LARGE TIME ASYMPTOTICS OF SOLUTIONS TO THE ANHARMONIC OSCILLATOR MODEL FROM NONLINEAR OPTICS *

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Abstract. The anharmonic oscillator model describing the propagation of electromagnetic waves in an exterior domain containing a nonlinear dielectric medium is investigated. The system under consideration consists of a generally nonlinear second order differential equation for the dielectrical polarization coupled with Maxwell's equations for the electromagnetic field. Local decay of the electromagnetic field for $t \to \infty$ in the charge free case is shown for a large class of potentials. (This paper has been accepted for publication in the SIAM Journal on Mathematical Analysis.)

Key words. nonlinear optics, Maxwell's equations, exterior boundary value problem, asymptotic behaviour

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1. Introduction. The subject of this paper is the anharmonic oscillator model from nonlinear optics consisting of Maxwell's equations

$$\partial_t \mathbf{E} = \text{ curl } \mathbf{H} - \partial_t \tilde{\mathbf{P}} - \mathbf{j}, \quad \partial_t \mathbf{H} = - \text{ curl } \mathbf{E}, \tag{1.1}$$

on $I\!R^+ \times \Omega$ coupled with the equation

$$\alpha \partial_t^2 \mathbf{P} + \partial_t \mathbf{P} + \nabla_P V(x, \mathbf{P}) = \gamma \mathbf{E} \tag{1.2}$$

on $I\!R^+ \times G$. The initial boundary conditions

$$\vec{n} \wedge \mathbf{E} = 0 \text{ on } (0,\infty) \times \Gamma_1 \text{ and } \vec{n} \wedge \mathbf{H} = 0 \text{ on } (0,\infty) \times \Gamma_2 \tag{1.3}$$

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), \mathbf{H}(0,x) = \mathbf{H}_0(x). \tag{1.4}$$

and

$$\mathbf{P}(0,x) = \mathbf{P}_0(x), \quad \partial_t \mathbf{P}(0,x) = \mathbf{P}_1(x) \text{ on } G \tag{1.5}$$

are imposed. This system describes the propagation of electromagnetic waves in a dielectric medium occupying the set G, see [3], [12]. Here $\Omega \subset I\!R^3$ is an exterior domain, $G \subset \Omega$ a certain subset and $\Gamma_1 \subset \partial\Omega$, $\Gamma_2 \stackrel{\text{def}}{=} \partial\Omega \setminus \Gamma_1$. The unknown functions are the electric and magnetic field $\mathbf{E}, \mathbf{H}$, which depend on the time $t \geq 0$ and the space-variable $x \in \Omega$, and the dielectric polarization $\mathbf{P}$ defined on $I\!R^+ \times G$. In 1.1 the function $\tilde{\mathbf{P}}$ is the extension of $\mathbf{P}$ on $I\!R^+ \times \Omega$ defined by zero on the set $I\!R^+ \times (\Omega \setminus G)$. The physical meaning of the boundary condition 1.3 is that Γ_1 is perfectly conducting, such that the tangential component of the electric field must vanish.

The coefficients $\alpha, \gamma \in L^\infty(G)$ depending on the space variables take into account the possibly variable mass , electrical charge and density of the oscillating charged particles. An external current $\mathbf{j} \in L^1((0,\infty), L^2(\Omega))$ is included also. The potential energy function $V : G \times I\!R^3 \to [0,\infty)$ causes a spring force $\nabla_P V(x, \mathbf{P})$, which may

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depend nonlinearily on $\mathbf{P}$. It is assumed that the potential V satisfies the attraction condition

$$0 \leq V(x,y) \leq Ky(\nabla_P V)(x,y) \text{ for all } x \in G, y \in I\!R^3 \tag{1.6}$$

with some constant $K > 0$.
In particular it is allowed that $|(\nabla_P V)(x,y)|$ tends to zero for $|y| \to \infty$ as in [12]. The linear case $(\nabla_P V)(x,y) = ay$ with some $a > 0$ is included also.

In [12], where $G = \Omega = I\!R^3$ and the coefficients and the potential do not depend on x, it is shown that 1.1, 1.2 admits a unique strong strong solution in $C([0,\infty), H^s(I\!R^3))$ for $s \geq 2$. Note that in our case system 1.1 does not admit classical solutions on all of $(0,\infty) \times \Omega$ due to the discontinuity of $\tilde{\mathbf{P}}$ on $\Sigma \overset{\text{def}}{=} (\partial G) \cap \Omega$, the interface between the polarizable medium and the vacuum-region $\Omega \setminus G$. But if the solution is smooth on $(0,\infty) \times G$ and on $(0,\infty) \times (\Omega \setminus G)$ then 1.1 involves a transmission condition, which requires the continuity of the tangential components of $\mathbf{E}$ and $\mathbf{H}$, as well as a linking condition for the normal components of $\mathbf{D} = \mathbf{E} + \tilde{\mathbf{P}}$ and $\mathbf{H}$ on Σ. Therefore a suitable weak formulation of 1.1, 1.2 will be given in section 2, which admits discontinuous solutions. In [4] the Landau-Lifschitz equation for the magnetic moment coupled with Maxwell's equations is handled analogously. The magnetic moment is located in a bounded domain, whereas Maxwell's equations are posed on the whole space. It is shown in [4] that all points of the weak ω-limit set are solutions of the corresponding stationary equations.

The main topic of this paper is the investigation of the long time asymptotic behaviour of the solutions. For this purpose it is assumed that

$$\gamma \in L^{3/2}(G) \text{ and } (1+|x|)\gamma \in L^{r_0}(G) \text{ with some } r_0 \in (3/2, \infty).$$

Since $\gamma \in L^\infty(G)$, this assumption is fulfilled for example if $\int_G (1+|x|)^{r_0} dx < \infty$, in particular if the set G is bounded.

Let X_0 denote the set of all $(\mathbf{f}, \mathbf{g}) \in X \overset{\text{def}}{=} L^2(\Omega, I\!R^6)$, which satisfy

$$\text{curl } \mathbf{f} = \text{ curl } \mathbf{g} = 0 \text{ on } \Omega, \quad \vec{n} \wedge \mathbf{f} = 0 \text{ on } \Gamma_1, \vec{n} \wedge \mathbf{g} = 0 \text{ on } \Gamma_2.$$

The basic goal is to prove the decay property

$$\int_{\{x \in \Omega : |x| \leq \alpha t\}} |\mathbf{E}|^2 + |\mathbf{H}|^2 dx \overset{t\to\infty}{\longrightarrow} 0 \text{ for all } \alpha < 1, \tag{1.7}$$

in particular local energy decay, provided that the initial data satisfy

$$\int_\Omega (\mathbf{D}_1 \mathbf{f} + \mathbf{H}_0 \mathbf{g}) dx = 0 \text{ for all } (\mathbf{f}, \mathbf{g}) \in X_0. \tag{1.8}$$

Here $\mathbf{D}_1 \overset{\text{def}}{=} \mathbf{E}_0 + \tilde{\mathbf{P}}_0 - \int_0^\infty \mathbf{j}(s) ds$, where $\tilde{\mathbf{P}}_0$ denotes the extension of $\mathbf{P}_0$ by zero on $\Omega \setminus G$. (Note that the propagation speed of electromagnetic waves in vacuum is normalized to 1 in 1.1.) Furthermore it is shown that

$$\int_\Omega |\mathbf{E}(t,x) + |x|^{-1} x \wedge \mathbf{H}(t,x)|^2 + |\mathbf{H}(t,x) - |x|^{-1} x \wedge \mathbf{E}(t,x)|^2 dx \overset{t\to\infty}{\longrightarrow} 0. \tag{1.9}$$

The physical meaning of 1.7 is that the wave-packet $(\mathbf{E}(t), \mathbf{H}(t))$ is concentrated near the sphere $|x| = t$ for large times. In section 4 it is also shown that the solution

$(\mathbf{E}(t), \mathbf{H}(t))$ behaves asymptotically like a solution of the linear homogeneous Maxwell equations in $\mathbb{R}^3$ as $t \to \infty$.

Condition 1.8 includes

$$\text{div } \mathbf{D}_1 = 0 \text{ and } \text{div } \mathbf{H}_0 = 0 \text{ on } \Omega$$

and the boundary conditions

$$\vec{n}\mathbf{D}_1 = 0 \text{ on } \Gamma_2 \text{ and } \vec{n}\mathbf{H}_0 = 0 \text{ on } \Gamma_1.$$

By 1.1 the function $\mathbf{D} \stackrel{\text{def}}{=} \mathbf{E} + \tilde{\mathbf{P}}$ and $\mathbf{H}$ obey $\text{div } \mathbf{H}(t) = \text{div } \mathbf{H}_0 = 0$ and

$$\text{div } \mathbf{D}(t) = \text{div } \left[\mathbf{E}_0 + \tilde{\mathbf{P}}_0 - \int_0^t \mathbf{j}(s)ds\right] \stackrel{t\to\infty}{\longrightarrow} \text{div } \mathbf{D}_1 = 0 \text{ in } \mathcal{D}'(\Omega)$$

if condition 1.8 is fulfilled. Physically this means that the space charge $\rho \stackrel{\text{def}}{=} \text{div } \mathbf{D}$ determined by the initial-state $(\mathbf{E}_0, \mathbf{H}_0)$ and the prescribed current $\mathbf{j}$ vanishes as $t \to \infty$.

The proof of the decay property 1.7 uses a result in [11]. In particular it is shown in section 3 that for arbitrary initial states $(\mathbf{E}_0, \mathbf{H}_0)$, $\mathbf{P}_0$ and $\mathbf{P}_1$ not necessarily satisfying 1.8 the weak ω-limit set of $(\mathbf{E}, \mathbf{H})$ is contained in X_0.

2. Basic definitions, assumptions and preliminaries. For an arbitrary open set $K \subset \mathbb{R}^3$ the space of all infinitely differentiable functions with compact support contained in K is denoted by $C_0^\infty(K)$. For $p \in [1, \infty)$ the dual exponent p^* is given by $p^{-1} + (p^*)^{-1} = 1$.

Let $\Omega \subset \mathbb{R}^3$ be a (connected) domain with bounded complement, such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is a Lipschitz domain and $G \subset \Omega$ a measurable set with nonempty interior. Throughout this paper the following assumptions are imposed on $V : G \times \mathbb{R}^3 \to [0, \infty)$. First $V(\cdot, y) \in L^\infty(G)$ for all $y \in \mathbb{R}^3$,

$$V(x, \cdot) \in C^2(\mathbb{R}^3, \mathbb{R}), \quad V(x, 0) = 0 \text{ and } (\nabla_P V)(x, 0) = 0 \tag{2.1}$$

for all $x \in G$. It is assumed that $(\nabla_P V)$ Lipschitz-continuous with respect to y, i.e. there exits some $L_0 \in (0, \infty)$, such that

$$|(\nabla_P V)(x, y) - (\nabla_P V)(x, z)| \le L_0|y - z| \text{ for all } x \in G, y, z \in \mathbb{R}^3. \tag{2.2}$$

This condition is also required in [12], since the second- and third order derivatives of V are assumed to be globally bounded there.

Next, let $\alpha \in L^\infty(G)$ be a uniformly positive and $\gamma \in L^\infty(G)$ be a positive, but not necessarily uniformly positive function on G. Now $\mathcal{G} \subset L^2(G)$ is the weighted $L^2(G)$-space consisting of all measurable functions $\mathbf{f} : G \to \mathbb{R}^3$ with $\int_G \gamma^{-1}(x)|\mathbf{f}(x)|^2 dx < \infty$ endowed with the norm

$$||\mathbf{f}||_{\mathcal{G}}^2 \stackrel{\text{def}}{=} \int_G \gamma^{-1}(x)|\mathbf{f}(x)|^2 dx.$$

In the sequel we denote by $\underline{\mathbf{w}}_1 \in \mathbb{C}^3$ the first three and by $\underline{\mathbf{w}}_2 \in \mathbb{C}^3$ the last three components of a vector $\mathbf{w} \in \mathbb{C}^6$ and $S\mathbf{w} \stackrel{\text{def}}{=} (-x \wedge \underline{\mathbf{w}}_2, x \wedge \underline{\mathbf{w}}_1)$.

Next, some function-spaces related to Maxwell's equations with mixed boundary conditions are introduced.

First W_H denotes the closure of $C_0^\infty(\mathbb{R}^3 \setminus \overline{\Gamma_2}, \mathbb{C}^3)$ in $H_{curl}(\Omega)$, where $H_{curl}(\Omega)$, is the space of all $\mathbf{E} \in L^2(\Omega, \mathbb{C}^3)$ with curl $\mathbf{E} \in L^2(\Omega)$ in the sense of distributions. Next, W_E denotes the set of all $\mathbf{E} \in H_{curl}(\Omega)$, such that

$$\int_\Omega \mathbf{E} \text{ curl } \mathbf{F} - \mathbf{F} \text{ curl } \mathbf{E} dx = 0 \text{ for all } \mathbf{F} \in W_H,$$

which includes a weak formulation of the boundary condition $\vec{n} \wedge \mathbf{E} = 0$ on Γ_1, see [7].

Now, the following operators are defined.
Let $D(B) \stackrel{\text{def}}{=} W_E \times W_H$ and

$$B(\mathbf{E}, \mathbf{H}) \stackrel{\text{def}}{=} (\text{ curl } \mathbf{H}, - \text{ curl } \mathbf{E}) \text{ for } (\mathbf{E}, \mathbf{H}) \in D(B).$$

Then B is a densely defined skew self-adjoint operator in the Hilbert-space $X \stackrel{\text{def}}{=} L^2(\Omega, \mathbb{C}^6)$ endowed with the usual scalar product. The space X_0 in 1.8 is defined as the kernel of B, i.e.

$$X_0 \stackrel{\text{def}}{=} \{(\mathbf{E}, \mathbf{H}) \in D(B) : B(\mathbf{E}, \mathbf{H}) = 0\}$$

$$= \{(\mathbf{E}, \mathbf{H}) \in W_E \times W_H : \text{ curl } \mathbf{E} = \text{ curl } \mathbf{H} = 0\}.$$

Let Q be the orthogonal projector on $X_0^\perp = (\ker B)^\perp = \overline{\text{ran } B}$.

