Virtual intersection numbers

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August 1, 2000

Dedicated to H. Kurke

Abstract

We attempt to present the intersection theory which is required to understand the work of Kontsevich and Manin. Finally, we repeat their computations of intersection numbers in a concrete example. To do so, we study the moduli stack of stable maps of degree two from rational curves to $\mathbb{P}^1$. We show that its Picard group is infinite cyclic. We give an étale map from M to $\mathbb{P}^2$ of degree $\frac{1}{2}$. Eventually, we compute an intersection number which arises in Kontsevich's computation of the number of rational curves on the quintic threefold.

Introduction

The computation of the (expected) numbers of rational curves on a general quintic threefold in $\mathbb{P}^4$ is an important result in algebraic geometry. One of the main tools for these calculations are moduli stacks of stable maps of curves to a target space. We have to compute certain intersection numbers on these moduli stacks. It turns out that these intersection numbers give only the virtual number of rational curves on a quintic threefold. Using residual intersection theory we can compute the real numbers from these virtual numbers.
In section 1 we give the results on intersection theory we need in the sequel. We show how well known numbers can be computed using residual intersection theory in section 2. In the next section we give the results of Kontsevich and Manin. The main object of this article is to study the moduli stack M of stable maps of degree two from rational curves to $\mathbb{P}^1$. However, in section 4 we first study an example of a smooth stack N of dimension one. We will see later on that this stack N shares many features with our moduli stack M. We included this example, because the computations are less technical in this case. Finally, we come in section 5 to the moduli stack M. We compute its Picard group, the canonical line bundle K_M, an étale map from M to $\mathbb{P}^2$, and last but not least the intersection number a_2.
None of the presented results is new. The author is responsible for all mistakes and inaccuracies.

1 Preliminaries

1.1 Residual intersection theory

The standard reference for intersection theory we follow here is W. Fulton's book [1]. Let X be a smooth variety of dimension n over $\mathrm{Spec}(k)$. Furthermore, let E be a X-vector bundle with $\mathrm{rk}(E) = n = \dim(X)$. The vanishing locus $V(s)$ of a global section $s \in H^0(E)$ is locally given by n equations. Thus, we expect it to be of codimension n which means of dimension zero. If this is the case, the Euler characteristic $\chi(\mathcal{O}_{V(s)})$ is the n-th Chern number of the vector bundle E. However, in many situations there arise components of higher dimension in $V(s)$. Since their expected dimension is zero, we say that the virtual dimension of $V(s)$ is zero. Let Z be

a connected component of $V(s)$, we want to know what is its contribution $(E,s)^Z$ to the n-th Chern number $\int_X c_n(E)$? For simplicity, we assume that Z is regularly embedded. The answer is given by residual intersection theory (cf. [1], §9):

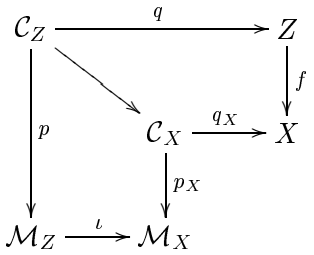
$$(E,s)^Z = \int_Z \big(c(E|_Z)\cdot c^{-1}(N_{Z|X})\big)$$

where $N_{Z|X}$ denotes the normal bundle of Z in X, and $c(-)$ denotes the total Chern class of a vector bundle. Suppose we have $V(s) = \coprod_{j=1}^k Z_j$, then we have

$$\int_X c_n(E) = \sum_{j=1}^k (E,s)^{Z_j}\,.$$

1.2 The normal bundle of smooth substacks

We suppose the reader to be familiar with the theory of moduli spaces of stable curves. A good introduction to this subject is the article of W. Fulton and R. Pandharipande [2]. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{M}_X$ and $\mathcal{M}_Z$ be the moduli stacks of curves with given numerical invariants in X and Z respectively. As usual, we denote the universal curves with $\mathcal{C}_X$, and $\mathcal{C}_Z$ (respectively). We have the following diagram



In the sequel we need the normal bundle $N_{\mathcal{M}_Z|\mathcal{M}_X}$ of the closed subscheme $\mathcal{M}_Z$ of $\mathcal{M}_X$. For simplicity we assume Z is regularly embedded in X and both moduli stacks are smooth of the expected dimension. From the sequence $0 \to T_Z \to f^*T_X \to N_{Z|X} \to 0$, and the fact that the tangent bundle $T_{\mathcal{M}_X}$ of $\mathcal{M}_X$ is given by $(p_X)_* q_X^* T_X$ (analogously for $T_{\mathcal{M}_Z}$) we derive the normal bundle equation:

$$N_{\mathcal{M}_Z|\mathcal{M}_X} = p_* q^* N_{Z|X}\,.$$

2 First examples

Here we consider two examples where we can illustrate the above techniques. They provide nice and applicable tools for many problems in intersection theory. To be honest, this section is included as a compensation for the lack of explicit computations in the next section. However, it might be interesting to see how well-known numbers are computed in the virtual way.

