AN UNIFIED APPROACH TO LINEAR DIFFERENTIAL ALGEBRAIC EQUATIONS AND THEIR ADJOINT EQUATIONS

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Abstract. Instead of a single matrix occurring in the standard setting, the leading term of the linear differential algebraic equation is composed of a pair of well matched matrices. This problem formulation enables an unified treatment of the original equation and its adjoint one. An index notion is proposed for the equations and solvability statements for the lower index case are proven. The original equation and its adjoint are shown to have the same index. Their fundamental solution matrices satisfy a relation that generalizes the classical Lagrange identity. The coefficients are assumed to be just continuous and only certain subspaces have to be continuously differentiable additionally. Solution representations are given that base on the solutions of certain inherent regular ODEs which are uniquely determined by the problem data.

Key words. differential algebraic equations, adjoint equations, index, Lagrange identity

AMS subject classifications. 34A09, 34A30

1. Introduction. For the implicit regular linear ordinary differential equation (ODE)

$$Ax' + Bx = 0 \tag{1.1}$$

with continuous coefficients $A, B : \mathcal{I} \subseteq \mathbb{R} \to L(\mathbb{C}^m)$, A nonsingular, the adjoint equation is

$$(A^*y)' - B^*y = 0. \tag{1.2}$$

One obtains (1.2) in three steps. First, (1.1) is transformed to the form $x' + A^{-1}Bx = 0$. Then, one takes its adjoint pair $z' - B^*A^{-*}z = 0$. Finally, $A^{-*}z = y$ is set. The standard theory dealing mostly with *explicit* regular ODEs provides the Lagrange identity (e.g.[4]) $Z^*(t)X(t) \equiv Z^*(t_0)X(t_0)$. This identity connects the pair of fundamental solution matrices X and Z associated with the explicit equations. Both matrices may be normalized at some common $t_0 \in \mathcal{I}$, i.e. $X(t_0) = Z(t_0) = I$ can be chosen. In terms of (1.1) and (1.2), one obtains

$$Y^*(t)A(t)X(t) \equiv A(t_0), \tag{1.3}$$

provided that simultaneously with X the relevant fundamental matrix Y is normalized. The explicit form of this connection is

$$Y(t) = \{A(t_0)[X(t)]^{-1}[A(t)]^{-1}\}^* = \{[A(t)]^*\}^{-1}[X(t)]^{-*}[A(t_0)]^*. \tag{1.4}$$

For (1.1), the natural solution regularity to be met is $x \in C^1(\mathcal{I})$. For (1.2), however, a solution y has to be a continuous function with A^*y being differentiable, only (e.g. [5]).

In the case of differential algebraic equations (DAEs), the leading coefficient A in (1.1) is singular everywhere on $\mathcal{I}$. As a consequence, the standard formulation (1.1)

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becomes somehow incomplete since it does not say anything about suitable solution regularity. Following [7], a more precise reformulation of (1.1) is

$$A(Px)' + (B - AP')x = 0, \tag{1.5}$$

where $P : \mathcal{I} \to L(\mathbb{C}^m)$ denotes an arbitrary C^1-projector function along the nullspace $\ker A$, say, $P := A^+A$ may be set. Clearly, the continuous functions x that have a component $Px \in C^1$ naturally form the solution space for (1.5), i.e. for (1.1).

For DAEs, the adjoint equation to (1.1) has the form (1.2), too, as it was claimed first in [3]. However, (1.2) is not a standard form DAE, that is, standard DAE theory (index notions, etc., see [7, 2])does not apply unless assuming A to be smooth and changing to $A^*y' - (B^* - A^{*\prime})y = 0$ instead of (1.2). As we could note in the case of a nonsingular coefficient A we had started with, there was nor reason for assuming A to be smooth. To obtain solvability results in case of DAEs (1.1), resp. (1.5) there is no need of smoothness of A, too, with the exception of that for certain subspaces, typically for $\ker A$ (see [7, 10]). As concerns the adjoint equation (1.1), in [1] solvability is reached under the same conditions in the index-1 case. Behind the approach in [1] one may become aware of the reformulation of the adjoint equation yielding

$$P^*(A^*y)' - (B^* - P^{*\prime}A^*)y = 0, \tag{1.6}$$

that corresponds to (1.5). With the notations introduced below, in later sections, one of the results of [1] states the connection of normalized maximal fundamental solution matrices

$$Y(t) = [A(t)]_c^{*-}[X(t)]^{-*}[A(t_0)]^*, \tag{1.7}$$

which generalizes (1.4). Here, $A(t), X(t)$ and $Y(t)$ are singular matrices and $[X(t)]^-$ and $[A(t)]_c^{*-}$ denote special reflexive generalized inverses.

Let us further mention that in optimal control, when DAEs are to be controlled, both types of equations (1.1) and (1.2) (or (1.5) and (1.6)) are coupled together into one large system (see [9]). The standard DAE theory does not apply to those mixed systems, either.