For $\mathbf{f} \in L^1_{loc}([0,\infty), X)$ a function $\mathbf{u} \in C([0,\infty), X)$ is called a weak solution to the initial boundary value problem

$$\partial_t \underline{\mathbf{u}}_1 = \text{ curl } \underline{\mathbf{u}}_2 + \underline{\mathbf{f}}_1, \quad \partial_t \underline{\mathbf{u}}_2 = - \text{ curl } \underline{\mathbf{u}}_1 + \underline{\mathbf{f}}_2, \tag{2.3}$$

supplemented by the initial-boundary conditions

$$\vec{n} \wedge \underline{\mathbf{u}}_1 = 0 \text{ on } (0,\infty) \times \Gamma_1, \text{ and } \vec{n} \wedge \underline{\mathbf{u}}_2 = 0 \text{ on } (0,\infty) \times \Gamma_2 \tag{2.4}$$

if

$$\frac{d}{dt}\langle \mathbf{u}(t), \mathbf{a}\rangle_X = -\langle \mathbf{u}(t), B\mathbf{a}\rangle_X + \langle \mathbf{f}(t), \mathbf{a}\rangle_X \text{ for all } \mathbf{a} \in D(B). \tag{2.5}$$

This means that 2.3 is fulfilled in the sense of distributions, whereas the boundary conditions 2.4 are satisfied in the sense that $\int_0^t \mathbf{u}(s)ds \in D(B) = W_E \times W_H$ for all $t \geq 0$. It is well known that 2.5 is equivalent to the variation of constant formula

$$\mathbf{u}(t) = \exp(tB)\mathbf{u}(0) + \int_0^t \exp((t-s)B)\mathbf{f}(s)ds \tag{2.6}$$

where $(\exp(tB))_{t \in \mathbb{R}}$ is the unitary group generated by B, see [13]. 2.6 yields the energy estimate

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|_X^2 = \langle \mathbf{f}(t), \mathbf{u}(t)\rangle_X \tag{2.7}$$

Next $\mathcal{R} : L^2(G) \to X$ is defined by

$$(\mathcal{R}\mathbf{p})(x) \stackrel{\text{def}}{=} (\mathbf{p}(x), 0) \text{ if } x \in G \text{ and } (\mathcal{R}\mathbf{p})(x) \stackrel{\text{def}}{=} 0 \text{ if } x \in \Omega \setminus G.$$

Let

$$\mathbf{j} \in L^1((0,\infty), L^2(\Omega, \mathbb{R}^3)), \quad (\mathbf{E}_0, \mathbf{H}_0) \in X, \quad \mathbf{P}_0 \in \mathcal{G} \text{ and } \mathbf{P}_1 \in \mathcal{G}. \tag{2.8}$$

By 2.1 and 2.2 the nonlinear composition operator $\mathbf{p} \in \mathcal{G} \to (\nabla_y V)(\cdot, \mathbf{p}(\cdot))$ is globally Lipschitz continuous as a map from $\mathcal{G}$ to $\mathcal{G}$. Therefore the initial value problem

$$\alpha \partial_t^2 \mathbf{P} + \partial_t \mathbf{P} + (\nabla_y V)(x, \mathbf{P}) = \gamma \mathbf{E} \text{ on } (0,\infty) \times G \tag{2.9}$$

supplemented by the initial-conditions

$$\mathbf{P}(0) = \mathbf{P}_0, \quad \partial_t \mathbf{P}(0) = \mathbf{P}_1 \tag{2.10}$$

admits for all $\mathbf{E} \in C([0,\infty), L^2(\Omega, \mathbb{R}^3))$ a unique weak solution $\mathbf{P} \in C^2([0,\infty), \mathcal{G}) \subset C^2([0,\infty), L^2(G))$. If $\mathbf{E} \in C([0,\infty), L^2(\Omega, \mathbb{R}^3))$ and $\mathbf{F} \in C([0,\infty), L^2(\Omega, \mathbb{R}^3))$ then the Lipschitz continuity of $\nabla_y V$ yields the estimate

$$\frac{1}{2}\frac{d}{dt}\left[\|\alpha^{1/2}(\partial_t \mathbf{P}(t) - \partial_t \mathbf{Q}(t))\|_{\mathcal{G}}^2 + \|\mathbf{P}(t) - \mathbf{Q}(t)\|_{\mathcal{G}}^2\right] = \int_G \gamma^{-1}(\partial_t \mathbf{P} - \partial_t \mathbf{Q})$$

$$[\gamma \mathbf{E} - \partial_t \mathbf{P} - \nabla_P V(x, \mathbf{P}) - \gamma \mathbf{F} + \partial_t \mathbf{Q} + \nabla_P V(x, \mathbf{Q}) + \mathbf{P} - \mathbf{Q}]\, dx$$

$$\leq C_1 \|\alpha^{1/2}(\partial_t \mathbf{P}(t) - \partial_t \mathbf{Q}(t))\|_{\mathcal{G}} \left(\|\mathbf{E}(t) - \mathbf{F}(t)\|_{L^2(\Omega)} + \|\mathbf{P}(t) - \mathbf{Q}(t)\|_{\mathcal{G}}\right)$$

$$\leq C_2 \left[\|\alpha^{1/2}(\partial_t \mathbf{P}(t) - \partial_t \mathbf{Q}(t))\|_{\mathcal{G}}^2 + \|\mathbf{P}(t) - \mathbf{Q}(t)\|_{\mathcal{G}}^2 + \|\mathbf{E}(t) - \mathbf{F}(t)\|_{L^2(\Omega)}^2\right]$$

with constants C_1, C_2 independent of $\mathbf{E}, \mathbf{F}$ and t. Here $\mathbf{Q} \in C^2([0,\infty), \mathcal{G})$ is the solution of 2.9 and 2.10 with $\mathbf{E}$ replaced by $\mathbf{F}$. By Gronwall's lemma one obtains

$$\|\partial_t(\mathbf{P}(t) - \mathbf{Q}(t))\|_{L^2(G)} \leq \|\gamma^{1/2}\|_{L^\infty(G)} \|\partial_t(\mathbf{P}(t) - \mathbf{Q}(t))\|_{\mathcal{G}} \tag{2.11}$$

$$\leq C_3 \int_0^t \exp\left(L(t-s)\right) \|\gamma(\mathbf{E}(t) - \mathbf{F}(s))\|_{L^2(\Omega)} ds$$

with some $L, C_3 > 0$ independent of $\mathbf{E}$, $\mathbf{F}$ and t.

Let $\mathcal{A} : C([0,\infty), X) \to C([0,\infty), X)$ be defined by

$$(\mathcal{A}(\mathbf{E}, \mathbf{H}))\,(t) \stackrel{\text{def}}{=} \exp(tB)(\mathbf{E}_0, \mathbf{H}_0) - \int_0^t \exp((t-s)B)\,[\mathcal{R}\partial_t \mathbf{P}(s) + (\mathbf{j}(s), 0)]\, ds,$$

where $\mathbf{P}$ solves 2.9 and 2.10.

Now $(\mathbf{E}, \mathbf{H}) \in C([0,\infty), X)$ and $\mathbf{P} \in C^2([0,\infty), \mathcal{G})$ solve 1.1-1.5 (in the sense of 2.5), if

$$(\mathbf{E}(t), \mathbf{H}(t)) = \exp(tB)(\mathbf{E}_0, \mathbf{H}_0) \tag{2.12}$$

$$- \int_0^t \exp((t-s)B)\,[\mathcal{R}\partial_t \mathbf{P}(s) + (\mathbf{j}(s), 0)]\, ds,$$

i.e.

$$\mathcal{A}(\mathbf{E},\mathbf{H}) = (\mathbf{E},\mathbf{H}). \tag{2.13}$$

and $\mathbf{P}$ solves 2.9 and 2.10. It follows from the estimates 2.7 and 2.11 and the contraction mapping principle in the space $C([0,T],X)$ with arbitrary large $T>0$ that the fixed point problem 2.13 has a unique solution on each finite time interval $(0,T)$ and hence a unique global solution on $(0,\infty)$.

THEOREM 2.1. *Problem 1.1-1.5 has a unique weak solution* $(\mathbf{E},\mathbf{H},\mathbf{P})$ *with the properties* $(\mathbf{E},\mathbf{H}) \in C([0,\infty),X)$ *and* $\mathbf{P} \in C^2([0,\infty),\mathcal{G})$.

Further regularity of the solution can be obtained under the additional regularity assumption

$$(\mathbf{E}_0,\mathbf{H}_0) \in D(B) \text{ and } \mathbf{j} \in W^{1,1}((0,\infty),L^2(\Omega)). \tag{2.14}$$

Then $\mathcal{R}\partial_t\mathbf{P}(\cdot) + (\mathbf{j},0) \in W^{1,1}_{loc}([0,\infty),X)$. By the result in [13, Corollary 2.5, sect. 4.2] it follows that

$$(\mathbf{E},\mathbf{H}) \in C^1([0,\infty),X) \cap C([0,\infty),D(B)) \tag{2.15}$$

is a strong solution of

$$\partial_t(\mathbf{E}(t),\mathbf{H}(t)) = B(\mathbf{E}(t),\mathbf{H}(t)) - \mathcal{R}\partial_t\mathbf{P}(t) - (\mathbf{j}(t),0).$$

REMARK 1. *It follows from 2.15 that all partial derivatives occurring in 1.1 and 1.2 belong to the space* $L^\infty_{loc}([0,\infty),L^2(\Omega))$. *In this sense the solution is strong. As described in the introduction* $(\mathbf{E}(t),\mathbf{H}(t))$ *is not in* $H^1(\Omega)$ *in general due to the mixed boundary conditions and the possible discontinuity of the polarization. However the divergence free part of the electromagnetic field satisfies by 2.15 curl* $\underline{[Q(\mathbf{E}(\cdot),\mathbf{H}(\cdot))]}_k \in L^\infty_{loc}([0,\infty),L^2(\Omega))$ *and div* $\underline{[Q(\mathbf{E}(\cdot),\mathbf{H}(\cdot))]}_k = 0$ *for* $k \in \{1,2\}$. *Hence* $Q(\mathbf{E}(\cdot),\mathbf{H}(\cdot)) \in L^\infty_{loc}([0,\infty),H^1(\mathcal{U}))$ *for all subdomains* $\mathcal{U} \subset \Omega$, *which have positive distance to* $\partial\Omega$.

It follows from 2.12 and the energy estimate 2.7 that

$$\frac{1}{2}\frac{d}{dt}\|(\mathbf{E}(t),\mathbf{H}(t))\|_X^2 = -\langle \mathcal{R}\partial_t\mathbf{P}(t) + (\mathbf{j}(t),0)), (\mathbf{E}(t),\mathbf{H}(t))\rangle_X \tag{2.16}$$

$$= -\int_G \mathbf{E}\partial_t\mathbf{P}dx - \int_\Omega \mathbf{E}\mathbf{j}dx,$$

whereas 2.9 yields

$$\frac{d}{dt}\left(1/2\|\alpha^{1/2}\partial_t\mathbf{P}(t)\|_{\mathcal{G}}^2 + \int_G \gamma^{-1}V(x,\mathbf{P})dx\right) \tag{2.17}$$

$$= -\int_G \gamma^{-1}|\partial_t\mathbf{P}|^2 dx + \int_G \mathbf{E}\partial_t\mathbf{P}dx$$

By 2.16 and 2.17 one obtains the energy estimate

$$\frac{1}{2}\frac{d}{dt}\left(\|(\mathbf{E}(t),\mathbf{H}(t))\|_X^2 + \|\alpha^{1/2}\partial_t\mathbf{P}(t)\|_{\mathcal{G}}^2 + 2\int_G \gamma^{-1}V(x,\mathbf{P})dx\right) \tag{2.18}$$

$$= -\int_G \gamma^{-1}|\partial_t \mathbf{P}|^2 dx - \int_\Omega \mathbf{E}\mathbf{j} dx \le \|(\mathbf{E}(t), \mathbf{H}(t))\|_X \|\mathbf{j}(t)\|_{L^2(\Omega)} - \|\partial_t \mathbf{P}(t)\|_{\mathcal{G}}^2$$

In the next lemma elementary properties of the solution are shown.

LEMMA 2.2. *i)*

$$(\mathbf{E}, \mathbf{H}) \in L^\infty((0,\infty), X), \quad \partial_t \mathbf{P} \in L^\infty((0,\infty), \mathcal{G}) \cap L^2((0,\infty), \mathcal{G})$$

and

$$\gamma^{-1} V(x, \mathbf{P}(\cdot)) \in L^\infty((0,\infty), L^1(G)).$$

ii) If 2.14 holds one has

$$(\mathbf{E}, \mathbf{H}) \in W^{1,\infty}((0,\infty), X) \cap L^\infty((0,\infty), D(B))$$

Proof. Let

$$\mathcal{E}_t \stackrel{\text{def}}{=} \left(\|(\mathbf{E}(t), \mathbf{H}(t))\|_X^2 + \|\alpha^{1/2}\partial_t \mathbf{P}(t)\|_{\mathcal{G}}^2 + 2\int_G \gamma^{-1} V(x, \mathbf{P}) dx \right). \tag{2.19}$$

By 2.18 one has

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_t \le \mathcal{E}_t(\mathbf{w})^{1/2} \|\mathbf{j}(t)\|_{L^2(\Omega)} - \|\partial_t \mathbf{P}(t)\|_{\mathcal{G}}^2$$

Since $\|\mathbf{j}(\cdot)\|_{L^2(\Omega)} \in L^1(0,\infty)$, this inequality yields i).

