements have all order 2 commute because $stst = (st)^2 = 1$, $s^{-1} = s$, $t^{-1} = t$, hence $ts = st$. So each group of this type is abelian. If it is finite, it must be isomorphic to a power of the additive group of $\mathbb{F}_2$. Now (35) follows immediately via comparison of order. By a similar argument we get

(36) $$\Gamma(\pi^m)/\Gamma(\pi^{m+1}) \cong Z_2^r \; with \; 2^r = [\Gamma(\pi^m) : \Gamma(\pi^{m+1})] \; for \, m \geq 1.$$

Now let $2\mathbb{O}$ be the *binary octaeder group* defined via central Z_2-extension

$$0 \longrightarrow Z_2 \longrightarrow 2\mathbb{O} \longrightarrow S_4 \cong \mathbb{O} \longrightarrow 1.$$

Geometrically, S_4 can be represented as motion group $\mathbb{O} \subset \mathbb{O}(3,\mathbb{R})$ of the (regular) octahedron or cube. There is a canonical group homomorphism $\mathbb{SU}(2,\mathbb{C}) \longrightarrow \mathbb{O}(3,\mathbb{R})$ with central kernel $\{\pm 1\} \cong Z_2$, see e.g. [Hal]. It represents $2\mathbb{O}$ as preimage of $\mathbb{O}$.

Proposition 8.3

(i) $[\Gamma(\pi) : \Gamma(\pi^2)] = 2^3$, *hence* $\Gamma(\pi)/\Gamma(\pi^2) \cong Z_2 \times Z_2 \times Z_2$;

(ii) $\Gamma/\Gamma(\pi^2)$ *is isomorphic to the binary octaeder group* $2\mathbb{O}$.

Proof. Let $\mathfrak{O}_\pi$ be the π-adic completion of $\mathfrak{O}$ and $\Gamma_\pi = \mathbb{SU}((2,1),\mathfrak{O}_\pi) \supset \Gamma$. It is well-defined because the complex conjugation $a \mapsto \bar{a}$ on $\mathfrak{O}$ extends to $\mathfrak{O}_\pi$. We develop each element of $A \in Mat_3(\mathfrak{O}_\pi)$ componentwise in a π-adic series

$$A = A_0 + \pi A_1 + \pi^2 A_2 + \pi^3 A_3 + ..., A_i \in Mat_3(\mathbb{F}_2) \tag{37}$$

identifying $\mathbb{F}_2$ with the representatives $0, 1$ of $\mathbb{F}_2 = \mathfrak{O}/\pi\mathfrak{O}$. The congruence subgroups $\Gamma(\pi^k)$ are canonically embedded into the π-adic congruence subgroups $\Gamma_\pi(\pi^k)$, their π-adic completions, for all $k \in \mathbb{N}$. Since $\Gamma(\pi^k) = \Gamma \cap \Gamma_\pi(\pi^k)$ we have also canonical embeddings

$$\Gamma/\Gamma(\pi^k) \subseteq \Gamma_\pi/\Gamma_\pi(\pi^k) \subseteq \mathbb{SU}((2,1),\mathfrak{O}/\mathfrak{O}_\pi) \subset \mathbb{S}l_3(\mathfrak{O}/\mathfrak{O}_\pi). \tag{38}$$

For $A \in \Gamma$ the correspondences $A \mapsto A_0$ defines a group homomorphism

$$\Gamma/\Gamma(\pi) \xrightarrow{\sim} S_3 \cong \mathbb{O}(3,\mathbb{F}_2), \tag{39}$$

and $A \mapsto A_1$ an injective group homomorphism

$$c_1 : \Gamma(\pi)/\Gamma(\pi)^2 \longrightarrow Mat_3(\mathbb{F}_2)_0 = \{M \in Mat_3(\mathbb{F}_2);\ TrM = 0\} \cong \mathbb{F}_2^8. \tag{40}$$

The image is a linear code $C = C_1$ in the non-degenerate space $\mathbb{F}_2^8$ endowed with the $\mathbb{F}_2$-bilinear form defined by the traces of products of two elements, which is non-degenerate on $Mat_3(\mathbb{F}_2)_0$. C is contained in the 5-dimensional subspace $Symm_3(\mathbb{F}_2)_0$ of symmetric matrices with zero trace. This follows from the relations ${}^t(E+\pi A_1+...)\Phi(E+\bar{\pi}A_1+...) = \Phi$ with $\Phi = diag(1,1,-1)$ implying ${}^tA_1 = A_1$. The diagonal matrices with zero trace belong to C because $\tau_1 = diag(-1,i,i)$, $\tau_2 = diag(i,-1,i)$, $\tau_3 = diag(i,i,-1)$ belong to $\Gamma(\pi)$. Therefore C is at least 2-dimensional. Moreover, $\left(\begin{smallmatrix} 0&1&1\\ 1&0&1\\ 1&1&0 \end{smallmatrix}\right) \in C$. This is $A_1 = A_1(\tau_0)$ of a reflection τ_0 of $\Gamma(\pi)$ with unimodular reflection lattice $E_0 = \Lambda_{\mathfrak{c}}$ of even type, for instance take $\mathfrak{c} = (1,1,1)$, $\tau_0 = \left(\begin{smallmatrix} 1&1-i&-1+i\\ 1-i&1&-1+i\\ 1-i&1-i&-1+2i \end{smallmatrix}\right)$. Denote by

$$T = \mathbb{F}_2 t_0 + \mathbb{F}_2 t_1 + \mathbb{F}_2 t_2 + \mathbb{F}_2 t_3 \tag{41}$$

the 3-dimensional subspace generated by the images of the above reflections τ_j, $j = 0,1,2,3$.

The $\mathbb{F}_2$-dimension of C cannot be greater than 3. Namely, let $c \in C$ be arbitrary. Then it is the image of a product of reflections of order 4 by Theorem 8.1. It suffices to show that the image of each 4-reflection ρ belongs to T. By the classification of indefinite unimodular $\mathfrak{O}^3$- sublattices ρ is $\Gamma(\pi)$-conjugated to one of the τ_j. Now it is easy to see that ρ and τ_j have the same image in C, hence $C = T$. Together with (36) we proved (i).

The inclusions $\Gamma(\pi^2) \subset \Gamma(\pi) \subset \Gamma$ yield the exact sequence

$$0 \longrightarrow C \cong Z_2 \times Z_2 \times Z_2 \longrightarrow \Gamma/\Gamma(\pi^2) \longrightarrow \Gamma/\Gamma(\pi) = S_3 \longrightarrow 1 \tag{42}$$

We check that the action of S_3 on $Z_2 \times Z_2 \times Z_2$ via pull-back and conjugation is the same as that of the exact sequence

$$0 \longrightarrow Z_2 \times K_4 \longrightarrow 2\mathbb{O} \longrightarrow S_3 \longrightarrow 1$$

lifted from

$$0 \longrightarrow K_4 \longrightarrow S_4 \longrightarrow S_3 \longrightarrow 1,$$

where $K_4 \cong Z_2 \times Z_2$ is the normal subgroup of S_4 consisting of the elements of order 1 or 2 with positive signature. These are the products of two tranpositions. S_3 acts effectively on K_4 and ineffectively on Z_2. Now look back to the code $C \cong Z_2 \times K_4$, where Z_2 is identified with $\mathbb{F}_2 t_0$ and K_4 with

$$\mathbb{F}_2 t_1 + \mathbb{F}_2 t_2 + \mathbb{F}_2 t_3 = \mathbb{F}_2 t_1 + \mathbb{F}_2 t_2.$$

Each group $\Sigma_3 \subset \Gamma$ isomorphic to S_3 permutes the three $\Gamma(\pi)$-orbits of odd unimodular $\mathfrak{O}^3$-sublattices of rank 2 represented by the images t_1, t_2, t_3 of their 4-reflections, see Proposition 6.10. Since there is only one even $\Gamma(\pi)$-orbit of even unimodular $\mathfrak{O}^3$-sublattices of rank 2, the group S_3 acts trivially on the first factor $\mathbb{F}_2 t_0$. So both S_3-actions on $Z_2 \times K_4$ coincide, which proves the second part of the proposition. $\square$

Let

$$\gamma_1 : \Gamma(\pi) \twoheadrightarrow C_1 = Z_2 \times K_4$$

be the lift of the code embedding c_1 defined in (40) with the S_3- invariant subgroups $Z_2 = \mathbb{F}_2 t_0$ and $K_4 = \mathbb{F}_2 t_1 + \mathbb{F}_2 t_2 + \mathbb{F}_2 t_3$, see (41). The preimage $\Gamma_2 := \gamma_1(K_4)$ is a normal subgroup of $\Gamma(\pi)$ of index 2. Since K_4 is $S_3 = \Gamma/\Gamma(\pi)$-invariant, Γ_2 is also a normal subgroup of Γ of index 12. Any embedding $S_3 \hookrightarrow \Gamma$ defines an embedding $S_3 \hookrightarrow \Gamma/\Gamma_2$. Therefore the exact sequence

$$0 \longrightarrow Z_2 = \Gamma(\pi)/\Gamma_2 \longrightarrow \Gamma/\Gamma_2 \longrightarrow S_3 = \Gamma/\Gamma(\pi) \longrightarrow 1$$

splits, hence $\Gamma/\Gamma_2 \cong Z_2 \times S_3 =: 2S_3$. The splitting can be realized by finite subgroups of Γ in the following manner: Let $\mathbb{D}_0 \subset \mathbb{B}$ be an Σ_3-invariant $\Gamma(\pi)$- reflection disc coming from an even unimodular rank-2 sublattice of $\mathfrak{O}^3$, $\Gamma \supset \Sigma_3 \cong S_3$, see Proposition 7.11. The cyclic reflection group $\langle \tau_0 \rangle$ of $\mathbb{D}_0$ of order 4 is normalized by Σ_3. It defines the finite subgroup $4\Sigma_3 := \langle \tau_0 \rangle \times \Sigma_3$ of Γ of order 24. Since $\tau^2 \in \Gamma(\pi^2)$ we get $2S_3$ as image in the binary octahedron group $\Gamma/\Gamma(\pi^2)$.