In this paper we are going to state a linear DAE as

$$A(Dx)' + Bx = q \tag{1.8}$$

with well matched continuous leading coefficients $A, D : \mathcal{I} \to L(\mathbb{C}^m)$. None of these coefficients has to be a projector. D is a kind of incidence matrix that figures out which derivatives are involved in fact. A natural solution of (1.8) is a continuous function x having a differentiable part Dx. We believe in proposing the right form to deal with, since it seems to bring much more symmetry, transparency and beauty. In particular, the adjoint equation turns to be of the same shape as the original one, namely,

$$D^*(A^*y)' - B^*y = p. \tag{1.9}$$

The pair (1.5) and (1.6) appears to be a special case.

The paper is organized as follows. Basic notions like well matchedness and index are given in Section 2. In Section 3, solvability statements for index-1 and index-2 tractable DAEs are proved. In both cases, the geometric solution spaces are described in detail and a uniquely determined *inherent* regular ODE is figured out. Let us stress that we do with continuous coefficients but only certain subspaces, typically im D are assumed to be spanned by C^1 functions. Section 4 contains special observations on fundamental matrices. For normalized maximal fundamental matrices, group properties are provided. Section 5 deals with adjoint equations. Index-1 and index-2 tractability as well as the relevant subspace–smoothness are shown to take place always simultaneously for (1.8) and (1.9). We end up with an explicit representation

$$Y(t) = [A(t)]_c^{*-}[D(t)]^{*-}[X(t)]^{-*}[D(t_0)]^*A(t_0)^*, \tag{1.10}$$

which again is valid for index-1 and index-2 tractable DAEs with special generalized inverses involved. When $D := P$ and the DAE is of index 1, we are in full coincidence with 1.7.

2. Basic notions. Consider linear equations (1.8) with continuous matrix coefficients $A, D, B : \mathcal{I} \subseteq \mathbb{R} \to L(\mathbb{C}^m)$.

DEFINITION 2.1. *A solution of (1.8) is a continuous function $x : \mathcal{I} \to \mathbb{C}^m$ that has a continuously differentiable product Dx and satisfies (1.8) pointwisely. Let*

$$C_D^1 := \{x \in C : Dx \in C^1\}$$

denote the respective function space.

As a counterpart of (1.8), we introduce equation (1.9). Later we will realize this equation to be rightfully called the adjoint DAE to the DAE (1.8). As the first glance we see that a kind of Lagrange identity becomes true. Namely, for each pair of solutions $x \in C_D^1$, $y \in C_{A^*}^1$ of the homogeneous equations (1.8), (1.9) it holds that

$$y^*(t)A(t)D(t)x(t) = \text{const} \quad t \in \mathcal{I}. \tag{2.1}$$

Indeed, we have

$$\begin{aligned}(y^*ADx)' &= ((A^*y)^*Dx)' = (A^*y)^{*'}Dx + (A^*y)^*(Dx)' \\ &= (D^*(A^*y)')^*x + y^*A(Dx)' = y^*Bx - y^*Bx = 0.\end{aligned}$$

For obtaining solvability and deeper relations, the leading term $A(Dx)'$ in (1.8) is assumed to be properly formed so that condition C1 below is satisfied.

Condition C1:

$$\ker A(t) \oplus \text{im } D(t) = \mathbb{C}^m, t \in \mathcal{I}, \tag{2.2}$$

and there are functions $\eta_i \in C^1(\mathcal{I}, \mathbb{C}^m)$, $i = 1, \ldots, m$, which pointwisely span im $D =$ span $\{\eta_1, \ldots, \eta_r\}$ and $\ker A =$ span $\{\eta_{r+1}, \ldots, \eta_m\}$.

Note that condition C1 requires both matrices $A(t)$ and $D(t)$ to be of constant rank r. Due to C1, there is a uniquely determined continuously differentiable matrix function $R : \mathcal{I} \to L(\mathbb{C}^m)$ such that

$$[R(t)]^2 = R(t), \quad \text{im } R(t) = \text{im } D(t), \quad \ker R(t) = \ker A(t), \quad t \in \mathcal{I}. \tag{2.3}$$

Let us say shortly, that the ordered pair $\{A, D\}$ *provides a smooth $\mathbb{C}^m$-decomposition*, if C1 is valid. The projector function R realizes this smooth decomposition.

NOTE. At this place it must be stressed that Condition C1 is not a restriction but an extension with respect to earlier assumptions. If we start up with a standard form DAE

$$\tilde{A}(t)x'(t) + \tilde{B}(t)x(t) = q(t), \tag{2.4}$$

where the leading coefficient has constant rank r, we may reformulate this DAE to obtain (1.8) with $A := \tilde{A}$, $D := \tilde{P}$ and $B := \tilde{B} - \tilde{A}\tilde{P}'$ supposed there is a C^1-projector P with $\ker \tilde{P} = \ker \tilde{A}$, as we did when deriving (1.5) proceeding from (1.1). Provided that $\tilde{A}$ itself is continuously differentiable, another reformulation is possible. Namely, let $\tilde{R}$ be a C^1-projector with im $\tilde{R} =$ im $\tilde{A}$, say $R = AA^+$. Set $A := \tilde{R}$, $D := \tilde{A}$ and $B := \tilde{B} - \tilde{R}\tilde{A}'$. In both cases condition C1 is fulfilled trivially.

The following assertion confirms in some sense that we are on a right way to an unified treatment of DAEs and their adjoint equations.