If 2.14 holds it follows from 2.15 that

$$\partial_t^2(\mathbf{E}(t), \mathbf{H}(t)) = B\partial_t(\mathbf{E}(t), \mathbf{H}(t)) - \mathcal{R}\partial_t^2 \mathbf{P}(t) - \partial_t(\mathbf{j}(t), 0) \tag{2.20}$$

is satisfied weakly in the sense of 2.5. With a similar estimate as before one obtains using the global boundedness of $(D_P^2 V)$ by 2.2

$$\frac{1}{2}\frac{d}{dt}\left(\|\partial_t(\mathbf{E}(t), \mathbf{H}(t))\|_X^2 + \|\alpha^{1/2}\partial_t^2 \mathbf{P}(t)\|_{\mathcal{G}}^2 \right) \tag{2.21}$$

$$= -\int_\Omega \partial_t \mathbf{E} \partial_t \mathbf{j} dx - \int_G \gamma^{-1}|\partial_t^2 \mathbf{P}|^2 dx - \int_G \gamma^{-1}\partial_t^2 \mathbf{P} \cdot (D_P^2 V)(x, \mathbf{P}) \cdot \partial_t \mathbf{P} dx$$

$$\le \|\partial_t(\mathbf{E}(t), \mathbf{H}(t))\|_X \|\partial_t \mathbf{j}(t)\|_{L^2(\Omega)} + C_1\|\partial_t \mathbf{P}\|_{\mathcal{G}}^2 - b_0/2\|\partial_t^2 \mathbf{P}\|_{\mathcal{G}}^2$$

With part i) and $\|\partial_t \mathbf{j}(\cdot)\|_{L^2(\Omega)} \in L^1(0,\infty)$ it follows that

$$\partial_t(\mathbf{E}, \mathbf{H}) \in L^\infty((0,\infty), X) \tag{2.22}$$

By part i) one has $\partial_t \mathbf{P} \in L^\infty((0,\infty), \mathcal{G}) \subset L^\infty((0,\infty), L^2(G))$, in particular $\mathcal{R}\partial_t \mathbf{P} \in L^\infty((0,\infty), X)$. Since also $(\mathbf{j}, 0) \in L^\infty((0,\infty), X)$ by 2.14 and $(\mathbf{E}, \mathbf{H}) \in C^1([0,\infty), X) \cap C([0,\infty), D(B))$ solves

$$\partial_t(\mathbf{E}(t), \mathbf{H}(t)) = B(\mathbf{E}(t), \mathbf{H}(t)) - \mathcal{R}\partial_t \mathbf{P}(t) - (\mathbf{j}(t), 0),$$

one obtains $(\mathbf{E}, \mathbf{H}) \in L^\infty([0,\infty), D(B))$. This completes the proof of part ii). ◻

By 2.18, 2.19, the previous lemma and $\|\mathbf{j}(\cdot)\|_{L^2(\Omega)} \in L^1(0,\infty)$ one has $\frac{d}{dt}\mathcal{E}(t) \in L^1(0,\infty)$, which implies the existence of the limit

$$\mathcal{E}_\infty \stackrel{\text{def}}{=} \lim_{t\to\infty} \mathcal{E}(t) \tag{2.23}$$

$$= \lim_{t\to\infty} \left(\|(\mathbf{E}(t),\mathbf{H}(t))\|_X^2 + \|\alpha^{1/2}\partial_t \mathbf{P}(t)\|_{\mathcal{G}}^2 + 2\int_G \gamma^{-1} V(x,\mathbf{P})dx \right).$$

The physical meaning of $\mathcal{E}(t)$ is the total energy of the system, i.e. the sum of the potential and kinetic energy of the oscillating particles and the energy of the electromagnetic field. The dissipation term $-\|\partial_t \mathbf{P}(t)\|_{\mathcal{G}}^2 = -\int_G \gamma^{-1}|\partial_t \mathbf{P}|^2 dx$ in the energy estimate 2.18 describes the dielectric losses of the medium. This energy dissipation does not result from an electrical conductivity. It also occurs in insulating materials if they are exposed to a rapidly oscillating electric field.

3. A weak convergence property of the solutions. In what follows the additional regularity assumption 2.14 will be imposed on the data for convenience. The following 'unique continuation' principle is proved in [11], which holds even for arbitrary, not necessarily bounded spatial domains. As in [11] it will be used in the investigation of the weak ω-limit set of the solution of 1.1- 1.5.

THEOREM 3.1. *Suppose that* $\mathbf{g} \in X$ *obeys*

$$\underline{(\exp(tB)\mathbf{g})}_1 = 0 \textit{ on } G \textit{ for all } t \in \mathbb{R}. \tag{3.1}$$

Then $\mathbf{g} \in \ker B$.

This is a generalization of the unique continuation principle for the scalar wave equation in bounded domains, which is used in [5], [6] and [15].
Theorem 3.1 says that each solution $(\mathbf{e},\mathbf{f}) \in C(\mathbb{R}, L^2(\Omega,\mathbb{R}^{M+N}))$ of the evolution equation $\partial_t(\mathbf{e},\mathbf{f}) = B(\mathbf{e},\mathbf{f})$ with the property that $\mathbf{e}(t,x) = 0$ for all $t \in \mathbb{R}$ and $x \in G$ satisfies $(\mathbf{e}(0),\mathbf{f}(0)) \in \ker B$. In contrast to the unique continuation principle for bounded domains it is necessary to require the condition $\mathbf{e}(t,x) = 0$ on G for all $t \in \mathbb{R}$ and not only for positive times. The basic idea of the proof of Theorem 3.1 is to show that for each $f \in C_0^\infty(\mathbb{R}\setminus\{0\})$ the function $f(iB)\mathbf{g}$ is real-analytic and vanishes on G. This implies $f(iB)\mathbf{g} = 0$ for all $f \in C_0^\infty(\mathbb{R}\setminus\{0\})$ and hence $\mathbf{g} \in \ker B$. Here the operator $f(iB)$ can be defined by the spectral theorem, since iB is self-adjoint in $L^2(\Omega,\mathbb{C}^6)$. If $f \in C_0^\infty(\mathbb{R})$, then bounded operator $f(iB)$ has the representation

$$f(iB)\mathbf{u} = (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{f}(t)\exp(-tB)\mathbf{u}dt \text{ for all } \mathbf{u} \in X. \tag{3.2}$$

Here $\hat{f}$ denotes the Fourier-transform of f.

In the sequel let ω_0 denote the ω-limit-set of the trajectory $(\mathbf{E},\mathbf{H})$ with respect to the weak topology of X, i.e. the set of all $\mathbf{g} \in X$, such that there exists a sequence $t_n \stackrel{n\to\infty}{\longrightarrow} \infty$ with $(\mathbf{E}(t_n),\mathbf{H}(t_n)) \stackrel{n\to\infty}{\longrightarrow} \mathbf{g}$ in X weakly.

THEOREM 3.2. $Q(\mathbf{E}(t),\mathbf{H}(t)) \stackrel{t\to\infty}{\longrightarrow} 0$ *in* X *weakly.*

Proof. Suppose $\mathbf{g} \in \omega_0$ and $t_n \stackrel{n\to\infty}{\longrightarrow} \infty$ with

$$(\mathbf{E}(t_n),\mathbf{H}(t_n)) \stackrel{n\to\infty}{\longrightarrow} \mathbf{g} \tag{3.3}$$

in X weakly. Let $\mathbf{u}_n(t) \stackrel{\text{def}}{=} (\mathbf{E}(t_n+t),\mathbf{H}(t_n+t)) \in X$ and $\mathbf{f}_n(t) \stackrel{\text{def}}{=} (\nabla_P V)(x,\mathbf{P}(t_n+t)) \in \mathcal{G}$ for $n \in \mathbb{N}$. First, we have by 2.6

$$\mathbf{u}_n(t) = \exp(tB)\mathbf{u}_n(0) - \int_{t_n}^{t_n+t} \exp((t_n+t-s)B)\left(\partial_s \mathcal{R}\mathbf{P}(s) + (\mathbf{j}(s),0)\right)ds,$$

which implies by Lemma 2.2 i) that

$$\|\mathbf{u}_n(t) - \exp(tB)\mathbf{u}_n(0)\|_X \le \int_{t_n}^{t_n+t} \|\mathcal{R}\partial_s \mathbf{P}(s)\|_X + \|\mathbf{j}(s)\|_{L^2(\Omega)} ds \overset{\mathrm{n}\to\infty}{\longrightarrow} 0$$

for all $t \in I\!R$ and hence by 3.3 with $\mathbf{u}_n(0) = (\mathbf{E}(t_n), \mathbf{H}(t_n))$

$$\mathbf{u}_n(t) \overset{\mathrm{n}\to\infty}{\longrightarrow} \mathbf{u}_\infty(t) \overset{\mathrm{def}}{=} \exp(tB)\mathbf{g} \text{ in } X \text{ weakly for all } t \in I\!R. \tag{3.4}$$

Lemma 2.2yields

$$\|\mathbf{f}_n(t) - \mathbf{f}_n(0)\|_{\mathcal{G}} \le C_1 \int_{[t_n, t_n+t]} \|\partial_t \mathbf{P}(s)\|_{\mathcal{G}} ds \overset{\mathrm{n}\to\infty}{\longrightarrow} 0 \text{ for all } t \in I\!R. \tag{3.5}$$

Suppose $T > 0$. Then 3.5 implies for all $\varphi \in C_0^\infty((-T,T), \mathcal{G})$

$$\lim_{n\to\infty} \int_{-T}^{T} \langle \mathbf{f}_n(t), \partial_t \varphi(t)\rangle_{\mathcal{G}} dt = \lim_{n\to\infty} \int_{-T}^{T} \langle \mathbf{f}_n(0), \partial_t \varphi(t)\rangle_{\mathcal{G}} dt = 0 \tag{3.6}$$

Using $\partial_t \mathbf{P} \in L^2((0,\infty), \mathcal{G})$ again one obtains from 2.9, 3.4 and 3.6 that

$$\int_{-T}^{T} \int_G \underline{(\mathbf{u}_\infty)}_1 \partial_t \varphi dx dt = \lim_{n\to\infty} \int_{-T}^{T} \int_G \underline{(\mathbf{u}_n)}_1 \partial_t \varphi dx dt \tag{3.7}$$

$$= \lim_{n\to\infty} \int_{-T}^{T} \langle \gamma \mathbf{E}(t_n + t), \partial_t \varphi(t)\rangle_{\mathcal{G}} dt$$

$$= \lim_{n\to\infty} \int_{-T}^{T} \langle \alpha \partial_t^2 \mathbf{P}(t_n + t) + \partial_t \mathbf{P}(t_n + t) + \mathbf{f}_n(t), \partial_t \varphi(t)\rangle_{\mathcal{G}} dt = 0$$

for all $\varphi \in C_0^\infty((-T,T), \mathcal{G})$, in particular

$$\partial_t \underline{(\mathbf{u}_\infty)}_1 (t,x) = 0 \text{ for all } x \in G, t \in (-T,T) \tag{3.8}$$

Since $T > 0$ is chosen arbitrarily, 3.8 holds for all $t \in I\!R$.
With $(\mathbf{E},\mathbf{H}) \in L^\infty((0,\infty), D(B))$ by Lemma 2.2 ii), it follows $\mathbf{g} \in D(B)$ and

$$\underline{(\exp(tB)B\mathbf{g})}_1 (x) = \partial_t \underline{(\mathbf{u}_\infty)}_1 (t,x) = 0 \text{ for all } t \in I\!R \text{ and } x \in G$$

by 3.8. Invoking Theorem 3.1 one obtains $B\mathbf{g} \in \ker B$, and hence
$\|B\mathbf{g}\|_X^2 = -\langle \mathbf{g}, B^2 \mathbf{g}\rangle_X = 0$, whence $\mathbf{g} \in \ker B$. Hence

$$\omega_0 \subset \ker B. \tag{3.9}$$

Since $(\mathbf{E}(t), \mathbf{H}(t))$ is bounded in X as $t \to \infty$ by Lemma 2.2 i) and zero is the only possible accumulation point of $Q(\mathbf{E}(\cdot), \mathbf{H}(\cdot))$ by 3.9, the assertion follows. □

4. Decay of the electromagnetic field. For all $\mathbf{a} \in \ker B$ one has by 2.12

$$\langle (\mathbf{E}(t), \mathbf{H}(t)), \mathbf{a} \rangle_X$$

$$= \left\langle \exp(tB)(\mathbf{E}_0, \mathbf{H}_0) - \int_0^t \exp((t-s)B) \left(\mathcal{R}\partial_s \mathbf{P}(s) + (\mathbf{j}(t), 0) \right) ds, \mathbf{a} \right\rangle_X$$

$$= \left\langle (\mathbf{E}_0, \mathbf{H}_0) + \mathcal{R}\mathbf{P}(0) - \mathcal{R}\mathbf{P}(t) - \int_0^t (\mathbf{j}(s), 0) ds, \mathbf{a} \right\rangle_X$$

and hence

$$(1-Q) \left((\mathbf{E}(t), \mathbf{H}(t)) + \mathcal{R}\mathbf{P}(t) + \int_0^t (\mathbf{j}(s), 0) ds - (\mathbf{E}_0, \mathbf{H}_0) - \mathcal{R}\mathbf{P}(0) \right) = 0. \tag{4.1}$$

Recall that $1 - Q$ is the orthogonal projector on $X_0 = \ker B$.

Throughout this section it is assumed that the initial-state $(\mathbf{E}_0, \mathbf{H}_0) \in X$ satisfies

$$(\mathbf{D}_1, \mathbf{H}_0) = (\mathbf{E}_0, \mathbf{H}_0) + \mathcal{R}\mathbf{P}(0) - \int_0^\infty (\mathbf{j}(s), 0) ds \in X_0^\perp,$$

i.e.

$$(1-Q) \left((\mathbf{E}_0, \mathbf{H}_0) + \mathcal{R}\mathbf{P}(0) - \int_0^\infty (\mathbf{j}(s), 0) ds \right) = 0. \tag{4.2}$$

This is condition 1.8 on the initial states. It follows from 4.1 and 4.2 that

$$(1-Q)(\mathbf{E}(t), \mathbf{H}(t)) = (1-Q)(\mathbf{J}(t) - \mathcal{R}\mathbf{P}(t)). \tag{4.3}$$

with $\mathbf{J}(t) \stackrel{\text{def}}{=} \int_t^\infty (\mathbf{j}(s), 0) ds$.

The main goal of this section is the proof of the decay property 1.7. The basic steps are summarized now. By a standard energy estimate it follows that roughly speaking that the asymptotic propagation speed of the wave-packet $(\mathbf{E}(t), \mathbf{H}(t))$ does not exceed 1 as $t \to \infty$, i.e.

$$\int_{\{|x| \geq bt\}} |(\mathbf{E}(t), \mathbf{H}(t))|^2 dx \stackrel{t\to\infty}{\longrightarrow} 0 \text{ for all } b > 1. \tag{4.4}$$

Next it is shown that the potential energy and the energy of the curl free part of the electromagnetic field decay in time mean, i.e.

$$t^{-1} \int_0^t \left(\int_G \gamma^{-1} V(x, \mathbf{P}(s)) dx + \|(1-Q)(\mathbf{E}(s), \mathbf{H}(s))\|_X^2 \right) ds \stackrel{t\to\infty}{\longrightarrow} 0. \tag{4.5}$$

Here assumption 1.6, condition 1.8 and a $L^2 - L^6$-estimate for a vector potential are used. Theorem 3.2 and 4.5 yield the local decay of the electromagnetic field at least in time mean, i.e.

$$t^{-1} \int_0^t \|(\mathbf{E}(s), \mathbf{H}(s))\|_{L^2(\Omega \cap B_R)}^2 ds \stackrel{t\to\infty}{\longrightarrow} 0 \text{ for all } R > 0. \tag{4.6}$$

The main step of the proof of 1.7 is a description of the asymptotic energy $\mathcal{E}_\infty$ in 2.23. Due to the fact that Ω is an exterior domain one has $\mathcal{E}_\infty > 0$ in general, even if condition 1.8 is satisfied. It is shown that for all $b > 1$

$$(4.7)\qquad t^{-1}\int_{\{|x|\le bt\}} [SQ_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))]\cdot Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))dx \overset{t\to\infty}{\longrightarrow} \mathcal{E}_\infty,$$

where $S\mathbf{u} \overset{\text{def}}{=} (-x\wedge \underline{\mathbf{u}}_2, x\wedge \underline{\mathbf{u}}_1)$, $\chi_0 \in C^\infty(I\!R^3)$ is a cut off function with supp $\chi_0 \subset \Omega$ and $\chi_0(x) = 1$ outside some bounded set. Furthermore Q_0 denotes the orthogonal projector on the space of all $\mathbf{u} \in L^2(I\!R^3)$ with div $\underline{\mathbf{u}}_j = 0$. The proof of 4.7 relies on 4.4, 4.5, 4.6 and some L^p-estimates for Q_0.

