The polyhedral Hodge number $h^{2,1}$ and vanishing of obstructions

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Abstract

We prove a vanishing theorem for the Hodge number $h^{2,1}$ of projective toric varieties provided by a certain class of polytopes. We explain how this Hodge number also gives information about the deformation theory of the toric Gorenstein singularity derived from the same polytope. In particular, the vanishing theorem for $h^{2,1}$ implies that these deformations are *unobstructed.*

1 Introduction

(1.1) For an arbitrary polytope Δ, Brion has introduced in [Br] certain invariants $h^{p,q}(\Delta)$. These are defined as dimensions of cohomology groups $H^{p,q}$ of complexes which are associated directly to the polytope Δ. In case that Δ is a rational polytope, these invariants are exactly the Hodge numbers $\dim H^p(X, \Omega^q_X)$ of the corresponding projective toric variety $X = \mathbb{P}(\Delta)$.

In this paper, we focus on the K-vector spaces $D^k(\Delta) := H^{k,1}$ $(k \geq 2)$ and $D^1(\Delta) := H^{1,1}/K$. The notation is suggested by a second interpretation of these vector spaces: In §6 we will show that there is a close relation between $D^k(\Delta)$ and the vector spaces T^k describing the deformation theory of the toric Gorenstein singularity $X_{\mathrm{cone}(\Delta)}$ associated to the lattice polytope Δ.

Our main result is a vanishing theorem for $D^2(\Delta)$ for a certain class polytopes. An important special case is

Theorem (cf. (4.7)) *Let Δ be an n-dimensional, compact, convex polytope such that every three-dimensional face is a pyramid. If no vertex is contained in more than $(n-3)$ two-dimensional, non-triangular faces, then $D^2(\Delta) = 0$.*

There is a natural class of polytopes that arises from *quivers* (see [AH]) to which these seemingly strange conditions apply. Special examples of such quiver polytopes appeared in [BCKvS] as a description of toric degenerations of Grassmannians and partial flag manifolds that appeared in the works of Strumfels [St] and Lakshmibai [La]. In a forthcoming paper [AvS], we will apply the above vanishing result to show that the Gorenstein singularities provided by so-called flag like quivers are *unobstructed* and *smoothable* in codimension three.

(1.2) The paper is organized as follows:

In §2 we recall some notions of homological and cohomological systems on polyhedral complexes. For the special case of simplicial sets, these can be found in [GM1] or [GM2]. We quote the definition of the polyhedral Hodge numbers and review their basic properties.

In §3, we introduce the D-invariants from a slightly different point of view as above and show their relation to the polyhedral Hodge numbers. We present some examples as well as elementary properties, such as the relation of $D^1(\Delta)$ to the Minkowski decomposition of polytopes.

The paragraphs §4 and §5 contain the vanishing theorem for D^2 and its proof. The result is obtained from a spectral sequence relating the D-invariants of a polytope to those of its faces; $D^2(\Delta)$ is represented as the kernel of some differential on the E_2-level. In Theorem (4.5), this description is transformed into an explicit set of equations describing $D^2(\Delta)$.

The final §6 deals with the relations of the D-invariants to deformation theory that was mentioned before. In the paper [AS], a combinatorial description of the cotangent cohomology modules $T^k(X_{\text{cone}(\Delta)})$ was given. From this description it appears that the T^k are very sensitive to the interaction of the polytope Δ with the lattice structure of the ambient space. As a consequence, these invariants are often very difficult to calculate explicitly.
On the other hand, the invariants $D^k(\Delta)$ are rather coarse; they only depend on the polytope Δ *up to projective equivalence*, and the lattice structure is not involved at all. Nevertheless, in Theorem (6.6) we formulate a sufficient conditions on Δ ensuring that the T^k *are* determined by $D^k(\Delta)$. In particular, the vanishing Theorem (4.7) yields a vanishing theorem for $T^2(X_{\text{cone}(\Delta)})$ as well.

(1.3) *Acknowledgement*: We would like to thank M. Brion and P. McMullen for valuable comments and discussions.

2 Hodge numbers for polytopes

(2.1) Let $\Sigma = \cup_{k\geq 0}\Sigma_k$ be a finite, *polyhedral complex* in a K-vector space V ($\mathbb{Q} \subseteq K \subseteq \mathbb{R}$), i.e. a set of polyhedra in V that is closed under the face operation and with the additional property that for any two $\sigma, \tau \in \Sigma$, the intersection $\sigma \cap \tau$ is either empty or a common face of both polyhedra. Here Σ_k denotes the subset of k-dimensional elements of Σ. Examples for such polyhedral complexes are *simplicial sets* as well as *fans*.

Definition: A *cohomological system* $\mathcal{F}$ on Σ is a covariant functor from Σ to the category $\mathcal{AB}$ of abelian groups (or to any other abelian category $\mathcal{A}$).

Here Σ becomes a small category by declaring the face relations "$\tau \leq \sigma$" to be the morphisms. So a cohomological system is nothing else than a collection of abelian groups $\mathcal{F}(\sigma)$ for $\sigma \in \Sigma$ together with compatible face maps $\mathcal{F}(\tau) \to \mathcal{F}(\sigma)$.
Similarly, a *homological system* is defined as a contravariant functor from Σ to $\mathcal{AB}$.

We fix for each polyhedron $\sigma \in \Sigma$ an orientation. This enables us to introduce for each pair (τ, σ) of elements of Σ a number $\varepsilon(\tau, \sigma)$ as follows:

- If τ is a facet (i.e. a codimension-one face) of σ, then we may compare the original orientation of τ with that inherited from σ. Depending on the result, we define $\varepsilon(\tau, \sigma) := \pm 1$.
- If τ is *not* a facet of σ, then we simply set $\varepsilon(\tau, \sigma) := 0$.

Each cohomological system $\mathcal{F}$ on Σ gives rise to a complex $C^\bullet(\Sigma, \mathcal{F})$ of abelian groups:

$$C^k(\Sigma, \mathcal{F}) := \oplus_{\sigma \in \Sigma_k} \mathcal{F}(\sigma) = \oplus_{[\sigma \in \Sigma;\, \dim \sigma = k]} \mathcal{F}(\sigma) \,.$$

The differential $d : C^k(\Sigma, \mathcal{F}) \to C^{k+1}(\Sigma, \mathcal{F})$ is defined in the obvious way, using the $\varepsilon(\tau, \sigma)$ introduced above. The associated cohomology is denoted by

$$H^k(\Sigma, \mathcal{F}) := H^k\big(C^\bullet(\Sigma, \mathcal{F})\big).$$

Note that there is an analogous construction for homological systems.

(2.2) The cohomology groups of a cohomological system $\mathcal{F}$ can sometimes be computed using certain subcomplexes of Σ. To be more precise, let $M^i \subseteq \Sigma$ be subcomplexes with $\cup_i M^i = \Sigma$. The *nerve* $\mathcal{M}$ of this covering is the simplicial set defined as

$$\mathcal{M}_p := \{i_0 \le \ldots \le i_p \mid M^{i_0} \cap \ldots \cap M^{i_p} \neq \emptyset\}.$$

We obtain cohomological systems $\mathcal{H}^q(\mathcal{F})$ on $\mathcal{M}$ via

$$\mathcal{H}^q(\mathcal{F}) : (i_0 \le \ldots \le i_p) \mapsto H^q\big(M^{i_0} \cap \ldots \cap M^{i_p}, \mathcal{F}\big)\,.$$

Proposition: *There is a degenerating spectral sequence $E_2^{p,q} = H^p\big(\mathcal{M}, \mathcal{H}^q(\mathcal{F})\big) \Rightarrow H^{p+q}(\Sigma, \mathcal{F})$ with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.*

Proof: Consider the double complex

$$C^{p,q} := \oplus_{\sigma \in [M^{i_0} \cap \ldots \cap M^{i_p}]_q} \mathcal{F}(\sigma) \;\text{ with }\; d_I : C^{p,q} \to C^{p+1,q}\,,\; d_{II} : C^{p,q} \to C^{p,q+1}\,.$$

The first spectral sequence yields $E_2^{p,q} = H_I^p H_{II}^q(C^{\bullet,\bullet}) = H^p\big(\mathcal{M}, \mathcal{H}^q(\mathcal{F})\big)$; the other one provides the complex $C^\bullet(\Sigma, \mathcal{F})$ at the E_1-level, i.e. $H^\bullet(\Sigma, \mathcal{F})$ is the cohomology of the total complex. □

(2.3) Now assume that Σ is a *fan* in the d-dimensional vector space V, i.e. its elements are polyhedral cones with 0 as their common vertex. Note that the intersection of cones from Σ is always non-empty. Another special feature of fans is that they come with an important cohomological system for free: $\mathcal{F}(\sigma) := \operatorname{span}_k(\sigma)$. From this cohomological system "span" one derives various other systems like V/span and its exterior powers. These give rise to the so called *Hodge spaces* of Σ, a notion which is due to Brion:

$$H^{p,q}(\Sigma) := H^{d-p}\Big(\Sigma, \Lambda^q V/\text{span}\Big)^*\,.$$

For rational fans, Danilov has shown in §12 of [Da] that $H^{p,q}(\Sigma)$ is $H^p(X, \Omega_X^q)$ where $X = X_\Sigma$ denotes the toric variety induced by Σ, and Ω_X^q is the reflexive hull of the Kähler q-differentials on X_Σ. For general fans Σ, Brion has obtained the following vanishing results:

Proposition: (cf. §1 of [Br])

(i) *$H^{p,q}(\Sigma) = 0$ for $p < q$.*

(ii) *If $|\Sigma| := \cup_{\sigma\in\Sigma}\sigma$ is not contained in any hyperplane, then $H^{d,q}(\Sigma) = 0$ for $q < d$, and $H^{d,d}(\Sigma)$ is isomorphic to K.*

(iii) *If $|\Sigma| = V$, and if e is a positive integer such that cones with dimension at most e are simplicial, then $H^{p,q}(\Sigma) = 0$ for $p - q > d - e$.*

(iv) *Assume that $|\Sigma| = V$ and that any two non-simplicial cones in Σ intersect only at the origin. Then $H^{2,1}(\Sigma) = 0$.*

Note that the assumption of (iv) implies that any $(d-1)$-dimensional cone in Σ is simplicial. Hence, by (iii), it follows that $H^{p,1}(\Sigma) = 0$ for $p \ge 3$.

(2.4) Let $\Delta \subseteq K^n$ be a compact, convex polytope. It gives rise to the (inner) *normal fan* $\Sigma(\Delta)$ in the dual space $(K^n)^* \cong K^n$. Brion has shown that the diagonal Hodge spaces $H^{p,p}(\Sigma(\Delta))$ have then a special combinatorial meaning; they coincide with the spaces of the so-called Minkowski p-weights of Δ.
In this paper we will focus on the spaces $H^{p,1}(\Sigma(\Delta))$ which sit close to the boundary of the Hodge diamond.