2.1 Quadrics in $\mathbb{P}^2$ tangent to 5 lines

We want to answer the following question: How many quadrics in $\mathbb{P}^2$ are tangent to 5 lines $\{l_i\}_{i=1\ldots5}$ in general position (i.e., through any point in $\mathbb{P}^2$ there pass at most two of them)? See also [1], Example 9.1.8.

We easily see that the space of all quadrics is a $\mathbb{P}^5$. The discriminant of a quadric on a line $l_i \cong \mathbb{P}^1$ is a quadratic form in the coefficients. Hence, the quadrics tangent to l_i form a divisor D_i in $\mathbb{P}^5$ of degree 2. Thus, our task is just to determine the intersection $D_1 \cap \ldots \cap D_5$. Bezout's

theorem persuades us to expect that there are $32 = 2^5$ quadrics tangent to all l_i. Unfortunately, the family Z of double counting lines is contained in the intersection $D_1 \cap \ldots \cap D_5$. Obviously, Z is the Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$. The five divisors D_i define a global section of the vector bundle $E := \mathcal{O}_{\mathbb{P}^5}(2)^{\oplus 5}$. From Bezout's theorem, we obtain its fifth Chern number $\int_X c_5(E) = 32$. Using the residual intersection theory, we can compute the contribution of Z to the intersection number 32:

$$\begin{aligned}(E,s)^Z &= \int_Z c(E|_Z)\cdot c(N_{Z|X})^{-1} \\ &= \int_Z (1+4h)^5\cdot(1+h)^3(1+2h)^{-6} \\ &= \int_Z 1+11h+31h^2 = 31.\end{aligned}$$

Here h denotes the hyperplane class on $Z \cong \mathbb{P}^2$. Hence, there is only one smooth quadric tangent to five lines in general position. It should be remarked that considering the dual situation this can be easily derived.

2.2 Virtual number of lines on a cubic in $\mathbb{P}^3$

Here we try to compute the number of lines on a cubic. We took this example because the reader presumably knows the answer.
Let X be the grassmannian $G(1,3)$, i.e., the space of all lines in $\mathbb{P}^3$. Let $U(1,3)$ be the universal curve over $G(1,3)$. We look at the following diagram:

$$G(1,3) \xleftarrow{p} U(1,3) \xrightarrow{q} \mathbb{P}^3 \ .$$

We have to consider the vector bundle $E = p_*q^*\mathcal{O}(3)$ on $G(1,3)$. Since $\mathrm{rk}(E) = \dim(G)$ and E is globally generated[1] a general global section s of E vanishes in $\int_{G(1,3)} c_4(E)$ different points. At the other hand, a global section s of E corresponds to a global section $\tilde{s}$ of the $\mathbb{P}^3$-line bundle $\mathcal{O}(3)$. The vanishing locus $V(s)$ consists of those lines which are on the divisor $V(\tilde{s}) \subset \mathbb{P}^3$. We could compute the number $\int_{G(1,3)} c_4(E)$ directly, but we want to practice residual intersection theory. We regard a concrete cubic in $\mathbb{P}^3$ by taking a union of a plane H with a smooth quadric Q intersecting each other in a smooth conic.
The lines contained in $Q \cup H$ are in three disjoint families. The family of dimension two of lines in H, and the two one-dimensional families of lines in $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$.
Let us compute the contribution of $\mathbb{P}^1 \subset X$ parameterizing one ruling of $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. To do so, we have to consider the following situation:

$$\mathbb{P}^1 \xleftarrow{f} Q \xrightarrow{g} \mathbb{P}^3 \ .$$

It turns out that we need only the following facts:

$$\begin{aligned} f_*g^*\mathcal{O}_{\mathbb{P}^3}(2) &\cong \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3} \\ f_*g^*\mathcal{O}_{\mathbb{P}^3}(3) &\cong \mathcal{O}_{\mathbb{P}^1}(3)^{\oplus 4}\,. \end{aligned}$$

[1] To see that E is globally generated we consider the exact sequence $0 \to \ker(\psi) \to H^0(\mathcal{O}(3)) \to \mathcal{O}(3) \to 0$ on $\mathbb{P}^3$. Applying the functor p_*q^* to this sequence yields an exact sequence on $G(1,3)$ as long as $R^1p_*q^*\ker(\psi)$ vanishes. To see that this coherent sheaf vanishes it is enough to show that the restriction of $\ker(\psi)$ to any line l does not contain a direct summand of type $\mathcal{O}_l(a)$ with $a \leq -2$. This is an easy computation.