For this purpose the following additional assumptions are imposed on V, the set G and γ in the sequel.

$$(4.8)\qquad V(x,\mathbf{y}) \le K_2\mathbf{y}(\nabla_y V)(x,\mathbf{y}) \text{ for all } x\in G, \mathbf{y}\in I\!R^3.$$

with some $K_2 \in (0,\infty)$ independent of $x, \mathbf{y}$, and

$$(4.9)\qquad \gamma \in L^{3/2}(G) \text{ and } (1+|x|)\gamma \in L^{r_0}(G) \text{ with some } r_0 \in (3/2,\infty).$$

Finally it is assumed that the external current $\mathbf{j}$ is located in a fixed finite ball, i. e. there is some $R_1 > 0$ with

$$(4.10)\qquad \mathbf{j}(t,x) = 0 \text{ for all } t\in(0,\infty), x\in I\!R^3\setminus B_{R_1}.$$

First it is shown that the convergence in Theorem 3.2 is strong on bounded subsets of Ω.

LEMMA 4.1. *For all $R > 0$ one has*

$$\|Q(\mathbf{E}(t),\mathbf{H}(t))\|_{L^2(\Omega\cap B_R)} \overset{t\to\infty}{\longrightarrow} 0.$$

Proof. Each $\mathbf{u} \in (\ker B)^\perp$ satisfies

$$(4.11)\qquad \text{div } (\underline{\mathbf{u}}_1) = 0, \quad \text{div } (\underline{\mathbf{u}}_2) = 0$$

$$\text{with } \vec{n}\underline{\mathbf{u}}_1 = 0 \text{ on } \Gamma_2 \text{ and } \vec{n}\underline{\mathbf{u}}_2 = 0 \text{ on } \Gamma_1$$

in the sense that

$$\int_\Omega (\underline{\mathbf{u}}_1\nabla\varphi + \underline{\mathbf{u}}_2\nabla\psi)\,dx = 0 \text{ for all } \varphi\in C_0^\infty(I\!R^3\setminus\overline{\Gamma_1}) \text{ and } \psi\in C_0^\infty(I\!R^3\setminus\overline{\Gamma_2}).$$

This follows from the fact that $(\nabla\varphi,\nabla\psi) \in \ker B$ for all $\varphi \in C_0^\infty(I\!R^3\setminus\overline{\Gamma_1})$ and $\psi\in C_0^\infty(I\!R^3\setminus\overline{\Gamma_2})$.

Suppose $\mathbf{u} \in (\ker B)^\perp \cap D(B)$. Then $\underline{\mathbf{u}}_1 \in W_E$, whereas $\underline{\mathbf{u}}_2 \in W_H$. Therefore 4.11 and the compactness theorem in [7], a generalization of the result in [16], see also [10], implies that

$$(4.12)\ (\ker B)^\perp\cap D(B) \text{ is compactly embedded in } L^2(\Omega\cap B_R) \text{ for all } R>0.$$

Now,the result follows from Lemma 2.2ii), Theorem 3.2 and 4.12. □

THEOREM 4.2. *Suppose $b > 1$. Then*

$$\int_{\{|x|\geq bt\}} |(\mathbf{E}(t), \mathbf{H}(t))|^2 dx \overset{t\to\infty}{\longrightarrow} 0.$$

Proof. The proof is based on an energy estimate. Let $g \in C^\infty(I\!R)$ with $g(u) = 1$ for $u \geq (1+b)/2$ and $g(u) = 0$ for $u \leq 1$. For $R > R_1$ define

$$\mathcal{E}^{(R)}(t) \overset{\text{def}}{=} \int_\Omega g((t+R)^{-1}|x|) \left[|\mathbf{E}(t)|^2 + |\mathbf{H}(t)|^2\right] dx$$

$$+ \int_G \gamma^{-1} g((t+R)^{-1}|x|) \left(\alpha|\partial_t \mathbf{P}|^2 + 2V(x, \mathbf{P})\right) dx.$$

Then one obtains from the basic equations using 2.15 and assumption 4.10 for all $t \geq 0$

$$\frac{d}{dt}\mathcal{E}^{(R)}(t) = 2\left\langle g((t+R)^{-1}|x|)(\mathbf{E}(t), \mathbf{H}(t)), B(\mathbf{E}(t), \mathbf{H}(t)) - \mathcal{R}\partial_t\mathbf{P}(t) - (\mathbf{j}(t), 0)\right\rangle_X$$

$$-(t+R)^{-2} \int_\Omega |x| g'((t+R)^{-1}|x|) \left[|\mathbf{E}(t)|^2 + |\mathbf{H}(t)|^2\right] dx$$

$$+ \int_G 2g((t+R)^{-1}|x|) \left(\mathbf{E}\partial_t\mathbf{P} - \gamma^{-1}|\partial_t\mathbf{P}|^2\right) dx$$

$$-(t+R)^{-2} \int_G \gamma^{-1}|x| g'((t+R)^{-1}|x|) \left(\alpha|\partial_t\mathbf{P}|^2 + 2V(x, \mathbf{P})\right) dx$$

$$\leq 2\left\langle g((t+R)^{-1}|x|)(\mathbf{E}(t), \mathbf{H}(t)), B(\mathbf{E}(t), \mathbf{H}(t))\right\rangle_X$$

$$-(t+R)^{-2} \int_\Omega |x| g'((t+R)^{-1}|x|) \left[|\mathbf{E}(t)|^2 + |\mathbf{H}(t)|^2\right] dx$$

$$\leq 2(t+R)^{-1} \int_\Omega |x|^{-1} g'((t+R)^{-1}|x|)\mathbf{E}(t) \cdot (x \wedge \mathbf{H}(t)) dx$$

$$-(t+R)^{-2} \int_\Omega |x| g'((t+R)^{-1}|x|) \left[|\mathbf{E}(t)|^2 + |\mathbf{H}(t)|^2\right] dx$$

Since $g(u) = 0$ for $u \leq 1$ and $g'(u) \geq 0$ it follows $\frac{d}{dt}\mathcal{E}^{(R)}(t) \leq 0$ and hence

$$\mathcal{E}^{(R)}(t) \leq \mathcal{E}^{(R)}(0) \text{ for all } R > R_1. \tag{4.13}$$

Since $b > 1$ one has by 4.13 for all $R > R_1$

$$\limsup_{t\to\infty} \int_{\{|x|\geq bt\}} |(\mathbf{E}(t), \mathbf{H}(t))|^2 dx \leq \limsup_{t\to\infty} \int_{\{|x|\geq(b+1)(t+R)/2\}} |(\mathbf{E}(t), \mathbf{H}(t))|^2 dx$$

$$\leq \limsup_{t\to\infty} \mathcal{E}^{(R)}(t) \leq \mathcal{E}^{(R)}(0).$$

Since $g(0) = 0$ it follows $\mathcal{E}^{(R)}(0) \overset{\mathrm{R}\to\infty}{\longrightarrow} 0$. Hence the assertion follows from the previous estimate letting $R \to \infty$. □

In the sequel let $R_0 > 0$, such that $\mathbb{R}^3 \setminus \Omega \subset B_{R_0} \overset{\mathrm{def}}{=} \{x \in \mathbb{R}^3 : |x| < R_0\}$ and choose $\chi_0 \in C^\infty(\mathbb{R}^3)$ with

$$\text{supp } \chi_0 \subset \Omega \quad \text{ and } \chi_0(x) = 1 \text{ on } \mathbb{R}^3 \setminus B_{R_0}. \tag{4.14}$$

For $\mathbf{w} \in X$ or $\mathbf{w} \in L^2(\mathbb{R}^3)$ define

$$\mathcal{C}_0\mathbf{w} \overset{\mathrm{def}}{=} \left((\nabla\chi_0) \wedge \underline{\mathbf{w}}_2, -(\nabla\chi_0) \wedge \underline{\mathbf{w}}_1\right). \tag{4.15}$$

For convenience $\chi_0\mathbf{w}$ and $\mathcal{C}_0\mathbf{w}$ will be regarded as elements of $L^2(\mathbb{R}^3)$ by extending them by zero outside supp χ_0 if $\mathbf{w} \in X$.

In what follows $W_{E,0}$ denotes the space of all $\mathbf{F} \in W_E$ with curl $\mathbf{F} = 0$. Since $\nabla\varphi \in W_{E,0}$ for all $\varphi \in C_0^\infty(\Omega)$ one has

$$\text{div } \mathbf{A} = 0 \text{ for all } \mathbf{A} \in W_{E,0}^{\perp}. \tag{4.16}$$

By the boundedness of supp $\nabla\chi_0$ it follows from 4.16 that curl $(\chi_0\mathbf{A}) = (\nabla\chi_0) \wedge \mathbf{A} + \chi_0$ curl $\mathbf{A} \in L^2(\mathbb{R}^3)$ and div $(\chi_0\mathbf{A}) = (\nabla\chi_0) \cdot \mathbf{A} \in L^2(\mathbb{R}^3)$ for all $\mathbf{A} \in W_{E,0}^{\perp} \cap W_E$. Here $\chi_0\mathbf{A}$ is extended by zero on $\mathbb{R}^3 \setminus$ supp χ_0 and χ_0 as in 4.14. From Sobolev's inequality one obtains $\chi_0\mathbf{A} \in L^6(\mathbb{R}^3)$ and hence $\mathbf{A} \in L^6(\mathbb{R}^3 \setminus B_{R_0})$.

The aim of the following considerations is to prove the estimate

LEMMA 4.3. *There exists a constant $K_3 \in (0,\infty)$, such that for all $\mathbf{A} \in W_E \cap W_{E,0}^{\perp}$ the estimate*

$$\|\mathbf{A}\|_{L^2(\Omega\cap B_{R_0})} + \|\mathbf{A}\|_{L^6(\mathbb{R}^3\setminus B_{R_0})} \leq K_3\| \; \mathit{curl}\; \mathbf{A}\|_{L^2(\Omega)}$$

holds.

LEMMA 4.4. *i) The set of all $\mathbf{F} \in W_{E,0}$ with bounded support is dense in $W_{E,0}$ (with respect to the $L^2(\Omega)$ -norm).*

ii) Let $\mathbf{A} \in L^2(\Omega \cap B_{R_0}) \cap L^6(\mathbb{R}^3 \setminus B_{R_0})$ with

$$\int_\Omega \mathbf{A} \;\mathit{curl}\; \mathbf{h} dx = 0 \textit{ for all } \mathbf{h} \in C_0^\infty(\mathbb{R}^3 \setminus \overline{\Gamma_2}, \mathbb{C}^3) \tag{4.17}$$

and

$$\int_\Omega \mathbf{A}\mathbf{F} dx = 0 \textit{ for all } \mathbf{F} \in W_{E,0} \textit{ with bounded support.} \tag{4.18}$$

Then $\mathbf{A} = 0$.

Proof. i) Suppose $\mathbf{F} \in W_{E,0}$. Since curl $\mathbf{F} = 0$ there exists some $\varphi \in L^6(\mathbb{R}^3 \setminus B_{R_0})$ with

$$\mathbf{F} = \nabla\varphi \text{ on } \mathbb{R}^3 \setminus B_{R_0} \tag{4.19}$$

Let $\psi_1 \in C_0^\infty(B_2)$ with $\psi_1 = 1$ on B_1 and $\psi_n \overset{\mathrm{def}}{=} \psi_1(x/n)$. Now define $\mathbf{F}_n(x) \overset{\mathrm{def}}{=} \mathbf{F}(x)$ if $x \in \Omega \cap B_n$ and $\mathbf{F}_n(x) \overset{\mathrm{def}}{=} \psi_n(x)\mathbf{F}(x) + \varphi(x)\nabla\psi_n(x)$ if $|x| \geq n$. Then $\mathbf{F}_n$ has bounded support and curl $\mathbf{F}_n = (\nabla\psi_n) \wedge \mathbf{F} + (\nabla\varphi) \wedge \nabla\psi_n = 0$ by 4.19. Since also $\mathbf{F}_n = \mathbf{F}$ near $\partial\Omega$ it follows easily that $\mathbf{F}_n \in W_{E,0}$. Next it follows from Hölder's inequality

$$\|\mathbf{F}_n - \mathbf{F}\|_{L^2(\Omega)} \leq \|(1-\psi_n)\mathbf{F}\|_{L^2(\Omega)} + \|\varphi\nabla\psi_n\|_{L^2(\Omega)}$$

$$\leq \|(1-\psi_n)\mathbf{F}\|_{L^2(\Omega)} + \|\varphi\|_{L^6(\{|x|>n\})}\|\nabla\psi_n\|_{L^3(\mathbb{R}^3)}$$

$$\leq \|(1-\psi_n)\mathbf{F}\|_{L^2(\Omega)} + \|\varphi\|_{L^6(\{|x|>n\})}\|\nabla\psi_n\|_{L^\infty(\mathbb{R}^3)}|B_{2n}|^{1/3}$$

$$\leq \|(1-\psi_n)\mathbf{F}\|_{L^2(\Omega)} + C_1\|\varphi\|_{L^6(\{|x|>n\})} \stackrel{n\to\infty}{\longrightarrow} 0$$

with some C_1 independent of n. This completes the proof of i).