We want to classify the ball quotient surface $\hat{Y} = \widehat{\mathbb{B}/\Gamma_2}$. It is the double cover of $\hat{X} = \mathbb{P}^2$ branched precisely along the quadric $\hat{C}_0$. The Galois group is realized by $2S_3/S_3 = Z_2 = \langle \tau_0 \rangle \; mod \; \Gamma(\pi^2)$. The degree formulas for orbital hights applied to the finite orbital double covering $\mathbf{f} : \hat{Y} \longrightarrow (\mathbb{P}^2, 2\hat{C}_0)$, see [H98] , compare with (13), (14), yield

$$\begin{aligned}
e(\hat{Y}) &= H_e(\hat{Y}) = 2H_e(\mathbb{P}^2, 2\hat{C}_0) \\
&= 2[e(\mathbb{P}^2) - (1 - \frac{1}{2})e(\hat{C}_0)] = 2[3 - \frac{1}{2} \cdot 2] = 4, \\
\tau(\hat{Y}) &= H_\tau(\hat{Y}) = 2H_\tau(\mathbb{P}^2, 2\hat{C}_0) \\
&= 2[\tau(\mathbb{P}^2) - \frac{1}{3}(2 - \frac{1}{2}) \cdot \frac{1}{2}(\hat{C}_0^2)] = 2[1 - \frac{1}{2} \cdot 2] = 0.
\end{aligned} \tag{43}$$

Now we calculate Euler numbers and selfintersections of irreducible preimage curves $\hat{D}_i$ of $\hat{C}_i$, $i = 0, 1, 2, 3$, respectively, by the degree formulas for local orbital hights, see [H98], compare with (9),(10). Since $\hat{C}_0$ is the branch locus we get immediately $\hat{D}_0 \cong \hat{C}_0 \cong \mathbb{P}^1$. We have to change to the double covering $Y' \longrightarrow X'$ for getting a locally abelian situation. With the ramification indices $v_0 = 2$, $v = v_j = 1$, $j = 1, 2, 3$, of f along D'_j covering C'_j, we get

$$\begin{aligned}
e(D'_j) &= h_e(D'_j) = [D'_j : C'_j] \cdot h_e(\mathbf{C}'_j) \\
&= [e(C'_j) - 2(1 - \frac{1}{v})] \cdot [D'_j : C'_j] = 2 \cdot [D'_j : C'_j],
\end{aligned}$$

hence

$$\begin{aligned}
&e(D'_i) = 2, D'_i \cong \mathbb{P}^1, \; [D'_i : C'_i] = 1, \; i = 0, 1, 2, 3; \\
&(D\prime_j^2) = h_\tau(D'_j) = [D'_j : C'_j] \cdot h_\tau(\mathbf{C}'_j) = (C\prime_j^2) = -1, \; j = 1, 2, 3; \\
&(D\prime_0^2) = h_\tau(D\prime_0) = [D'_0 : C'_0] \cdot h_\tau(\mathbf{C}'_0) = \frac{1}{2} \cdot (C\prime_0^2) = -1.
\end{aligned}$$

Since $f' : Y' \longrightarrow X'$ is not ramified and not inert at D'_j, each of the curves C'_j has precisely two irreducible preimage curves D_j^+ and D_j^-. Let $F_j \subset Y'$ denote the preimage of $E_j \subset X'$. Locally Z_2 acts around each fixed point on Y' with smooth image on X' as a reflection group. Starting from a preimage of the intersection point of E_j and C'_0 we see that Z_2 acts effectively on F_j because it acts trivially on D'_0, see Figure 2. This means that $[F_j : E_j] = 2$. We calculate

$$\begin{aligned}
e(F_j) &= h_e(F_j) = [F_j : E_j] \cdot h_e(\mathbf{E}_j) = 2 \cdot [e(E_j) - 2(1 - \frac{1}{2})] = 2; \\
(F_j^2) &= h_\tau(F_j) = [D'_j : C'_j] \cdot h_\tau(\mathbf{E}_j) = 2 \cdot [(E_j^2) + \frac{1}{2}] = -1.
\end{aligned}$$

Forgetting for a moment D_2^+ and D_2^- we get the following configuration on Y':

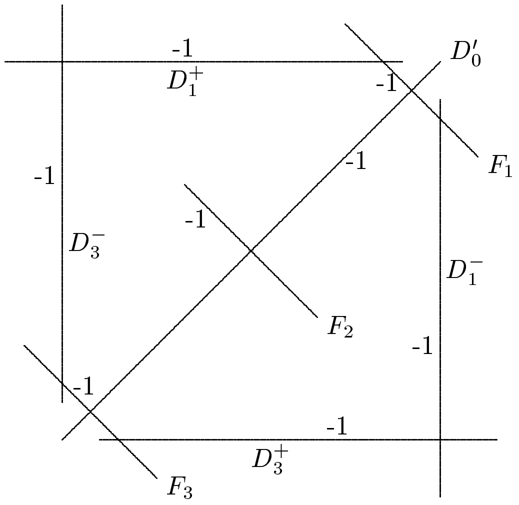

Figure13.

From (43) follows that

$$\chi(\hat{Y}) = \frac{1}{4}(e+\tau) = 1,\ c_1^2(\hat{Y}) = 12\chi - e = 8. \tag{44}$$

Blowing down the curves F_1, F_2, F_3 we get two crossing smooth rational curves with selfintersection 0 on $\hat{Y}$, for instance D_1^+ and D_3^-. There is up to isomorphy only one smooth compact surface with Chern numbers $\chi = 1$ and $c_1^2 = 8$ and such crossing curve pair, namely $\mathbb{P}^1 \times \mathbb{P}^1$. For this fact we refer to [H98], end of V.2 (blow up the intersection point of the curves and blow down the two curves to get a smooth rational surface with $c_1^2 = 9$, which must be $\mathbb{P}^2$, see [H98], V.2, Proposition 5.2.4). Taking in consideration now also D_2^+ and D_2^- we get the following branch configuration on $\hat{Y} = \widehat{\mathbb{B}/\Gamma_2} = \mathbb{P}^1 \times \mathbb{P}^1$

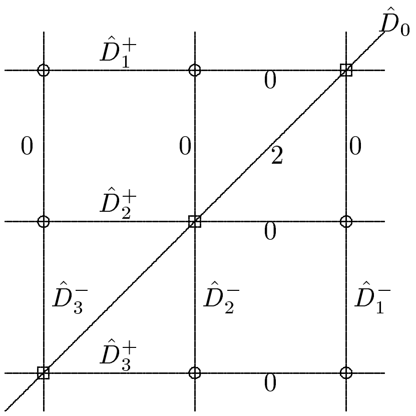

Figure14.

The six circles mark the (abelian) quotient points (images of all Γ_2-elliptic points on $\mathbb{B}$); the three boxes mark the compactifying cusp points. $\hat{D}_0$ crosses each of the three marked horizontal and three vertical fibres in precisely one point. Therefore $\hat{D}_0$ is a section for both canonical projections of $\mathbb{P}^1 \times \mathbb{P}^1$. Blowing up the central cusp point in Figure 14 and blowing down the two $\hat{D}_2^+$ and $\hat{D}_2^-$ after, then $\hat{D}_0$ becomes a smooth rational curve on $\mathbb{P}^2$ with selfintersection 1, hence a projective line. It is uniquely determined as line through the remaining two cusp points. Therefore $\hat{D}_0$ coincides with the diagonal line on $\mathbb{P}^1 \times \mathbb{P}^1$.

Altogether we get the following

Theorem 8.4 *The compactified ball quotient surface $\widehat{\mathbb{B}/\Gamma_2}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The compactified branch locus of the quotient map $p : \mathbb{B} \longrightarrow \mathbb{B}/\Gamma_2$ consists of three horizontal fibres $\hat{D}_j^+$, three vertical fibres $\hat{D}_j^-$ and the diagonal $\hat{D}_0$. The configuration is $Z_2 \times S_3$-invariant, where the generator of Z_2 changes the $\mathbb{P}^1$-components of each point $(P,Q) \in \mathbb{P}^1 \times \mathbb{P}^1$ and S_3 acts by simultaneous permutation of* *natural homogeneous $\mathbb{P}^2$-coordinates $(x:y:z)$ with sum zero ($x+y+z=0$), on both components. The cusp points are the three intersection points of the diagonal curve $\hat{D}_0$ with the other curves $\hat{D}_j^{\pm}$. The ramification indices of p at $\hat{D}_0$ or $\hat{D}_j^{\pm}$ are 2 or 4, respectively, for $j=1,2,3$.*

The $2S_3$-invariance comes from the factor group Γ/Γ_2. Cusp points and the branch indices are simply lifted from those of

$$\mathbb{B} \longrightarrow \mathbb{B}/\Gamma(\pi) = Y/Z_2 = \mathbb{P}^2 \setminus \{K_1, K_2, K_3\},$$

with obvious notation. Only at D_0 we loose the factor 2, while the other branch indices remain to be 4.

□

We want to interprete the ball quotient surface $\widehat{\mathbb{B}/\Gamma} = \mathbb{P}^2/S_3 = \mathbb{P}^1 \times \mathbb{P}^1/2S_3$ as compactified moduli space of a special curve family. Following Shimura [Sm64] we consider plane curves of affine equation type $Y^4 = p_2(X)p_3(X)^2$, where $p_n(X) \in \mathbb{C}[X]$ denotes a normalized polynomial of degree n. The normalization $\tilde{C}$ of its projective closure $C \subset \mathbb{P}^2$ has genus 3 in general. This happens, if the the five zeros of the X-polynomial of

$$\begin{aligned} C_{\alpha,\beta} \,:\, Y^4 &= (X-\alpha_1)(X-\alpha_2)(X-\beta_1)^2(X-\beta_2)^2(X-\beta_3)^2, \\ \alpha &= (\alpha_1,\alpha_2),\ \beta = (\beta_1,\beta_2,\beta_3), \end{aligned}$$

on the right-hand side are pairwise different. In the appendix the corresponding parameter space is denoted by Λ. We defined a curve families $\mathfrak{C}_\Lambda = \mathfrak{C}/\Lambda$ and $\tilde{\mathfrak{C}}_\Lambda = \tilde{\mathfrak{C}}/\Lambda$ on this way. The group $2S_3 = Z_2 \times S_3 = \langle\tau\rangle \times S_3$ acts in obvious manner on these families, where τ transposes α_1 and α_2 and S_3 permutes the indices of β_1,β_2,β_3. The curves over $2S_3$-equivalent points are the same.

Matsumoto [Mat] and van Geemen [vGm] work with the following family $\mathfrak{C}_M/M$

$$C_\gamma \,:\, Y^4 = X^2(X-1)^2(X-\gamma_1)(X-\gamma_2),\ \gamma = (\gamma_1,\gamma_2) \in M,$$

over $M = (\mathbb{C}^{**} \times \mathbb{C}^{**}) \setminus \Delta$, $\mathbb{C}^{**} = \mathbb{C} \setminus \{0,1\}$, Δ the diagonal, respectively with its fibrewise normalization $\tilde{\mathfrak{C}}_M/M$. Here the $2S_3$-action on M respecting isometry of fibre curves is more hidden, but the connection with the ball quotient surfaces is immediate. Namely, M can be identified with the complement of the seven lines on $\mathbb{P}^1 \times \mathbb{P}^1$ described in Figure 14. The connection with both families of hyperelliptic curves of genus 3 with $\mathbb{Q}(i)$-multiplication is given in the appendix by A. Piñeiro: The first family is the pull back of the second along the surjective morphism

$$\begin{aligned} &\gamma \,:\, \Lambda \longrightarrow \mathbb{C}^{**} \times \mathbb{C}^{**}, \\ &(\alpha,\beta) \mapsto \gamma(\alpha,\beta) = \left(\frac{\beta_3-\alpha_1}{\beta_1-\alpha_1} : \frac{\beta_3-\beta_2}{\beta_1-\beta_2}, \frac{\beta_3-\alpha_2}{\beta_1-\alpha_2} : \frac{\beta_3-\beta_2}{\beta_1-\beta_2}\right). \end{aligned}$$