Thus, we have $c(N_{\mathbb{P}^1|G(1,3)}) = (1+2h)^3$ with h the divisor class of a point. Since $h^2 = 0$, we have $c(N_{\mathbb{P}^1|G(1,3)})^{-1} = 1-6h$. Eventually, we compute the contribution of the component $\mathbb{P}^1$ of $G(1,3)$ to be

$$(E,s)^{\mathbb{P}^1} = \int_{\mathbb{P}^1} (1+3h)^4 \cdot (1-6h) = \int_{\mathbb{P}^1} (1+6h) = 6\,.$$

Now we compute the contribution of the lines on a plane $\mathbb{P}^2 \subset \mathbb{P}^3$. These lines are parameterized by the so called dual projective plane which we denote by $\mathbb{P}_2$. The universal family is denoted by $\mathcal{C}$. Thus, we have the following diagram:

$$\mathbb{P}_2 \xleftarrow{f} \mathcal{C} \xrightarrow{g} \mathbb{P}^2 \hookrightarrow \mathbb{P}^3$$

From the Euler sequence we see that

$$f_* g^* \mathcal{O}_{\mathbb{P}^2}(1) = T_{\mathbb{P}_2}(-1) \text{ and } c(T_{\mathbb{P}_2}(-1))^{-1} = 1-h\,,$$

with h the hyperplane class in $\mathbb{P}_2$. Thus, we have $c(T_{\mathbb{P}_2}(-1)) = 1+h+h^2$. The $\mathbb{P}_2$-vector bundle $f_* g^* \mathcal{O}_{\mathbb{P}^2}(3)$ is isomorphic to the threefold symmetric product $S^3(T_{\mathbb{P}_2}(-1))$. This allows an direct computation of its total Chern class using Chern roots. Indeed, let α and β be the Chern roots of $T_{\mathbb{P}_2}(-1)$, i.e., $c(T_{\mathbb{P}_2}(-1)) = (1+\alpha)(1+\beta)$. This implies $\alpha+\beta = h$ and $\alpha\cdot\beta = h^2$. Consequently, the Chern roots of $S^3(T_{\mathbb{P}_2}(-1))$ are given by 3α, $2\alpha+\beta$, $\alpha+2\beta$, and 3β. Hence, we compute the total Chern $c(S^3(T_{\mathbb{P}_2}(-1)))$ class to be

$$\begin{aligned} c(S^3(T_{\mathbb{P}_2}(-1))) &= (1+3\alpha)(1+2\alpha+\beta)(1+\alpha+2\beta)(1+3\beta) \\ &= 1+6(\alpha+\beta)+11(\alpha^2+\beta^2)+32\alpha\cdot\beta \\ &= 1+6(\alpha+\beta)+11(\alpha+\beta)^2+10\alpha\cdot\beta \\ &= 1+6h+21h^2 \end{aligned}$$

Finally, we are able to compute the contribution of $\mathbb{P}_2$:

$$(E,s)^{\mathbb{P}_2} = \int_{\mathbb{P}_2} (1+6h+21h^2)\cdot(1-h) = \int_{\mathbb{P}_2} (1+5h+15h^2) = 15\,.$$

This gives us the virtual number of lines on our cubic $27 = 6+6+15$. (We have two rulings on the quadric surface.)

3 The computations of Kontsevich and Manin

3.1 The virtual number of rational curves on a quintic threefold

Kontsevich considers in [5] the following situation: Let $\bar{\mathcal{M}}(\mathbb{P}^4, d)$ be the compactified moduli space of degree d-morphisms of $\mathbb{P}^1$ to $\mathbb{P}^4$, and $\mathcal{C}(d)$ the universal curve over the moduli space $\bar{\mathcal{M}}(\mathbb{P}^4, d)$. We see that the moduli space $\bar{\mathcal{M}}(\mathbb{P}^4, d)$ is a stack of dimension $5d+1$. For $d > 1$ this stack is no variety. We have the following universal diagram

$$\begin{array}{ccc} \mathcal{C}(d) & \xrightarrow{q} & \mathbb{P}^4 \\ \downarrow{\scriptstyle p} & & \\ \bar{\mathcal{M}}(\mathbb{P}^4, d) & & \end{array}$$