LEMMA 2.2. *$\{A, D\}$ provides a smooth $\mathbb{C}^m$-decomposition if and only if $\{D^*, A^*\}$ does so. If R realizes the decomposition for $\{A, D\}$ then R^* does it for $\{D^*, A^*\}$.*

Proof. Let C1 be true for $\{A, D\}$, $R^2 = R$, im $R =$ im D $\ker R = \ker A$. We have $R^{*2} = R^*$, im $R^* = \ker R^\perp = \ker A^\perp =$ im $A, \ker R^* =$ im $R^\perp =$ im $D^\perp = \ker D^*$. R^* is a projector, thus the decomposition $\mathbb{C}^m =$ im $R^* \oplus \ker R^* =$ im $A^* \oplus \ker D^*$ becomes true. Since R is C^1 so is R^* and thus im R^* and $\ker R^*$ are C^1-subspaces.

Starting with condition C1 valid for $\{D^*, A^*\}$ and applying similar arguments, we are done. ▯ For computational reasons the following assertion that can be checked easily may be useful.

LEMMA 2.3. *For a pair of arbitrary matrices $H, J \in L(\mathbb{C}^m)$,*

$$\ker H \oplus \text{im } J = \mathbb{C}^m \tag{2.5}$$

is true if and only if the next three relations

1. $\ker H \cap \text{im } J = \{0\}$,
2. $\ker HJ = \ker J$,
3. $\text{im } H = \text{im } HJ$

are satisfied.

REMARK 2.1. *Setting $H := A(t)$ and $J := D(t)$, one arrives at a triple of conditions equivalent to (2.2).*

Now we introduce for (1.8) a chain of matrix functions and time-varying subspaces to be used throughout this paper. The argument t is dropped, all relations are ment pointwisely.

$$\left.\begin{array}{l} G_0 := AD, \quad B_0 := B, \\ Q_i \text{ is a projector onto } \ker G_i, \quad P_i := I - Q_i, \\ W_i \text{ is a projector}, \ker W_i = \text{im } G_i, \\ G_{i+1} := G_i + B_i Q_i, \qquad B_{i+1} = B_i P_i, \\ \boldsymbol{N}_i := \ker G_i = \text{im } Q_i, \\ \boldsymbol{S}_i := \{z \in \mathbb{C}^m : B_i z \in \text{ im } G_i\} = \ker W_i B_i, \end{array}\right\} \quad i = 0, 1. \tag{2.6}$$

By construction it holds that

$$\begin{gathered} \text{im } G_0 \subseteq \text{im } G_1, \qquad \boldsymbol{S}_0 \subseteq \boldsymbol{S}_1, \quad \boldsymbol{N}_0 \subseteq \boldsymbol{S}_1, \\ \boldsymbol{S}_1 = P_0 \boldsymbol{S}_1 \oplus \boldsymbol{N}_0, \quad \dim \boldsymbol{N}_1 = \dim(\boldsymbol{N}_0 \cap \boldsymbol{S}_0). \end{gathered}$$

For a given P_0, we denote by D^- the reflexive generalized inverse of D with

$$DD^- = R, \quad D^- D = P_0.$$

By this, D^- is uniquely determined and depends only on the choice of P_0. Obviously, it is always possible to choose projectors Q_0, P_0 to be continuous and we will do so. Then, D^- also is continuous.

When the argument is indicated, the shorter notation $D^-(t)$ is considered equivalent to $[D(t)]^-$. Moreover, for all operations $\dagger$ on arbitrary matrix functions H and on the values $H(t)$, $t \in \mathcal{I}$, notations $H^\dagger(t)$ and $[H(t)]^\dagger$ are considered equivalent, where $\dagger$ is -1, $-, +, *$ and the corresponding operations are the following: taking the inverse (when exists), taking a reflexive generalized inverse and the Moore-Penrose generalized inverse and the adjoint.

In the next sections we will make of use of the reflexive generalized inverse A^- of A such that $A^-A = R$, $AA^- = I - W_0$. Again, W_0 may and will be chosen continuous, then so is A^-. Further, auxiliary matrices $\mathcal{G}_{i+1} := G_i + W_iB_iQ_i$, $i = 0, 1$ will be useful, as well.

DEFINITION 2.4. *Let C1 be valid.*

1. *Equation (1.8) is said to be an* index-1 tractable *DAE if*

$$\boldsymbol{N}_0(t) \cap \boldsymbol{S}_0(t) = \{0\}, \qquad t \in \mathcal{I}. \tag{2.7}$$

2. *Equation (1.8) is said to be an* index-2 tractable *DAE if*

$$\dim \boldsymbol{N}_0(t) \cap \boldsymbol{S}_0(t) = \text{const} > 0, \tag{2.8}$$

$$\boldsymbol{N}_1(t) \cap \boldsymbol{S}_1(t) = \{0\}, \qquad t \in \mathcal{I}. \tag{2.9}$$

REMARK 2.2. For DAEs in the form (2.4) by the reformulations mentioned in the above NOTE we get the same tractability-index μ, $\mu = 1, 2$ as with the former definitions (see [10], [1]).

REMARK 2.3. If all matrices are smooth enough for defining the differentiation index of (2.4), then tractability index μ, $\mu \in \{1, 2\}$ leads to differentiation index μ.