The α-transposition τ goes down to the transposition of γ_1 and γ_2, and the S_3-generatiing β-transpositions of $(1,3)$, $(2,3)$ on Λ go down to $(\gamma_1,\gamma_2) \mapsto (\frac{1}{\gamma_1},\frac{1}{\gamma_2})$ or $(\gamma_1,\gamma_2) \mapsto (1-\gamma_1, 1-\gamma_2)$, respectively. These three transpositions generate the subgroup $Aut_{hol} M \supset T \cong 2S_3$ of the appendix. We have a commutative *moduli diagram* of algebraic morphisms

$$\begin{array}{ccccccc} \Lambda & \longrightarrow & M & \hookrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 & = & \widehat{\mathbb{B}/\Gamma_2} \\ \downarrow & & \downarrow & & \downarrow & & \\ \Lambda/\langle\tau\rangle & \longrightarrow & M/\langle\tau\rangle & \hookrightarrow & \mathbb{P}^2 & = & \widehat{\mathbb{B}/\Gamma(\pi)} \\ \downarrow & & \downarrow & & \downarrow & & \\ \Lambda/2S_3 & \longrightarrow & M/2S_3 & \hookrightarrow & \mathbb{P}^2/S_3 & = & \widehat{\mathbb{B}/\Gamma} \end{array} \tag{45}$$

$M/2S_3$ is the moduli space of the curve family $\tilde{\mathfrak{C}}_M$ by Proposition 10.2 of the appendix. But $\mathbb{P}^2/S_3 = \widehat{\mathbb{B}/\Gamma}$ is also the moduli space of abelian 3-folds with $\mathbb{Q}(i)$-multiplication of type $(2,1)$, see [Sm63]. The Jacobians of the above curves $\tilde{C}_{\alpha,\beta}$ or $\tilde{C}_\gamma$ are obviously abelian threefolds of this type, see [Sm64]. It follows that

Theorem 8.5 *The compactified moduli spaces of of the curve families $\tilde{\mathfrak{C}}_\Lambda$, $\tilde{\mathfrak{C}}_M$ and of (principally polarized) abelian 3-folds with $\mathbb{Q}(i)$-multiplication of type $(2,1)$ coincide with $\mathbb{P}^2/S_3 \cong \widehat{\mathbb{B}/\Gamma}$.*

□

By Tschirnhaus transformation we can restrict our family $\mathfrak{C}_\Lambda$ to the family $\mathfrak{C}_0$ over

$$\Lambda_0 := \{(\alpha, \beta) \in \Lambda\,;\ \alpha_1 + \alpha_2 = 0\}$$

without loosing isomorphy classes of the fibre curves. Setting $a = \alpha_1 = -\alpha_2$ we have over Λ_0 curves of equation type

$$C_{a,\beta}\ :\ Y^4 = (X-a)(X+a)(X-\beta_1)^2(X-\beta_2)^2(X-\beta_3)^2$$

It is also obvious that two curves over two points on each line $\mathbb{C}(\alpha, \beta)$ are isomorphic. Therefore the isomorphy classes of our curves are completely represented by the curves

$$C_b:\ Y^4 = (X-1)(X+1)(X-b_1)^2(X-b_2)^2(X-b_3)^2,\ b = (b_1, b_2, b_3) \tag{46}$$

defining a family $\mathfrak{C}_1$ over $\Lambda_1 \subset \Lambda_0$ defined by the equation $a = 1$. With $b_i = \frac{\beta_i}{a}$ the restrictions γ_0, γ_1 of γ to Λ_0 or Λ_1 are the correspondences

$$(a, \beta)\ or\ b \mapsto (\frac{b_3 - 1}{b_1 - 1} : \frac{b_3 - b_2}{b_1 - b_2}, \frac{b_3 + 1}{b_1 + 1} : \frac{b_3 - b_2}{b_1 - b_2}),$$

respectively. Knowing the image (γ_1, γ_2) we can reconstruct $\beta_1, \beta_2, \beta_3$ up to a common factor. Namely,

$$\frac{\gamma_1}{\gamma_2} = \frac{b_3 - 1}{b_1 - 1} : \frac{b_3 + 1}{b_1 + 1};$$

fixing one of the numbers b_1, b_3 we get the other one, and finally b_2 from γ_1 or γ_2. So there is a well-defined rational map

$$\Lambda_1 \ni (\beta_1, \beta_2, \beta_3) \mapsto (\beta_1 : \beta_2 : \beta_3) \longrightarrow (\gamma_1, \gamma_2) \in M$$

We use the order of zeros of $p(X) = (X-a)(X+a)(X-\beta_1)^2(X-\beta_2)^2(X-\beta_3)^2$ to distinguish twice the corresponding curves $C_{a,\beta}$: $Y^4 = p(X)$ of Shimura equation type: firstly by the order of a, $-a$, secondly by the order of $\beta_1, \beta_2, \beta_3$. Observe that the order of γ_1, γ_2 determines the order of the linear factors $X-1$, $X+1$. Forgetting the order of γ_1, γ_2 means to forget the order of the two linear factors. Then we say that our curves are (only simply) distinguished.

Theorem 8.6 *The sufaces $\mathbb{P}^1 \times \mathbb{P}^1 = \widehat{\mathbb{B}/\Gamma_2}$ and $\mathbb{P}^2 = \widehat{\mathbb{B}/\Gamma(\pi)}$ are the (compactified) moduli spaces of double distinguished respectively distinguished curves of Shimura equation type. More precisely: The correspondence*

$$C_{a,\beta} \mapsto \mathbb{P}\beta = (\beta_1 : \beta_2 : \beta_3)$$

defines a map to the moduli space $\mathbb{P}^1 \times \mathbb{P}^1 \supset M$ of distinguished curves, which restricts to the set of curves $C_b^+ = C_{1,b}$ and $C_b^- = C_{-1,b}$. Via $\langle\tau\rangle \cong Z_2$- equivalence interchanging the curves C_b^+ and C_b^- we get a map to the moduli space $\mathbb{P}^2 = \mathbb{P}^1 \times \mathbb{P}^1/\langle\tau\rangle \supset M/\langle\tau\rangle$ of distinguished curves, which restricts in isomorphy-compatible manner to the curves $C_b^+ = C_b$ defined in (46).

□

9 Class fields corresponding to simple abelian CM threefolds of $\mathbb{Q}(i)$- type $(2,1)$

Consider the uniformizing analytic morphism

$$p = p_{\Gamma(\pi)}\ \mathbb{B} \longrightarrow \mathbb{P}^2,\ \tau \mapsto t = p(\tau).$$

again. Fix projective coordinates on $\mathbb{P}^2$ such that $S_3 = \Gamma/\Gamma(\pi)$ acts by permutations of them. This means that the branch locus coincides with the normalized symmetric Apollonius configuration

$$T_1 = 0,\ T_2 = 0,\ T_3 = 0,\ T_1^2 + T_2^2 + T_3^2 - 2T_1T_2 - 2T_1T_3 - 2T_2T_3 = 0,$$

see 2.4, 2.3.

Let V be the hermitian vector space $(\mathbb{C}^3, <\ ,\ >)$ with the hermitian diagonal form $<\ ,\ >$ of signature $(2,1)$. Take $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in V$ such that $\mathfrak{a}^2 :=< \mathfrak{a}, \mathfrak{a} >< 0$ and $\mathfrak{b}, \mathfrak{c}$ is a base of the orthogonal complement $\mathfrak{a}^\perp$ of $\mathfrak{a}$ in V. The corresponding *Picard matrix* is defined as

$$\Pi = \Pi(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = \begin{pmatrix} {}^t\mathfrak{a} \\ {}^t\bar{\mathfrak{b}} \\ {}^t\bar{\mathfrak{c}} \end{pmatrix}.$$

Using the (real) involution $\hat{}$: $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix}$ on $\mathbb{C}^n$ we define a $\mathbb{Z}$-lattice

$$\Lambda_\Pi = \Lambda(\mathfrak{a}, \bar{\mathfrak{b}}, \bar{\mathfrak{c}}) = \mathfrak{O} \circ \mathfrak{a} + \mathfrak{O} \circ \mathfrak{b} + \mathfrak{O} \circ \mathfrak{c} \subset V$$

with $\mathfrak{O} = \mathbb{Z} + \mathbb{Z}i$ and the (real) operation $\lambda \circ \mathfrak{x} := \widehat{\lambda \mathfrak{x}}$ on V. Checking Riemann period relations it turns out that $\Lambda(\mathfrak{a}, \bar{\mathfrak{b}}, \bar{\mathfrak{c}})$ is a period lattice of an abelian 3-fold. This means that

$$A_\Pi = A(\mathfrak{a}, \bar{\mathfrak{b}}, \bar{\mathfrak{c}}) := V/\Lambda(\mathfrak{a}, \bar{\mathfrak{b}}, \bar{\mathfrak{c}})$$

is an abelian variety with (principal) polarization given by the imaginary part of the hermitian form $<\ ,\ >$ restricted to the lattice. The $\circ$-operation of $\mathbb{C}$ on V restricted to $K = \mathbb{Q}(i)$ goes down to the $(2,1)$-multiplication of K on A_Π. Two of our principally polarized abelian 3-folds A_Π and $A(\mathfrak{a}', \bar{\mathfrak{b}}', \bar{\mathfrak{c}}')$ are isomorphic if the ball points $\tau = \mathbb{P}\mathfrak{a}$ and $\tau' = \mathbb{P}\mathfrak{a}'$ (both on $\mathbb{B} \subset \mathbb{P}^2 = \mathbb{P}V$) are $\Gamma = \mathbb{SU}((2,1), \mathfrak{O})$-equivalent. Especially, the isomorphy class A_τ is well-defined. Moreover, for almost all $\tau \in \mathbb{B}$, that means for all points τ outside of a thin (but dense) analytic subset R of $\mathbb{B}$, also the inverse conclusion holds. This is a special way to recognize $\widehat{\mathbb{B}/\Gamma}$ as moduli space (Shimura surface) of (principally polarized) abelian 3-folds with K-multiplication of type $(2,1)$. Together with Theorem 8.6 we connect curve moduli space via Jacobians with the $\pi = (1+i)$-level Shimura surface and get the following

Theorem 9.1 *Let*

$$R = \Gamma(\pi)\mathbb{D}_0 \cup \Gamma(\pi)\mathbb{D}_1 \cup \Gamma(\pi)\mathbb{D}_2 \cup \Gamma(\pi)\mathbb{D}_3$$

be the ramification locus of $p = p_{\Gamma(\pi)} : \mathbb{B} \longrightarrow \mathbb{P}^2$ consisting of the $\Gamma(\pi)$-shifts of four complete linear subdiscs $\mathbb{D}_i$ of $\mathbb{B}$, which is the preimage of the Apollonius configuration of $\mathbb{P}^2 = \widehat{\mathbb{B}/\Gamma(\pi)}$. For all $\tau \in \mathbb{B}\setminus R$ it holds that:

$$A_\tau \cong A'_\tau\ if\, and\, only\, if\ \tau' \in \Gamma(\pi)\tau.$$

□

On this way the (principally polarized) abelian 3-folds $A_t := A_\tau = Jac(\tilde{C}_t)$, $t = p(\tau)$ are well-defined up to isomorphy. We say that A_t has *decomposed complex multiplication*, if it splits up to isogeny into simple abelian varieties with complex multiplication. The corresponding points $\tau \in \mathbb{B}$ or $t = p(\tau) \in \mathbb{P}^2$ are called DCM-*points*. In more precise language we have to distuinguish between DCM period points on $\mathbb{B}$ and DCM moduli points on $\mathbb{P}^2$. A famous general theorem of Shiga-Wolfahrt [S-W], applied to our situation, states that

Theorem 9.2 *The ball point τ (or plane point t) is a DCM-point if and only if both, τ and $t = p(\tau)$, are points with algebraic coordinates.*