We now define the $\bar{\mathcal{M}}(\mathbb{P}^4, d)$-vector bundle E_d by:

$$E_d := p_* q^* (\mathcal{O}_{\mathbb{P}^4}(5))\,.$$

If E_d were globally generated, and $\bar{\mathcal{M}}(\mathbb{P}^4, d)$ were a smooth scheme, then the number

$$N_d := \int_{\bar{\mathcal{M}}(\mathbb{P}^4,d)} c_{5d+1}(E_d)$$

would be the number of degree d rational curves on a general quintic in $\mathbb{P}^4$. In particular, this number would be finite. Since this does not hold, the number N_d is the *virtual number* of degree d rational curves on a quintic. However, Kontsevich computes these virtual numbers. We give from [5] the first of them:

d	1	2	3	4
N_d	2875	$\frac{4876875}{8}$	$\frac{8564575000}{27}$	$\frac{15517926796875}{64}$

3.2 The real number of rational curves on a quintic threefold

We assume from now on that the Clemens conjecture holds, i.e., the number N_d^0 of rational curves of degree d on the general quintic is finite and all these curves are irreducible. We will (still following Kontsevich) derive a formula which relates the virtual numbers N_d with the true numbers[2] N_d^0.

We consider a global section s of E_d corresponding to a smooth quintic $X(s)$ in $\mathbb{P}^4$. We assume the quintic to be general in the sense of the Clemens conjecture, i.e., the number of rational curves of degree less or equal d on $X(s)$ is finite. The vanishing locus $V(s)$ in $\bar{\mathcal{M}}(\mathbb{P}^4, d)$ consists of all those points corresponding to a mapping $\psi : \mathbb{P}^1 \to \mathbb{P}^4$ with $\psi^*\mathcal{O}_{\mathbb{P}^4}(1) \cong \mathcal{O}_{\mathbb{P}^1}(d)$ with $\psi(\mathbb{P}^1) \subset X(s)$. If the degree of ψ is 1, then the Clemens conjecture implies that ψ is an isolated point in $V(s)$.

If ψ is a morphism of degree k, then of course all morphisms ψ' form $\mathbb{P}^1$ to the image of ψ belong to the zero set $V(s)$.

Let $M_{0,0}(\mathbb{P}^1, k)$ be the compactified moduli space of mappings from $\mathbb{P}^1$ to $\mathbb{P}^1$ of degree k. Then any rational curve of degree $\frac{d}{k}$ in $\mathbb{P}^4$ defines a natural inclusion $\iota_{k,d} : M_{0,0}(\mathbb{P}^1, k) \hookrightarrow \bar{\mathcal{M}}(\mathbb{P}^4, d)$. Using the results of 1.1 we find the connection between the virtual and the true numbers

$$\begin{aligned} N_d &= \sum_{k|d} a_{k,d} N^0_{d/k} \qquad \text{where} \\ a_{k,d} &= \int_{M_{0,0}(\mathbb{P}^1,k)} \left(c(\iota_{k,d}^* E_d) \cdot c(N_{M_{0,0}(\mathbb{P}^1,k)|\bar{\mathcal{M}}(\mathbb{P}^4,d)})^{-1} \right) . \end{aligned}$$

Thus, we have to study the numbers $a_{k,d}$. To do so, we consider the following morphisms.

$$M_{0,0}(\mathbb{P}^1, k) \xleftarrow{f} \mathcal{C}_{0,0} \xrightarrow{g} \mathbb{P}^1 \xrightarrow{\psi} \mathbb{P}^4$$

Here ψ is an embedding of degree $\frac{d}{k}$, and $\mathcal{C}_{0,0}$ is the universal curve. By definition, we have

$$\iota_{k,d}^* E_d \cong f_* g^* \psi^* \mathcal{O}_{\mathbb{P}^4}(5) \cong f_* g^* \mathcal{O}_{\mathbb{P}^1}(\frac{5d}{k}) .$$

Using the normal bundle equation (cf. 1.2), we obtain

$$N_{M_{0,0}(\mathbb{P}^1,k)|\bar{\mathcal{M}}(\mathbb{P}^4,d)} = f_* g^* N_{\mathbb{P}^1|\mathbb{P}^4} .$$

[2] It should be remarked that the Clemens conjecture is still open. Thus, the true numbers are a little bit virtual too.