3 The D-invariants

(3.1) Let $\Delta \subseteq K^n$ be a *compact, convex polytope*; the cone over it, denoted by $\operatorname{cone}(\Delta)$, generates a (non-complete) fan $\underline{\operatorname{cone}}(\Delta)$ in K^{n+1}. This gives rise to the following invariants of the polytope Δ:

$$D^k(\Delta) := H^k\Big(\underline{\operatorname{cone}}(\Delta),\, K^{n+1}\!/\text{span}\Big) = H^{k+1}\big(\underline{\operatorname{cone}}(\Delta),\, \text{span}\big)\,.$$

The equality is a result of the exactness of the complex $C^\bullet(\underline{\operatorname{cone}}(\Delta),\, K^{n+1})$ sitting in the middle of the short exact sequence of cohomological systems

$$0 \to \text{span} \longrightarrow K^{n+1} \longrightarrow K^{n+1}\!/\text{span} \to 0\,.$$

Up to isomorphisms, the vector spaces $D^k(\Delta)$ depend only on the *projective equivalence class* of the given polytope Δ. However, as examples from McMullen and Smilansky show, they are not combinatorial invariants of Δ.

From now on, we will always assume that $\Delta \subseteq K^n$ has the full dimension n.

Lemma: *Denote by $\Delta^\vee$ the polytope that is* polar *to Δ, i.e. the face lattice of $\Delta^\vee$ is opposite to that of Δ, and the cones* $\operatorname{cone}(\Delta^\vee)$ *and* $\operatorname{cone}(\Delta)$ *are mutually dual. Then, there is a perfect pairing*

$$D^k(\Delta^\vee) \times D^{n-k}(\Delta) \longrightarrow K\,.$$

Proof: If $\sigma \le \operatorname{cone}(\Delta)$ is an $(n+1-k)$-dimensional face, then $\big[\sigma^\perp \cap \operatorname{cone}(\Delta^\vee)\big] \le \operatorname{cone}(\Delta^\vee)$ is a face of dimension k with $\operatorname{span}\big[\sigma^\perp \cap \operatorname{cone}(\Delta^\vee)\big] = \sigma^\perp$. Moreover, all faces of $\operatorname{cone}(\Delta^\vee)$ arise in this way. Hence,

$$\begin{aligned} D^k(\Delta^\vee) &= H^k\Big(\underline{\operatorname{cone}}(\Delta^\vee),\, K^{n+1}\!/\text{span}\Big) = H^k\Big(\underline{\operatorname{cone}}(\Delta^\vee),\, \big((\bullet)^\perp\big)^*\Big) \\ &= H_k\big(\underline{\operatorname{cone}}(\Delta^\vee),\, (\bullet)^\perp\big)^* = H^{n+1-k}\big(\underline{\operatorname{cone}}(\Delta), \text{span}\big)^* = D^{n-k}(\Delta)^*\,. \end{aligned}$$

□

(3.2) The following remarks are intended to obtain a better feeling for the meaning of the invariants $D^k(\Delta)$.

(i) For $\Delta = \emptyset$ we define $\operatorname{cone}(\emptyset) := 0$, hence $D^k(\emptyset) = 0$ for every $k \in \mathbb{Z}$.

(ii) If Δ is a point, then $\operatorname{cone}(\Delta) = K_{\ge 0}$. In particular, $D^0(\text{point}) = K$ is the only non-trivial D-space.

(iii) Let $\dim(\Delta) \ge 1$. Then, the defining complex for the $D^k(\Delta)$ looks like

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & \dots \longrightarrow & C^n & \longrightarrow & C^{n+1} & \longrightarrow\; 0 \\ & & \| & & \| & & & \| & & \| & \\ & & K^{n+1} & & \oplus_{a\in\Delta} K^{n+1}\!/K\cdot a & & & \oplus_{f<\Delta} K^{n+1}\!/\text{span}\, f & & 0 & \end{array}$$

with $a \in \Delta$ and $f < \Delta$ running through the vertices and facets of Δ, respectively. In particular, the injectivity of $C^0 \hookrightarrow C^1$ implies $D^0(\Delta) = 0$ and, by the previous lemma, $D^n(\Delta) = D^0(\Delta^\vee)^* = 0$.
Hence, $D^1(\Delta), \dots, D^{n-1}(\Delta)$ are the only non-trivial D-invariants of a polytope $\Delta \subseteq K^n$.

Denote by $f_j(\Delta)$ the number of j-dimensional faces of Δ with $f_{-1} := 1$, i.e. the Euler equation says $\sum_{j=-1}^{n}(-1)^j f_j = 0$. Then $\dim C^k = (n+1-k)\cdot f_{k-1}$.

$n = 2$:
The only non-trivial invariant is D^1 with $\dim D^1(\Delta) = -\dim C^0 + \dim C^1 - \dim C^2 = f_0(\Delta) - 3$.

$n = 3$:
D^1 and D^2 may be non-trivial with $\dim D^2(\Delta) - \dim D^1(\Delta) = \sum_k (-1)^k \dim C^k = f_2(\Delta) - f_0(\Delta)$.

(3.3) We would like to compare the D-invariants with Brion's Hodge spaces. First, there are the straightforward relations

$$D^k(\Delta) = H^k\big(\underline{\mathrm{cone}}(\Delta),\, K^{n+1}/\mathrm{span}\big) = H^{n+1-k,1}\big(\underline{\mathrm{cone}}(\Delta)\big)^*$$

and

$$D^k(\Delta) = D^{n-k}(\Delta^\vee)^* = H^{k+1,1}\big(\underline{\mathrm{cone}}(\Delta^\vee)\big)\,.$$

The D-invariants have also a direct description in terms of the normal fan $\Sigma(\Delta)$ of Δ.

Proposition: *Let $\Delta \subseteq K^n$ be a compact, convex polytope of dimension n and denote by $\Sigma(\Delta)$ its inner normal fan. Then, there is an exact sequence*

$$0 \to K \longrightarrow H^{1,1}\big(\Sigma(\Delta)\big) \longrightarrow D^1(\Delta) \to 0\,.$$

For the remaining indices $k \neq 1$ we have $H^{k,1}\big(\Sigma(\Delta)\big) = D^k(\Delta)$.

Proof: Assume that both Δ and $\Delta^\vee$ contain the origin as an interior point. Then, the projection $K^{n+1} \twoheadrightarrow K^n$ induces an isomorphism of fans $\pi : \partial\,\mathrm{cone}(\Delta^\vee) \to K_{\geq 0} \cdot \partial\Delta^\vee \cong \Sigma(\Delta)$. Moreover, we obtain the following diagram of cohomological systems:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & & K \arrow[r, "\sim"] \arrow[d] & K \arrow[d] & \\
\text{on the fan } \partial\,\mathrm{cone}(\Delta^\vee): & 0 \arrow[r] & \mathrm{span} \arrow[r] \arrow[d, "\sim"] & K^{n+1} \arrow[r] \arrow[d] & K^{n+1}/\mathrm{span} \arrow[r] \arrow[d] & 0 \\
\text{on the fan } \Sigma(\Delta): & 0 \arrow[r] & \mathrm{span} \arrow[r] & K^n \arrow[r] \arrow[d] & K^n/\mathrm{span} \arrow[r] \arrow[d] & 0 \\
 & & & 0 & 0 &
\end{tikzcd}$$

Since $H^{k,1}\big(\Sigma(\Delta)\big) = H^{n-k}\big(\Sigma(\Delta), K^n/\mathrm{span}\big)^*$ and

$$D^k(\Delta) = D^{n-k}(\Delta^\vee)^* = H^{n-k}\big(\underline{\mathrm{cone}}(\Delta^\vee), K^{n+1}/\mathrm{span}\big)^* = H^{n-k}\big(\partial\,\mathrm{cone}(\Delta^\vee), K^{n+1}/\mathrm{span}\big)^*,$$

the last column of the above diagram implies the long exact sequence

$$\ldots \to H^{n-k+1}(\partial\,\mathrm{cone}(\Delta^\vee), K)^* \to H^{k,1}\big(\Sigma(\Delta)\big) \to D^k(\Delta) \to H^{n-k}(\partial\,\mathrm{cone}(\Delta^\vee), K)^* \to \ldots\,.$$

On the other hand, by comparison with the cohomology groups $H^\bullet(\underline{\mathrm{cone}}(\Delta^\vee), K) = 0$, we obtain that $H^\bullet(\partial\,\mathrm{cone}(\Delta^\vee), K)$ is also trivial – with the only exception $H^n(\partial\,\mathrm{cone}(\Delta^\vee), K) = K$. $\square$

(3.4) It is well-known that the vector space $H^{1,1}\big(\Sigma(\Delta)\big)$ of Minkowski 1-weights is generated (as an abelian group) by the semi-group of Minkowski summands of $K_{\geq 0}$-multiples of Δ. It is useful to see this fact directly:

$H^{1,1}\big(\Sigma(\Delta)\big) = H^{n-1}\big(\Sigma(\Delta), K^n\!/\!\text{span}\big)^* = H_{n-1}\big(\Sigma(\Delta), (\bullet)^\perp\big)$ equals the kernel

$$\ker\Big[\oplus_{[\dim\sigma=n-1]}\,\sigma^\perp \longrightarrow \oplus_{[\dim\tau=n-2]}\tau^\perp\Big] = \ker\Big[\oplus_{d<\Delta}\,K\cdot d \longrightarrow \oplus_{[f<\Delta,\,\dim f=2]}\,\text{span}\,f\Big]$$

with $d \in K^n$ running through the edges of Δ. The latter space encodes Minkowski summands of $K_{\geq 0}\cdot\Delta$ just by keeping track of the dilatation factors of the Δ-edges, cf. [Al], Lemma (2.2).

Note that the trivial Minkowski summand Δ itself induces the element $\underline{1}$ in $H^{1,1}\big(\Sigma(\Delta)\big) \subseteq \oplus_{d<\Delta}K\cdot d$. It is exactly this element which is killed in the projection $H^{1,1}\big(\Sigma(\Delta)\big) \twoheadrightarrow D^1(\Delta)$ from the previous proposition.

Corollary: *Polytopes Δ with only triangles as two-dimensional faces have a trivial $D^1(\Delta)$. In particular, for simplicial three-dimensional polytopes, the only non-trivial D-invariant is $D^2(\Delta)$; it has dimension $f_0(\Delta)-4$.*

Proof: The first claim is clear. The dimension of $D^2(\Delta)$ for three-dimensional polytopes follows from (3.2), the Euler equation, and the fact that $3\,f_2(\Delta) = 2\,f_1(\Delta)$ if Δ is simplicial. □

Examples: 1) Since the icosahedron I is simplicial, one obtains $D^1(I)=0$ and $\dim D^2(I)=8$.
2) Consider three-dimensional pyramids P^m and double pyramids $\diamond^m$ over an m-gon; in both cases we have a trivial D^1 focusing again the interest on D^2. Whereas $D^2(P^m)$ is also trivial, we do have $\dim D^2(\diamond^m) = m-2$.