□

Things are well-defined using canonical coordinates of the embedded ball $\mathbb{B} \subset \mathbb{C}^2$ respectively homogeneous coordinates on $\mathbb{P}^2$ with Apollonius configuration defined over $\bar{\mathbb{Q}}$. A DCM-point τ or $t = p(\tau)$ is a CM-*point*, iff the corresponding abelian 3-fold is simple. The corresponding curves $\tilde{C}_t$ normalizing

$$C_t : Y^4 = (X^2-1)(X^3+g_1X^2+g_2X+g_3)^2 = (X^2-1)((X-t_1)(X-t_2)(X-t_3))^2,$$

or abelian 3-fold $A_t = Jac(\tilde{C}_t)$ are called CM- *curves* or abelian CM-*threefolds*, respectively. The set of CM-points is dense on $\mathbb{B}$ or $\mathbb{P}^2$, respectively. A K-*line* L on $\mathbb{P}^2$ is a projective line through two different points belonging to $\mathbb{P}^2(K)$. The intersections of K-lines with $\mathbb{B}$ are called K- *discs on* $\mathbb{B}$. In [Ho94] we proved

Theorem 9.3 *A DCM period point $\tau \in \mathbb{B}$ is a CM-point if and only if $F_\tau := K(\tau)$ is a cubic extension of K. The locus of all period points $\tau \in \mathbb{B}$, where A_τ is not simple coincides with the union of all K-discs on $\mathbb{B}$. If the completion $\bar{\mathbb{D}}$ of a K-disc $\mathbb{D}$ contains a cusp $\kappa \in \mathbb{B}(K) = \mathbb{B} \cap \mathbb{P}^2(K)$, then $p(\mathbb{D})$ is a plane modular curve. If this is not the case, then $p(\mathbb{D})$ is a (compact plane) Shimura curve corresponding to abelian surfaces B with End^0B isomorphic to a fixed indefinite quaternion field.*

□

Let τ be a CM-point. The field F_τ is isomorphic to the endomorphism algebra $End^0A_\tau = \mathbb{Q} \otimes EndA_\tau$. The diagonalized representation of F_t on the tangent space of A_τ (at O) yields three different embeddings $\varphi_1, \varphi_2, \varphi_3$ of F_τ into $\mathbb{C}$ extending the $(2,1)$-embedding of the K-multiplication. $\Phi = (\varphi_1, \varphi_2, \varphi_3)$ is known as *type* of the F_τ- multiplication. Up to isomorphy we have for each CM-point $t = p(\tau) \in \mathbb{P}^2$ the complex multiplication field $F_t \cong F_\tau$ with unique action type Φ on (the tangent space of) A_t. The *reflex field* F_t^* of F_t is the field generated over $\mathbb{Q}$ by the Φ-traces $Tr_\Phi = \varphi_1(f) + \varphi_2(f) + \varphi_3(f)$ of all elements $f \in F$.

We want to define a *Picard modular function* $\mathfrak{j} : \mathbb{B} \longrightarrow \mathbb{P}^2$. For

$$\mathfrak{o} \neq \mathfrak{t} := (t_1, t_2, t_3) \in \mathbb{C}^3, \; t = \mathbb{P}\mathfrak{t} = (t_1 : t_2 : t_3),$$

set

$$f_\mathfrak{t}(X) = (X-t_1)(X-t_2)(X-t_3) = X^3 - g_1X^2 + g_2X - g_3, \qquad \begin{aligned} g_1 &= g_1(\mathfrak{t}) = t_1+t_2+t_3, \\ g_2 &= g_2(\mathfrak{t}) = t_1t_2+t_1t_3+t_2t_3, \\ g_3 &= g_3(\mathfrak{t}) = t_1t_2t_3. \end{aligned}$$

Consider the homogeneous symmetric polynomials in t_1, t_2, t_3

(47)
$$\begin{aligned} H_1 = H_1(\mathfrak{t}) &:= g_1g_2 - 3g_3 = t_1t_2(t_1+t_2) + t_2t_3(t_2+t_3) + t_1t_3(t_1+t_3), \\ H_2 = H_2(\mathfrak{t}) &:= g_1^3 - 2g_1g_2 + 3g_3 = t_2(t_1+t_2)(t_2+t_3) + t_3(t_1+t_3)(t_2+t_3) + t_1(t_1+t_2)(t_1+t_3), \\ H_3 = H_3(\mathfrak{t}) &:= H_1 + 2g_3 = g_1g_2 - g_3 = (t_1+t_2)(t_1+t_3)(t_2+t_3) \end{aligned}$$

and the corresponding functions

(48)
$$\begin{aligned} J_1 = J_1(t) &:= H_1/g_3 = \frac{g_1g_2}{g_3} - 3 = \frac{t_1+t_2}{t_3} + \frac{t_1+t_3}{t_2} + \frac{t_2+t_3}{t_1}, \\ J_2 = J_2(t) &:= H_2/g_3 = \frac{g_1^3 - 2g_1g_2}{g_3} + 3 = \frac{(t_1+t_2)(t_2+t_3)}{t_1t_3} + \frac{(t_1+t_3)(t_2+t_3)}{t_1t_2} + \frac{(t_1+t_2)(t_1+t_3)}{t_2t_3}, \\ J_3 = J_3(t) &:= H_3/g_3 = \frac{g_1g_2}{g_3} - 1 = J_1 + 2 = \frac{(t_1+t_2)(t_1+t_3)(t_2+t_3)}{t_1t_2t_3}. \end{aligned}$$

We have the polynomial relation

$$\begin{aligned} (Z - \frac{t_1+t_2}{t_3})(Z - \frac{t_2+t_3}{t_1})(Z - \frac{t_3+t_1}{t_2}) &= Z^3 - \frac{H_1}{g_3}Z^2 + \frac{H_2}{g_3}Z - \frac{H_3}{g_3} \\ &= Z^3 - J_1Z^2 + J_2Z - J_3 = Z^3 - J_1Z^2 + J_2Z - (J_1+2) \end{aligned} \tag{49}$$

The rational functions $J_i(t)$ on $\mathbb{P}^2$ define a rational map and the lifted meromorphic map

$$(50)\qquad \begin{aligned} \mathfrak{J}(t) &= (J_1(t), J_2(t)),\\ \mathfrak{j}(\tau) &= \mathfrak{J}(p(\tau)) = \mathfrak{J}(t) \end{aligned}$$

on $\mathbb{P}^2$ or on $\mathbb{B}$, respectively. Writing $\mathfrak{J}$ in homogeneous coordinates,

$$\mathfrak{J} : \mathbb{P}^2 \cdots \to \mathbb{P}^2, t \mapsto (H_1(\mathfrak{t}) : H_2(\mathfrak{t}) : g_3(\mathfrak{t})),$$

we see that it factorizes through $\mathbb{P}^2/S_3$ and is precisely defined (as morphism) outside the three points

$$Q_1 = (0:1:-1),\ Q_2 = (1:0:-1),\ Q_3 = (0:1:-1) \quad (defined\, by\, g_1 = g_3 = 0).$$

Then the meromorphic lift $\mathfrak{j} = \mathfrak{J} \circ p$ is a well-defined analytic map outside of

$$p^{-1}(Q_1, Q_2, Q_3) \subset \Gamma(\pi)\mathbb{D}_1 \cup \Gamma(\pi)\mathbb{D}_2 \cup \Gamma(\pi)\mathbb{D}_3 .$$

Especially, $\mathfrak{j}$ is defined at each CM-point of $\mathbb{B}$ because all the discs appearing on the right-hand side of this inclusion are $\Gamma(\pi)$-reflection discs, hence K-discs. Here we use Theorem 9.3.
For a number field F we denote the Galois group $Gal(\bar{\mathbb{Q}}/F)$ of all automorphisms of the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ fixing F elementwise by G_F. Let $\tilde{C} \subset \mathbb{P}^N$ be a smooth complex projective curve defined over an algebraic number field L. The group $G_{\mathbb{Q}}$ acts via projective coordinates on the set $\tilde{C}(\bar{\mathbb{Q}})$ of algebraic points of $\tilde{C}$. The application of $\sigma \in G_{\mathbb{Q}}$ on the coefficients of the defining equations of $\tilde{C}$ defines the curve $\tilde{C}^\sigma$. In general the curves $\tilde{C}^\sigma$ is not isomorphic to $\tilde{C}$. The *moduli field* of $\tilde{C}$ is defined as

$$M(\tilde{C}) := \{\sigma \in \mathbb{G}_{\mathbb{Q}};\, \tilde{C}^\sigma \cong \tilde{C}\}.$$

Fix $\mathfrak{t} \in \bar{\mathbb{Q}}^3$, $\tilde{C} = \tilde{C}_t$ and its moduli field $M = M(\tilde{C})$.

Proposition 9.4 *If $t = \mathbb{P}\mathfrak{t} \in \mathbb{P}^2$ does not belong to the Apollonius configuration and $g_1(t) \neq 0$, then $M(\tilde{C}_t) = \mathbb{Q}(\mathfrak{J}(t))$.*

Proof. We have $t_1, t_2, t_3 \neq 0$, hence $g_3 = g_3(\mathfrak{t}) \neq 0$. To show the equivalence of the two conditions

(i) $\sigma \in \mathbb{G}_{\mathbb{Q}}$ fixes the isomorphy class of $\tilde{C}_t$;

(ii) σ fixes $\mathbb{Q}(\mathfrak{J}(t))$ elementwise.

The condition (ii) can be transformed successively to the equivalent conditions:

$$\begin{gathered} \sigma \text{ fixes } J_1(t) \text{ and } J_2(t) \\ \Longleftrightarrow \\ \sigma \text{ fixes the polynomial } Z^3 - J_1 Z^2 + J_2 Z - (J_1 + 2) \in \bar{\mathbb{Q}}[Z] \\ \Longleftrightarrow \\ \text{(by (49)} \\ \sigma \text{ fixes } \tfrac{H_1}{g_3},\ \tfrac{H_2}{g_3} \text{ and } \tfrac{H_3}{g_3} \\ \Longleftrightarrow \\ \sigma \text{ permutes } u_3 := \tfrac{t_1+t_2}{t_3},\ u_2 := \tfrac{t_2+t_3}{t_1},\ u_1 := \tfrac{t_3+t_1}{t_2}) \\ \Longleftrightarrow \\ \sigma \text{ permutes } u_3 + 1 = g_1/t_3,\ u_2 + 1 = g_1/t_2,\ u_1 = g_1/t_1 \\ \Longleftrightarrow \\ \sigma \text{ permutes } t_1, t_2, t_3 \text{ up to a common factor} \\ \Longleftrightarrow \\ \text{(by Piñeiro's Proposition 10.2 pushed down to } \mathbb{P}^2\text{, see 8.4)} \\ \sigma \text{ preserves the isomorphy class of } \tilde{C}_{\mathfrak{t}}. \end{gathered}$$

$\square$

Remark 9.5 *Moduli points $t \in \mathbb{P}^2$ on the line $L :\ T_1 + T_2 + T_3 = 0$ correspond to curves of equation type*

$$(51)\qquad C_{\mathfrak{t}} :\ Y^4 = (X^2 - 1)p(X)^2 = (X^2 - 1)(X^3 + g_2(\mathfrak{t})X + g_3(\mathfrak{t}))^2.$$