The normal bundle $N_{\mathbb{P}^1|\mathbb{P}^4}$ is of degree $\frac{5d}{k}-2$, and of rank 3. We have a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \to N_{\mathbb{P}^1|\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^1}(\frac{5d}{k}) \to 0\,.$$

Since $f_*g^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}=0$, we obtain the following short exact sequence

$$0 \to f_*g^*N_{\mathbb{P}^1|\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^1}(\frac{5d}{k}) \to R^1f_*g^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \to 0\,.$$

Since the total Chern class is multiplicative for short exact sequences, we conclude the formula

$$a_{k,d} = \int_{M_{0,0}(\mathbb{P}^1,k)} c(R^1f_*g^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})\,.$$

Thus, the numbers $a_{k,d}$ do not depend from the integer d. The vector bundle $R^1f_*g^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ is the pull back of the normal bundle of $\mathbb{P}^1$ in the quintic $X(s)$. Manin gives a formula for the numbers $a_k = a_{k,d}$ which says $a_k = k^{-3}$ (cf. [6]). This gives us the connection between the virtual number (N_d) and the number (N_d^0) of degree d rational curves on a general quintic in $\mathbb{P}^4$:

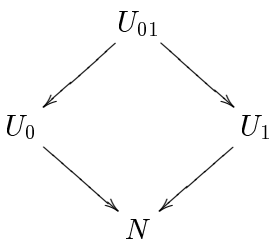
$$N_d = \sum_{k|d} \frac{N^0_{d/k}}{k^3}\,.$$

This provides us with a list of the first true numbers of rational curves of degree d on a quintic:

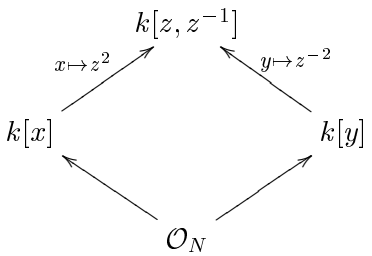
d	1	2	3	4
N_d^0	2875	609250	317206375	242467530000

4 A strange $\mathbb{P}^1$

Let k be a field of characteristic different from two. We consider the following étale cover of the Deligne-Mumford stack N.



Where the open subsets U_i are the spectra of the coordinate rings on the right hand side.



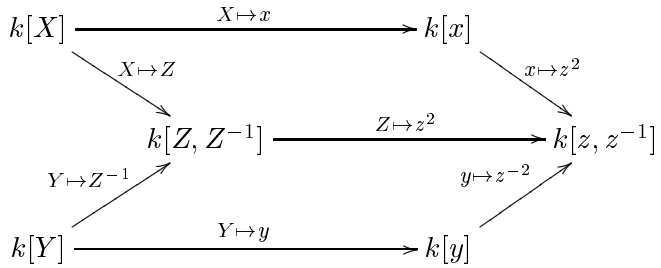
A line bundle on N is given by a unit on U_{01} modulo the units on U_j. Thus the line bundle L corresponding to z^{-1} generates the Picard group of N. The transition function of the canonical bundle K_N is calculated by computing $\frac{dx}{-dy}$ on U_{01}. We directly compute, that $\frac{dx}{-dy} = \frac{d(z^2)}{-d(z^{-2})} = \frac{2zdz}{2z^{-3}dz} = z^4$. Thus, $K_N \cong L^{-4}$. To compute the space of global sections of L^n, we consider the Čech complex

$$\begin{array}{rcl} k[x]\oplus k[y] & \to & k[z,z^{-1}] \\ (f,g) & \mapsto & z^{-n}\cdot\alpha(f)-\beta(g) \end{array}$$

where α and β are the ring homomorphisms defined by $\alpha(x)=z^2$ and $\beta(y)=z^{-2}$. We see that the cokernel $H^1(N,L^n)$ is infinite dimensional, whereas

$$h^0(N,L^n) = \begin{cases} \frac{n}{2}+1 & \text{for even non-negative } n\,; \\ 0 & \text{else.} \end{cases}$$

The global sections of L^2 define a morphism from N to $\mathbb{P}^1$. The next diagram gives a local description of this morphism.



The morphism $\psi : N \to \mathbb{P}^1$ is an étale morphism of stacks. Using the fact that the map $U_0 \to N$ is étale of degree two, we obtain that ψ is of degree $\frac{1}{2}$. We directly see from the above diagram that $\psi^*\mathcal{O}_{\mathbb{P}^1}(1) \cong L^2$. Thus, we obtain the equalities of Chern numbers

$$\int_N c_1(L) = \frac{1}{2}\int_N c_1(L^2) = \frac{\deg(\psi)}{2}\int_{\mathbb{P}^1} c_1(\mathcal{O}_{\mathbb{P}^1}(1)) = \frac{1}{4}\,.$$

5 The moduli stack $M = M_{0,0}(\mathbb{P}^1, 2)$

5.1 Why do we need a stack?

We fix a ground field k with $\mathrm{char}(k) \neq 2$. Let M be the space of all stable maps of degree two to $\mathbb{P}^1$. It is easy to see that M is the disjoint union of the following two sets:

M_0 the set of mappings of a smooth rational curve to $\mathbb{P}^1$ ramified over two different points;

M_1 the curves we consider here are two rational curves identified in one point and a degree one map of each one to $\mathbb{P}^1$.