(3.5) We finish this chapter by an extension of the previous example. We denote by $\diamond(\Delta) \subseteq K^{n+1}$ the *double pyramid* over the polytope $\Delta \subseteq K^n$. On the polar level, this means that $\diamond(\Delta)^\vee = \Delta^\vee \times I$ with $I := [0,1] \subseteq K^1$.

Proposition: *The natural inclusion $D^1(\Delta^\vee) \hookrightarrow D^1(\Delta^\vee \times I)$ has a one-dimensional cokernel. For $k \geq 2$, there are isomorphisms $D^k(\Delta^\vee) \xrightarrow{\sim} D^k(\Delta^\vee \times I)$.*
Thus, the D-invariants of a double pyramid depend on those of the base via

$$D^k\big(\diamond(\Delta)\big) = D^{k-1}(\Delta) \;\; \textit{for}\;\; k \neq n \qquad \textit{and} \qquad \dim\, D^n\big(\diamond(\Delta)\big) = \dim\, D^{n-1}(\Delta) + 1\,.$$

Proof: Just to impress the reader, we are going to use the language of triangulated categories. The normal fan $\Sigma(\Delta^\vee \times I)$ can be easily expressed by $\Sigma(\Delta^\vee)$; if $N, S \in K^{n+1}$ denote the "poles" $\pm e^{n+1}$, then

$$\Sigma(\Delta^\vee \times I) = \Sigma(\Delta^\vee) \sqcup \Sigma^N(\Delta^\vee) \sqcup \Sigma^S(\Delta^\vee) \;\;\text{with}\;\; \Sigma^{N/S}(\Delta^\vee) := \big\{\langle\sigma, N/S\rangle \subseteq K^{n+1} \mid \sigma \in \Sigma(\Delta^\vee)\big\}.$$

Since the complex $C^\bullet\big(\Sigma(\Delta^\vee \times I),\, K^{n+1}\!/\!\text{span}\big)$ is isomorphic to the shifted mapping cone $C^\bullet_{(\pi,\pi)}[-1]$ with

$$\pi\,:\, C^\bullet\Big(\Sigma(\Delta^\vee),\, K^{n+1}\!/\!\text{span}\Big) \twoheadrightarrow C^\bullet\Big(\Sigma(\Delta^\vee),\, K^n\!/\!\text{span}\Big)$$

and $(\pi,\pi) : \bullet \to \bullet \oplus \bullet$, we obtain that $C^\bullet\big(\Sigma(\Delta^\vee \times I),\, K^{n+1}\!/\!\text{span}\big)[1]$ and $C^\bullet\big(\Sigma(\Delta^\vee),\, K\big)[1]$ are on top of the distinguished triangles over the maps (π,π) and π, respectively. Hence, the octahedral axiom for triangulated categories yields a new distinguished triangle

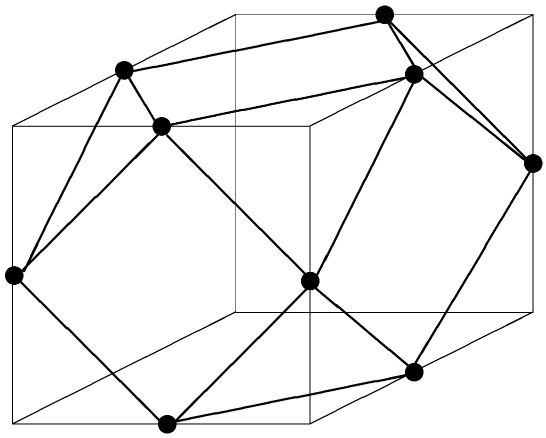
$$\begin{tikzcd}[column sep=small] & C^\bullet\Big(\Sigma(\Delta^\vee),\, K^n\!/\!\text{span}\Big) \arrow[dl, "{[1]}"'] & \\ C^\bullet\Big(\Sigma(\Delta^\vee),\, K\Big)[1] \arrow[rr] & & C^\bullet\Big(\Sigma(\Delta^\vee \times I),\, K^{n+1}\!/\!\text{span}\Big)[1] \arrow[ul] \end{tikzcd}$$

inducing the long exact sequence

$$\ldots \to H^{k,1}\big(\Sigma(\Delta^\vee \times I)\big)^* \to H^{k,1}\big(\Sigma(\Delta^\vee)\big)^* \to H^{n-k+2}\big(\Sigma(\Delta^\vee),\, K\big) \to H^{k-1,1}\big(\Sigma(\Delta^\vee \times I)\big)^* \to \ldots\,.$$

As already mentioned at the end of the proof of Proposition (3.3), the spaces $H^{n-k+2}\big(\Sigma(\Delta^\vee),\, K\big)$ vanish unless $k = 2$. Thus, it remains to use Proposition (3.3) itself, and to remark that the injection $D^1(\Delta^\vee) \hookrightarrow D^1(\Delta^\vee \times I)$ cannot be an isomorphism. $\square$

Example: Let Δ be the three-dimensional cuboctahedron obtained by cutting the eight corners of a cube.



Then, Δ may be decomposed into a Minkowski sum of two tetrahedra – the summands are formed by the triangles in every other corner. In particular, $\dim D^1(\Delta) = 1$. Moreover, since $f_0(\Delta) = 12$, $f_1(\Delta) = 24$, $f_2(\Delta) = 14$, Example (3.2) implies $\dim D^2(\Delta) = 3$.
The four-dimensional double pyramid $\diamond(\Delta)$ has two kinds of facets: 16 tetrahedra and 12 pyramids over squares. From the previous proposition, we obtain $D^1\big(\diamond(\Delta)\big) = 0$, $\dim D^2\big(\diamond(\Delta)\big) = 1$, and $\dim D^3\big(\diamond(\Delta)\big) = 4$.

4 The vanishing theorem

(4.1) Let $\Delta \subseteq K^n$ be an n-dimensional, compact, convex polytope with $n \geq 1$. We would like to find conditions under which some of the spaces $D^k(\Delta)$ vanish.
The first idea is to check Brion's properties (iii) and (iv) of (2.3) for this purpose. However, we do not find surprising results in this way – for instance the first two claims of the following proposition are already contained in [Br]. The third assertion generalizes the observation made in Corollary (3.4).

Proposition:

(1) *If Δ is a simple polytope, then $D^k(\Delta) = 0$ for $k \geq 2$. In particular, for the space of Minkowski summands we obtain* $\dim D^1(\Delta) = \sum_j (-1)^{j+1}\, j \cdot f_j(\Delta)$.

(2) *If each face of Δ contains at most one non-simple vertex, then we have still $D^2(\Delta) = 0$.*

(3) *If any ℓ-face of Δ is a simplex, then $D^k(\Delta) = 0$ for $k < \ell$.*

(A vertex of Δ is called *simple* if it sits in exactly $n = \dim \Delta$ different facets. Moreover, the whole polytope is said to be simple if every vertex is.)

Proof: The first two claims exploit the fact that $D^k(\Delta) = H^{k,1}\big(\Sigma(\Delta)\big)$ for $k \geq 2$. The dimension of $D^1(\Delta)$ follows from $\dim D^1(\Delta) = \sum_k (-1)^{k+1} \dim D^k(\Delta) = \sum_k (-1)^{k+1}(n+1-k)\cdot f_{k-1}$ as in (3.2).
On the other hand, the proof of the third assertion uses (2.3)(iii) for the "dual" fan $\Sigma(\Delta^\vee)$:
$D^k(\Delta) = D^{n-k}(\Delta^\vee)^* = H^{n-k,1}\big(\Sigma(\Delta^\vee)\big)^* = 0$ if $(n-k)-1 > n-(\ell+1)$. □

(4.2) More interesting results can be obtained from working with the spectral sequence introduced in (2.2). When applied to the "affine" fan $\Sigma := \underline{\text{cone}}(\Delta)$, it provides us with a nice tool for studying the spaces $D^k(\Delta)$ up to a certain bound $k < \ell$:

Write $M^i < \Delta$ for the ℓ-dimensional faces and denote by $\Delta^{(\ell)} := \cup_i M^i$ the ℓ-skeleton of our polytope Δ. Then, the simplicial complex $\mathcal{M}$ with $\mathcal{M}_p := \{i_0 \leq \ldots \leq i_p\}$ carries the cohomological system

$$\mathcal{D}^q : (i_0 \leq \ldots \leq i_p) \mapsto D^q\big(M^{i_0} \cap \ldots \cap M^{i_p}\big)\,.$$

Proposition: *There is a degenerating spectral sequence with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $E_2^{p,q} = H^p\big(\mathcal{M},\, \mathcal{D}^q\big) \Rightarrow D^{p+q}(\Delta)$ for $p+q < \ell$.*

Proof: Let $\Sigma := \underline{\text{cone}}(\Delta)^{(\ell+1)}$ be the union of cones with dimension at most $(\ell+1)$. Then, using the cohomological system

$$\mathcal{H}^q(\text{span cone}) : (i_0 \leq \ldots \leq i_p) \mapsto H^q\big(\text{cone}\, M^{i_0} \cap \ldots \cap \text{cone}\, M^{i_p},\, \text{span}\big)\,,$$

section (2.2) yields a degenerating spectral sequence with

$$E_2^{p,q} = H^p\big(\mathcal{M},\, \mathcal{H}^q(\text{span cone})\big) \Rightarrow H^{p+q}\big(\Sigma,\, \text{span}\big).$$

On the other hand, by (3.1) we have $H^q\big(\text{cone}\, M^{i_0} \cap \ldots \cap \text{cone}\, M^{i_p},\, \text{span}\big) = D^{q-1}\big(M^{i_0} \cap \ldots \cap M^{i_p}\big)$ and, for $p+q < \ell+1$, $H^{p+q}\big(\Sigma,\, \text{span}\big) = H^{p+q}\big(\underline{\text{cone}}(\Delta),\, \text{span}\big) = D^{p+q-1}(\Delta)$. Hence, an index shift by one completes the proof. □

(4.3) The cohomology groups $E_2^{p,q} = H^p\big(\mathcal{M},\, \mathcal{D}^q\big)$ remain unchanged when we build the complex $C^\bullet\big(\mathcal{M},\, \mathcal{D}^q\big)$ only from the strict tuples $(i_0 < \ldots < i_p)$. In particular, besides $E_2^{0,0} = 0$, we obtain at the first glance that $E_2^{p,q} = 0$ for $q \geq \ell$ or $p \geq 1\,, q = \ell - 1$. However, since the previous proposition restricts us to the region $p+q < \ell$ anyway, these first vanishings do not help.

For vertices $a \in \Delta$ we denote by $\Delta(a)$ the corresponding vertex figure; it is the polytope obtained by cutting Δ with a hyperplane sufficiently close to a. The faces of $\Delta(a)$ are exactly the vertex figures of those Δ-faces containing a.

Lemma: *Unless $p = \ell$, the vector spaces $E_2^{p,0}$ on the buttom row vanish. The remaining one may be expressed by singular cohomology groups with values in K as*

$$E_2^{\ell,0} = \oplus_{a \in \Delta\, \text{vertex}} H^\ell\big(\Delta(a),\, \Delta(a)^{(\ell-1)}\big) = \oplus_{a\in\Delta\,\text{vertex}} \widetilde{H}^{\ell-1}\big(\Delta(a)^{(\ell-1)}\big)$$

with $\Delta(a)^{(\ell-1)}$ denoting the $(\ell-1)$-skeleton of the vertex figure $\Delta(a)$.