Looking at towers of function fields around the smooth projective curves $\tilde{C} = \tilde{C}_t$ for general parameters $t \in L$

$$\mathbb{C}(x) \subset \mathbb{C}(\tilde{C}) = \mathbb{C}(C) = \mathbb{C}(x,y) \subset \mathbb{C}(x)(v,w)\ ,\ v^4 = x^2 - 1, w^2 = p(x), \tag{52}$$

we see that a 2-sheeted cover of $\tilde{C}$ coincides with a 4-cyclic cover of the elliptic curve with Weierstraß equation $W^2 = p(X)$. Unfortunately, L is not a quotient of a K-disc $\mathbb{D} \subset \mathbb{B}$ because the signature hight of the orbital line $\mathbf{L}$ on $\widehat{\mathbb{B}/\Gamma(\pi)}$ is equal to 1 by (10) but it must be negative for orbital disc quotients, see 4.7. Therefore the Jacobians of the curves (51) are simple in general by Theorem 9.3. The intersection points of L with the quadric $X^2 + Y^2 + Z^2 - 2XY - 2XZ - YZ = 0$ of our normalized Apollonius configuration are $(1 : \rho : \rho^2)$ and $(\rho : 1 : \rho^2)$, $\rho = e^{2\pi i/3}$, which are isolated fixed points of elements of order 3 of S_3. Since S_3 lifts to isomorphic subgroups in Γ, these are the images of Γ-elliptic points on $\mathbb{B}$ of same quality, represented by $(1 : -\rho : \rho - 1)$, $(\rho : -1 : 1 - \rho)$ lying on the disc $\mathbb{D} = \mathbb{B} \cap \mathbb{P}(1,1,-1)^{\perp}$, whose p-image $\mathbb{D}/\Gamma \subset \mathbb{B}/\Gamma$ has nothing to do with L. Its compactification goes through cusp points because $(1 : 0 : -1)$ and $(0 : 1 : -1)$ are Γ-cusps on the boundary of $\mathbb{D}$. It is a separate interesting question to find and investigate CM-points/-curves of the special subfamily (52) but also of the Shimura curve $\mathbb{D}/\Gamma$.

Theorem 9.6 *Let $\tau \in \mathbb{B} \setminus p^{-1}(L)$ be a CM-point with respect to our curve family, $K = \mathbb{Q}(i)$, $F = K(\tau)$ the corresponding cubic extension of K and F^* the reflex field with respect to $A_\tau \cong Jac\,\tilde{C}_{p(\tau)}$. Then $F^*(j(\tau))/F^*$ is an abelian field extension (class field) of F^*.*

Proof. It is a consequence of the first Main Theorem of complex multiplication (see [H95], Ch.IV, and the basic text books of Shimura-Taniyama [S-T] and Lang [La]) that F^*- extended moduli field $F^*M(A_\tau) = F^*M(\tilde{C}_{p(\tau)})$ is abelian extension (Shimura class field) of F^*. By (50) and Proposition 9.4 the moduli field coincides with $\mathbb{Q}(\mathfrak{J}(t)) = \mathbb{Q}(\mathrm{j}(\tau))$.

□

It is clear that $F^*(t) = F^*(p(\tau))$ is a definition field of the (isomorphy class of) A_τ containing F^*. Over such definition fields the torsion points of the abelian CM-variety yield abelian extensions over the definition field by the second Main Theorem of complex multiplication (see [S-T], [La], [Sm71], Proposition 7.41). So we get the

Proposition 9.7 *For CM-points $\tau \in \mathbb{B}$, $t = p(\tau) \in \mathbb{P}^2$, and a projective model $A = A_t$ of $A_\tau = Jac(\tilde{C}_t)$ the field extension $F^*(t)(A_{tor})$ is abelian.*

□

Problem 9.8 *Find explicitly $\Gamma(\pi)$-modular forms (maybe of Nebentypus) θ_1, θ_2, θ_3 on $\mathbb{B}$ such that $(\theta_1 : \theta_2 : \theta_3) : \mathbb{B} \longrightarrow \mathbb{P}^2$ coincides with the canonical quotient map onto $\widehat{\mathbb{B}/\Gamma(\pi)}$, up to compactification. This would solve the explicit Schottky-problem to find the curve $Y^4 = (X-1)(X+1)(X-\theta_1(B))^2(X-\theta_2(B))^2(X-\theta_3(B))^2$ with given period point $B \in \mathbb{B}$.*

Hints. For Eisenstein numbers and the family of Picard curves $Y^3 = p_4(X)$ this problem has been solved in terms of theta constants by Shiga [Shg], Feustel [Feu], see also [H86], [H95], [H98] for a basic approach and further connections, especially with some Hilbert problems. Tobias Finis [Fin] found explicitly the Fourier-Jacobi series of the correspondung modular forms.

For Gauß numbers and the congruence subgroup $\Gamma(\pi^2)$ we refer to van Geemen's article [vGm] presenting theta constants without precise knowledge of the corresponding surface $\widehat{\mathbb{B}/\Gamma(\pi^2)}$. As next step this orbital surface should be classified knowing that it is the biquadratic covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched with ramification index 2 along the three horizontal and three vertical fibres drawn in Figure 14.

10 Appendix 1 (by A. Piñeiro): The moduli space of hyperellitic genus 3 curves with $\mathbb{Q}(i)$-multiplication

In this section we will study the space of moduli of the family F of curves with affine model

$$C_{\alpha_1\alpha_2\beta_1\beta_2\beta_3} : y^4 = (x-\alpha_1)(x-\alpha_2)(x-\beta_1)^2(x-\beta_2)^2(x-\beta_3)^2$$

where the vector of parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) \in \mathbb{C}^5$. For fixing some notation we put for $i = 1, 2$; $j = 1, 2, 3$; $k = 1, 2, 3$ and $j \neq k$

$$\begin{aligned} \Lambda_{\alpha_1=\alpha_2} &= \{(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) \in \mathbb{C}^5 \ / \ \alpha_1 = \alpha_2\} \\ \Lambda_{\beta_k=\beta_j} &= \{(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) \in \mathbb{C}^5 \ / \ \beta_k = \beta_j\} \\ \Lambda_{\alpha_i=\beta_j} &= \{(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) \in \mathbb{C}^5 \ / \ \alpha_i = \beta_j\} \end{aligned}$$

and

$$\Lambda = \mathbb{C}^5 \setminus (\Lambda_{\alpha_1=\alpha_2} \bigcup_{k\neq j} \Lambda_{\beta_k=\beta_j} \bigcup_{1\leq j\leq 3} \Lambda_{\alpha_1=\beta_j} \bigcup_{1\leq j\leq 3} \Lambda_{\alpha_2=\beta_j})$$

and firstly concentrate ourselves to the open part

$$F_0 = \{C_{\alpha_1\alpha_2\beta_1\beta_2\beta_3} \in F \ / \ (\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) \in \Lambda\}$$

Making birational transformations we can obtain an affine model for a curve $C_{\alpha_1\alpha_2\beta_1\beta_2\beta_3} \in F_0$ as

$$C_{\gamma_1\gamma_2} : y^4 = (x-\gamma_1)(x-\gamma_2)x^2(x-1)^2$$

where the new parameters are exactly

$$\gamma_1 = \frac{(\beta_1-\beta_2)(\beta_3-\alpha_1)}{(\beta_1-\alpha_1)(\beta_3-\beta_2)} \qquad \gamma_2 = \frac{(\beta_1-\beta_2)(\beta_3-\alpha_2)}{(\beta_1-\alpha_2)(\beta_3-\beta_2)}$$

We observe that $\tau = y^2/x(x-1)(x-\gamma_1)$ defines a rational map $\tau : C_{\gamma_1\gamma_2} \xrightarrow{2:1} \mathbb{P}^1$. Taking the normalization $(\widetilde{C_{\gamma_1\gamma_2}}, \sigma)$ of $C_{\gamma_1\gamma_2}$ we obtain a degree two morphism π with commutative diagram

$$\begin{tikzcd} C_{\gamma_1\gamma_2} \arrow[d, "\tau"] & \widetilde{C_{\gamma_1\gamma_2}} \arrow[l, "\sigma"'] \arrow[dl, "\pi"] \\ \mathbb{P}^1 & \end{tikzcd}$$

The curve $\widetilde{C_{\gamma_1\gamma_2}}$ is therefore hyperelliptic with the set

$$\sigma^{-1}\{(\gamma_1, 0); (\gamma_2, 0); (0, 0); (1, 0); (\infty, \infty)\}$$

as branch locus.
The values of π at this points

$$\begin{aligned} &\pi(\sigma^{-1}(\gamma_1, 0)) = \infty_{\mathbb{P}^1} \\ &\pi(\sigma^{-1}(\gamma_2, 0)) = 0 \\ &\pi(\sigma^{-1}(0, 0)) = \pm\sqrt{\tfrac{\gamma_2}{\gamma_1}} \\ &\pi(\sigma^{-1}(1, 0)) = \pm\sqrt{\tfrac{1-\gamma_2}{1-\gamma_1}} \\ &\pi(\sigma^{-1}(\infty)) = \pm 1 \end{aligned}$$

determine the equation:

$$\widetilde{C_{\gamma_1\gamma_2}} : w^2 = u(u^2 - \frac{\gamma_2}{\gamma_1})(u^2 - \frac{1-\gamma_2}{1-\gamma_1})(u^2-1)$$

In orther to study the moduli space of the curves $\widetilde{C_{\gamma_1\gamma_2}}$ we will intensively use the following well-known theorem, which permits us to express the moduli space of any family of hyperelliptic curves $C_{\gamma_1,..,\gamma_r}$ of fixed genus as a quotient of $\mathbb{C}^r$ througt a subset of $\mathrm{Aut}(\mathbb{P}^1)$.

Theorem 10.1 *Let be (C,π), (C',π') two hyperelliptic curves and $\varphi : C \to C'$ a morphism of them. Then φ is an isomorphism if and only if there exists an automorphism ρ of $\mathbb{P}^1$ such that the following diagram is commutative:*

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\pi"'] & C' \arrow[d, "\pi'"] \\ \mathbb{P}^1 \arrow[r, "\rho"] & \mathbb{P}^1 \end{tikzcd}$$

Proposition 10.2 *Let's consider the group T generated by the following three automorphisms of $\mathbb{C}^2$*

- $t_1(\gamma_1,\gamma_2) = (\gamma_2,\gamma_1)$
- $t_2(\gamma_1,\gamma_2) = (1-\gamma_1, 1-\gamma_2)$
- $t_3(\gamma_1,\gamma_2) = (\frac{1}{\gamma_1}, \frac{1}{\gamma_2})$

Then it holds that:

(i) T has 12 elements

(ii) $\widetilde{C_{\gamma_1\gamma_2}} \cong \widetilde{C_{\gamma_1'\gamma_2'}} \Leftrightarrow (\gamma_1',\gamma_2') = t(\gamma_1,\gamma_2)$ for some $t \in T$.