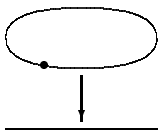
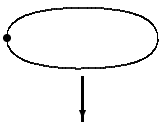
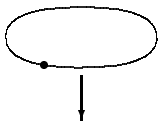
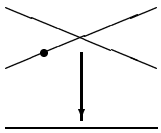
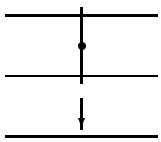
At the first glance it seems that our space M is the linear system of two points on $\mathbb{P}^1$. Furthermore, to give an algebra structure on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ we need to give a section of $\mathcal{O}_{\mathbb{P}^1}(2)$.
There exists a universal reason why a moduli scheme does not suffice for this situation: There are non trivial automorphisms.
The point is that our moduli space is $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \setminus 0$ modulo the natural action of $k^* = \mathrm{Iso}(\mathcal{O}_{\mathbb{P}^1}(-1))$. In coordinates we identify (x, y, z) with $(xt^{-2}, yt^{-2}, zt^{-2})$, for any $t \in k^*$. The short exact sequence

$$0 \longrightarrow \mu_2 \longrightarrow G_m \xrightarrow{x\to x^2} G_m \longrightarrow 0$$

from étale cohomology yields that our stack is an étale cover of $\mathbb{P}^2$ with fibre $\frac{1}{\mu_2}$. This shows that M can not be an ordinary scheme.
If we take $\mathbb{C}$ as our ground field, then we see that as a topological space M is the same as $\mathbb{P}^2$. Whereas taking a finite field k we see that $M(k)$ is twice as big as $\mathbb{P}^2(k)$. Thus, the existence of the moduli scheme would contradict the Weyl conjectures.

5.2 Computing the Betti numbers using the Weyl conjectures

We assume here, that k is a finite field with q elements. By $A_M(q^l)$ we denote the number of $\mathbb{F}_{q^l}$-points of M each counted with multiplicity $\frac{1}{\#\mathrm{Aut}}$. The divisors of degree two are parameterized by $\mathbb{P}^2_k$. To such an divisor correspond two different curves ramified over that divisor. However,

description	picture	number of different curves	number of markings	number of automorphisms	contribution to $A_{\mathcal{C}}(q^l)$
a smooth point on a smooth curve ramified over two $\mathbb{F}_{q^l}$-points		$q^l(q^l+1)$	$\frac{q^l-1}{2}$	1	$\frac{q^l(q^l+1)(q^l-1)}{2}$
a ramification point on a smooth curve ramified over two $\mathbb{F}_{q^l}$-points		$q^l(q^l+1)$	2	2	$q^l(q^l+1)$
a point on a smooth curve ramified over two $\mathbb{F}_{q^{2l}}$-points which are conjugated		$q^l(q^l-1)$	$\frac{q^l+1}{2}$	1	$\frac{q^l(q^l+1)(q^l-1)}{2}$
a smooth point on a reducible curve consisting of two projective lines		q^l+1	q^l	1	$q^l(q^l+1)$
a smooth point on a contracted curve connecting two projective lines		$2(q^l+1)$	1	2	q^l+1

both have two automorphisms. Hence, we conclude that $A_M(q^l) = A_{\mathbb{P}^2}(q^l) = q^{2l} + q^l + 1$. This implies for the ζ-function ζ_M

$$\zeta_M(t) = \zeta_{\mathbb{P}^2}(t) = \exp\left(\sum_{l>0} A_M(q^l)\frac{t^l}{l}\right) = \frac{1}{(1-t)(1-qt)(1-q^2t)}\,.$$

We conclude that the Betti numbers of M are $b_0(M) = b_2(M) = b_4(M) = 1$ and zero otherwise. To compute the Betti numbers of the universal curve $\mathcal{C}$ over M, we use Knudsen's result (see [4]) that $\mathcal{C}$ is the moduli stacks of stable 1-pointed degree two maps from a rational curve to $\mathbb{P}^1$ (see also [3]).
Summing up the last column of the above table we obtain the number of $\mathbb{F}_{q^l}$-points in the moduli space $M_{0,1}(\mathbb{P}^1, 2)$ which is the universal curve $\mathcal{C}$ over M.

$$A_{\mathcal{C}}(q^l) = q^{3l} + 2q^{2l} + 2q^l + 1$$

As before we calculate the ζ-function $\zeta_{\mathcal{C}}$ of $\mathcal{C}$ and obtain

$$\zeta_{\mathcal{C}}(t) = \frac{1}{(1-t)(1-qt)^2(1-q^2t)^2(1-q^3t)}\,.$$

Eventually, we can give the Betti numbers of the universal curve $\mathcal{C}$ to be $b_0(\mathcal{C}) = b_6(\mathcal{C}) = 1$, $b_2(\mathcal{C}) = b_4(\mathcal{C}) = 2$, and $b_k = 0$ otherwise.