Next let $\mathbf{A} \in L^2(\Omega \cap B_{R_0}) \cap L^6(\mathbb{R}^3 \setminus B_{R_0})$ satisfy 4.17 and 4.18. Then one has in analogy to 4.16

$$\text{curl } \mathbf{A} = 0 \text{ and } \text{div } \mathbf{A} = 0 \text{ on } \Omega. \tag{4.20}$$

Since supp $\nabla\chi_0$ is bounded, it follows from 4.20 that curl $(\chi_0\mathbf{A}) = (\nabla\chi_0) \wedge \mathbf{A} \in L^{6/5}(\mathbb{R}^3)$ and div $(\chi_0\mathbf{A}) = (\nabla\chi_0)\cdot\mathbf{A} \in L^{6/5}(\mathbb{R}^3)$, where $\chi_0\mathbf{A}$ is extended by zero on $\mathbb{R}^3\setminus$ supp χ_0. From Lemma 1 in [8] one obtains $\chi_0\mathbf{A} \in L^2(\mathbb{R}^3)$ and hence $\mathbf{A} \in L^2(\Omega)$. By the definition of W_E and 4.17 we have $\mathbf{A} \in W_{E,0}$. Since $\mathbf{A} \in L^2(\Omega)$, equation 4.18 holds for all $\mathbf{F} \in W_{E,0}$ by assertion i). But this means $\mathbf{A} \in W_{E,0}^{\perp}$ and therefore $\mathbf{A} = 0$. □

LEMMA 4.5. *Let $\{\mathbf{A}_n\}_{n\in\mathbb{N}}$ be a sequence in $W_E \cap W_{E,0}^{\perp}$, which is bounded in $L^2(\Omega \cap B_{R_0}) \cap L^6(\mathbb{R}^3 \setminus B_{R_0})$, such that $\{$ curl $\mathbf{A}_n\}_{n\in\mathbb{N}}$ is precompact in $L^2(\Omega)$. Then $\{\mathbf{A}_n\}_{n\in\mathbb{N}}$ is precompact in $L^2(\Omega \cap B_{R_0}) \cap L^6(\mathbb{R}^3 \setminus B_{R_0})$.*

Proof. Let $\tilde{\Omega} \stackrel{\text{def}}{=} B_{2R_0} \cap \Omega$ and choose $\chi_1 \in C_0^\infty(B_{2R_0})$ with $\chi_1(x) = 1$ on B_{R_0}, in particular

$$\chi_1(x) = 1 \text{ on supp } (\nabla\chi_0) \tag{4.21}$$

Let $S_1 \stackrel{\text{def}}{=} \Gamma_1 \cup \partial B_{2R_0}$ and $S_2 \stackrel{\text{def}}{=} \Gamma_2 = \partial\tilde{\Omega} \setminus S_1$. Recall that $\mathbb{R}^3 \setminus \Omega \subset B_{R_0} \subset B_{2R_0}$. In analogy to the definition of W_E let $\mathcal{W}_E$ be the space of all $\mathbf{e} \in H_{curl}(\tilde{\Omega})$ with $\vec{n}\wedge\mathbf{e} = 0$ on S_1 in the sense that

$$\int_{\tilde{\Omega}} \mathbf{e} \text{ curl } \mathbf{f} - \mathbf{f} \text{ curl } \mathbf{e}dx = 0 \text{ for all } \mathbf{f} \in C_0^\infty(\mathbb{R}^3 \setminus \overline{S_2}, \mathbb{C}^3).$$

Now, it follows from the assumptions that

$$\{\chi_1\mathbf{A}_n\}_{n\in\mathbb{N}} \text{ is bounded in } \mathcal{W}_E. \tag{4.22}$$

Since $\mathbf{A}_n \in W_{E,0}^{\perp}$ one has also

$$\{\text{ div } [\chi_1\mathbf{A}_n]\}_{n\in\mathbb{N}} = \{\mathbf{A}_n\nabla\chi_1\}_{n\in\mathbb{N}} \text{ is bounded in } L^2(\tilde{\Omega}) \tag{4.23}$$

and $\chi_1\vec{n}\mathbf{A} = 0$ on S_2, in the sense that

$$-\int_{\tilde{\Omega}} \chi_1\mathbf{A}_n\nabla\varphi dx = \int_{\tilde{\Omega}} (\text{ div } [\chi_1\mathbf{A}_n])\varphi dx = \int_{\tilde{\Omega}} (\mathbf{A}_n\nabla\chi_1)\varphi dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3 \setminus \overline{S_1})$.
Since the Lipschitz domain $\tilde{\Omega} = B_{2R_0} \cap \Omega$ and the decomposition of its boundary $\partial\tilde{\Omega} = S_1 \cup S_2$ satisfy the assumptions in [7], it follows from 4.22, 4.23 and the result in [7] that the sequence

$$\{\chi_1 \mathbf{A}_n\}_{n\in\mathbb{N}} \text{ is precompact in } L^2(\tilde{\Omega}) = L^2(B_{2R_0} \cap \Omega). \tag{4.24}$$

Let $\mathbf{f}_n(x) \overset{\text{def}}{=} \chi_0 \mathbf{A}_n(x)$ if $x \in \Omega$ and $\mathbf{f}_n(x) \overset{\text{def}}{=} 0$ if $x \in \mathbb{R}^3 \setminus \Omega$.
Next, 4.21, 4.24 and the compactness assumption on $\{\text{ curl } \mathbf{A}_n\}_{n\in\mathbb{N}}$ imply that the sequences

$$\{\text{ curl } \mathbf{f}_n\}_{n\in\mathbb{N}} = \{(\nabla\chi_0) \wedge \mathbf{A}_n + \chi_0 \text{ curl } \mathbf{A}_n\}_{n\in\mathbb{N}} \tag{4.25}$$

and

$$\{\text{ div } \mathbf{f}_n\}_{n\in\mathbb{N}} = \{\mathbf{A}_n \nabla\chi_0\}_{n\in\mathbb{N}} \text{ are precompact in } L^2(\mathbb{R}^3). \tag{4.26}$$

Recall that supp $\chi_0 \subset \Omega$. By 4.25 and 4.26 it follows from Sobolev's inequality or directly Lemma 1 in [8] that the sequence $(\mathbf{f}_n)_{n\in\mathbb{N}}$ is precompact in $L^6(\mathbb{R}^3)$ and hence

$$(\mathbf{A}_n)_{n\in\mathbb{N}} \text{ is precompact in } L^6(\Omega \setminus B_{R_0}), \tag{4.27}$$

since $\chi_0(x) = 1$ for $|x| > R_0$. □

Proof of Lemma 4.3

Suppose that the estimate was not correct, i.e. would exist a sequence $\mathbf{A}_n \in W_E \cap W_{E,0}^{\perp}, n \in \mathbb{N}$ with

$$1 = \|\mathbf{A}_n\|_{L^2(\Omega\cap B_{R_0})} + \|\mathbf{A}_n\|_{L^6(\mathbb{R}^3\setminus B_{R_0})} \geq n\|\text{ curl } \mathbf{A}_n\|_{L^2} \text{ for all } n \in \mathbb{N} \tag{4.28}$$

By Lemma 4.5 the sequence $\{\mathbf{A}_n\}_{n\in\mathbb{N}}$ is precompact in $L^2(\Omega \cap B_{R_0}) \cap L^6(\mathbb{R}^3 \setminus B_{R_0})$, i.e. there exist $\mathbf{A} \in L^2(\Omega \cap B_{R_0}) \cap L^6(\mathbb{R}^3 \setminus B_{R_0})$ and a subsequence $\mathbf{A}_{n_k}, k \in \mathbb{N}$ with

$$\|\mathbf{A}_{n_k} - \mathbf{A}\|_{L^2(\Omega\cap B_{R_0})} + \|\mathbf{A}_{n_k} - \mathbf{A}\|_{L^6(\mathbb{R}^3\setminus B_{R_0})} \overset{k\to\infty}{\longrightarrow} 0, \tag{4.29}$$

in particular

$$\|\mathbf{A}\|_{L^2(\Omega\cap B_{R_0})} + \|\mathbf{A}\|_{L^6(\mathbb{R}^3\setminus B_{R_0})} = 1. \tag{4.30}$$

From 4.28 and 4.29 it follows that

$$\int_\Omega \mathbf{A} \text{ curl } \mathbf{h} dx = \lim_{k\to\infty} \int_\Omega \mathbf{A}_{n_k} \text{ curl } \mathbf{h} dx \tag{4.31}$$

$$= \lim_{k\to\infty} \int_\Omega \mathbf{h} \text{ curl } \mathbf{A}_{n_k} dx = 0 \text{ for all } \mathbf{h} \in C_0^\infty(\mathbb{R}^3 \setminus \overline{\Gamma_2}, \mathbb{C}^3).$$

Furthermore

$$\int_\Omega \mathbf{A}\mathbf{F} dx = \lim_{k\to\infty} \int_\Omega \mathbf{A}_{n_k} \mathbf{F} dx = 0 \tag{4.32}$$

for all $\mathbf{F} \in W_{E,0}$ with bounded support. Now 4.31, 4.32 and Lemma 4.4 ii) would implie $\mathbf{A} = 0$. This contradicts 4.30.

The aim of the following considerations is to show decay of the potential energy and the local electromagnetic energy at least in time mean, i.e. for all $R > 0$

$$t^{-1}\int_0^t \left(\int_G \gamma^{-1}V(x,\mathbf{P}(s))dx + \|(\mathbf{E}(s),\mathbf{H}(s))\|^2_{L^2(\Omega\cap B_R)}\right) ds \overset{t\to\infty}{\longrightarrow} 0.$$

LEMMA 4.6. *There holds*

$$t^{-1}\int_0^t \langle Q(\mathbf{E}(s),\mathbf{H}(s)),\mathcal{R}\mathbf{P}(s)\rangle_X ds \overset{t\to\infty}{\longrightarrow} 0.$$

Proof. Let $\mathbf{u}(t) \overset{\text{def}}{=} Q(\mathbf{E}(t),\mathbf{H}(t))$ and $\mathbf{A}(t) \overset{\text{def}}{=} \int_0^t \underline{\mathbf{u}}_1(s)ds$.
Since $\mathbf{u}(t) \in (\ker B)^\perp$ one has $\mathbf{A}(t) \in W^\perp_{E,0}$. With

$$\text{curl } \underline{\mathbf{u}}_1(s) = -\underline{(B\mathbf{u}(s))}_2 = -\underline{[B(\mathbf{E}(t),\mathbf{H}(t))]}_2 = \text{ curl } \mathbf{E}(s) = -\partial_t\mathbf{H}(s)$$

one gets by using Lemma 4.3

$$\|\mathbf{A}(t)\|_{L^2(\Omega\cap B_{R_0})} + \|\mathbf{A}(t)\|_{L^6(\mathbb{R}^3\setminus B_{R_0})} \le K_3\| \text{ curl } \mathbf{A}(t)\|_{L^2(\Omega)}$$

$$= K_3\|\int_0^t \text{ curl } \underline{\mathbf{u}}_1(s)ds\|_{L^2(\Omega)} = K_3\|\mathbf{H}(0)-\mathbf{H}(t)\|_{L^2(\Omega)}$$

Now, it follows from Lemma 2.2 and the previous estimate that

$$\|\mathbf{A}(t)\|_{L^2(\Omega\cap B_{R_0})} + \|\mathbf{A}(t)\|_{L^6(\mathbb{R}^3\setminus B_{R_0})} \le C_1 \text{ for all } t \in (0,\infty). \tag{4.33}$$

with some constant C_1 independent of t. Next,

$$t^{-1}\int_0^t \langle Q(\mathbf{E}(s),\mathbf{H}(s)),\mathcal{R}\mathbf{P}(s)\rangle_X ds = t^{-1}\int_0^t\int_G \underline{\mathbf{u}}_1(s)\mathbf{P}(s)dxds \tag{4.34}$$

$$= t^{-1}\int_0^t\int_G \partial_t\mathbf{A}(s)\mathbf{P}(s)dxds = t^{-1}\int_G \mathbf{A}(t)\mathbf{P}(t)dx - t^{-1}\int_0^t\int_G \mathbf{A}(s)\partial_t\mathbf{P}(s)dxds$$

$$\le C_1t^{-1}\left(\|\mathbf{P}(t)\|_{L^2(G)} + \|\mathbf{P}(t)\|_{L^{6/5}(G)}\right)$$

$$+C_1t^{-1}\int_0^t \left(\|\partial_t\mathbf{P}(s)\|_{L^2(G)} + \|\partial_t\mathbf{P}(s)\|_{L^{6/5}(G)}\right) ds$$

$$\le C_1t^{-1}\left(\|\gamma^{1/2}\|_{L^\infty(G)} + \|\gamma^{1/2}\|_{L^3(G)}\right)\left(\|\mathbf{P}_0\|_{\mathcal{G}} + 2\int_0^t \|\partial_t\mathbf{P}(s)\|_{\mathcal{G}}ds\right)$$

by assumption 4.9 and Hölder's inequality. With 4.34 and Lemma 2.2 one obtains

$$t^{-1}\int_0^t \langle Q(\mathbf{E}(s),\mathbf{H}(s)),\mathcal{R}\mathbf{P}(s)\rangle_X ds$$

$$\le C_1\left(\|\gamma^{1/2}\|_{L^\infty(G)} + \|\gamma^{1/2}\|_{L^3(G)}\right)\left(t^{-1}\|\mathbf{P}_0\|_{\mathcal{G}} + 2t^{-1/2}\|\partial_s\mathbf{P}\|_{L^2((0,\infty),\mathcal{G})}\right) \overset{t\to\infty}{\longrightarrow} 0.$$

□

LEMMA 4.7. *There holds*

$$t^{-1}\int_0^t \left(\int_G \gamma^{-1}V(x,\mathbf{P}(s))dx + \|(1-Q)(\mathbf{E}(t),\mathbf{H}(t))\|_X^2\right) ds \overset{\mathrm{t}\to\infty}{\longrightarrow} 0,$$

in particular

$$\lim_{t\to\infty} t^{-1}\int_0^t \|(\mathbf{E}(s),\mathbf{H}(s))\|_X^2 ds = \lim_{t\to\infty} t^{-1}\int_0^t \|Q(\mathbf{E}(s),\mathbf{H}(s))\|_X^2 ds = \mathcal{E}_\infty,$$

where $\mathcal{E}_\infty$ as in 2.23.