Proof: According to the remark at the beginning of the present section (4.3), the vector space $E_2^{p,0} = H^p\big(\mathcal{M}, \mathcal{D}^0\big)$ is the p-th cohomology of the complex

$$0 \longrightarrow \oplus_{i_0} D^0(M^{i_0}) \longrightarrow \oplus_{i_0<i_1} D^0(M^{i_0} \cap M^{i_1}) \longrightarrow \oplus_{i_0<i_1<i_2} D^0(M^{i_0} \cap M^{i_1} \cap M^{i_2}) \longrightarrow \ldots$$

which will also be denoted by $\mathcal{D}^0$ (creating a slight abuse of notation). On the other hand, for any vertex $a \in \Delta$ we call $\mathcal{D}^0(a)$ the complex built similarly as $\mathcal{D}^0$, but using only those faces $M^i < \Delta$ containing a. Since D^0 is trivial unless its argument is a point, the canonical projection

$$\mathcal{D}^0 \twoheadrightarrow \oplus_{a\in\Delta} \mathcal{D}^0(a)$$

yields an isomorphism of complexes.
This splitting enables us to fix an arbitrary vertex $a \in \Delta$ and to forget about faces M^i which do not contain a. Then, using the vertex figures $M^i(a) < \Delta(a)$, the whole story may be translated into singular cohomology with values in K via

$$D^0\big(M^{i_0} \cap \ldots \cap M^{i_p}\big) = H^0\big(\Delta(a),\, M^{i_0}(a) \cap \ldots \cap M^{i_p}(a)\big).$$

Denoting by $C_q(\bullet)$ the singular q-chains, the Mayer-Vietoris spectral sequence yields

$$\begin{aligned} H^p\big(\mathcal{D}^0(a)\big) = H^p\Big(\Big[C_\bullet(\Delta(a))\Big/\textstyle\sum_i C_\bullet(M^i(a))\Big]^*\Big) &= H^p\big(\Delta(a),\, \cup_i M^i(a)\big) \\ &= H^p\big(\Delta(a),\, \Delta(a)^{(\ell-1)}\big) = \widetilde{H}^{p-1}\big(\Delta(a)^{(\ell-1)}\big), \end{aligned}$$

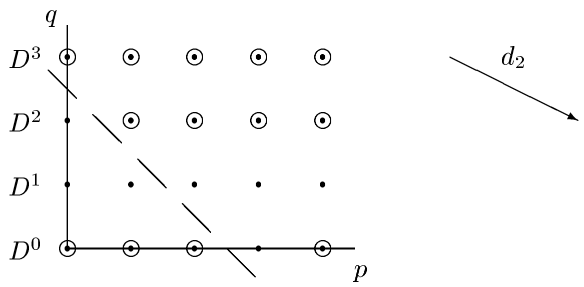
cf. (5.3) for more details. Now, the observation that the latter groups vanish unless $p = \ell$ finishes the proof. $\Box$

Corollary: *(1) Let $\ell \geq 2$. If any at most ℓ-dimensional face $M < \Delta$ satisfies $D^k(M) = 0$ for $0 < k < \ell$, then so does the polytope Δ itself.*
(2) If there is an $\ell \geq 2$ such that $D^1(M) = 0$ for every ℓ-face $M < \Delta$, then $D^1(\Delta) = 0$.

Proof: (1) This generalization of the last claim of Proposition (4.1) follows directly from the spectral sequence (4.2): The assumption means that $E_2^{p,q} = 0$ for $p + q < \ell$, $q \neq 0$, and the previous lemma takes care of the case $q = 0$.
(2) Here, the assumption translates into the vanishing $E_2^{0,1} = 0$. $\Box$

(4.4) For the rest of this chapter, we focus on the case $\ell = 3$, i.e. we would like to investigate $D^1(\Delta)$ and $D^2(\Delta)$ by studying the 3-dimensional faces of Δ. Here comes the actual situation of the second layer of our spectral sequence (the big circles stand for the vanishing of the corresponding E_2-term):

$E_2^{\bullet,\bullet}$ for $\ell = 3$



Proposition: *Denote by $M^i < \Delta$ the three-dimensional faces of Δ. Then*

(1) $D^1(\Delta) = \ker\Big[\oplus_i D^1(M^i) \longrightarrow \oplus_{i<j} D^1(M^i \cap M^j)\Big]$ *and*

(2) if $D^2(M^i) = 0$ for every i, then $D^2(\Delta) = \ker\Big[d_2 : E_2^{1,1} \longrightarrow E_2^{3,0}\Big]$.

Proof: The claims follow from $D^1(\Delta) = E_\infty^{0,1} = E_2^{0,1} = H^0\big(\mathcal{M}, \mathcal{D}^1\big)$ and, since $E_2^{0,2} = 0$ in (2), from $D^2(\Delta) = E_\infty^{1,1} = E_3^{1,1}$. $\Box$

(4.5) We are going to apply the previous properties to obtain an explicit description of $D^2(\Delta)$ by equations. In the following we will use the symbols a, V, M, F to denote Δ-faces of dimension 0, 2, 3, and 4, respectively.

Notation: Whenever (V, F) is a flag with dimension vector $(2, 4)$, then we denote by $M^{(V,F)}$ and $M_{(V,F)}$ the two unique three-dimensional faces sitting in between. Their order depends on the

orientation of the whole configuration.
For any two-dimensional face $V \leq \Delta$ we fix some three-dimensional face $M(V)$ containing V.

Theorem: *Assume that $D^1(M) = D^2(M) = 0$ for every three-dimensional face $M \leq \Delta$. Then, $D^2(\Delta) \subseteq K^{\#\{(0,2,3)-\text{flags}\}}$ is given by the following equations in the variables called $s(a, V, M)$:*

(1) If (a, F) is a flag with dimension $(0, 4)$, then

$$\sum_{a \in V \subseteq F} \Big[s(a, V, M^{(V,F)}) - s(a, V, M_{(V,F)}) \Big] = 0 \,. \qquad (1)_{(a,F)}$$

(2) For every flag (V, M), the coordinates $s(\bullet, V, M)$ provide an affine relation among the vertices of V, i.e.

$$\sum_{a \in V} s(a, V, M) \cdot [a, 1] = 0 \,. \qquad (2)_{(V,M)}$$

(3) Finally, for each $(0,2)$-flag (a, V), we simply have

$$s(a, V, M(V)) = 0 \,. \qquad (3)_{(a,V)}$$

Note that the equations $(2)_{(V,M)}$ imply that we can completly forget about the triangular faces V; they provide only trivial coordinates $s(\bullet, V, \bullet) = 0$.

The *proof of the previous theorem* consists of a detailed, but straightforward analysis of the differential map $d_2 : E_2^{1,1} \to E_2^{3,0}$. Since it is quite long and technical, we postpone these calculations to their own section §5. In the rest of §4, we continue with a discussion of the consequences and applications.

(4.6) **Corollary:** *If the polytope Δ is four-dimensional, then $D^2(\Delta) \subseteq K^{\#\{(0,2)-\text{flags}\}}$ is given by the easier equations*

(1) $\sum_{V \ni a} s(a, V) = 0$ for every vertex $a \in \Delta$, and

(2) $\sum_{a \in V} s(a, V) \cdot [a, 1] = 0$ for the two-dimensional faces $V \leq \Delta$.

Proof: Since $F = \Delta$, we may just set $M(V) := M_{(V,\Delta)}$ and $s(a, V) := s(a, V, M^{(V,\Delta)})$. □

Example: Consider the double pyramid $\diamond(\Delta)$ of Example (3.5). A non-trivial element of the one-dimensional $D^2\big(\diamond(\Delta)\big)$ may be obtained by assigning ± 1 to the vertices of each rectangle such that adjacent vertices obtain opposite signs.

(4.7) The main point of the present paper is to provide a vanishing theorem for $D^2(\Delta)$ for polytopes whose three-dimensional faces are not assumed to be simplices.

Definition: We define an inductive process of *"cleaning" vertices and two-dimensional faces* of Δ. At the beginning, all faces are assumed to be "contaminated", but then one may repeatedly apply the following rules (i) and (ii) in an arbitrary order:

(i) A two-dimensional m-gon $V < \Delta$ is said to be clean if at least $(m-3)$ of its vertices are so. (In particular, every triangle is automatically clean.)

(ii) A vertex of Δ is declaired to be clean if it is contained in no more than $(n-3)$ two-dimensional faces that are not cleaned yet.

Examples: (1) If no vertex of Δ is contained in more than $(n-3)$ two-dimensional, non-triangular faces, then every vertex and every two-dimensional face may be cleaned.

(2) Each vertex of the four-dimensional double pyramid $\diamond(\Delta)$ shown in Example (3.5) sits in exactly $2 = (n-3)+1$ quadrangular, two-dimensional faces. In particular, it is not possible to clean any of them at all.

Theorem: *Let Δ be an n-dimensional, compact, convex polytope such that every three-dimensional face is a pyramid. If every vertex (or, equivalently, every two-dimensional face) may be cleaned in the sense of the previous definition, then $D^2(\Delta) = 0$.*

Remarks: (1) Pyramids are the easiest three-dimensional solids with trivial D-invariants. Moreover, polytopes with only pyramids as three-dimensional faces do naturally arise from quivers, cf. [AvS] for more details.

(2) The double pyramid $\diamond(\Delta)$ from Example (3.5) has a non-trivial D^2. This shows that the assumption concerning the cleaning of vertices cannot be dropped.

Proof: Using the dictionary

$$\begin{array}{rcl} \text{“the vertex } a \text{ is clean”} & \longleftrightarrow & s(a,V,M)=0 \text{ for every } V,M \\ \text{“the 2-face } V \text{ is clean”} & \longleftrightarrow & s(a,V,M)=0 \text{ for every } a,M, \end{array}$$

the vanishing of $D^2(\Delta)$ is a consequence of Theorem (4.5) and the following two facts:

(i) If V is an m-gon, then, for any M, the equation $(2)_{(V,M)}$ of Theorem (4.5) says that the coordinates $s(\bullet,V,M)$ describe an $(m-3)$-dimensional vector space. Hence, if $(m-3)$ of them vanish, then they do all. In particular, as already mentioned in (4.5), we do not have to care about triangular faces V.