5.3 Construction of M

Let k be a field not of characteristic two. We assume that $k^* \overset{x\mapsto x^2}{\longrightarrow} k^*$ is surjective. Furthermore, let V be a k-vector space of dimension two with base vectors X and Y. We denote the projective space $\mathbb{P}(V)$ by $\mathbb{P}^1$.
To construct M we add a marking to get rid of the automorphisms. One possibility is to consider the following moduli spaces:

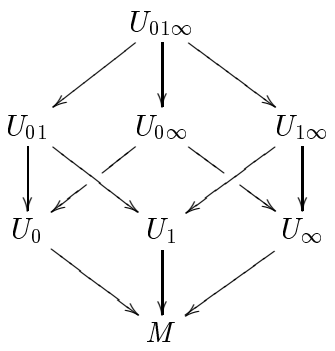
$$U_j := \left\{ \begin{array}{l} \text{Stable } 2:1 \text{ finite maps } \phi : X \to \mathbb{P}^1 \text{ from a rational curve} \\ X \text{ to } \mathbb{P}^1 \text{ for which } j \text{ is no critical value plus the choice of a} \\ \text{point in the fibre of } j \in \mathbb{P}^1. \end{array} \right\}$$

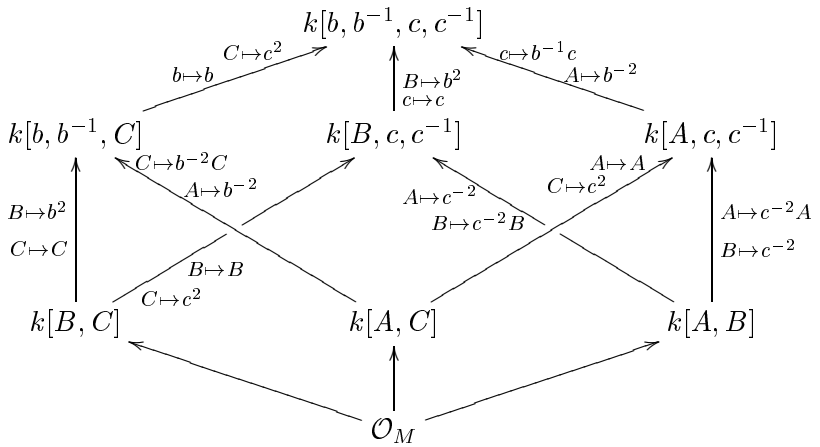
It is obvious that M is covered by the open sets U_0, U_1, and U_∞. Furthermore, we see that these U_j are étale neighbourhoods of M of degree two.
Next we will construct U_0 and the universal family over U_0. Let $V_0 = \mathbb{P}^1 \setminus \infty$ and $V_1 = \mathbb{P}^1 \setminus 0$. Let e_0 and e_1 be trivializing sections of $\mathcal{O}_{\mathbb{P}^1}(-1)$ restricted to V_0 respectively to V_1 with $e_0 = xe_1$ on $V_{01} = V_0 \cap V_1$. We obtain an algebra structure on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ by setting

$$e_0^2 = A(1-x) + Bx + C(x-1)x \quad \text{and} \quad e_1^2 = A(y-1)y + By + C(1-y)\,.$$

The requirement that $0 \in \mathbb{P}^1$ is not a critical value is equivalent to $A \neq 0$. The marking corresponds to a surjection π of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ to $k = \mathcal{O}_0$. This is given by $1 \mapsto u$ and $e_0 \mapsto a$. However, since π has to be an algebra homomorphism, we must have $u = 1$ and $a^2 = A$. Thus, a point in U_0 can be parameterized by a triple (a, B, C). If we replace e_0 and e_1 by λe_0 and λe_1 respectively, then we have to pass from the triple (a, B, C) to $(\lambda^{-1}a, \lambda^{-2}B, \lambda^{-2}C)$. Thus in every equivalence class there exits a unique representative with $a = 1$. We end up with $U_0 \cong \mathbb{A}^2$.

Now we are able to describe our moduli space M. All arrows are étale morphisms of degree two. We have
$U_0 \cong U_1 \cong U_\infty \cong \mathbb{A}^2$,
$U_{01} \cong U_{0\infty} \cong U_{1\infty} \cong \mathbb{A}^1 \times k^*$,
and $U_{01\infty} \cong k^* \times k^*$.
In the next diagram we give the corresponding coordinate rings and ring homomorphisms.