Proof. It follows from Lemma 2.2 and 4.3 that

$$\|(1-Q)\left(\mathbf{E}(t),\mathbf{H}(t)\right)\|_X^2 = \langle(\mathbf{E}(t),\mathbf{H}(t)),(1-Q)\left[(\mathbf{J}(t),0)-\mathcal{R}\mathbf{P}(t)\right]\rangle_X$$

$$= \langle Q(\mathbf{E}(t),\mathbf{H}(t)),\mathcal{R}\mathbf{P}(t)\rangle_X + \langle(\mathbf{E}(t),\mathbf{H}(t)),(1-Q)(\mathbf{J}(t),0)\rangle_X - \int_G \mathbf{E}(t)\mathbf{P}(t)dx$$

$$\leq \langle Q(\mathbf{E}(t),\mathbf{H}(t)),\mathcal{R}\mathbf{P}(t)\rangle_X + C_1\|\mathbf{J}(t)\|_{L^2(\Omega)}$$

$$-\int_G \gamma^{-1}\left[\alpha\partial_t^2\mathbf{P}(t)+\partial_t\mathbf{P}(t)+(\nabla_y V)(x,\mathbf{P}(t))\right]\mathbf{P}dx$$

$$\leq \langle Q(\mathbf{E}(t),\mathbf{H}(t)),\mathcal{R}\mathbf{P}(t)\rangle_X + C_1\|\mathbf{J}(t)\|_{L^2(\Omega)}$$

$$-\int_G \gamma^{-1}\left(\alpha\partial_t^2\mathbf{P}(t)+\partial_t\mathbf{P}(t)\right)\mathbf{P}(t)dx - K_2^{-1}\int_G \gamma^{-1}V(x,\mathbf{P}(t))dx$$

by assumption 4.8. Now,

$$t^{-1}\int_0^t \left(\|(1-Q)\left(\mathbf{E}(s),\mathbf{H}(s)\right)\|_X^2 + K_2^{-1}\int_G \gamma^{-1}V(x,\mathbf{P}(s))dx\right) ds \tag{4.35}$$

$$\leq t^{-1}\int_0^t \left(\langle Q(\mathbf{E}(s),\mathbf{H}(s)),\mathcal{R}\mathbf{P}(s)\rangle_X + C_1\|\mathbf{J}(s)\|_{L^2(\Omega)}\right) ds$$

$$+C_2/t - 1/2t^{-1}\int_G \gamma^{-1}|\mathbf{P}(t)|^2dx - t^{-1}\int_G \alpha\gamma^{-1}\partial_t\mathbf{P}(t)\mathbf{P}(t)dx$$

$$+t^{-1}\int_0^t\int_G \alpha\gamma^{-1}|\partial_t\mathbf{P}(s)|^2dxds$$

$$\leq t^{-1}\int_0^t \left(\langle Q(\mathbf{E}(s),\mathbf{H}(s)),\mathcal{R}\mathbf{P}(s)\rangle_X + C_1\|\mathbf{J}(s)\|_{L^2(\Omega)}\right) ds$$

$$+C_3/t + C_3 t^{-1}\|\partial_t \mathbf{P}(t)\|_{\mathcal{G}}^2 + t^{-1}\|\alpha^{1/2}\partial_t \mathbf{P}\|_{L^2((0,\infty),\mathcal{G})}^2$$

$$\leq t^{-1}\int_0^t \left(\langle Q(\mathbf{E}(s),\mathbf{H}(s)),\mathcal{R}\mathbf{P}(s)\rangle_X + C_1\|\mathbf{J}(s)\|_{L^2(\Omega)}\right) ds + C_4/t$$

by Lemma 2.2 i) again.

In the previous estimates C_j are constants independent of t. Now, it follows from Lemma 4.6 and 4.35 that

$$t^{-1}\int_0^t \left(\int_G \gamma^{-1} V(x,\mathbf{P}(s))dx + \|(1-Q)(\mathbf{E}(t),\mathbf{H}(t))\|_X^2\right) ds \stackrel{t\to\infty}{\longrightarrow} 0. \tag{4.36}$$

Since

$$t^{-1}\int_0^t \|\alpha^{1/2}\partial_t \mathbf{P}(s)\|_{\mathcal{G}}^2 ds \leq t^{-1}\|\alpha^{1/2}\partial_t\mathbf{P}\|_{L^2((0,\infty),\mathcal{G})}^2 dx ds \stackrel{t\to\infty}{\longrightarrow} 0$$

by Lemma 2.2 i), 2.23 and 4.36 yield

$$\lim_{t\to\infty} t^{-1}\int_0^t \|Q(\mathbf{E}(s),\mathbf{H}(s))\|_X^2 ds = \lim_{t\to\infty} t^{-1}\int_0^t \|(\mathbf{E}(s),\mathbf{H}(s))\|_X^2 ds = \mathcal{E}_\infty,$$

which completes the proof. □

From Lemma 4.1 and the previous lemma one obtains now easily

Corollary 4.8. *For all $R>0$ one has*

$$t^{-1}\int_0^t \|(\mathbf{E}(s),\mathbf{H}(s))\|_{L^2(\Omega\cap B_R)} ds \stackrel{t\to\infty}{\longrightarrow} 0.$$

In what follows let

$$D(B_0) \stackrel{\text{def}}{=} H_{curl}(\mathbb{R}^3)\times H_{curl}(\mathbb{R}^3) \text{ and } B_0(\mathbf{e},\mathbf{h}) \stackrel{\text{def}}{=} (\text{ curl }\mathbf{h}, -\text{ curl }\mathbf{e}).$$

Furthermore, let Q_0 be the orthogonal projector on $(\ker B_0)^\perp$, which consists of all $\mathbf{u}\in L^2(\mathbb{R}^3)$ with div $\underline{\mathbf{u}}_j = 0$. The following estimate will be used in the proof of 4.7.

Lemma 4.9. *Let $s\in[0,1]$. Then there exists a constant $K_1\in(0,\infty)$ such that*

$$|\langle(1+|x|)^s\mathbf{f}, Q_0\mathbf{g}\rangle_{L^2(\mathbb{R}^3)}| \leq K_1\|\mathbf{f}\|_{H^1(\mathbb{R}^3)}\|(1+|x|)^s\mathbf{g}\|_{L^{q_0}(\mathbb{R}^3)}$$

for all $\mathbf{f}\in H^1(\mathbb{R}^3)$ and $\mathbf{g}\in L^2(\mathbb{R}^3)$ with $(1+|x|)^s\mathbf{f}\in L^2(\mathbb{R}^3)$ and $(1+|x|)^s\mathbf{g}\in L^{q_0}(\mathbb{R}^3)$. Here $1/q_0 = 1/(2r_0)+1/2$, where r_0 as in assumption 4.9.

Proof. By a standard density argument it suffices to consider $\mathbf{f},\mathbf{g}\in C_0^\infty(\mathbb{R}^3)$. Recall that $2r_0>3$. Let $p_1 \stackrel{\text{def}}{=} (\frac{1-s}{3}-\frac{1-s}{2r_0})^{-1}\in(\frac{3}{1-s},\infty]$ ($p_1=\infty$ for $s=1$) and $p_2 \stackrel{\text{def}}{=} (\frac{s}{3}-\frac{s}{2r_0})^{-1}$ Since $(s-1)p_1<-3$ and $sp_2>3$ one has

$$(1+|x|)^{s-1}\in L^{p_1}(\mathbb{R}^3) \text{ and } (1+|x|)^{-s}\in L^{p_2}(\mathbb{R}^3). \tag{4.37}$$

Now $\mathbf{F}\stackrel{\text{def}}{=}(1+|x|)^s Q_0\mathbf{f} - Q_0\left((1+|x|)^s\mathbf{f}\right)$ obeys

$$B_0\mathbf{F} = s(1+|x|)^{s-1}|x|^{-1}SQ_0\mathbf{f} + (1+|x|)^s B_0\mathbf{f} - B_0\left((1+|x|)^s\mathbf{f}\right)$$

$$= s(1+|x|)^{s-1}|x|^{-1}S[Q_0\mathbf{f}-\mathbf{f}],$$

where $S\mathbf{w} \stackrel{\text{def}}{=} (-x \wedge \underline{\mathbf{w}}_2, x \wedge \underline{\mathbf{w}}_1)$. Hence

$$\|(1+|x|)^{1-s} B_0 \mathbf{F}\|_{L^2(I\!R^3)} \le s\|(Q_0 - 1)\mathbf{f}\|_{L^2(I\!R^3)} \le s\|\mathbf{f}\|_{L^2(I\!R^3)}.$$

A similar estimate using div $\underline{(Q_0\mathbf{f})}_j = 0$ yields

$$\|(1+|x|)^{1-s} \text{ div } \underline{\mathbf{F}}_j\|_{L^2(I\!R^3)} \le s\|\underline{(Q_0\mathbf{f})}_j\|_{L^2(I\!R^3)} \le s\|\mathbf{f}\|_{L^2(I\!R^3)}.$$

With $1/q \stackrel{\text{def}}{=} 1/p_1 + 1/2$ we obtain by 4.37 and Hölder's inequality

$$\| \text{ curl } \underline{\mathbf{F}}_j\|_{L^q(I\!R^3)} \le C_1\|B_0\mathbf{F}\|_{L^q(I\!R^3)}$$

$$\le C_1\|(1+|x|)^{s-1}\|_{L^{p_1}(I\!R^3)}\|(1+|x|)^{1-s}B_0\mathbf{F}\|_{L^2(I\!R^3)} \le C_2\|\mathbf{f}\|_{L^2(I\!R^3)}$$

and

$$\| \text{ div } \underline{\mathbf{F}}_j\|_{L^q(I\!R^3)} \le C_2\|\mathbf{f}\|_{L^2(I\!R^3)}$$

By Sobolev's inequality or directly Lemma 1 in [8] one obtains

$$\|\mathbf{F}\|_{L^r(I\!R^3)} \le C_3\|D\mathbf{F}\|_{L^q(I\!R^3)} \tag{4.38}$$

$$\le C_4\left(\| \text{ curl } \underline{\mathbf{F}}_j\|_{L^q(I\!R^3)} + \| \text{ div } \underline{\mathbf{F}}_j\|_{L^q(I\!R^3)}\right) \le C_3\|\mathbf{f}\|_{L^2(I\!R^3)}$$

Here $r \stackrel{\text{def}}{=} (\frac{1}{q} - \frac{1}{3})^{-1} = (1/6 + \frac{1-s}{3} - \frac{1-s}{2r_0})^{-1} \in (2,6)$. Now, $\frac{1}{r} + \frac{1}{p_2} + \frac{1}{q_0} = 1$. Therefore 4.37, 4.38 and Hölder's inequality yield

$$|\langle \mathbf{F}, \mathbf{g}\rangle_{L^2(I\!R^3)}| \le \|\mathbf{F}\|_{L^r(I\!R^3)}\|(1+|x|)^{-s}\|_{L^{p_2}(I\!R^3)}\|(1+|x|)^s\mathbf{g}\|_{L^{q_0}(I\!R^3)} \tag{4.39}$$

$$\le C_5\|\mathbf{f}\|_{L^2(I\!R^3)}\|(1+|x|)^s\mathbf{g}\|_{L^{q_0}(I\!R^3)}$$

Since $q_0 \ge 6/5$ one has $q_0^* \le 6$. Therefore, it follows from Hölder's inequality, 4.39 and the embedding $H^1(I\!R^3) \hookrightarrow L^{q_0^*}(I\!R^3)$ that

$$|\langle (1+|x|)^s\mathbf{f}, Q_0\mathbf{g}\rangle_{L^2(I\!R^3)}| \le |\langle Q_0\mathbf{f}, (1+|x|)^s\mathbf{g}\rangle_{L^2(I\!R^3)} + \langle \mathbf{F}, \mathbf{g}\rangle_{L^2(I\!R^3)}| \tag{4.40}$$

$$\le C_5\|Q_0\mathbf{f}\|_{H^1(I\!R^3)}\|(1+|x|)^s\mathbf{g}\|_{L^{q_0}(I\!R^3)} + |\langle \mathbf{F}, \mathbf{g}\rangle_{L^2(I\!R^3)}|$$

$$\le C_6\|\mathbf{f}\|_{H^1(I\!R^3)}\|(1+|x|)^s\mathbf{g}\|_{L^{q_0}(I\!R^3)}$$

□

Since supp $\chi_0 \subset \Omega$, Lemma 2.2 yields $\chi_0(\mathbf{E}(\cdot), \mathbf{H}(\cdot)) \in L^\infty(0,\infty), D(B_0))$ and hence

$$Q_0\chi_0(\mathbf{E}(\cdot), \mathbf{H}(\cdot)) \in L^\infty((0,\infty), D(B_0) \cap (ker B_0)^\perp) \tag{4.41}$$

$$\subset L^\infty((0,\infty), H^1(I\!R^3)),$$

where $\chi_0 \in C^\infty(\mathbb{R}^3)$ as in 4.14.

LEMMA 4.10. *There holds i)*

$$t^{-1}\int_0^t \|(1-Q_0)\chi_0(\mathbf{E}(s),\mathbf{H}(s))\|_{L^2(\mathbb{R}^3)}ds \overset{t\to\infty}{\longrightarrow} 0.$$

ii)

$$t^{-1}\int_0^t \|Q_0\chi_0(\mathbf{E}(s),\mathbf{H}(s))\|^2_{L^2(\mathbb{R}^3)}ds \overset{t\to\infty}{\longrightarrow} \mathcal{E}_\infty.$$

and iii)

$$t^{-1}\int_0^t \|Q_0\chi_0(\mathbf{E}(s),\mathbf{H}(s))\|_{L^2(\{|x|\geq as\})}ds \overset{t\to\infty}{\longrightarrow} 0 \textit{ for all } a\in(1,\infty).$$

Proof. First the following estimate is proved.

$$\|(1-Q_0)\chi_0\mathbf{u}\|_{L^2(\mathbb{R}^3)} \leq K_1\left(\|(1-Q)\mathbf{u}\|_X + \|\mathbf{u}\|_{L^2(\Omega\cap B_{R_0})}\right) \tag{4.42}$$

for all $\mathbf{u}\in X$ with some constant independent of $\mathbf{u}$. For this purpose let $\mathbf{u}\in X$ and define $\mathbf{f} \overset{\text{def}}{=} (1-Q_0)\chi_0\mathbf{u} \in \ker B_0$, i.e. curl $\underline{\mathbf{f}}_j = 0$ on $\mathbb{R}^3$. Hence there exist $\varphi_j \in L^6(\mathbb{R}^3)$ with $\nabla\varphi_j \in L^2(\mathbb{R}^3)$ such that

$$\underline{\mathbf{f}}_j = \nabla\varphi_j \tag{4.43}$$

Hence

$$\|\mathbf{f}\|^2_{L^2(\mathbb{R}^3)} = \langle\chi_0\mathbf{u},\mathbf{f}\rangle_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3}\chi_0\cdot(\underline{\mathbf{u}}_1\nabla\varphi_1 + \underline{\mathbf{u}}_2\nabla\varphi_2)\,dx$$

$$= \langle\mathbf{u},(\nabla[\chi_0\varphi_1],\nabla[\chi_0\varphi_2])\rangle_X - \int_{\mathbb{R}^3}(\underline{\mathbf{u}}_1(\nabla\chi_0)\varphi_1 - \underline{\mathbf{u}}_2(\nabla\chi_0)\varphi_2)\,dx$$

Since $(\nabla[\chi_0\varphi_1],\nabla[\chi_0\varphi_2]) \in \ker B$ and supp $(\nabla\chi_0)$ is bounded, it follows

$$\|(1-Q_0)\chi_0\mathbf{u}\|^2_{L^2(\mathbb{R}^3)} = \|\mathbf{f}\|^2_{L^2(\mathbb{R}^3)} \tag{4.44}$$

$$\leq C_1\|(1-Q)\mathbf{u}\|_X\left(\|\nabla(\chi_0\varphi_1)\|_{L^2(\mathbb{R}^3)} + \|\nabla(\chi_0\varphi_2)\|_{L^2(\mathbb{R}^3)}\right)$$

$$+C_1\|\mathbf{u}\|_{L^2(\Omega\cap B_{R_0})}\left(\|\varphi_1\|_{L^2(B_{R_0})} + \|\varphi_2\|_{L^2(B_{R_0})}\right)$$

$$\leq C_1\left(\|(1-Q)\mathbf{u}\|_X + \|\mathbf{u}\|_{L^2(\Omega\cap B_{R_0})}\right)$$

$$\left(\|\nabla\varphi_1\|_{L^2(\mathbb{R}^3)} + \|\varphi_1\|_{L^6(\mathbb{R}^3)} + \|\nabla\varphi_2\|_{L^2(\mathbb{R}^3)} + \|\varphi_1\|_{L^6(\mathbb{R}^3)}\right)$$

$$\leq C_2\left(\|(1-Q)\mathbf{u}\|_X + \|\mathbf{u}\|_{L^2(\Omega\cap B_{R_0})}\right)\|\mathbf{f}\|_{L^2(\mathbb{R}^3)}.$$

This completes the proof of 4.42. Now, assertion i) follow immediately from Lemma 4.7, Corollary 4.8 and inequality 4.42.