Proof. We begin with the proof of $(ii) \Rightarrow$. Let $\varphi : \widetilde{C_{\gamma_1\gamma_2}} \to \widetilde{C_{\gamma_1'\gamma_2'}}$ be an isomorphism. By Theorem 10.1 there exist a projective transformation ρ such that the following diagram is commutative:

$$\begin{tikzcd} \widetilde{C_{\gamma_1\gamma_2}} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \widetilde{C_{\gamma_1'\gamma_2'}} \arrow[d, "\pi'"] \\ \mathbb{P}^1 \arrow[r, "\rho"] & \mathbb{P}^1 \end{tikzcd}$$

Denoting by B or B' the branch locus of $\widetilde{C_{\gamma_1\gamma_2}}$ or $\widetilde{C_{\gamma_1'\gamma_2'}}$, respectively, we have $\varphi(B) = B'$.
But we also have for some birational map δ the commutative diagram

$$\begin{tikzcd} \widetilde{C_{\gamma_1\gamma_2}} \arrow[r, "\varphi"] \arrow[d, "\sigma"'] & \widetilde{C_{\gamma_1'\gamma_2'}} \arrow[d, "\sigma'"] \\ C_{\gamma_1\gamma_2} \arrow[r, "\delta"] & C_{\gamma_1'\gamma_2'} \end{tikzcd}$$

Setting

$$o = (0,0,1) \in \widetilde{C_{\gamma_1\gamma_2}} \qquad \infty = (0,1,0) \in \widetilde{C_{\gamma_1\gamma_2}}$$

$$o' = (0,0,1) \in \widetilde{C_{\gamma_1'\gamma_2'}} \qquad \infty' = (0,1,0) \in \widetilde{C_{\gamma_1'\gamma_2'}}$$

we get $\sigma'(\varphi(\infty, o)) = \delta(\sigma(\infty, o))$ and $\varphi\{\infty, o\} = \{\infty', o'\}$ because the set $\sigma(\infty, o)$ consists of simple points of $C_{\gamma_1\gamma_2}$, the set $\sigma'(B' - \{\infty', o'\})$ consists of singular points of $C_{\gamma_1'\gamma_2'}$ and the birational transformation δ maps simple points to simple points.
Now the first diagram gives

$$\rho(\pi\{o,\infty\}) = \pi^{'}(\varphi\{o,\infty\})$$

$$\rho\{0, \infty_{\mathbb{P}^1}\} = \{0, \infty_{\mathbb{P}^1}\}$$

At this point we have obtained that our ρ's are only allowed to have the form $\rho(u) = lu$ or $\rho(u) = l/u$. In order to simplify our arguments we define

Definition 10.3 *We will call a vector $(l, \gamma_1^{'}, \gamma_2^{'}) \in \mathbb{C}^3$ admissible, if there exist a vector $(\gamma_1, \gamma_2) \in \mathbb{C}^2$ and a projective transformation $\rho_{l\gamma_1\gamma_2}$ of the form $\rho = lu$ or $\rho = l/u$ such that*

$$\rho_{l\gamma_1\gamma_2}\{\pm\sqrt{\frac{\gamma_2}{\gamma_1}}, \pm\sqrt{\frac{1-\gamma_2}{1-\gamma_1}}, 1, -1\} = \{\pm\sqrt{\frac{\gamma_2'}{\gamma_1'}}, \pm\sqrt{\frac{1-\gamma_2'}{1-\gamma_1'}}, 1, -1\}$$

By our calculations for $\rho(u) = lu$ we obtain the vectors

$$(l, \gamma_1^{'}, \gamma_2^{'}) = (\pm 1, \gamma_1, \gamma_2) = (\pm 1, id(\gamma_1, \gamma_2))$$

$$(l, \gamma_1^{'}, \gamma_2^{'}) = (\pm 1, 1-\gamma_1, 1-\gamma_2) = (\pm 1, t_3(\gamma_1, \gamma_2))$$

$$(l, \gamma_1^{'}, \gamma_2^{'}) = (\pm\sqrt{\tfrac{\gamma_1}{\gamma_2}}, 1/\gamma_1, 1/\gamma_2) = (\pm\sqrt{\tfrac{\gamma_1}{\gamma_2}}, t_2(\gamma_1, \gamma_2))$$

$$(l, \gamma_1^{'}, \gamma_2^{'}) = (\pm\sqrt{\tfrac{\gamma_1}{\gamma_2}}, 1-1/\gamma_1, 1-1/\gamma_2) = (\pm\sqrt{\tfrac{\gamma_1}{\gamma_2}}, t_3 \circ t_2(\gamma_1, \gamma_2))$$

$$(l, \gamma_1^{'}, \gamma_2^{'}) = (\pm\sqrt{\tfrac{1-\gamma_1}{1-\gamma_2}}, \tfrac{\gamma_1-1}{\gamma_1}, \tfrac{\gamma_2-1}{\gamma_2}) = (\pm\sqrt{\tfrac{1-\gamma_1}{1-\gamma_2}}, t_3 \circ t_2 \circ t_3(\gamma_1, \gamma_2))$$

$$(l, \gamma_1^{'}, \gamma_2^{'}) = (\pm\sqrt{\tfrac{1-\gamma_1}{1-\gamma_2}}, \tfrac{1}{1-\gamma_1}, \tfrac{1}{1-\gamma_2}) = (\pm\sqrt{\tfrac{1-\gamma_1}{1-\gamma_2}}, t_2 \circ t_3(\gamma_1, \gamma_2))$$

And for $\rho(u) = l/u$

$$(l, \gamma_1^{'}, \gamma_2^{'}) = (\pm 1, \gamma_2, \gamma_1) = (\pm 1, t_1(\gamma_1, \gamma_2))$$

$$(l, \gamma_1^{'}, \gamma_2^{'}) = (\pm 1, 1-\gamma_2, 1-\gamma_1) = (\pm 1, t_3 \circ t_1(\gamma_1, \gamma_2))$$

$$(l, \gamma_1^{'}, \gamma_2^{'}) = (\pm\sqrt{\tfrac{\gamma_2}{\gamma_1}}, 1/\gamma_2, 1/\gamma_1) = (\pm\sqrt{\tfrac{\gamma_2}{\gamma_1}}, t_2 \circ t_1(\gamma_1, \gamma_2))$$

$$(l, \gamma_1^{'}, \gamma_2^{'}) = (\pm\sqrt{\tfrac{\gamma_2}{\gamma_1}}, 1-1/\gamma_2, 1-1/\gamma_1) = (\pm\sqrt{\tfrac{\gamma_2}{\gamma_1}}, t_3 \circ t_2 \circ t_1(\gamma_1, \gamma_2))$$

$$(l, \gamma_1^{'}, \gamma_2^{'}) = (\pm\sqrt{\tfrac{1-\gamma_2}{1-\gamma_1}}, \tfrac{\gamma_2-1}{\gamma_2}, \tfrac{\gamma_1-1}{\gamma_1}) = (\pm\sqrt{\tfrac{1-\gamma_2}{1-\gamma_1}}, t_3 \circ t_2 \circ t_3 \circ t_1(\gamma_1, \gamma_2))$$

$$(l, \gamma_1^{'}, \gamma_2^{'}) = (\pm\sqrt{\tfrac{1-\gamma_2}{1-\gamma_1}}, \tfrac{1}{1-\gamma_2}, \tfrac{1}{1-\gamma_1}) = (\pm\sqrt{\tfrac{1-\gamma_2}{1-\gamma_1}}, t_2 \circ t_3 \circ t_1(\gamma_1, \gamma_2))$$

We have already proved (ii)$\Leftarrow$, and we know that $(\#T) \geq 12$.
Now we obtain (i) as a consequence of the followings facts

- (a) $\quad t_i^2 = 1 \qquad (1 \leq i \leq 3)$
- (b) $\quad t \circ t_1 = t_1 \circ t \qquad \forall t \in T$
- (c) $\quad t_2 \circ t_3 \circ t_2 = t_3 \circ t_2 \circ t_3$

In order to prove the other direction of (ii), we suppose that two points (γ_1, γ_2) and $(\gamma_1^{'}, \gamma_2^{'})$ are related by $t(\gamma_1, \gamma_2) = (\gamma_1^{'}, \gamma_2^{'})$ for some $t \in T$. Choose an admissible vector $(l, \gamma_1^{'}, \gamma_2^{'})$ and apply Theorem 10.1 with $\rho = \rho_{l\gamma_1\gamma_2}$ and $\varphi(x, y) = (\rho_{l\gamma_1\gamma_2}(x), y)$.

□

The moduli space of the curves belonging to the family F_0 can be now expressed as

Corollary 10.4 $F_0 \cong \mathbb{C}^2/T$

11 Appendix 2 (by N. Vladov): Determination of all proportional Apollonius cycles by MAPLE package "Picard"

Picard is a MAPLE package for creating and working with orbital invariants. We explain how it works on Apollonius configuration (see the end of the programm).

Let us consider again Figure 2. There are points, curves and the surface $\mathbf{X}'$. These are our basic orbital objects. We create orbital object for $\mathbf{C}_0'$ with `orbital(C0, type=curv, init=[2,-2], member={S1,S2,S3}, weight=c0)`. Here `C0` is the curve name; `curv` is the type of the object. Similar for points `R1,R2,R3` type is `abel` and for surface `X` type is `surf`. By surface and curves `init` is the vector [euler-number, selfintersection] and [euler-number, signature] respectively. The curve `C0` intersects exceptional curves `E1,E2,E3` at `S1,S2,S3` and `C0` has `member={S1,S2,S3}`. For a surface object as `X` member has another sence. It is a list of curves and a list of points. If any point belongs to a curve (not isolated point) one can skip this point.

For abelian points with resolution different from $\langle 1,0\rangle$ it is important to create separate orbital objects. Such a point is `R1` and one writes `orbital(R1, type=abel, resol=[2], member=[E1], weight=[e1,1])`. Here `resol=[2]` is the resolution of the contracted curve with selfintersection -2; `R1` belongs to the curve `E1` and have weighs `e1` and 1 on `E1` and on the contracted curve.

One creates orbital objects looking at atomic graphs (Figure 3 and Figure 2). We do not use the weights 4 on $\circ$. The programm will look for possible weights.

Now all objects are on the computer memory. Using `maincheckorbital(X)` we check whether everything is correct and `mainproporinv(X)` calculate the basic invariants for points, curves and surface. Obtaining these invariants we do not use that `E1,E2,E3` are exceptional curves. Using `makesingular(cusp, X, {E1,E2,E3})` we contract three exceptional curves to cusp points. After computer calculation of all finite invariants $[h_e, h_\tau]$. Now with `makeequations(X)` one obtains equations: $4c_0 = 1$, $c_0 + c_j = 1/2$, $j = 1,2,3$. Obviously this system has unique solution $c_0 = c_1 = c_2 = c_3 = 1/4$ (the programm uses inverse weights).

We can try another possibilities. First with `cleanornital(X)` all finite invariants are deleted from the memory and with `makesingular(cusp,X,{E1,E2})`, `makesingular(triple,X,{E3})` one contracts `E1,E2` to cusp points and `E3` to triple. Using `makeequations(X)` we obtain $c_1 = c_2$, $3c_2 + c_3 = 1$, $4c_2 + e_3 = 1$, $c_0 + c_2 = 1/2$. It is easy to see that there is unique solution: $\{c0 = 1/3, c1 = 1/6, c2 = 1/6, c3 = 1/2, e3 = 1/3\} \in 1/\mathbb{N}$. In the similar way we consider the another cases: 1,0 cusp and 2,3 triple points respectively. In case 0 cusp points (see the last example) it is not easy to solve the equations in $1/\mathbb{N}$.

I am preparing an extension to Picard package which will solve inverse linear systems $Ax = B$, A - matrix, B -vector with rational coefficients, $\det A = 0$, $\mathrm{rank}A = \mathrm{rank}B$. The solutions we are looking for must be integers. Here we present the four unique *proportional solutions* for Apollonius configuration satisfying the conditions (Prop 1), (Prop 2) of the Proportionality Theorem 1.1.