5.4 The Picard group and the universal family

A line bundle on M is given by units φ_{01}, $\varphi_{1\infty}$, and $\varphi_{\infty 0}$ satisfying the cocycle condition $\varphi_{01} \cdot \varphi_{1\infty} \cdot \varphi_{\infty 0} = 1$ on $U_{01\infty}$. Using the above coordinates we can write $\varphi_{01} = b^k$, $\varphi_{1\infty} = c^l$, and $\varphi_{\infty 0} = c^m$. The cocycle condition reads $b^k \frac{c^l}{b^l} c^m = 1$. This implies $l = k$ and $m = -k$. Thus, the line bundle L corresponding to $k = -1$, $l = -1$, and $m = 1$ generates the Picard group $\mathrm{Pic}(M)$ of M.

Analogously to section 4 we can compute the space of global sections of L^n and obtain:

$$h^0(M, L^n) = \begin{cases} \frac{n^2+6n+8}{8} & \text{, for even } n \geq 0; \\ 0 & \text{else.} \end{cases}$$

We conclude that $\mathrm{Pic}(M) \cong \mathbb{Z}$. Repeating the computations of section 4 we obtain $K_m \cong L^{-6}$. As before the global sections of L^2 define a morphism $\psi : M \to \mathbb{P}^2$ of degree $\frac{1}{2}$. The corresponding morphism of Picard groups is given by $\psi^* \mathcal{O}_{\mathbb{P}^2}(1) = L^2$. Eventually, we obtain

$$\int_M c_1(L)^2 = \frac{1}{4} \int_M c_1(L^2)^2 = \frac{\deg(\psi)}{4} \int_{\mathbb{P}^2} c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2 = \frac{1}{8}.$$

Now we glue the universal families of 5.3 to obtain a finite map $\rho : \mathcal{C} \to M \times \mathbb{P}^1$. It turns out that we do not need the marking we introduced before. However, we need that we have s square root of the ample line bundle $\psi^* \mathcal{O}_{\mathbb{P}^2}(1)$. We define $\mathcal{C}$ locally as before by putting

$$\begin{array}{rcll} e_{00}^2 & = & (1-x) + Bx + C(x-1)x & \text{on } U_0 \times V_0; \\ e_{01}^2 & = & (y-1)y + By + C(1-y) & \text{on } U_0 \times V_1; \\ e_{10}^2 & = & A(1-x) + x + C(x-1)x & \text{on } U_1 \times V_0; \\ e_{11}^2 & = & A(y-1)y + y + C(1-y) & \text{on } U_1 \times V_1; \\ e_{\infty 0}^2 & = & A(1-x) + Bx + (x-1)x & \text{on } U_\infty \times V_0; \\ e_{\infty 1}^2 & = & A(y-1)y + By + (1-y) & \text{on } U_\infty \times V_1. \end{array}$$

These local data glue well if we set $e_{j0} = x \cdot e_{j1}$, for all $j \in \{0, 1, \infty\}$, and

$$e_{0k} = b \cdot e_{1k} \quad e_{1k} = c \cdot e_{\infty k} \quad e_{\infty k} = c^{-1} e_{0k} \quad \text{for } k \in \{0, 1\}.$$

Considering the morphisms $M \xleftarrow{p_M} M \times \mathbb{P}^1 \xrightarrow{p_{\mathbb{P}^1}} \mathbb{P}^1$. We see that this defines an algebra structure on the direct sum $\mathcal{O}_{M \times \mathbb{P}^1} \oplus \left(p_m^* L^{-1} \otimes p_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(-1)\right)$.

5.5 Drawing the conclusion: $a_2 = \frac{1}{8}$

Now we are able to compute the direct image bundles $p_! q^* \mathcal{O}_{\mathbb{P}^1}(m)$ for the morphisms $M \xleftarrow{p} \mathcal{C} \xrightarrow{q} \mathbb{P}^1$.

$$\begin{array}{rcl} p_! q^* \mathcal{O}_{\mathbb{P}^1}(m) & = & p_{M!} \left(p_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(m) \oplus (p_M^* L^{-1} \otimes p_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(m-1))\right) \\ & = & \left(H^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) \otimes \mathcal{O}_M\right) \oplus \left(H^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m-1)) \otimes L^{-1}\right) \end{array}$$

In particular, this yields $R^1 p_* q^* \mathcal{O}_{\mathbb{P}^1}(-1) \cong L^{-1}$. Coming back to our formula from 3.2 we conclude

$$a_2 = \int_M c(L^{-1} \oplus L^{-1}) = \int_M c_1(L)^2 = \frac{1}{8}.$$

Thus, we were able to reproduce a very small part of an important calculation.

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