REMARK 2.4. We stress that the solvability assertions below will show an index-μ tractable DAE ($\mu = 1, 2$) to have also perturbation index μ without assuming the coefficients or parts of them to be smooth. Continuity of the coefficients plus a C^1-basis of certain subspaces will do.

REMARK 2.5. For the case of constant coefficients A, D, B, it can be checked ([7]) that using the chain (2.2) up to $i = \mu - 1$, $1 \leq \mu \leq m$, the matrix pencil $\{AD, B\}$ is regular with Kronecker index μ if and only if $\dim \boldsymbol{N}_i \cap \boldsymbol{S}_i > 0$, $i = 0, \ldots, \mu - 2$, but $\boldsymbol{N}_{\mu-1} \cap \boldsymbol{S}_{\mu-1} = \{0\}$.

We recall that index-1 tractability holds if and only if G_1 is nonsingular ([7, Theorem A13]). In this case, $G_2 = G_1$. In the index-2 case, G_1 is singular and has constant rank, say r_1. Now G_2 becomes nonsingular. It follows also that the index definition is independent of the choice of Q_0 and Q_1 in the matrix chain (2.6).

Index-2 DAEs are characterized by hidden constraints and a lower degree of freedom. Formally, index-2 tractability provides the decomposition $\boldsymbol{N}_1(t) \oplus \boldsymbol{S}_1(t) = \mathbb{C}^m$.

LEMMA 2.5. *For an index-2 tractable DAE the decomposition*

$$D\boldsymbol{S}_1 \oplus D\boldsymbol{N}_1 \oplus \ker A = \mathbb{C}^m, \quad t \in \mathcal{I} \tag{2.10}$$

holds true. The dimensions of these subspaces are $r_1 + r - m$, $m - r_1$ and $m - r$, respectively. If $\hat{Q}_1(t)$ is the special projector onto $\boldsymbol{N}_1(t)$ along $\boldsymbol{S}_1(t)$, $\quad t \in \mathcal{I}$, then

$$D\hat{P}_1D^-, \quad D\hat{Q}_1D^-, \quad I - R \tag{2.11}$$

are projectors that realize decomposition (2.10).

Proof. Choose $\hat{Q}_1(t)$ to be the projector onto $\boldsymbol{N}_1(t)$ along $\boldsymbol{S}_1(t)$. One can verify that $\hat{Q}_1$ has a representation $\hat{Q}_1 = Q_1G_2^{-1}BP_0$, where Q_1 is an arbitrary projector onto $\boldsymbol{N}_1$ and G_2 is associated with Q_1 as (2.6) dictates. By construction it holds that $\hat{Q}_1 = \hat{Q}_1\hat{G}_2^{-1}BP_0$ ([7, Lemma 14]), where $\hat{G}_2 := G_1 + BP_0\hat{Q}_1$ thus $\hat{Q}_1Q_0 = 0$, and with $\hat{P}_1 := I - \hat{Q}_1$, $P_0\hat{P}_1Q_0 = 0$ takes place. Therefore

$$D\hat{Q}_1D^-D\hat{Q}_1D^- = D\hat{Q}_1P_0\hat{Q}_1D^- = D\hat{Q}_1D^-,$$
$$D\hat{P}_1D^-D\hat{P}_1D^- = D(I-\hat{Q}_1)D^-D(I-\hat{Q}_1)D^- = D(I-\hat{Q}_1)D^- = D\hat{P}_1D^-.$$

Obviously, $D\hat{Q}_1D^-D\hat{P}_1D^- = 0, \quad D\hat{P}_1D^-D\hat{Q}_1D^- = 0$. Moreover, $R = DD^-$ leads to

$$\begin{aligned} D\hat{Q}_1D^-(I-R) &= 0, \quad (I-R)D\hat{Q}_1D^- = 0,\\ D\hat{P}_1D^-(I-R) &= 0, \quad (I-R)D\hat{Q}_1D^- = 0,\\ \operatorname{im} D\hat{P}_1 &\subseteq D\boldsymbol{S}_1, \qquad \operatorname{im} D\hat{Q}_1 \subseteq D\boldsymbol{N}_1. \end{aligned}$$

Finally, because of the relation $P_0\hat{Q}_1 = (I - Q_0\hat{Q}_1)\hat{Q}_1$, where $(I - Q_0\hat{Q}_1)$ is nonsingular, $P_0\boldsymbol{N}_1$ has the same dimension $m - r_1$ as $\boldsymbol{N}_1$ has. Hence, $\dim D\boldsymbol{N}_1 = \dim P_0\boldsymbol{N}_1 = \dim \boldsymbol{N}_1 = m - r_1$. For reasons of dimensions it follows that

$$\operatorname{im} D\hat{P}_1D^- = D\boldsymbol{S}_1, \quad \operatorname{im} D\hat{Q}_1D^- = D\boldsymbol{N}_1.$$

□

The subspace $D\boldsymbol{S}_1$ will arise later to be relevant for the *inherent regular ODE* while $D\boldsymbol{N}_1$ is somewhat responsible for the *hidden constraint.* This is why we now consider a refinement of the standard condition C1.

Condition C2: Both subspaces $D\boldsymbol{S}_1$ and $D\boldsymbol{N}_1$ are spanned by continuously differentiable functions.