(ii) Assume that M^A, M^B are two pyramids with common facet $V < \Delta$.
We denote by $\Delta(V)$ the $(n-3)$-dimensional vertex figure of a slice of Δ transversal to V. In particular, the faces of $\Delta(V)$ correspond to those of Δ containing V: While $\bar{V} := V(V) = \emptyset$, the two pyramids turn into vertices $\bar{M}^A := M^A(V)$ and $\bar{M}^B := M^B(V)$. Moreover, any four-dimensional face $F < \Delta$ containing V corresponds to an edge $\bar{F}$ in $\Delta(V)$.
The important feature about pyramids as three-dimensional faces is the following: Any two non-triangular, two-dimensional faces of Δ span an at least four-dimensional space. Hence, for any two-dimensional V', different from V, there is *at most one* four-dimensional $F' < \Delta$ containing both V and V'.
Thus, if there are given $(n-4)$ (contaminated) faces V^k additional to V, then they induce at most $(n-4)$ four-dimensional faces F^k in this way. Since $\dim \Delta(V) = n-3$, this means that it is possible to find a path along the edges of $\Delta(V)$ connecting the vertices $\bar{M}^A$ and $\bar{M}^B$, but avoiding $\bar{F}^k$ $(k = 1, \dots, n-4)$.
Let us, w.l.o.g., assume that $\bar{M}^A$ and $\bar{M}^B$ are directly connected via an edge $\bar{F}$ with F not containing the $(n-4)$ faces $V^k \neq V$. Hence, $\bar{M}^A = M^{(V,F)}$, $\bar{M}^B = M_{(V,F)}$, and in the equation $(1)_{(a,F)}$ of Theorem (4.5)

$$\sum_{a \in \bullet \subseteq F} \left[s(a, \bullet, M^{(\bullet,F)}) - s(a, \bullet, M_{(\bullet,F)}) \right] = 0\,,$$

we automatically sum only over V itself and, additionally, over two-dimensional faces which are already clean. $\square$

5 The proof of the D^2-equations

Here, we present the proof of Theorem (4.5). It consists of a detailed, but straightforward analysis of the differential map $d_2 : E_2^{1,1} \to E_2^{3,0}$.

(5.1) *Describing $E_2^{1,1}$:*
According to the remark at the beginning of section (4.3), the vector space $E_2^{1,1} = H^1\big(\mathcal{M}, \mathcal{D}^1\big)$ equals the kernel

$$E_2^{1,1} = \ker \Big[\oplus_{i_0<i_1} D^1(M^{i_0} \cap M^{i_1}) \longrightarrow \oplus_{i_0<i_1<i_2} D^1(M^{i_0} \cap M^{i_1} \cap M^{i_2}) \Big] .$$

Denote by $V^1, \ldots, V^M$ the two-dimensional faces of Δ which are no triangles; each V^k is contained in some three-dimensional faces $M_k^0, \ldots, M_k^{N_k}$. Note that certain M's might occur in more than one of these lists. Nevertheless,

$$E_2^{1,1} \;\cong\; \oplus_{k=1}^M D^1\big(V^k\big)^{\oplus N_k}$$

with the i-th summand in $D^1\big(V^k\big)^{\oplus N_k}$ being identified with $D^1(M_k^0 \cap M_k^i)$; the remaining entries in $D^1(M_k^{i_0} \cap M_k^{i_1})$ may be obtained in the usual way as differences from those of $D^1(M_k^0 \cap M_k^{i_1})$ and $D^1(M_k^0 \cap M_k^{i_0})$. On the other hand, if the intersection $M^{i_0} \cap M^{i_1}$ is less than two-dimensional, then $D^1(M^{i_0} \cap M^{i_1}) = 0$, anyway.
We choose the special three-dimensional face $M(V^k)$ mentioned in (4.5) to be M_k^0.

(5.2) *Describing d_2:*
From (2.2) we recall that the double complex inducing the spectral sequence we are dealing with, looks as follows:

$$C^{p,q} = \oplus_{A \in [M^{i_0} \cap \ldots \cap M^{i_p}]_q} \operatorname{span}\big(\operatorname{cone}(A)\big) \quad \text{with} \quad d_I : C^{p,q} \to C^{p+1,q} \,,\; d_{II} : C^{p,q} \to C^{p,q+1} \,.$$

We fix one of the two-dimensional faces V^k and call it V. During (5.2), we abbreviate the three-dimensional faces $M_k^0, \ldots, M_k^{N_k}$ containing $V^k = V$ by $M^0, \ldots, M^N$. The index i will be reserved for these M^i, while $j \notin \{0, \ldots, N\}$ points to those three-dimensional faces $M^j < \Delta$ belonging *not* to this list.
Assume that V is an m-gon with vertices a^ν ($\nu \in \mathbb{Z}/m\mathbb{Z}$). Then, by (3.4), an element of $D^1(V)$ may be represented as an m-tuple $(t_1, \ldots, t_m) \in K^m$ with t_ν being the dilatation factor assigned to the edge $\overline{a^\nu a^{\nu+1}} < V$. In particular, we may start our tour through the double complex with an

$$x = (t^1, \ldots, t^N) \in D^1\big(V\big)^{\oplus N} \subseteq E_2^{1,1} \;\text{ with each } t^i \text{ represented as }\; t^i = (t_1^i, \ldots, t_m^i) \in K^m .$$

The corresponding element $x \in C^{1,1}$ looks like

$$x_{i_0 i_1}\big(\overline{a^\nu a^{\nu+1}}\big) = (t_\nu^{i_1} - t_\nu^{i_0}) \cdot \overrightarrow{a^\nu a^{\nu+1}} \in \operatorname{span}(a^\nu, a^{\nu+1}) \quad \text{with} \;\; t_\nu^0 := 0 \,,$$

and we have to walk through $C^{\bullet,\bullet}$ along the following path:

$$\text{Differential } d_2: \qquad \begin{array}{rccccl} & \odot & & \bullet & & \bullet \\ & \uparrow & & & & \\ x \in C^{1,1} & \bullet & \mapsto & \bullet & & \bullet \\ & & & \uparrow & & \\ & \bullet & & \bullet & \mapsto & \bullet \quad d_2(x) \in C^{3,0} \\ & & & y & & \end{array} \qquad\qquad \begin{array}{c} \xrightarrow{d_I} \\[2em] \big\uparrow d_{II} \end{array}$$

The components of the image $d_I(x) \in C^{2,1}$ vanish unless exactly two of the three indices belong to faces M^i containing V. In this case, we obtain

$$\begin{aligned}
d_I(x)_{i_0 i_1 j_2}\left(\overline{a^\nu a^{\nu+1}}\right) &= x_{i_1 j_2}\left(\overline{a^\nu a^{\nu+1}}\right) - x_{i_0 j_2}\left(\overline{a^\nu a^{\nu+1}}\right) + x_{i_0 i_1}\left(\overline{a^\nu a^{\nu+1}}\right) \\
&= x_{i_0 i_1}\left(\overline{a^\nu a^{\nu+1}}\right) \\
&= (t_\nu^{i_1} - t_\nu^{i_0}) \cdot \overrightarrow{a^\nu a^{\nu+1}}
\end{aligned}$$

if $(M^{i_0} \cap M^{i_1}) \cap M^{j_2} = V \cap M^{j_2} = \overline{a^\nu a^{\nu+1}}$.
Now, we lift this result to an element $y \in C^{2,0}$, i.e. we solve the equation $d_{II}(y) = d_I(x)$. Obviously, the following y does the job:

$$y_{i_0 i_1 j_2}(a^\nu) := \begin{cases} (t_{\nu-1}^{i_1} - t_{\nu-1}^{i_0}) \cdot a^\nu & \text{if } V \cap M^{j_2} = \overline{a^{\nu-1} a^\nu} \\ (t_\nu^{i_1} - t_\nu^{i_0}) \cdot a^\nu & \text{if } V \cap M^{j_2} = \overline{a^\nu a^{\nu+1}} \end{cases}$$

and $y_{\ldots}(\bullet) := 0$ for any other constellation. Its image $d_I(y) \in C^{3,0}$ asks for quadrupels (i_0, i_1, j_2, j_3) with still exactly two indices belonging to V-solids. Up to antisymmetric permutation of the four indices, we have

$$d_I(y)_{i_0 i_1 j_2 j_3}(a^\nu) = \begin{cases} (t_{\nu-1}^{i_1} - t_{\nu-1}^{i_0} - t_\nu^{i_1} + t_\nu^{i_0}) \cdot a^\nu & \text{if } V \cap M^{j_2} = \overline{a^\nu a^{\nu+1}}\,,\ V \cap M^{j_3} = \overline{a^{\nu-1} a^\nu} \\ (t_{\nu-1}^{i_1} - t_{\nu-1}^{i_0}) \cdot a^\nu & \text{if } V \cap M^{j_2} = \{a^\nu\}\,,\ V \cap M^{j_3} = \overline{a^{\nu-1} a^\nu} \\ -(t_\nu^{i_1} - t_\nu^{i_0}) \cdot a^\nu & \text{if } V \cap M^{j_2} = \overline{a^\nu a^{\nu+1}}\,,\ V \cap M^{j_3} = \{a^\nu\} \end{cases}$$

and zero for any other constellation. The element $d_I(y)$ represents the cohomology class

$$d_2(x) \in E_2^{3,0} = H^3\left(\mathcal{D}^0\right) = \oplus_{a \in \Delta} H^3\left(\mathcal{D}^0(a)\right).$$

Hence, the only non-trivial components $d_2(x)(a \in \Delta)$ occur for $a = a^\nu \in V$, and they look like $d_I(y)(a^\nu)$ in the formula above.

(5.3) *Transfer from $H^3\left(\mathcal{D}^0(a)\right)$ to singular cohomology:*
Let $a \in \Delta$ be an arbitrary vertex. As already indicated in the proof of Lemma (4.3), we have to use the Mayer-Vietoris spectral sequence to describe the isomorphism

$$H_2\left(\Delta(a)^{(2)}\right) = H_3\left(\Delta(a), \Delta(a)^{(2)}\right) \xrightarrow{\sim} H_3\left(\mathcal{D}_0(a)\right)$$

with $\mathcal{D}_0(a)$ meaning the complex built by the same recipe as $\mathcal{D}^0(a)$ in (4.3), but using homology

$$D_0\left(M^{i_0} \cap \ldots \cap M^{i_p}\right) := H_0\left(\Delta(a),\, M^{i_0}(a) \cap \ldots \cap M^{i_p}(a)\right)$$

instead of D^0. Denoting by $C_q(\bullet)$ the singular q-chains and abbreviating the vertex figures $M^i(a) < \Delta(a)$ simply by $\bar{M}^i < \bar{\Delta}$, we define

$$K_{p,q} := \oplus_{i_0 \le \ldots \le i_p} C_q(\bar{\Delta}) \big/ C_q(\bar{M}^{i_0} \cap \ldots \cap \bar{M}^{i_p}) \quad \text{with} \quad \begin{array}{rl} d_I: & K_{p,q} \to K_{p-1,q} \\ d_{II}: & K_{p,q} \to K_{p,q-1}\,. \end{array}$$