Next, part i), Corollary 4.8 and Lemma 4.7 yield by the boundedness of supp $(1-\chi_0)$

$$\lim_{t\to\infty} t^{-1}\int_0^t \|Q_0\chi_0(\mathbf{E}(s),\mathbf{H}(s))\|^2_{L^2(I\!R^3)}ds = \lim_{t\to\infty} t^{-1}\int_0^t \|\chi_0(\mathbf{E}(s),\mathbf{H}(s))\|^2_{L^2(I\!R^3)}ds$$

$$= \lim_{t\to\infty} t^{-1}\int_0^t \|(\mathbf{E}(s),\mathbf{H}(s))\|^2_X ds = \mathcal{E}_\infty,$$

whence ii). Finally, part iii) follows from i) and Theorem 4.2. ◻

Next 4.7 is proved.

THEOREM 4.11. *Suppose* $g \in C_0^\infty(I\!R)$ *with* $g(u) = 1$ *on a neighbourhood of* $[0,1]$. *Then*

$$t^{-1}\langle Sg(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\rangle_{L^2(I\!R^3)} \overset{t\to\infty}{\longrightarrow} \mathcal{E}_\infty,$$

where $\mathcal{E}_\infty$ *as in 2.23 and* $S\mathbf{u} \overset{\text{def}}{=} (-x\wedge \underline{\mathbf{u}}_2, x\wedge \underline{\mathbf{u}}_1)$.

Proof. Define

$$F(t) \overset{\text{def}}{=} \langle Sg(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\rangle_{L^2(I\!R^3)}.$$

Then

$$F'(t) = 2\,\langle Sg(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), \tag{4.45}$$

$$Q_0\chi_0\left(B(\mathbf{E}(t),\mathbf{H}(t)) - (\mathbf{j}(t),0) - \mathcal{R}\partial_t\mathbf{P}(t)\right)\rangle_{L^2(I\!R^3)}$$

$$-t^{-2}\,\langle S|x|g'(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\rangle_{L^2(I\!R^3)}$$

$$= \sum_{j=0}^{2} h_j(t) + 2\,\langle Sg(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), B_0Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\rangle_{L^2(I\!R^3)}$$

$$-t^{-2}\,\langle S|x|g'(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\rangle_{L^2(I\!R^3)}$$

by 2.15. Here

$$h_0(t) \overset{\text{def}}{=} -2\,\langle Sg(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), Q_0\chi_0(\mathbf{j}(t),0)\rangle_{L^2(I\!R^3)}\,, \tag{4.46}$$

$$h_1(t) \overset{\text{def}}{=} -2\,\langle Sg(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), Q_0\chi_0\mathcal{R}\partial_t\mathbf{P}(t)\rangle_{L^2(I\!R^3)}\,, \tag{4.47}$$

$$h_2(t) \overset{\text{def}}{=} 2\,\langle Sg(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), Q_0\mathcal{C}_0(\mathbf{E}(t),\mathbf{H}(t))\rangle_{L^2(I\!R^3)}\,, \tag{4.48}$$

where $\mathcal{C}_0$ as in 4.15.

For $\mathbf{u} \in (\ker B_0)^\perp \cap D(B_0) \subset H^1(I\!R^3)$ one has $\operatorname{div} \underline{\mathbf{u}}_j = 0$. Therefore, it follows from the identity $x \wedge \operatorname{curl} \mathbf{a} = \nabla(x\mathbf{a}) - \mathbf{a} - (x\nabla)\mathbf{a}$ that

$$\langle Sg(|x|/t)\mathbf{u}, B_0\mathbf{v}\rangle_{L^2(I\!R^3)} + \langle Sg(|x|/t)B_0\mathbf{u}, \mathbf{v}\rangle_{L^2(I\!R^3)}$$

$$= \int_{I\!R^3} g(|x|/t)\mathbf{u} \cdot (x \wedge \operatorname{curl} \underline{\mathbf{v}}_1, x \wedge \operatorname{curl} \underline{\mathbf{v}}_2) dx$$

$$+ \int_{I\!R^3} g(|x|/t)(x \wedge \operatorname{curl} \underline{\mathbf{u}}_1, x \wedge \operatorname{curl} \underline{\mathbf{u}}_2) \cdot \mathbf{v} dx$$

$$= \int_{I\!R^3} g(|x|/t)\mathbf{u} \cdot (\nabla[x\underline{\mathbf{v}}_1], \nabla[x\underline{\mathbf{v}}_2])\, dx + \int_{I\!R^3} g(|x|/t)\, (\nabla[x\underline{\mathbf{u}}_1], \nabla[x\underline{\mathbf{u}}_2]) \cdot \mathbf{v} dx$$

$$-2\int_{I\!R^3} g(|x|/t)\mathbf{u} \cdot \mathbf{v} dx - \int_{I\!R^3} g(|x|/t)(x\nabla)[\mathbf{u}\mathbf{v}] dx$$

$$= -2t^{-1}\left\langle \tilde{S}g'(|x|/t)\mathbf{u}, \mathbf{v}\right\rangle_{L^2(I\!R^3)} + \left\langle [g(|x|/t) + t^{-1}|x|g'(|x|/t)]\mathbf{u}, \mathbf{v}\right\rangle_{L^2(I\!R^3)}$$

with $\tilde{S}\mathbf{u} \overset{\text{def}}{=} |x|^{-1}\left([x\underline{\mathbf{u}}_1]x, [x\underline{\mathbf{u}}_2]x\right)$.
Hence

$$2\left\langle Sg(|x|/t)Q_0\chi_0(\mathbf{E}(t), \mathbf{H}(t)), B_0Q_0\chi_0(\mathbf{E}(t), \mathbf{H}(t))\right\rangle_{L^2(I\!R^3)}$$

$$= \left\langle [g(|x|/t) + t^{-1}|x|g'(|x|/t)]Q_0\chi_0(\mathbf{E}(t), \mathbf{H}(t)), Q_0\chi_0(\mathbf{E}(t), \mathbf{H}(t))\right\rangle_{L^2(I\!R^3)}$$

$$-2t^{-1}\left\langle \tilde{S}g'(|x|/t)Q_0\chi_0(\mathbf{E}(t), \mathbf{H}(t)), Q_0\chi_0(\mathbf{E}(t), \mathbf{H}(t))\right\rangle_{L^2(I\!R^3)}$$

With 4.45 -4.48 it follows

$$F'(t) = \|Q_0\chi_0(\mathbf{E}(t), \mathbf{H}(t))\|^2_{L^2(I\!R^3)} + \sum_{j=0}^{3} h_j(t) \tag{4.49}$$

where

$$h_3(t) \tag{4.50}$$

$$\overset{\text{def}}{=} \left\langle [g(|x|/t) - 1 + t^{-1}|x|g'(|x|/t)]Q_0\chi_0(\mathbf{E}(t), \mathbf{H}(t)), Q_0\chi_0(\mathbf{E}(t), \mathbf{H}(t))\right\rangle_{L^2(I\!R^3)}$$

$$-t^{-1}\left\langle (2\tilde{S} + t^{-1}|x|S)g'(|x|/t)Q_0\chi_0(\mathbf{E}(t), \mathbf{H}(t)), Q_0\chi_0(\mathbf{E}(t), \mathbf{H}(t))\right\rangle_{L^2(I\!R^3)}$$

In the following estimates C_j are constants independent of s. Lemma 4.9 and 4.41 yield by Hölder's inequality and assumption 4.9

$$|h_1(s)| \le C_1\|(1 + |x|)^{-1/2}Sg(|x|/s)Q_0\chi_0(\mathbf{E}(s), \mathbf{H}(s))\|_{H^1(I\!R^3)}$$

$$\|(1+|x|)^{1/2}\chi_0\mathcal{R}\partial_s\mathbf{P}(s)\|_{L^{q_0}(I\!R^3)}$$

$$\leq C_1\|(1+|x|)^{-1/2}Sg(|x|/s)Q_0\chi_0(\mathbf{E}(s),\mathbf{H}(s))\|_{H^1(I\!R^3)}$$

$$\|(1+|x|)^{1/2}\gamma^{1/2}\|_{L^{2r_0}(I\!R^3)}\|\gamma^{-1/2}\partial_s\mathbf{P}(s)\|_{L^2(G)}$$

$$\leq C_2 s^{1/2}\|\partial_s\mathbf{P}(s)\|_{\mathcal{G}}$$

For all $T>0$ one obtains

$$t^{-1}\int_1^t |h_1(s)|ds \leq t^{-1}\int_1^T |h_1(s)|ds + C_2 t^{-1}\int_T^t s^{1/2}\|\partial_s\mathbf{P}(s)\|_{\mathcal{G}}ds$$

$$\leq t^{-1}\int_1^T |h_1(s)|ds + C_2\left(\int_T^t \|\partial_s\mathbf{P}(s)\|_{\mathcal{G}}^2 ds\right)^{1/2}$$

and hence by Lemma 2.2

$$\limsup_{t\to\infty} t^{-1}\int_1^t |h_1(s)|ds \leq C_2\left(\int_T^\infty \|\partial_s\mathbf{P}(s)\|_{\mathcal{G}}^2 ds\right)^{1/2}$$

for all $T>0$, which implies that

$$t^{-1}\int_1^t |h_1(s)|ds \overset{\mathrm{t}\to\infty}{\longrightarrow} 0. \tag{4.51}$$

Next

$$|h_0(t)| \leq C_3\|(1+|x|)^{-1}Sg(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\|_{H^1}\|(1+|x|)\chi_0(\mathbf{j}(t),0)\|_{L^{q_0}}$$

$$\leq C_4\|\mathbf{j}(t)\|_{L^2(B_{R_1})} \leq C_4\|\mathbf{j}(t)\|_{L^2(\Omega)}$$

by assumption 4.10 which implies that

$$t^{-1}\int_0^t |h_0(s)|ds \overset{\mathrm{t}\to\infty}{\longrightarrow} 0. \tag{4.52}$$

Similarely

$$|h_2(t)| \leq C_5\|(1+|x|)^{-1}Sg(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\|_{H^1}\|(1+|x|)\mathcal{C}_0(\mathbf{E}(t),\mathbf{H}(t))\|_{L^{q_0}}$$

$$\leq C_6\|(\mathbf{E}(t),\mathbf{H}(t))\|_{L^2(B_{R_0})}$$

and hence by Corollary 4.8

$$t^{-1}\int_0^t |h_2(s)|ds \overset{\mathrm{t}\to\infty}{\longrightarrow} 0. \tag{4.53}$$

Since $g'(|x|/t) = 0$ and $g(|x|/t) = 1$ if $|x| \leq at$ with some $a > 1$, Lemma 4.10 iii) yields

$$t^{-1}\int_0^t |h_3(s)|ds \leq C_7 t^{-1}\int_0^t \|Q_0\chi_0(\mathbf{E}(s),\mathbf{H}(s))\|_{L^2(\{|x|\geq as\})}ds \overset{t\to\infty}{\longrightarrow} 0. \tag{4.54}$$

Now, it follows from 4.49-4.54 and Lemma 4.10 that

$$\lim_{t\to\infty} t^{-1}F(t) = \lim_{t\to\infty} t^{-1}\int_1^t F'(s)ds$$

$$= \lim_{t\to\infty} t^{-1}\int_1^t \|Q_0\chi_0(\mathbf{E}(s),\mathbf{H}(s))\|^2_{L^2(I\!R^3)}ds = \mathcal{E}_\infty.$$

This completes the proof. □

Now the main results of this section 1.7 and 1.9 can be proved.