By construction, $\hat{P}_1, \hat{Q}_1$ are continuous, but we would like having all three projectors in (2.11) continuously differentiable.

LEMMA 2.6. *Given an index-2 tractable DAE and condition C2 be satisfied. Then all projectors in (2.11) are continuously differentiable.*

Proof. Denote the continuously differentiable functions that span $D\boldsymbol{S}_1$, $D\boldsymbol{N}_1$ and $\ker A$ by $Ds_j,\ j = 1, \ldots, r_1 - (m-r), Dn_j,\ j = 1, \ldots, m - r_1$ and $\eta_{r+1}, \ldots, \eta_m$, respectively. The matrix function

$$\Gamma := \left(Ds_1, \ldots, Ds_{r_1-(m-r)}, Dn_1 \ldots, Dn_{m-r_1}, \eta_{r+1}, \ldots, \eta_m\right)$$

is nonsingular and belongs to C^1. Obviously,

$$\Gamma I^{DS}\Gamma^{-1}, \quad \Gamma I^{DN}\Gamma^{-1}, \quad \Gamma I^{A}\Gamma^{-1} \tag{2.12}$$

with blockdiagonal projector matrices $I^{DS} = \operatorname{diag}(I_{r_1+r-m}, 0_{m-r_1}, 0_{m-r})$, $I^{DN} = \operatorname{diag}(0_{r_1+r-m}, I_{m-r_1}, 0_{m-r})$ and $I^A = \operatorname{diag}(0_{r_1+r-m}, 0_{m-r_1}, 1_{m-r})$ inside are C^1–projectors that realize the decomposition (2.10). The indices show the dimensions of the unit and zero matrices. The projectors (2.12) are uniquely defined by the decomposition, therefore, due to Lemma 2.5, they coincide with $D\hat{P}_1D^-$, $D\hat{Q}_1D^-$ and $I - R$, respectively.

In particular, $D\hat{P}_1D^-$ and $D\hat{Q}_1D^-$ belong to C^1. □

We postpone until Section 5 the formulation of a theorem concerning condition C2 that plays the same role for the pair of index-2 equations as Lemma 2.2 about condition C1 will do for the index-1 problems (1.8) and (1.9). The next section explains what can be gained by having conditions C1 and C2 for a DAE (1.8).

3. Solvability of index-1 and index-2 tractable DAEs.

3.1. Initial value problem for index-1 equations. The DAE (1.8) can be rewritten as

$$\underbrace{(AD + BQ_0)}_{G_1}\{D^-(Dx)' + Q_0x\} + BP_0x = q, \tag{3.1}$$

since $R = DD^-$, $A = AR = ADD^-$, $Q_0D^- = D^- - P_0D^- = D^- - D^-DD^- = 0$ and $DQ_0 = DD^-DQ_0 = DP_0Q_0 = 0$. Supposed we have an index-1 tractable DAE (1.8), G_1 is nonsingular and scaling by G_1^{-1} leads to

$$D^-(Dx)' + Q_0x + G_1^{-1}BP_0x = G_1^{-1}q \tag{3.2}$$

Multiplication by P_0 and Q_0 decouples equation (3.2) into

$$R(Dx)' + DG_1^{-1}BP_0x = DG_1^{-1}q, \tag{3.3}$$

$$Q_0x + Q_0G_1^{-1}BP_0x = Q_0G_1^{-1}q. \tag{3.4}$$

A solution $x \in C_D^1$ of (1.8), if exists, may be represented as

$$x = P_0x + Q_0x \tag{3.5}$$

$$= D^-Dx + Q_0x = (I - Q_0G_1^{-1}B)D^-Dx + Q_0G_1^{-1}q, \tag{3.6}$$

where Dx satisfies the regular ODE

$$(Dx)' - R'Dx + DG_1^{-1}BD^-Dx = DG_1^{-1}q. \tag{3.7}$$

The latter equation is an equivalent of (3.3) since $RD = D$. This fact moves us to take a close look at the regular ODE

$$u' - R'u + DG_1^{-1}BD^-u = DG_1^{-1}q. \tag{3.8}$$

Multiplication of (3.8) by $(I - R)$ leads to

$$\begin{array}{lrcl} & (I-R)u' - (I-R)R'u & = & 0, \\ \text{i.e.} & ((I-R)u)' - (I-R)'u + (I-R)'Ru & = & 0, \\ \text{thus,} & ((I-R)u)' - (I-R')(I-R)u & = & 0. \end{array}$$

Hence, for any solution u of (3.8), we have $u = Ru$, if $u(\tilde{t}) = R(\tilde{t})u(\tilde{t})$ holds at some $\tilde{t} \in \mathcal{I}$. In other words, im R = im D represents a (time varying) invariant subspace for (3.8).

DEFINITION 3.1. *Equation (3.8) is said to be* the inherent regular ODE *of index-1 equation (1.8).*

At this place we recall that the index–1 property $\boldsymbol{N}_0 \cap \boldsymbol{S}_0 = \{0\}$ is equivalent to $\boldsymbol{N}_0 \oplus \boldsymbol{S}_0 = \mathbb{C}^m$ ([7, Appendix A, Theorem 13]). This allows to consider a special (canonical) projector P_{0c} onto $\boldsymbol{S}_0$ along $\boldsymbol{N}_0$. Given a fixed P_0 along $\boldsymbol{N}_0$, a possible representation is

$$P_{0c} = I - Q_0G_1^{-1}B. \tag{3.9}$$

We see that P_{0c} is continuous.