The spectral sequence obtained by taking the vertical homology first yields the complex $\mathcal{D}_0(a)$ as $E^1_{\bullet,0}$ and zero elsewhere. The other one, beginning with the horizontal homology, has $E^1_{0,\bullet}$ as the only entries at the first level. They form the complex $C_\bullet(\bar{\Delta}) \big/ \sum_i C_\bullet(\bar{M}^i)$ which is quasiisomorphic to $C_\bullet(\bar{\Delta}) \big/ C_\bullet(\cup_i \bar{M}^i)$.
Hence, the existence of the isomorphism promised above is clear. However, we have to understand

what the isomorphism really does with $[\bar F] \in H_3(\bar\Delta, \cup_i \bar M^i)$. To see this we chase $[\bar F]$ along the following diagram:

$$
\begin{array}{ccccccccc}
H_3(\bar\Delta, \cup_i \bar M^i) & & & & & & & & \\
\big| & & & & & & & & \\
C_3(\bar\Delta)/\Sigma_i C_3(\bar M^i) & \leftarrow & \oplus_{i_0} C_3(\bar\Delta)/C_3(\bar M^{i_0}) & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\cdots & \leftarrow & \oplus_{i_0} C_2(\bar\Delta)/C_2(\bar M^{i_0}) & \leftarrow & \oplus_{i_0 \le i_1} C_2(\bar\Delta)/C_2(\bar M^{i_0} \cap \bar M^{i_1}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \cdots & \leftarrow & \oplus_{i_0 \le i_1} C_1(\bar\Delta)/C_1(\bar M^{i_0} \cap \bar M^{i_1}) & \leftarrow & \oplus_{i_0 \le i_1 \le i_2} C_1(\bar\Delta)/C_1(\bar M^{\underline{i}}) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \cdots & \leftarrow & \oplus_{i_0 \le i_1 \le i_2} C_0(\bar\Delta)/C_0(\bar M^{\underline{i}}) & \leftarrow & \cdots \\
 & & & & & & \downarrow & & \\
 & & & & & & \cdots & \leftarrow & \cdots
\end{array}
$$

Fixing an arbitrary index $i_0 := 0$, we arrive at the third row with $[\partial \bar F] \in C_2(\bar\Delta)/C_2(\bar M^0)$. Let $\partial \bar F = \bar M^1 \cup \ldots \cup \bar M^l$ and assume that the orientation of the $\bar M^i$ is inherited from some orientation of $\bar F$. Then, a possible lift to the right is

$$
-[\bar M^i] \in C_2(\bar\Delta)\Big/C_2(\bar M^0 \cap \bar M^i), \quad i = 1, \ldots, l\,.
$$

Applying the vertical boundary operator and lifting again to the right, we obtain

$$
[\bar M^i \cap \bar M^j] \in C_1(\bar\Delta)\Big/C_1(\bar M^0 \cap \bar M^i \cap \bar M^j)
$$

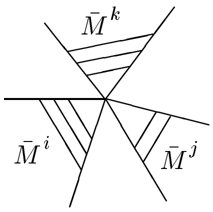
with $(\bar M^i, \bar M^j)$ running through the pairs of mutually adjacent faces of $\bar F$ with $i < j$. Our convention is that the edges $[\bar M^i \cap \bar M^j]$ inherit their orientation from the first argument, i.e. $[\bar M^i \cap \bar M^j] = -[\bar M^j \cap \bar M^i]$.

$$
\begin{array}{ccc}
\cdots \leftarrow \oplus_{i_0 \le i_1 \le i_2} C_1(\bar\Delta)/C_1(\bar M^{\underline{i}}) & & \\
\downarrow & & \\
\cdots \leftarrow \oplus_{i_0 \le i_1 \le i_2} C_0(\bar\Delta)/C_0(\bar M^{\underline{i}}) & \leftarrow & \oplus_{i_0 \le i_1 \le i_2 \le i_3} C_0(\bar\Delta)/C_0(\bar M^{\underline{i}}) \\
\downarrow & & \downarrow \\
\cdots & \leftarrow & \oplus_{i_0 \le i_1 \le i_2 \le i_3} H_0(\bar\Delta, \bar M^{\underline{i}}) \quad — \; H_3\big[\ldots \to \oplus_{\underline{i}} H_0(\bar\Delta, \bar M^{\underline{i}}) \to \ldots\big]
\end{array}
$$

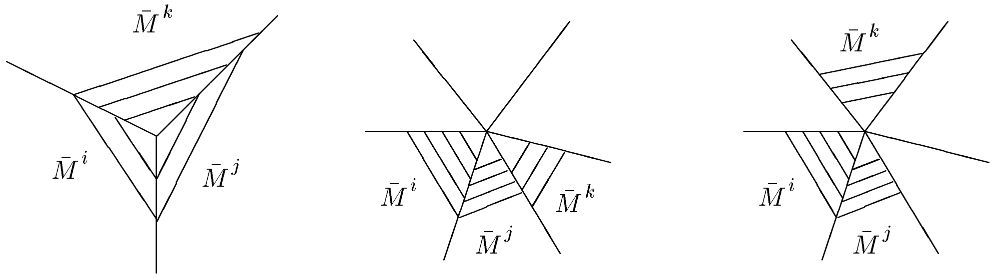
It is easy to apply the boundary operator to the edges $[\bar M^i \cap \bar M^j]$; but for doing the last lifting to $\oplus_{i_0 \le i_1 \le i_2 \le i_3} C_0(\bar\Delta)/C_0(\bar M^{\underline{i}})$, we have to introduce for each vertex $\bar X \in \bar F$ an auxillary function φ_X. Its arguments are triples of $\bar F$-facets containing $\bar X$, and it is determined by the following properties:

(i) $\varphi_X(\bar M^i, \bar M^j, \bar M^k)$ is antisymmetric in its arguments.

(ii) If any two of the arguments intersect only in $\{\bar X\}$, then $\varphi_X(\bar M^i, \bar M^j, \bar M^k) := 0$.



(iii) Denote by $u(\bar X)$ the number of two-dimensional $\bar F$-facets meeting in $\bar X$. Then, depending on $u(\bar X)$ and on the fact if there are "isolated" arguments or not, we distinguish between three cases:



$u(\bar X) = 3$ $\qquad\qquad\qquad\qquad$ $u(\bar X) \geq 4$

$\varphi_X(\bar M^i, \bar M^j, \bar M^k) := 1 \qquad \varphi_X(\bar M^i, \bar M^j, \bar M^k) := 2/u(\bar X) \qquad \varphi_X(\bar M^i, \bar M^j, \bar M^k) := 1/u(\bar X)$

Now, it is not difficult to check that a possible lifting of the tuple $\big(\partial[\bar M^i \cap \bar M^j]\big)_{i<j}$ to the vector space $\oplus_{i_0 \leq \dots \leq i_3} C_0(\bar\Delta)/C_0(\bar M^{\underline{i}})$ is given by

$$\varphi_X(\bar M^i, \bar M^j, \bar M^k) \cdot [\bar X] \;\in\; {C_0(\bar\Delta)}/{C_0(\bar M^0 \cap \bar M^i \cap \bar M^j \cap \bar M^k)}$$

with $(\bar M^i, \bar M^j, \bar M^k)$ running through all triples of facets of $\bar F$ with $\bar M^i \cap \bar M^j \cap \bar M^k = \{\bar X\}$ and $i < j < k$. The projection to $H_0(\bar\Delta, \bar M^0 \cap \bar M^i \cap \bar M^j \cap \bar M^k)$ does not change the shape of the element $\varphi_X(\bar M^i, \bar M^j, \bar M^k) \cdot [\bar X]$. However, if $\bar X \in \bar M^0$, then the element $[\bar X]$ spanning the whole homology group vanishes, anyway.
Note that the final result of the previous construction cannot depend on the choice of $\bar M^0$ made in the very beginning. In particular, one might exploit this freedom to take for $\bar M^0$ one of the $\bar F$-faces, or to do exactly the opposite.

(5.4) *Interpreting $d_2(x)(a^\nu)$ inside $H^3\big(\Delta(a^\nu), \Delta(a^\nu)^{(2)}\big)$:*
We apply the previous calculation to show how $d_2(x)(a^\nu)$ acts on a homology class $[\bar F] = [F(a^\nu)] \in H_3\big(\Delta(a^\nu), \Delta(a^\nu)^{(2)}\big)$ induced by a four-dimensional face $F < \Delta$ containing a^ν: Unless $V \subseteq F$, we have $\big\langle d_2(x)(a^\nu), [\bar F]\big\rangle = 0$. However, assuming $V \subseteq F$, then F contains exactly two faces M^i, M^k with common facet V, and the result is

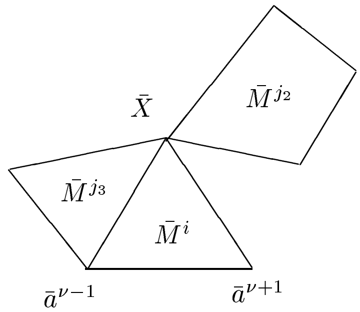
$$\big\langle d_2(x)(a^\nu), [\bar F]\big\rangle \;=\; \big(t^k_{\nu-1} - t^k_\nu\big) - \big(t^i_{\nu-1} - t^i_\nu\big)\,.$$

Proof: Using the notation of (5.2), we select the first V-solid $M^0 = M(V)$ as the face inducing the vertex figure $\bar M^0 < \bar\Delta = \Delta(a^\nu)$ being fixed in (5.3). Then, the only quadrupels having a chance to produce a non-trivial entry in both steps are

$[0, i, j_2, j_3]$ with

- $i \in \{0, \dots, N\}$ and $M^i,\, M^{j_2},\, M^{j_3} < F$
- $V \cap M^{j_2},\, V \cap M^{j_3} = \{a^\nu\},\ \overline{a^{\nu-1}a^\nu}$, or $\overline{a^\nu a^{\nu+1}}$
- $V \cap M^{j_2} \cap M^{j_3} = \{a^\nu\}$.

Focusing on the vertex figures $\bar V \leq \bar M^i \leq \bar F$ at a^ν, we see that $\bar M^i$ is a polygon with $\bar V = \overline{a^{\nu-1}a^{\nu+1}}$ as one of its edges. While $\bar V \cap \bar M^{j_2} \cap \bar M^{j_3} = \emptyset$, the result of (5.3) implies that the intersection $\bar M^i \cap \bar M^{j_2} \cap \bar M^{j_3}$ has to be some point $\bar X \neq \overline{a}^{\nu-1}, \overline{a}^{\nu+1}$.



Hence, fixing $\bar{M}^i$ and choosing the ordering of the Δ-faces and their corresponding indices well, we obtain contributions to $\langle d_2(x)(a^\nu), [\bar{F}]\rangle$ only from the arguments $\bar{M}^{j_2}$ and $\bar{M}^{j_3}$ running through the two-dimensional $\bar{F}$-faces fitting in one of the following cases:

(i) $\bar{M}^{j_2}$ has a common edge $\overline{a^{\nu+1}X}$ with $\bar{M}^i$. Then, if $\bar{M}^{j_3} \ni \bar{X}$ is adjacent to one of them, we obtain twice

$$d_2(x)_{0ij_2j_3}(a^\nu) \cdot \varphi_X(\bar{M}^i, \bar{M}^{j_2}, \bar{M}^{j_3}) \;=\; -t^i_\nu \cdot 2/u(\bar{X})\,.$$

Moreover, there are $\big(u(\bar{X}) - 4\big)$ possibilities such that $\bar{M}^{j_3} \ni \bar{X}$ is "isolated". Each constellation yields the contribution $-t^i_\nu \cdot 1/u(\bar{X})$.