THEOREM 4.12. *For all $a < 1$ and $b > 1$ one has*

$$\|(\mathbf{E}(t),\mathbf{H}(t))\|_{L^2(\Omega\cap B_{at})} \overset{t\to\infty}{\longrightarrow} 0 \tag{4.55}$$

and

$$\|(\mathbf{E}(t),\mathbf{H}(t)) - t^{-1}S\chi_{\{at\leq|x|\leq bt\}}(\mathbf{E}(t),\mathbf{H}(t))\|_X \overset{t\to\infty}{\longrightarrow} 0.$$

Furthermore

$$\|(1-Q_0)\chi_0(\mathbf{E}(t),\mathbf{H}(t))\|_{L^2(I\!R^3)} \overset{t\to\infty}{\longrightarrow} 0.$$

Proof. Suppose $\delta > 0$. Choose $g \in C_0^\infty(I\!R,[0,\infty))$ with $g(y) = 1$ on $[0, 1+\delta/2]$ and $g(u) = 0$ for all $u > 1+\delta$. Then

$$\|(1-Q_0)\chi_0(\mathbf{E}(t),\mathbf{H}(t))\|^2_{L^2(I\!R^3)}$$

$$= \|\chi_0(\mathbf{E}(t),\mathbf{H}(t))\|^2_{L^2(I\!R^3)} - \|Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\|^2_{L^2(I\!R^3)}$$

$$\leq \|(\mathbf{E}(t),\mathbf{H}(t))\|^2_X - \|Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\|^2_{L^2(I\!R^3)}$$

$$\leq \|(\mathbf{E}(t),\mathbf{H}(t))\|^2_X - (1+\delta)^{-1}t^{-1}$$

$$\langle Sg(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\rangle_{L^2(I\!R^3)}$$

Theorem 4.11 yields

$$\limsup_{t\to\infty} \|(1-Q_0)\chi_0(\mathbf{E}(t),\mathbf{H}(t))\|^2_{L^2(I\!R^3)} \leq \left(1-(1+\delta)^{-1}\right)\mathcal{E}_\infty,$$

since $\limsup_{t\to\infty}\|(\mathbf{E}(t),\mathbf{H}(t))\|^2_X \leq \mathcal{E}_\infty$ by 2.23. By letting $\delta \to 0$ this implies

$$\lim_{t\to\infty} \|(1-Q_0)\chi_0(\mathbf{E}(t),\mathbf{H}(t))\|^2_{L^2(I\!R^3)} = 0 \tag{4.56}$$

This improves assertion i) of Lemma 4.10. Next, one obtains from Theorem 4.2, the boundedness of supp $(1-\chi_0)$, Lemma 2.2 i), 4.56 and Theorem 4.11 that for all $\beta > 1$

$$\lim_{t\to\infty} t^{-1}\langle S\chi_{\{|x|\le\beta t\}}(\mathbf{E}(t),\mathbf{H}(t)),(\mathbf{E}(t),\mathbf{H}(t))\rangle_X \tag{4.57}$$

$$= \lim_{t\to\infty} t^{-1}\langle Sg(|x|/t)(\mathbf{E}(t),\mathbf{H}(t)),(\mathbf{E}(t),\mathbf{H}(t))\rangle_X$$

$$= \lim_{t\to\infty} t^{-1}\langle Sg(|x|/t)\chi_0(\mathbf{E}(t),\mathbf{H}(t)),\chi_0(\mathbf{E}(t),\mathbf{H}(t))\rangle_{L^2(I\!R^3)}$$

$$= \lim_{t\to\infty} t^{-1}\langle Sg(|x|/t)Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)),Q_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\rangle_{L^2(I\!R^3)} = \mathcal{E}_\infty.$$

Here a function $g \in C_0^\infty(I\!R,[0,\infty))$ with the properties $g(y) = 1$ on $[0,\beta]$ and $g(u) = 0$ for all $u > 2\beta$ is chosen.

Let $\beta > 1$. Then one obtains from 4.57

$$\int_{\Omega\cap B_{at}} |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx \le \int_{\Omega\cap B_{\beta t}} |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx$$

$$-\beta^{-1}t^{-1}\int_{\{at\le|x|\le\beta t\}} |x| \quad |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx$$

$$\le \|(\mathbf{E}(t),\mathbf{H}(t))\|_X^2 - \beta^{-1}t^{-1}\int_{\Omega\cap B_{\beta t}} |x| \quad |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx$$

$$+\beta^{-1}a\int_{\Omega\cap B_{at}} |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx$$

$$\le \|(\mathbf{E}(t),\mathbf{H}(t))\|_X^2 - \beta^{-1}t^{-1}\langle S\chi_{\{|x|\le\beta t\}}(\mathbf{E}(t),\mathbf{H}(t)),(\mathbf{E}(t),\mathbf{H}(t))\rangle_X$$

$$+a\int_{\Omega\cap B_{at}} |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx$$

Hence

$$(1-a)\int_{\Omega\cap B_{at}} |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx \le \|(\mathbf{E}(t),\mathbf{H}(t))\|_X^2$$

$$-\beta^{-1}t^{-1}\langle S\chi_{\{|x|\le\beta t\}}(\mathbf{E}(t),\mathbf{H}(t)),(\mathbf{E}(t),\mathbf{H}(t))\rangle_X$$

Invoking 4.57 one gets

$$(1-a)\limsup_{t\to\infty}\int_{\Omega\cap B_{at}} |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx \le (1-\beta^{-1})\mathcal{E}_\infty \text{ for all } \beta > 1.$$

By letting $\beta \to 1$ this impliess

$$\|(\mathbf{E}(t), \mathbf{H}(t))\|_{L^2(\Omega \cap B_{at})} \overset{t\to\infty}{\longrightarrow} 0. \tag{4.58}$$

This completes the proof of the first assertion 4.55.

Suppose $\beta > 1$. Then it follows from Theorem 4.2 that

$$\limsup_{t\to\infty} t^{-1} \|S\chi_{\{|x|\le bt\}}(\mathbf{E}(t), \mathbf{H}(t))\|_X \le \limsup_{t\to\infty} t^{-1} \|S\chi_{\{|x|\le \beta t\}}(\mathbf{E}(t), \mathbf{H}(t))\|_X$$

$$\le \beta \limsup_{t\to\infty} \|(\mathbf{E}(t), \mathbf{H}(t))\|_X \le \beta \mathcal{E}_\infty^{1/2}$$

Letting $\beta \to 1$ this yields

$$\limsup_{t\to\infty} t^{-1} \|S\chi_{\{|x|\le bt\}}(\mathbf{E}(t), \mathbf{H}(t))\|_X \le \mathcal{E}_\infty^{1/2}$$

By 4.57 one obtains

$$\limsup_{t\to\infty} \|t^{-1} S\chi_{\{|x|\le bt\}}(\mathbf{E}(t), \mathbf{H}(t)) - (\mathbf{E}(t), \mathbf{H}(t))\|_X^2$$

$$= \limsup_{t\to\infty} \big(t^{-2} \|S\chi_{\{|x|\le bt\}}(\mathbf{E}(t), \mathbf{H}(t))\|_X^2$$

$$-2t^{-1} \left\langle S\chi_{\{|x|\le bt\}}(\mathbf{E}(t), \mathbf{H}(t)), (\mathbf{E}(t), \mathbf{H}(t)) \right\rangle_X + \|(\mathbf{E}(t), \mathbf{H}(t))\|_X^2 \big) \le 0,$$

which completes the proof. □

REMARK 2. *The above theorem does not provide any information on the asymptotic behaviour of* $\mathbf{P}$. *But if the potential is quadratically coercive in the sense that*

$$\mathbf{p}(\nabla_P V)(x, \mathbf{p}) \ge a_0 |\mathbf{p}|^2 \text{ for all } \mathbf{p} \in \mathbb{R}^3$$

with some $a_0 > 0$ *it follows easily from a similar estimate as 2.17 that*

$$\|\mathbf{P}(t)\|_{L^2(G\cap B_R)} \overset{t\to\infty}{\longrightarrow} 0 \text{ for all } R > 0. \tag{4.59}$$

provided that $\mathbf{E}$ *satisfies*

$$\|\mathbf{E}(t)\|_{L^2(\Omega\cap B_R)} \overset{t\to\infty}{\longrightarrow} 0 \text{ for all } R > 0. \tag{4.60}$$

In particular 4.59 holds if condition 1.8 is fulfilled by Theorem 4.12. Furthermore it turns out that condition 1.8 is also necessary for the local decay of the electromagnetic field in this case. This can be seen as follows. If $\|(\mathbf{E}(t), \mathbf{H}(t))\|_{L^2(\Omega\cap B_R)} \overset{t\to\infty}{\longrightarrow} 0$ *for all* $R > 0$ *then also 4.59 holds and therefore*

$$(\mathbf{E}(t), \mathbf{H}(t)) \overset{t\to\infty}{\longrightarrow} 0 \text{ in } X \text{ weakly} \tag{4.61}$$

and

$$\mathbf{P}(t) \overset{t\to\infty}{\longrightarrow} 0 \text{ in } L^2(G) \text{ weakly.} \tag{4.62}$$

Hence one obtains from 4.1, 4.61and 4.62 by letting $t \to \infty$ that

$$(1-Q)(\mathbf{D}_1, \mathbf{H}_0) = (1-Q)\left((\mathbf{E}_0, \mathbf{H}_0) + \mathcal{R}\mathbf{P}(0) - \int_0^\infty (\mathbf{j}(s), 0)ds\right)$$

$$= w - \lim_{t\to\infty} (1-Q)\left((\mathbf{E}(t), \mathbf{H}(t)) + \mathcal{R}\mathbf{P}(t)\right) = 0,$$

whence 1.8.

Invoking a result in [9] concerning the linear inhomogeneous Maxwell equations without polarization it can be shown that the solution $(\mathbf{E}, \mathbf{H})$ of 1.1-1.5 is asymptotically free in the sense that there exists a uniquely determined pair of functions $(\mathbf{F}_0, \mathbf{G}_0) \in L^2(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{F}_0 = \operatorname{div} \mathbf{G}_0 = 0$, such that

$$\|(\mathbf{E}(t), \mathbf{H}(t)) - (\mathbf{F}(t), \mathbf{G}(t))\|_{L^2(\Omega)} \xrightarrow{t\to\infty} 0 \tag{4.63}$$

Here $(\mathbf{F}, \mathbf{G}) \in C(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^6))$ denotes the solution to Maxwell's equations in the whole space, that is

$$\partial_t \mathbf{F} = \operatorname{curl} \mathbf{G}, \quad \partial_t \mathbf{G} = -\operatorname{curl} \mathbf{F}, \tag{4.64}$$

supplemented by the initial-condition

$$\mathbf{F}(0,x) = \mathbf{F}_0(x), \mathbf{G}(0,x) = \mathbf{G}_0(x). \tag{4.65}$$

This means that the solution to 1.1-1.4 behaves asymptotically like a free space solution to equations 4.64, 4.65 as $t \to \infty$. In what follows suppose that in addition

$$(1+|x|)^{1+\alpha_0}\gamma^{1/2} \in L^{r_1}(G) \tag{4.66}$$

for some $\alpha_0 > 0$ and $r_1 \in (3, \infty)$. Again this condition is fulfilled in the case where the set G is bounded.

THEOREM 4.13. *The strong limit*

$$\mathbf{U} \stackrel{\text{def}}{=} \lim_{t\to\infty} \exp(-tB_0) J^*(\mathbf{E}(t), \mathbf{H}(t))$$

exists in $L^2(\mathbb{R}^3)$ and $\mathbf{U} \in (\ker B_0)^\perp$. Here $J^* : L^2(\Omega) \to L^2(\mathbb{R}^3)$ provides the extension by zero on $\mathbb{R}^3 \setminus \Omega$.

Proof. It follows from Theorem 4.12 that for all $a < 1 < b$

$$\lim_{t\to\infty} \|t^{-1} S\chi_{\{at\le|x|\le bt\}} Q_0 \chi_0(\mathbf{E}(t), \mathbf{H}(t)) - J^*(\mathbf{E}(t), \mathbf{H}(t))\|_{L^2} = 0. \tag{4.67}$$

Let g be defined as in [9], Theorem 8 by $g(t,u) \stackrel{\text{def}}{=} c_\alpha t^{-1-\alpha} u^\alpha$ for $u \le (1+\alpha)^{-1}\alpha t$ and $g(t,u) \stackrel{\text{def}}{=} u^{-1}$ for $u \ge (1+\alpha)^{-1}\alpha t$. Here $\alpha \stackrel{\text{def}}{=} \alpha_0/2 > 0$ with α_0 as in assumption 4.66 and $c_\alpha \stackrel{\text{def}}{=} (1+\alpha^{-1})^{1+\alpha}$.
Since $g(t,t) = t^{-1}$, it follows easily from 4.67, Theorem 4.2 and Theorem 4.12 that

$$\lim_{t\to\infty} \|Sg(t,|x|) Q_0 \chi_0(\mathbf{E}(t), \mathbf{H}(t)) - J^*(\mathbf{E}(t), \mathbf{H}(t))\|_{L^2} = 0. \tag{4.68}$$

Next Theorem 4.12 yields further $\lim_{t\to\infty} \|(1-Q_0)J^*(\mathbf{E}(t), \mathbf{H}(t))\|_{L^2} = 0$, and hence by 4.68

$$\lim_{t\to\infty} \|L(t)\chi_0(\mathbf{E}(t), \mathbf{H}(t)) - J^*(\mathbf{E}(t), \mathbf{H}(t))\|_{L^2} = 0, \tag{4.69}$$

where $L(t) \stackrel{\text{def}}{=} Q_0 S g(t,|x|) Q_0 \in B(L^2, L^2)$ with g defined as above.

The following result concerning the inhomogeneous linear Maxwell equations can be found in [9], Theorem 8.

THEOREM 4.14. *Suppose that* $\mathbf{u} \in L^\infty((0,\infty), D(B)) \cap C([0,\infty), X)$ *solves* $\partial_t \mathbf{u} = B\mathbf{u} + \mathbf{f}$, *where* $\mathbf{f} \in L^1_{loc}([0,\infty), X)$ *obeys* $(1+|x|)^{1+\alpha_0}\mathbf{f} \in L^1((0,\infty), L^{q_1}(\Omega)) + L^\infty((0,\infty), L^{q_1}(\Omega))$.
Then the strong limit

$$\lim_{t\to\infty} \exp(-tB_0) L(t) \chi_0 \mathbf{u}(t)$$

with respect to the $L^2(\mathbb{R}^3)$*-topology exists.*

Here $q_1 \in [6/5, 2)$ is defined by $1/q_1 = 1/2 + 1/r_1$, where $\alpha_0 > 0$ and $r_1 \in [3,\infty)$ are as in assumption 4.66.

In order to apply Theorem 4.14 let $\mathbf{u}(t) \stackrel{\text{def}}{=} (\mathbf{E}(t), \mathbf{H}(t))$ and $\mathbf{f}(t) \stackrel{\text{def}}{=} \partial_t \mathcal{R}\mathbf{P}(t) + (\mathbf{j}(t), 0)$. With the assumptions 4.66, 4.10 and Lemma 2.2 one has $(1+|x|)^{1+\alpha_0}\mathbf{f} \in L^\infty((0,\infty), L^{q_1}(\Omega))$. Hence $\mathbf{u}$ satisfies the conditions of Theorem 4.14, which implies the existence of the limit

$$\lim_{t\to\infty} \exp(-tB_0) L(t) \chi_0 (\mathbf{E}(t), \mathbf{H}(t)). \tag{4.70}$$

By 4.69 one obtains the existence of the limit

$$\mathbf{U} \stackrel{\text{def}}{=} \lim_{t\to\infty} \exp(-tB_0) J^* (\mathbf{E}(t), \mathbf{H}(t)) \text{ in } L^2(\mathbb{R}^3). \tag{4.71}$$

Since ran $L(t) \subset$ ran Q_0, it follows from 4.69 and 4.71 that
$\mathbf{U} \in \overline{\text{ran } B_0} = (\ker B_0)^\perp$, i.e. div $(\underline{\mathbf{U}}_j) = 0$ on $\mathbb{R}^3$. Now, it follows easily that $(\mathbf{F}, \mathbf{G}) \stackrel{\text{def}}{=} \exp(tB_0)\mathbf{U}$ satisfies 4.63. □

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