THEOREM 3.2. *Let an index-1 tractable DAE (1.8) be given.*

1. *For each* $q \in C(\mathcal{I}, \mathbb{C}^m)$, $d \in \operatorname{im} D(t_0)$, $t_0 \in \mathcal{I}$, *the initial value problem*

$$A(Dx)' + Bx = q, \quad D(t_0)x(t_0) = d \tag{3.10}$$

is uniquely solvable in C_D^1.

2. *(1.8) has perturbation index 1.*

3. *Through each pair* (t_0, x_0), $t_0 \in \mathcal{I}$, $x_0 \in \boldsymbol{S}_0(t_0)$, *passes exactly one solution of the homogeneous equation.*

Proof.

1. First we obtain function $u \in C^1$, the uniquely determined solution of the inherent regular ODE which satisfies the initial condition $u(t_0) = d$. Then we construct the function

$$x := P_{0c}D^- u + Q_0 G_1^{-1} q \in C. \tag{3.11}$$

Observe that $Dx = DP_{0c}D^- u = DD^- u = Ru = u \in C^1$, and $D(t_0)x(t_0) = u(t_0) = d$. Now it remains to check via (2.3), (3.4) that x satisfies the DAE itself.
Assume that $\tilde{x} \in C_D^1$ also is a solution of (3.10) different from the solution x constructed above. Then, $\hat{x} = \tilde{x} - x$ satisfies (3.10) with $q = 0$ and $d = 0$. Due to (3.7) we get $D\hat{x} = 0$, thus, $P_0\hat{x} = D^- D\hat{x} = 0$ and by (3.4) $Q_0\hat{x} = 0$, i.e. $\tilde{x} = x + \hat{x} = x + P_0\hat{x} + Q_0\hat{x} = 0$ in contrast to the assumption.

2. Given a compact interval $\mathcal{I}$. Compare the solutions x and $x_q \in C_D^1$ of the IVPs

$$\begin{array}{rclcrcl} A(Dx)' + Bx & = & 0, & & D(t_0)x(t_0) & = & d, \\ A(Dx)' + Bx & = & q, & & D(t_0)x(t_0) & = & d. \end{array} \tag{3.12}$$

For the corresponding solutions u and u_q of the inherent regular ODEs, we derive

$$\|u_q - u\|_\infty \le K_1 \|DG_1^{-1} q\|_\infty,$$

with some constant K_1, hence

$$\|x_q - x\|_\infty = \|P_{0c}D^-(u_q - u) + Q_0 G_1^{-1} q\|_\infty \le K_2 \|q\|_\infty$$

holds with some constant K_2.

3. $x_0 \in \boldsymbol{S}(t_0)$ means $x_0 = P_{0c}(t_0)x_0 = P_{0c}(t_0)D^-(t_0)D(t_0)x_0$. Solve the IVP

$$A(Dx)' + Bx = 0, \qquad D(t_0)x(t_0) = D(t_0)x_0.$$

The solution is $x = P_{0c}D^- u$, therefore

$$x(t_0) = P_{0c}(t_0)D^-(t_0)u(t_0) = P_{0c}(t_0)D^-(t_0)D(t_0)x_0 = x_0.$$

□

REMARK 3.1. The initial condition $D(t_0)x(t_0) = d \in \operatorname{im} D(t_0)$ can be replaced by

$$D(t_0)x(t_0) = D(t_0)x^0, \quad x^0 \in \mathbb{C}^m. \tag{3.13}$$

The latter choice has the advantage that x^0 remains in $\mathbb{C}^m$. In particular, the variational equation for $x'_{x_0} =: X \in C_D^1(\mathcal{I}, L(C^m))$ becomes

$$A(DX)' + BX = 0, \quad D(t_0)(X(t_0) - I) = 0. \tag{3.14}$$

This matrix problem will be addressed in the next section. However, applying (3.13) one has to take into account that in general $x(t_0) \neq x^0$ has to be expected. $x(t_0) = x^0$ holds only if x^0 is consistent, i.e. $Q_{0c}x_0 = Q_0 G_1^{-1} q(t_0)$.

REMARK 3.2. One could think that the inherent regular ODE (3.8) would depend on how the projector P_0 and, consequently, D^- are chosen. However, both terms DG_1^{-1} and $DG_1^{-1}BD^-$ remain invariant when changing P_0.

Namely, let $P_0, \tilde{P}_0$ be two projectors along $\boldsymbol{N}_0$, $D^-, \tilde{D}^-$ the respective generalized inverses of D, $G_1 = AD + BQ_0$, $\tilde{G}_1 = AD + B\tilde{Q}_0$. Because of $\tilde{Q}_0 = Q_0\tilde{Q}_0$, $\tilde{G}_1 = AD + BQ_0\tilde{Q}_0 = G_1(P_0 + \tilde{Q}_0)$ and $\tilde{G}_1^{-1} = (\tilde{P}_0 + Q_0)G_1^{-1}$, we have $D\tilde{G}_1^{-1} = DG_1^{-1}$. Further,

$$\begin{aligned} D\tilde{G}_1^{-1}B\tilde{D}^- &= D\tilde{G}_1^{-1}B\tilde{D}^-DD^- = D\tilde{G}_1^{-1}B\tilde{D}^-R = D\tilde{G}_1^{-1}B\tilde{D}^-DD^- \\ &= D\tilde{G}_1^{-1}B\tilde{P}_0D^- = D\tilde{G}_1^{-1}BD^- = DG_1^{-1}BD^- \end{aligned}$$

since $\tilde{G}_1^{-1}B\tilde{Q}_0 = \tilde{Q}_0$ holds.