(ii) $\bar{M}^{j_3}$ has a common edge $\overline{X a^{\nu-1}}$ with $\bar{M}^i$. Then, as in (i), we obtain twice $t^i_{\nu-1} \cdot 2/u(\bar{X})$ and $\big(u(\bar{X}) - 4\big)$-times $t^i_{\nu-1} \cdot 1/u(\bar{X})$.

(iii) If both $\overline{a}^{\nu+1} \notin \bar{M}^{j_2}$ and $\overline{a}^{\nu-1} \notin \bar{M}^{j_3}$, then the result of (5.2) shows that this case contributes nothing to $\langle d_2(x)(a^\nu), [\bar{F}]\rangle$.

Altogether, this adds up to $(t^i_{\nu-1} - t^i_\nu)$, and we should finally remark that the exceptional cases "$u(\bar{X}) = 3$" and "$\bar{M}^i$ is a triangle" yield the same result. In the latter situation, the cases (i) and (ii) might overlap. $(\square)$

Finally, we should remark that it is exactly the differences $t^i_{\nu-1} - t^i_\nu$ which are called $s(a^\nu, V, M^i)$ in (4.5). The equations $(2)_{(V,M)}$ of the theorem say nothing else than that these s-variables come from some t's satisfying the equations for Minkowski summands of V as mentioned in (3.4). $\square$

6 Applications to deformation theory

(6.1) Let N, M be two finitely generated, free abelian groups which are mutually dual; denote by $N_{\mathbb{R}}, M_{\mathbb{R}}$ the vector spaces obtained by extending the scalars. Each polyhedral, rational cone $\sigma \subseteq N_{\mathbb{R}}$ with apex in 0 gives rise to an affine toric variety $X_\sigma := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. It comes with an action of the torus $N_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}^* = \operatorname{Spec} \mathbb{C}[M]$, which leads to a stratification into orbits which are parametrized by the faces of σ. We refer to [Da] for more details.
In particular, the trivial face $\sigma \leq \sigma$ corresponds to a unique fixed point $\operatorname{orb}(\sigma) = 0$ of the torus action. It is the "most singular" point of X_σ, and we are going to study the deformation theory of the germ $(X_\sigma, 0)$.

The point that makes toric varieties so exciting is the fact that many algebro-geometric properties of X_σ (or its non-affine generalizations) translate directly into combinatorial properties of cones and their relation to the lattice structure $N \subseteq N_{\mathbb{R}}$. A first example of such a translation can be seen in (2.3). We will need in the future the following two further examples of such translations:

- X_σ is Gorenstein if and only if σ is the cone over a compact, convex lattice polytope $\Delta \subseteq \mathbb{R}^n$ sitting in an affine hyperplane of height one. This means that $N = \mathbb{Z}^n \times \mathbb{Z}$, and Δ is a polytope with vertices in $\mathbb{Z}^n \times \{1\}$.
- X_σ is, additionally, smooth in codimension two iff the edges of Δ do not contain any interior lattice points.

(6.2) If $X = \operatorname{Spec} A$ is an affine algebraic variety, then the cohomology of the cotangent complex produces A-modules T^k_X which play an important role in deformation theory: T^0_X describes infinitesimal automorphisms, T^1 describes infinitesimal deformations, and T^2_X contains the obstructions for extending deformations to larger base spaces. See [BC] for a nice survey, or [Lo]

for the details.
In the case that $X = X_\sigma$ is a toric variety, the ring $A = \mathbb{C}[\sigma^\vee \cap M]$ as well as the modules T^k_X are M-graded. It is possible to obtain combinatorial formulas for the homogeneous pieces $T^k_X(-R)$ with $R \in M$. This has been done in [AS], and we recall the result:

Assume we are given a rational, polyhedral cone $\sigma = \langle a^1, \dots, a^m \rangle \subseteq N_{\mathbb{R}}$ with $a^1, \dots, a^m \in N$ denoting its *primitive fundamental generators*, i.e. none of the a^ν is a proper multiple of an element of N. The dual cone is $\sigma^\vee := \{r \in M_{\mathbb{R}} \mid \langle \sigma, r \rangle \geq 0\} \subseteq M_{\mathbb{R}}$. For any degree $R \in M$ and face $\tau \leq \sigma$ we introduce a special subset of lattice points of $\sigma^\vee$:

$$K^R_\tau := \sigma^\vee \cap \big(R - \operatorname{int} \tau^\vee\big) \cap M \subseteq (\sigma^\vee \cap M)\,.$$

In particular, $K^R_0 = \sigma^\vee \cap M$, whereas K^R_σ consists of a finite set of lattice points. For an arbitrary subset $K \subset M$ we set:

$$\overline{\operatorname{Hom}}\,(K, \mathbb{C}) := \big\{ f : K \to \mathbb{C} \,\big|\, f(r) + f(s) = f(r+s) \;\text{ if }\; r, s, r+s \in K \big\}\,.$$

For each given $R \in M$, these sets give rise to a cohomological system $\overline{\operatorname{Hom}}\,\big(K^R_\bullet, \mathbb{C}\big)$ on the "affine" fan $\underline{\sigma}$.

Theorem: (cf. [AS], (5.3)) *For $k \leq 2$, one has*

$$T^k_X(-R) = H^k\big(\underline{\sigma},\, \overline{\operatorname{Hom}}\,(K^R_\bullet, \mathbb{C})\big)\,.$$

Moreover, if either $k \leq 1$, or if $k = 2$ and X_σ is Gorenstein in codimension two, then

$$T^k_X(-R) = H^k\big(\underline{\sigma},\, \operatorname{span}_{\mathbb{C}}(K^R_\bullet)^*\big)\,.$$

Remarks: 1) There is always a natural homomorphism of cohomological systems $\operatorname{span}_{\mathbb{C}}(K^R_\bullet)^* \to \overline{\operatorname{Hom}}\,(K^R_\bullet, \mathbb{C})$, but in general their cohomology groups are different. The second part of the theorem thus gives a condition under which we can replace the complicated system $\overline{\operatorname{Hom}}\,(K^R_\bullet, \mathbb{C}))$ by a slightly simpler system of *vector spaces.*

2) The module structure of T^k is the natural one: If $x^s \in \mathbb{C}[\sigma^\vee \cap M]$, then the multiplication with x^s is obtained from the map $T^k_X(-R) \to T^k_X(-R+s)$ provided by the inclusions $K^{R-s}_\tau \subseteq K^R_\tau$.

3) The property *Gorenstein in codimension two* translates into the following condition for the cone: For every two-dimensional face $\langle a^\nu, a^\mu \rangle < \sigma$ there is an $R_{\nu\mu} \in M$ with $\langle a^\nu, R_{\nu\mu} \rangle = \langle a^\mu, R_{\nu\mu} \rangle = 1$.

(6.3) Let $\Delta \subseteq \mathbb{R}^n$ be a lattice polytope; via $\sigma := \operatorname{cone}(\Delta)$ it gives rise to a *toric Gorenstein singularity* $X := X_\Delta$. For this special case, we are going to explain the relations between the vector spaces $T^k_X(-R)$ and the coarse D-invariants D^k defined in §3.

If $a^1, \dots, a^m \in \mathbb{Z}^n$ denote the vertices of Δ, then $a^\nu := (a^\nu, 1) \in N$ are the fundamental generators of σ. Moreover, there is a special degree $R^* := [\underline{0}, 1] \in M$; it recovers the polytope from the cone via $\Delta = \sigma \cap [R^* = 1]$.

Proposition: *Let Δ and $X := X_\Delta$ be as before. If $R \in M$ is a degree such that $R \leq 1$ holds everywhere on Δ, then $\Delta \cap [R = 1]$ is a face of Δ and*

$$T^k_X(-R) = D^k\Big(\Delta \cap [R = 1]\Big) \quad \textit{for} \quad k \leq 2.$$

Proof: The reader should convince her/himself from the fact that the property $R \leq 1$ in Δ imlpies

$$\operatorname{span}_{\mathbb{C}}(K^R_\tau) = \begin{cases} \tau^\perp & \text{if } \tau \leq \operatorname{cone}(\Delta \cap [R = 1]) \\ 0 & \text{otherwise}\,. \end{cases}$$

The claim then follows from Theorem (6.2). $\square$

(6.4) It is possible to describe $T^1_X(-R)$ in the Gorenstein case also for degrees with $R \not\leq 1$ on Δ. However, in the following three sections of the present paper, we look for sufficient conditions forcing $T^1_X(-R)$ and $T^2_X(-R)$ to vanish for those R.

Assume that $\sigma = \langle a^1, \ldots, a^m \rangle \subseteq N_{I\!R}$ is a rational, polyhedral cone as in (6.2). For any degree $R \in M$, we define another homological system $V^R_\bullet \supseteq \operatorname{span}_{\mathbb{C}}(K^R_\bullet)$ on $\underline{\sigma}$ by

$$V^R_\tau := \bigcap_{a^\nu \in \tau} V^R_{a^\nu} \qquad \text{with} \qquad V^R_{a^\nu} := \operatorname{span}_{\mathbb{C}}(K^R_{a^\nu}) = \begin{cases} M_{\mathbb{C}} & \text{if } \langle a^\nu, R\rangle \geq 2 \\ (a^\nu)^\perp & \text{if } \langle a^\nu, R\rangle = 1 \\ 0 & \text{if } \langle a^\nu, R\rangle \leq 0\,. \end{cases}$$

Let X_σ be *smooth in codimension two*, i.e. whenever $\langle a^\nu, a^\mu\rangle < \sigma$ is a two-dimensional face, then the set $\{a^\nu, a^\mu\}$ may be extended to a $\mathbb{Z}$-basis of the whole lattice N. In particular, for any $R \in M$, we have $V^R_{\langle a^\nu, a^\mu\rangle} = \operatorname{span}_{\mathbb{C}}(K^R_{\langle a^\nu, a^\mu\rangle})$ for these faces. Hence, for X_{sigma} smooth in codimension two one has

$$T^1_X(-R) = H^1\big(\underline{\sigma},\, (V^R_\bullet)^*\big).$$

Definition: If X_σ is smooth in codimension two, then we define the *local contribution* of a three-dimensional face $\tau \leq \sigma$ to $T^2_X(-R)$ as

$$T^2_{\tau,\,\mathrm{loc}}(-R) \;:=\; \left({V^R_\tau}\Big/{\operatorname{span}_{\mathbb{C}} K^R_\tau} \right)^* = \left({\bigcap_{a^\nu \in \tau} \big(\operatorname{span}_{\mathbb{C}} K^R_{a^\nu}\big)}\Big/{\operatorname{span}_{\mathbb{C}}\big(\bigcap_{a^\nu\in\tau} K^R_{a^\nu}\big)} \right)^* .$$

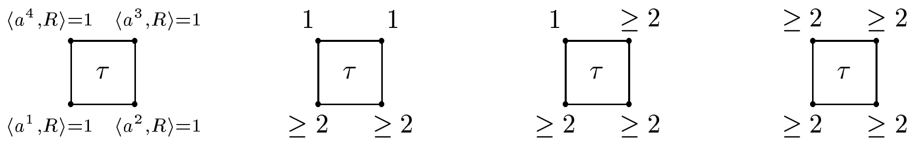
If $\dim \sigma = 3$ itself, then Theorem (6.2) tells us that $T^2_X(-R) = T^2_{\sigma,\,\mathrm{loc}}(-R)$. Moreover, for the general case, we obtain the straightforward

Proposition: *Let X_σ be smooth in codimension two. If there are no local contributions from three-dimensional faces to $T^2_X(-R)$ (i.e. if T^2_X sits in codimension at least four), then*

$$T^2_X(-R) = H^2\big(\underline{\sigma},\, (V^R_\bullet)^*\big)\,.$$

Application: If the three-dimensional faces of σ are either smooth (generated by a part of a $\mathbb{Z}$-basis of N) or isomorphic to cones over unit squares in $\mathbb{Z}^2$, then X_σ is a *conifold* in codimension three, i.e. it is smooth in codimension two and has at most A_1-singularities in codimension three. In particular, for thoses cones, the assumption of the previous proposition is satisfied for every multidegree $R \in M$.