C_0	C_1	C_2	C_3	E_1	E_2	E_3	cusps	triples
4	4	4	4	∞	∞	∞	E_1 E_2 E_3	—
3	6	6	2	∞	∞	3	E_1 E_2	E_3
3	6	3	3	∞	6	6	E_1	E_2 E_3
3	4	4	3	12	12	6	—	E_1 E_2 E_3

We denote the cusp type with graph 7 by $(\infty; \langle 2,1\rangle, \langle 4,0\rangle, \langle 4,0\rangle)$. The first number is the central weight and the pairs stand for the three weighted curve germes through the central resolution curve. This notation extends to all (smooth) orbital cusp and quotient points in obvious manner (see [H98]), where quotient points have finite wights. On this way we correspond to the three tangent points of the Apollonius configuration three such *weight tuples*. We read them off from the above table and get the following

Theorem 11.1 *Up to order there are precisely four triples of wight tuples attached to the tangent points of the Apollonius configuration such that the proportionality conditions for ball quotient surfaces are satisfied, namely*

- $[(\infty,\infty,\infty)]$:
  $(\infty; \langle 2,1\rangle, \langle 4,0\rangle, \langle 4,0\rangle)$, $(\infty; \langle 2,1\rangle, \langle 4,0\rangle, \langle 4,0\rangle)$, $(\infty; \langle 2,1\rangle, \langle 4,0\rangle, \langle 4,0\rangle)$
  (the case of Gauß numbers)

- $[(\infty,\infty,3)]$:
  $(\infty;\langle 2,1\rangle,\langle 3,0\rangle,\langle 6,0\rangle)\,,\,(\infty;\langle 2,1\rangle,\langle 3,0\rangle,\langle 6,0\rangle)\,,\,(3;\langle 2,1\rangle,\langle 3,0\rangle,\langle 2,0\rangle)$
- $[(\infty,6,6)]$:
  $(\infty;\langle 2,1\rangle,\langle 3,0\rangle,\langle 6,0\rangle)\,,\,(6;\langle 2,1\rangle,\langle 3,0\rangle,\langle 3,0\rangle)\,,\,(6;\langle 2,1\rangle,\langle 3,0\rangle,\langle 3,0\rangle)$
- $[(12,12,6)]$:
  $(12;\langle 2,1\rangle,\langle 3,0\rangle,\langle 4,0\rangle)\,,\,(12;\langle 2,1\rangle,\langle 3,0\rangle,\langle 4,0\rangle)\,,\,(6;\langle 2,1\rangle,\langle 3,0\rangle,\langle 3,0\rangle)$