We may add the identities

$$\tilde{Q}_0\tilde{G}_1^{-1} = \tilde{Q}_0(\tilde{P}_0 + Q_0)G_1^{-1} = Q_0G_1^{-1}, \tag{3.15}$$

$$P_{0c}\tilde{D}^- = P_{0c}\tilde{D}^-D\tilde{D}^- = P_{0c}\tilde{D}^-DD^- = P_{0c}\tilde{P}_0D^- = P_{0c}D^-. \tag{3.16}$$

In the latter we used that P_{0c} is defined geometrically and so it is independent of P_0 $(\tilde{P}_0)$. As a result, we get that not only the solution of (3.10) is unique, but its splitting (3.11) is term by term independent of the choice of P_0 (Q_0).

3.2. Initial value problem for index-2 equations. Now we assume that the DAE (1.8) is of index 2 and condition C2 holds. Again we rewrite (1.8) as

$$ADD^-(Dx)' + Bx = q.$$

Due to the chain construction, the index-2 properties and the special choice $\hat{Q}_1 = \hat{Q}_1\hat{G}_2^{-1}BP_0$, by simple calculations we obtain the relations

$$\hat{G}_2^{-1}AD = \hat{P}_1P_0, \quad \hat{G}_2^{-1}A = \hat{P}_1P_0D^-, \quad \hat{G}_2^{-1}B = \hat{G}_2^{-1}BP_0\hat{P}_1 + \hat{Q}_1 + Q_0. \tag{3.17}$$

Scaling the DAE by $\hat{G}_2^{-1}$ yields

$$\hat{P}_1P_0D^-(Dx)' + Q_0x + \hat{Q}_1x + \hat{G}_2^{-1}BP_0\hat{P}_1x = \hat{G}_2^{-1}q. \tag{3.18}$$

Equation (3.18) decouples into the following three parts when multiplying by $\hat{Q}_1, P_0\hat{P}_1$ and $Q_0\hat{P}_1$, respectively:

$$\hat{Q}_1x = \hat{Q}_1\hat{G}_2^{-1}q, \tag{3.19}$$

$$P_0\hat{P}_1D^-(Dx)' + P_0\hat{P}_1\hat{G}_2^{-1}BP_0\hat{P}_1x = P_0\hat{P}_1\hat{G}_2^{-1}q, \tag{3.20}$$

$$-Q_0\hat{Q}_1D^-(Dx)' + Q_0x + Q_0\hat{P}_1\hat{G}_2^{-1}BP_0\hat{P}_1x = Q_0\hat{P}_1\hat{G}_2^{-1}q. \tag{3.21}$$

Multiplying both (3.19) and (3.20) by D and using the relation $P_0 = D^-D$ once more we find

$$D\hat{Q}_1x = D\hat{Q}_1\hat{G}_2^{-1}q, \tag{3.22}$$

$$(D\hat{P}_1x)' - (D\hat{P}_1D^-)'Dx + D\hat{P}_1\hat{G}_2^{-1}BD^-D\hat{P}_1x = D\hat{P}_1\hat{G}_2^{-1}q. \tag{3.23}$$

Now we see, that each solution $x \in C_D^1$ of the DAE (1.8) has to satisfy (3.22), (3.23) and (3.21). Due to Lemma 2.6, we have $D\hat{Q}_1\hat{G}_2^{-1}q = D\hat{Q}_1x = D\hat{Q}_1D^-Dx \in C^1$.

Hence, we are allowed to insert into (3.21) the expression below:

$$\begin{aligned} Q_0\hat{Q}_1D^-(Dx)' &= Q_0\hat{Q}_1D^-[(D\hat{P}_1x)' + (D\hat{Q}_1\hat{G}_2^{-1}q)'] \\ &= Q_0\hat{Q}_1D^-D\hat{Q}_1D^-(D\hat{P}_1x)' + Q_0\hat{Q}_1D^-(D\hat{Q}_1\hat{G}_2^{-1}q)' \\ &= -Q_0\hat{Q}_1D^-(D\hat{Q}_1D^-)'D\hat{P}_1x + Q_0\hat{Q}_1D^-(D\hat{Q}_1\hat{G}_2^{-1}q)'. \end{aligned}$$

This leads to

$$\begin{aligned} Q_0x = &-Q_0\hat{P}_1\hat{G}_2^{-1}BD^-D\hat{P}_1x - Q_0\hat{Q}_1D^-(D\hat{Q}_1D^-)'D\hat{P}_1x \\ &+Q_0\hat{Q}_1D^-(D\hat{Q}_1\hat{G}_2^{-1}q)' + Q_0\hat{P}_1\hat{G}_2^{-1}q. \end{aligned} \tag{3.24}$$