Example: To get some familiarity with the sets K^R_τ, we explain the vanishing of the local contributions for conifolds on the combinatorial level.. Let τ be the cone over a unit square. Unless R is positive at the four vertices of this square, the space $\bigcap_{a^\nu\in\tau}\big(\operatorname{span}_{\mathbb{C}} K^R_{a^\nu}\big)$ vanishes, anyway. Now, focusing on these four positive values, there are only the following possibilities:



For these four cases we get:

- $\bigcap_{a^\nu\in\tau}\big(\operatorname{span}_{\mathbb{C}} K^R_{a^\nu}\big) = \tau^\perp = \operatorname{span}_{\mathbb{C}}\big(\bigcap_{a^\nu\in\tau} K^R_{a^\nu}\big)$,

- $\bigcap_{a^\nu\in\tau}\big(\operatorname{span}_{\mathbb{C}} K^R_{a^\nu}\big) = (a^3,a^4)^\perp = \tau^\perp + \begin{array}{|cc|}0 & 0\\ 1 & 1\end{array} = \operatorname{span}_{\mathbb{C}}\big(\bigcap_{a^\nu\in\tau} K^R_{a^\nu}\big)$,
- $\bigcap_{a^\nu\in\tau}\big(\operatorname{span}_{\mathbb{C}} K^R_{a^\nu}\big) = (a^4)^\perp = \tau^\perp + \begin{array}{|cc|}0 & 0\\ 1 & 1\end{array} + \begin{array}{|cc|}0 & 1\\ 0 & 1\end{array} = \operatorname{span}_{\mathbb{C}}\big(\bigcap_{a^\nu\in\tau} K^R_{a^\nu}\big)$, and
- $\bigcap_{a^\nu\in\tau} K^R_{a^\nu}$ contains $\begin{array}{|cc|}1 & 1\\ 0 & 0\end{array}$, $\begin{array}{|cc|}1 & 0\\ 1 & 0\end{array}$, and $\begin{array}{|cc|}0 & 0\\ 1 & 1\end{array}$; hence $\operatorname{span}_{\mathbb{C}}\big(\bigcap_{a^\nu\in\tau} K^R_{a^\nu}\big) = M_{\mathbb{C}}$.

So indeed $T^2_{\tau,\mathrm{loc}}(-R)=0$ for all R.

(6.5) Since we have related $T^k_X(-R)$ to the cohomolgy groups of $C_\bullet\big(\underline{\sigma}, V^R_\bullet\big)$, we are going to show the exactness of this complex for the degrees in question. Let us begin with a topological lemma stating the contractibility of certain subcomplexes of polytopes.

Lemma: *Let $\Delta\subseteq \mathbb{R}^n$ be a polytope. For a hyperplane $H\subseteq\mathbb{R}^{n+1}$ and any subfan $\mathcal{C}\subseteq\big\{\tau\le \operatorname{cone}(\Delta)\,\big|\,\tau\subseteq H\big\}$, we define*

$$\underline{\operatorname{cone}}(\Delta)^{H,\mathcal{C}} := \big\{\tau\le\operatorname{cone}(\Delta)\,\big|\,\tau\subseteq H^+ \text{ and } \tau\cap H\in\mathcal{C}\big\}$$

with H^+ denoting a closed half space corresponding to H. Then, if $\Delta\cap\operatorname{int}(H^+)$ is non-empty, the constant cohomological system is acyclic, i.e.

$$H^\bullet\Big(\underline{\operatorname{cone}}(\Delta)^{H,\mathcal{C}},\,\mathbb{Z}\Big)=0\,.$$

Proof: We have to check that the corresponding polyhedral subcomplex $\Delta^{H,\mathcal{C}}\subseteq\Delta$ is contractible. But this is a consequence of the following two points:

(i) $\Delta_{H,\mathcal{C}} := \Delta\cap\big[\operatorname{int}(H^+)\cup|\mathcal{C}|\big]\ \subseteq\Delta$ is star shaped, hence contractible.

(ii) We use the general fact that, if Q is a polytope and $\widetilde{H}^+$ is a subset of the closed halfspace H^+ containing $\operatorname{int}(H^+)$ with $Q\not\subseteq\widetilde{H}^+$, then $\partial Q\cap\widetilde{H}^+$ is a deformation retract of $Q\cap\widetilde{H}^+$. This enables us to successively get rid of "damaged" Δ-faces contained in $\Delta_{H,\mathcal{C}}$. In the end we get that $\Delta^{H,\mathcal{C}}$ is a deformation retract of $\Delta_{H,\mathcal{C}}$. $\square$

(6.6) We return to the situation of (6.3) and (6.4), i.e. $\Delta\subseteq\mathbb{R}^n$ is a lattice polytope giving rise to the Gorenstein cone $\sigma:=\operatorname{cone}(\Delta)\subseteq N_{\mathbb{R}}=\mathbb{R}^{n+1}$.

Proposition: *If $R\in M$ is a degree such that $R\not\le 1$ on Δ, then the complex induced by the homological system $V^R_\bullet$ is exact.*

Proof: The degree $R\in M$ induces a subfan

$$\underline{\operatorname{cone}}(\Delta)^{[R\ge1]} := \{\tau\le\operatorname{cone}(\Delta)\,|\,\langle a^\nu,R\rangle\ge1\ \text{ for every } a^\nu\in\tau\}\ \subseteq\underline{\operatorname{cone}}(\Delta).$$

For every $\tau\in\underline{\operatorname{cone}}(\Delta)^{[R\ge1]}$, we write $\tilde\tau\le\tau$ for the face spanned only by those generators $a^\nu\in\tau$ satisfying $\langle a^\nu,R\rangle=1$. The homological system $V^R_\bullet$ can more conveniently be described as

$$V^R_\tau=\begin{cases}\tilde\tau^\perp\subseteq M_{\mathbb{C}} & \text{if } \tau\in\underline{\operatorname{cone}}(\Delta)^{[R\ge1]}\\ 0 & \text{otherwise.}\end{cases}$$

We construct a homotopy between 0 and the identical map $\mathrm{id} : (V_\bullet^R)^* \to (V_\bullet^R)^*$. Hence, denoting by $\mathbb{Z}\big[\underline{\mathrm{cone}}(\Delta)_i^{[R\geq 1]}\big]$ the free abelian group generated by the k-dimensional cones, it remains to construct a homotopy

$$\begin{tikzcd} \mathbb{Z}\big[\underline{\mathrm{cone}}(\Delta)_{k+1}^{[R\geq 1]}\big] \arrow[r, "\partial"] \arrow[d, "\mathrm{id}"'] & \mathbb{Z}\big[\underline{\mathrm{cone}}(\Delta)_{k}^{[R\geq 1]}\big] \arrow[r, "\partial"] \arrow[d, "\mathrm{id}"'] \arrow[dl, "D^k"'] & \mathbb{Z}\big[\underline{\mathrm{cone}}(\Delta)_{k-1}^{[R\geq 1]}\big] \arrow[d, "\mathrm{id}"'] \arrow[dl, "D^{k-1}"'] \\ \mathbb{Z}\big[\underline{\mathrm{cone}}(\Delta)_{k+1}^{[R\geq 1]}\big] \arrow[r, "\partial"] & \mathbb{Z}\big[\underline{\mathrm{cone}}(\Delta)_{k}^{[R\geq 1]}\big] \arrow[r, "\partial"] & \mathbb{Z}\big[\underline{\mathrm{cone}}(\Delta)_{k-1}^{[R\geq 1]}\big] \end{tikzcd}$$

such that $D^k(\tau) \in \mathbb{Z}\big[\underline{\mathrm{cone}}(\Delta)_{k+1}^{[R\geq 1]}\big]$ may be written as $D^k(\tau) = \sum_v \lambda_v \phi_v$ with $\lambda_v \in \mathbb{Z}$ and $\phi_v \in \underline{\mathrm{cone}}(\Delta)^{[R\geq 1]}$ such that $\tilde{\phi} \subseteq \tilde{\tau}$:
Assume that D^{k-1} has been already constructed. If $\tau \in \underline{\mathrm{cone}}(\Delta)^{[R\geq 1]}$ is an k-dimensional cone, then we can apply Lemma (6.5) with $H := [R = 1]$ and $\mathcal{C}$ being the fan consisting of $\tilde{\tau}$ and its faces. Since $\big(\tau - D^{k-1}(\partial\tau)\big) \in \underline{\mathrm{cone}}(\Delta)^{H,\mathcal{C}}$ and

$$\partial\big(\tau - D^{k-1}(\partial\tau)\big) \;=\; \partial\tau - (\partial \circ D^{k-1})(\partial\tau) \;=\; (D^{k-2}\circ\partial)(\partial\tau) \;= 0,$$

there exists an element $D^k(\tau) \in \underline{\mathrm{cone}}(\Delta)_{k+1}^{H,\mathcal{C}}$ such that $\partial D^k(\tau) = \tau - D^{k-1}(\partial\tau)$. □

As a straightforward consequence of the Propositions (6.3), (6.4), (6.6) and of Theorem (4.7) we obtain the following

Theorem: *Assume that the two-dimensional faces of Δ are either squares or triangles with area 1 and 1/2, respectively, i.e. X_Δ is a conifold in codimension three. Then, if $R \in M$ is any degree, we have for $k \leq 2$*

$$T_X^k(-R) = \begin{cases} D^k\big(\Delta \cap [R=1]\big) & \text{if } R \leq 1 \text{ on } \Delta \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if Δ additionally satisfies the cleaning condition (4.7) and contains only pyramids as three-dimensional faces, then $T_X^2 = 0$.

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