□

```
# Picard Package for Proportional Orbital Invariants
# available by author: Nikola Vladov <vladov@fmi.uni-sofia.bg>

###    Orbital Functions:
# orbital
# mainproporinv      proporinv
# makesingular       makesingsurface    makesingcurve
# maincheckorbital   checkorbital
# cleanorbital
# showsolution
# makeequations

####################
orbital:=proc(X,a,b,c,d,e,f,g) local opt,str;
  if nargs=0 then RETURN(`usage: orbital(name,<options>)`) fi;
  if not(type(X,name)) then ERROR(`First arg must be name`, X) fi;

  if not(assigned(X[type_])) then
     X:=table();  X[name_] := args[1];  fi;

  for opt in [args[2..nargs]] do
    if type(opt,`=`) then
      str:=substring(op(1,opt),1..3);
      if str='typ' and type(op(2,opt),name) then X[type_]:=op(2,opt)
      elif str='sin' and type(op(2,opt),list) then X[singularity_]:=op(2,opt)
      elif str='res' and type(op(2,opt),list) then X[resolution_]:=op(2,opt)
      elif str='mem' and (type(op(2,opt),list) or type(op(2,opt),set)) then
               X[member_]:=op(2,opt)
      elif str='wei' then X[weight_]:=op(2,opt)
      elif str=`ini` and type(op(2,opt),list) then X[init_] := op(2,opt)
      fi;
    fi;  # type(opt,`=`) then
  od;

  if not(assigned(X[type_])) then ERROR(X, `type_ is not defined`) fi;
  userinfo(7,orbital, `Create object`, X, X[type_]);
end:

####################
mainproporinv:= proc(X) local c,p;          # X orbital surface
for p in X[points_] do proporinv(p) od;
   for c in X[curves_] do proporinv(c) od;
   proporinv(X);
end:

#####################
proporinv:=proc(X) local s,v,v1,v2,d,h1,h2,e,pp,cc,vi;
# X - orbital object
  if nargs=0 then RETURN(`usage: proporinv(X)`) fi;

  if X[type_] = 'abel' then
     v1:=X[weight_][1];     v2:=X[weight_][2];  d:=X[singularity_][1];
```

```
    if X[singularity_] = [1,0] then s:=0;
    else
     if not(assigned(X[resolution_])) then ERROR(`Need resolution_`, X) fi;
     s := resoltoEDHsum(X[resolution_]);
    fi;

    X[propor_] := [ 1 - v1/d - v2/d + v1*v2/d , s]: # ABEL

  elif X[type_] = 'curv' then
    h1:=X[init_][1];     h2:=X[init_][2];     v := X[weight_];

    for pp in X[member_] do               # each curve have members Points
       if not(pp[type_] = 'abel') then
            ERROR(`Curve must contain only abel Points on X'`, X) fi;
       d := pp[singularity_][1];

       if nops(pp[member_]) = 1 then
          vi:= pp[weight_][2];       e := pp[singularity_][2];
       else
          if pp[member_][1] = X[name_] then
               vi:=pp[member_][2][weight_];  e:=pp[singularity_][2];
          else vi:=pp[member_][1][weight_];  e:=etoeprim(pp[singularity_]);
          fi;
       fi;

       h1 := h1 - 1 + vi/d;        h2 := h2 +  e/d;
    od;
    X[propor_] := [expand(h1),expand(h2*v)];        # CURVE

  elif X[type_] = 'surf' then
    h1:=X[init_][1];     h2:=X[init_][2];

    for pp in X[points_] do
        h1:=h1-pp[propor_][1];   h2:=h2-pp[propor_][2]; od;

    for cc in X[curves_] do
        h1 := h1 - (1-cc[weight_]) * cc[propor_][1];
        h2 := h2 - (1/cc[weight_]-cc[weight_]) * cc[propor_][2] /3; od;
    X[propor_] := [expand(h1),expand(h2)];             # SURFACE

  else ERROR(`Unknown type_`, X, X[type_]);
  fi;

  userinfo(9, orbital, ` `,  X, X[type_]);
end:

###########
makesingular:= proc(type,X,CC) local out,scurv,tmp,c,e;
# type= cusp,triple, X=orbital surface,  CC=[C1,C2,C3,...] list of curves
   if nargs=0 then RETURN(`usage: makesingular(type,X,[E1,E2,E3,...])`) fi;

   scurv := {};
   out:= convert(X[curves_],set) minus convert(CC,set);

   for c in out do
      for e in CC do
         tmp:=  convert(e[member_],set) intersect convert(c[member_],set);
         if nops(tmp) > 0 then scurv:= scurv union {c} fi;
      od;
   od;
```

```
   makesingsurface(type,X,CC);
   for c in scurv do makesingcurve(type,c,CC) od;
end:

#################
makesingsurface := proc(type,X,CC)  local c,h1,h2,e,tmp;
# make triples or cusps points from curves and calculate Surface
# Proportional Invariants.
# type=cusp,triple, X= orbital surface, CC=[C1,C2,C3,...] list of curves.
   if nargs=0 then RETURN(`usage: makesingsurface(type,X,[E1,E2,E3,...])`) fi;

   if not(assigned(X[cusps_])) then X[cusps_] := {} fi;
   if not(assigned(X[triples_])) then X[triples_] := {} fi;

   if assigned(X[finite_]) then
      h1:= X[finite_][1];    h2:= X[finite_][2];
   else
      h1:= X[propor_][1]; h2:= X[propor_][2];
   fi;

   if type = 'cusp' then
      X[cusps_] := X[cusps_] union convert(CC,set);
      for c in CC do
          h1 := subs(c[weight_] = 0, h1);
          h2 := subs(c[weight_] = 0, h2); od;
   elif type = 'triple' then
      X[triples_] := X[triples_] union convert(CC,set);
      for c in CC do
         h1:= h1 - c[propor_][1]*c[weight_] / 2;
         h2:= h2 + c[propor_][1]*c[weight_] / 2;   od;
   else ERROR(`Unknown surface singularity`, type,XX,CC);
   fi;

   X[finite_] := [expand(h1),  expand(h2)];
   userinfo(9,orbital, ` `, `Surface`, type, CC);
end:

#####################
makesingcurve := proc(type,C,EE)  local p,e,h1,h2,tmpP,tmp;
# make finite invariant for curve C, [E1,E2,E3,...] list of central
# resolution curves, type=cusp,triple
   if nargs=0 then RETURN(`usage: makesingcurve(type,C,[E1,E2,E3,...])`) fi;

   if assigned(C[finite_]) then
      h1:= C[finite_][1];    h2:= C[finite_][2];
   else
      h1:= C[propor_][1];        h2:= C[propor_][2];
   fi;

   if type = 'cusp' then
      for e in EE do
          h1:= subs(e[weight_] = 0, h1);
          h2:= subs(e[weight_] = 0, h2);
      od;
   elif type = 'triple' then
      for e in EE do
         tmp := 0;
         tmpP:= convert(C[member_], set) intersect convert(e[member_], set);
         for p in tmpP do tmp:= tmp + e[weight_] / p[singularity_][1]; od;
         h2:= h2 + tmp;
      od;
   else ERROR(`Unknown curve singularity`, type,XX,CC);
```

```
   fi;

   C[finite_] := [h1, h2];
   userinfo(9,orbital, ` `, `Curve`, C,EE,type);
end:

######################
checkorbital:=proc(X) local tmp,c;
# check single orbital object
  if nargs=0 then RETURN(`usage: checkorbital(OrbitalName)`) fi;

  if not(type(X,name)) then ERROR(`First arg must be name`) fi;
  if not(type(X,table)) then ERROR(`Not type table`, X) fi;
  if not(assigned(X[type_])) then ERROR(`Unknown type_`, X) fi;

  if X[type_] = 'abel' then
     if assigned(X[resolution_]) and assigned(X[singularity_]) then
        tmp := resoltosing(X[resolution_]);
        if not(tmp = X[singularity_]) then
              ERROR(`singularity_ and resolution_ are uncompatible`, X) fi;
     elif assigned(X[resolution_]) and not(assigned(X[singularity_])) then
        X[singularity_] := resoltosing(X[resolution_]); fi;

     if not(assigned(X[singularity_])) then
              ERROR(`Unknown resolution_ or singularity_`, X) fi;

     if assigned(X[member_]) and not(assigned(X[weight_])) then
        if nops(X[member_]) > 2 then
             ERROR(`Abel point is member of maximal 2 curves`, X);
        elif nops(X[member_]) = 2 then
             X[weight_] := [ X[member_][1][weight_], X[member_][2][weight_] ];
        elif nops(X[member_]) = 1 then
             X[weight_] := [ X[member_][1][weight_], 1];
        fi;
     elif not(assigned(X[member_])) and not(assigned(X[weight_])) then
        X[weight_] := [1,1];
     fi;

  elif X[type_] = 'curv' then
     if not(assigned(X[init_])) then ERROR(`Curve init_ = ?`, X) fi;

  elif X[type_] = 'surf' then
     if not(assigned(X[init_])) then ERROR(`Surface init_ = ?`, X) fi;
     if not(assigned(X[member_])) then ERROR(`Surface member_ = ?`, X) fi;
     if nops(X[member_]) = 2 then
         X[curves_] := convert(X[member_][1],set);
         X[points_] := convert(X[member_][2],set);
         for c in X[curves_] do
            X[points_] := X[points_] union convert(c[member_],set);
         od;
         X[member_] := 'X[member_]';
     else    ERROR(`Member_ must be = [ {C1,C2,C3,C4}, {P1,P2,P3} ] `, X) fi;
   if not(assigned(X[cusps_])) then X[cusps_] := {} fi;
   if not(assigned(X[triples_])) then X[triples_] := {} fi;
  fi;

  userinfo(9,orbital, `Check `, X, X[type_]);
end:

##################
maincheckorbital := proc(X) local p,c,cp;
# check all objects of orbital surface
```

```
  checkorbital(X);
  for c in X[curves_] do checkorbital(c) od;

  for p in X[points_] do
     if not(assigned(p[singularity_])) and not(assigned(p[resolution_]))
         then orbital(p,type=abel,memb={}) fi;
  od;

  for c in X[curves_] do
     for cp in c[member_] do
        if not(assigned(cp[singularity_]))
           and not(assigned(cp[resolution_])) then
           cp[member_] := cp[member_] union {c[name_]};
        fi;
     od;
  od;

  for p in X[points_] do
     if type(p[member_],set) then p[member_]:=convert(p[member_],list) fi;
     if not(assigned(p[singularity_])) and not(assigned(p[resolution_])) then
           p[singularity_] := [1,0] fi;
  od;

  for p in X[points_] do checkorbital(p) od;
end:

########
cleanorbital := proc(X) local c;
# clean all finite objects
   for c in X[curves_] do   c[finite_]:='c[finite_]';   od;
   X[triples_]:={};    X[cusps_]:={};   X[finite_]:='X[finite_]';
   userinfo(9,orbital, `Clean Orbital Surface`, X);
end:

###############
showsolution := proc(X,V,S) local sol,fin,c,cusps,triples,j,tmp;
# check and show solution
# X=orbital surface, V={c0=1/4,c1=1/4,...} or
# V=[c0,c1,c2,...], S=[1/4,1/4,...],  V-list ot curves' weights  or
# V=[C0,C1,C2,...], S=[1/4,1/4,...],  V-list ot curves' names

 if nargs = 2 then sol:=V
 elif assigned(V[1][weight_]) then
    if not(nops(V) = nops(S)) then ERROR(`Uncompatible vectors`, V,S) fi;
    cusps:= {};    triples:= {};
    for j from 1 to nops(V) do
       if S[j] = 0 then cusps := cusps union {V[j]};
       elif S[j] < 0 then triples := triples union {V[j]}
       fi;
    od;
    cleanorbital(X);
    makesingular(`cusp`,X,cusps);
    makesingular(`triple`,X,triples);
    sol:= [seq(V[j][weight_] = abs(S[j]), j=1..nops(S))];
 else
    sol:= [seq(V[j] = S[j], j=1..nops(S))];
 fi;

 print(`weights`, sol);
 if nops(X[cusps_]) > 0 then
     print(`Cusps`);
     for c in X[cusps_] do
```

```
       tmp:= [subs(sol,c[propor_][1]), c[propor_][2]];
       if tmp[1] = 0 and tmp[2]/c[weight_] < 0 then print(` `.c, tmp);
       else print(`ERROR:        `.c, tmp);   fi;
     od;
 fi;

 if nops(X[triples_]) > 0  then
     print(`Triples`);
     for c in X[triples_] do
       tmp:= subs(sol,c[propor_]);
       if tmp[1] >0 and -tmp[1] = 2*tmp[2] then print(` `.c, tmp);
       else print(`ERROR:        `.c, tmp);   fi;
     od;
 fi;

  print(`Prop. 1`);
  fin:= X[curves_] minus X[cusps_] minus X[triples_];
  for c in fin do
     if assigned(c[finite_]) then  tmp:= subs(sol,c[finite_]);
     else tmp := subs(sol,c[propor_]);     fi;
     if tmp[1] < 0 and tmp[1] = 2*tmp[2] then print(` `.c, tmp);
     else print(`ERROR:        `.c, tmp);   fi;
  od;

  print(`Prop. 2`);
  if assigned(X[finite_]) then tmp:= subs(sol,X[finite_]);
  else tmp:= subs(sol,X[propor_]);       fi;
  if tmp[1] > 0 and tmp[1] = 3*tmp[2] then print(` `.X, tmp);
  else print(`ERROR:        `.X, tmp);   fi;

  userinfo(9,orbital, ` `);
end:

##########
makeequations := proc(X) local c,cusp,triple,p,other,i,A,AB, Ec,Et,Ef,fin;
global INEQ, EQUA, VARI, SOLU;
   EQUA:={}; INEQ:= {}; VARI:= {}; cusp:= {}; triple:= {};
   Ec:={}; Et:={}; Ef:= {};

   if not(assigned(X[finite_])) then X[finite_] := X[propor_] fi;

   for c in X[cusps_] do
      Ec:= Ec union {c[propor_][1]};
      INEQ:= INEQ union {-c[propor_][2]/c[weight_]};
      cusp:= cusp union {c[weight_]};
      od:

   for c in X[triples_] do
      Et:= Et union {c[propor_][1] + 2*c[propor_][2]};
      INEQ:= INEQ union {-c[propor_][2]};
      triple:= triple union {c[weight_]};
   od:

   fin:= X[curves_] minus X[cusps_] minus X[triples_];
   for c in fin do
      if assigned(c[finite_]) then
         Ef:= Ef union {c[finite_][1] - 2*c[finite_][2]};
         INEQ:= INEQ union {-c[finite_][2]};
      else
         Ef:= Ef union {c[propor_][1] - 2*c[propor_][2]};
         INEQ:= INEQ union {-c[propor_][2]};
      fi;
```

```
   od;

   EQUA:= [ [seq(Ec[i], i=1..nops(Ec)),
             seq(Et[i], i=1..nops(Et)),
             seq(Ef[i], i=1..nops(Ef))],
            expand(X[finite_][1]-3*X[finite_][2]) ];
   INEQ:= [INEQ, expand(X[finite_][2]) ];

   for p in X[points_] do
      if not(type(p[weight_][1],numeric)) then
                       VARI:=VARI union {p[weight_][1]} fi;
      if not(type(p[weight_][2],numeric)) then
                       VARI:=VARI union {p[weight_][2]} fi;
   od;
   for c in X[curves_] do VARI:= VARI union {c[weight_]} od;

   other:= VARI minus (cusp union triple);
   VARI:= [seq(cusp[i], i=1..nops(cusp)),
           seq(triple[i], i=1..nops(triple)),
           seq(other[i], i=1..nops(other))];
   VARI:=[VARI, convert(cusp,list), convert(triple,list)]:

            ###################
   print(`Picard ==>  INEQualities,  EQUAtions,   VARIables,  SOLUtions`);
   print(`VARI: `,  VARI);

   A := linalg[genmatrix](EQUA[1], VARI[1]);
   AB:= linalg[genmatrix](EQUA[1], VARI[1], flag);

   print(`#vars,  #cusps,  rank(Lin),  rank(Lin+)`
         ,nops(VARI[1]), nops(VARI[2]), linalg[rank](A), linalg[rank](AB));

   SOLU:= solve(convert(EQUA[1],set));
   print(`SOLU: `,  SOLU);
end:


####            SOME HELP FUNCTIONS               ######

#############
resoltosing := proc(XX) local TmpSum, b, a;
##   [b1,b2,b3,b4,b5,...] ==> <d,e>
    if nargs=0 then RETURN(`usage: resoltosing([b1,b2,b3,b4,b5,...])`) fi;
    b := nops(XX):
    if b = 1 then TmpSum := XX[1];
    else
       TmpSum :=XX[b]:
       for a from 1 to (b-1) do TmpSum := XX[b-a] - 1/TmpSum; od; fi;
    if numer(TmpSum) = 1 then [1,0];
    else [numer(TmpSum), denom(TmpSum)] fi;
end:

###################
resoltoEDHsum := proc(B) local l,Tr,de,ep,i;
## [b1,b2,b3,b4,b5 ...] ==> 1/3*(3*l + Tr -e/d - ep/d)
   l:= nops(B);
   Tr:= -sum('B[i]', 'i'=1..l);
   de := resoltosing(B);
   ep := etoeprim(de);
   1/3*(3*l + Tr -de[2]/de[1] - ep/de[1]);
end:
```

```
########################
etoeprim := proc(de) local e;       ##   <d,e> ==> e'
    if nargs=0 then RETURN(`usage: etoeprim([d,e]) ==> e'`) fi;
    if de[2] = 0 then e := 0;
    elif de[2] = 1 then e := 1;
    elif type(de[2],posint) and de[2] >1 then
          e := subs(msolve(de[2]*XXX=1,de[1]),'XXX');
    else ERROR(`something wrong with`, de);
    fi;
    eval(e);
end:
# end of Package Picard

#### Data for Apollonius Configuration
# infolevel[orbital]:=10;

orbital(C0,type=curv,init=[2,-2],member={S1,S2,S3},weight=c0) ;
orbital(C1,type=curv,init=[2,-1],member={T1,P2,P3},weight=c1);
orbital(C2,type=curv,init=[2,-1],member={T2,P1,P3},weight=c2);
orbital(C3,type=curv,init=[2,-1],member={T3,P1,P2},weight=c3);

orbital(E1,type=curv,init=[2,-1],member={R1,T1,S1},weight=e1);
orbital(E2,type=curv,init=[2,-1],member={R2,T2,S2},weight=e2);
orbital(E3,type=curv,init=[2,-1],member={R3,T3,S3},weight=e3);

orbital(R1,type=abel,resol=[2],member=[E1],weight=[e1,1]);
orbital(R2,type=abel,resol=[2],member=[E2],weight=[e2,1]);
orbital(R3,type=abel,resol=[2],member=[E3],weight=[e3,1]);

orbital(X,type=surf,init=[6,-2],member=[{C0,C1,C2,C3,E1,E2,E3}, {}]);

maincheckorbital(X);    mainproporinv(X);

makesingular(cusp,X,{E1,E2,E3});   makeequations(X);
showsolution(X,{c0=1/4,c1=1/4,c2=1/4,c3=1/4});

cleanorbital(X);
makesingular(cusp,X,{E1,E2}); makesingular(triple,X,{E3});
makeequations(X); showsolution(X,{c0=1/3,c1=1/6,c2=1/6,c3=1/2,e3=1/3});

cleanorbital(X);   makesingular(cusp,X,{E1});
makesingular(triple,X,{E2,E3});   makeequations(X);
showsolution(X,{c0=1/3,c1=1/6,c2=1/3,c3=1/3,e2=1/6,e3=1/6});

cleanorbital(X);
makesingular(triple,X,{E1,E2,E3});    makeequations(X);
showsolution(X,{c0=1/3,c1=1/4,c2=1/4,c3=1/3,e1=1/12,e2=1/12,e3=1/6});
```

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