Finally, each solution $x \in C_D^1$ of (1.8) may be represented as

$$\begin{aligned} x &= D^-D\hat{P}_1x + D^-D\hat{Q}_1x + Q_0x \\ &= KD^-D\hat{P}_1x + D^-D\hat{Q}_1\hat{G}_2^{-1}q + Q_0\hat{P}_1\hat{G}_2^{-1}q + Q_0\hat{Q}_1D^-(D\hat{Q}_1\hat{G}_2^{-1}q)', \end{aligned} \tag{3.25}$$

where the component $D\hat{P}_1x$ satisfies the regular ODE

$$(D\hat{P}_1x)' - (D\hat{P}_1D^-)'D\hat{P}_1x + D\hat{P}_1\hat{G}_2^{-1}BD^-D\hat{P}_1x = D\hat{P}_1\hat{G}_2^{-1}q + (D\hat{P}_1D^-)'D\hat{Q}_1\hat{G}_2^{-1}q,$$

and

$$K := I - Q_0\hat{P}_1\hat{G}_2^{-1}BP_0 - Q_0\hat{Q}_1D^-(D\hat{Q}_1D^-)'D \tag{3.26}$$

is a nonsingular matrix function, $K^{-1} = I + Q_0\hat{P}_1\hat{G}_2^{-1}BP_0 + Q_0\hat{Q}_1D^-(D\hat{Q}_1D^-)'D$. This solution representation gives the idea to consider the so-called *inherent regular ODE* of (1.8).

DEFINITION 3.3. *The equation*

$$u' - (D\hat{P}_1D^-)'u + D\hat{P}_1\hat{G}_2^{-1}BD^-u = D\hat{P}_1\hat{G}_2^{-1}q + (D\hat{P}_1D^-)'D\hat{Q}_1\hat{G}_2^{-1}q \tag{3.27}$$

is called the inherent regular ODE *of index-2 equation (1.8).*

Multiplying (3.27) with $(I - D\hat{P}_1D^-)$ we check that $D\boldsymbol{S}_1 = \operatorname{im} D\hat{P}_1$ is an invariant subspace of the inherent regular ODE.

As well as Remark 3.2 demonstrated the geometric origin of the inherent regular ODE for the index-1 equations, so does the next one for the index-2 case.

REMARK 3.3. Neither of the terms $D\hat{Q}_1D^-$, $D\hat{P}_1D^-$, $D\hat{Q}_1\hat{G}_2^{-1}$, $D\hat{P}_1\hat{G}_2^{-1}$ and $D\hat{P}_1\hat{G}_2^{-1}BD^-$ do depend on the choice of P_0.

Starting with projectors P_0, $\tilde{P}_0$ and the associated generalized inverses D^-, $\tilde{D}^-$, we stated $\tilde{G}_1 = G_1(P_0 + \tilde{Q}_0)$ in Remark 3.2. Hence, $\tilde{\boldsymbol{N}}_1 = (\tilde{P}_0 + Q_0)\boldsymbol{N}_1$, $\tilde{\boldsymbol{S}}_1 = \boldsymbol{S}_1$ and

$$\begin{aligned} &\hat{\tilde{Q}}_1 = (\tilde{P}_0 + Q_0)\hat{Q}_1, \quad \hat{Q}_1 = (P_0 + \tilde{Q}_0)\hat{\tilde{Q}}_1, \\ &\hat{\tilde{G}}_2 = \hat{G}_2[P_0 + \tilde{Q}_0 + (Q_0 - \tilde{Q}_0)\hat{Q}_1], \quad \hat{\tilde{G}}_2^{-1} = [\tilde{P}_0 + Q_0 - (Q_0 - \tilde{Q}_0)\hat{Q}_1]\hat{G}_2^{-1}. \end{aligned}$$

Thus,

$$D\hat{\tilde{Q}}_1\tilde{D}^- = D(\tilde{P}_0 + Q_0)\hat{Q}_1\tilde{D}^- = D\hat{Q}_1\tilde{D}^-DD^- = D\hat{Q}_1\tilde{P}_0D^- = D\hat{Q}_1D^-,$$

$$
\begin{aligned}
D\hat{\tilde{P}}_1\tilde{D}^- &= D\tilde{D}^- - D\hat{\tilde{Q}}_1\tilde{D}^- = DD^- - D\hat{Q}_1D^- = D\hat{P}_1D^-,\\
D\hat{\tilde{Q}}_1\hat{\tilde{G}}_2^{-1} &= D\hat{Q}_1\hat{\tilde{G}}_2^{-1} = DQ_1\hat{G}_2^{-1},\\
D\hat{\tilde{P}}_1\hat{\tilde{G}}_2^{-1} &= D\hat{\tilde{G}}_2^{-1} - D\hat{\tilde{Q}}_1\hat{\tilde{G}}_2^{-1} = D\hat{G}_2^{-1} - D\hat{Q}_1\hat{G}_2^{-1} = D\hat{P}_1\hat{\tilde{G}}_2^{-1},\\
D\hat{\tilde{P}}_1\hat{\tilde{G}}_2^{-1}B\tilde{D}^- &= D\hat{\tilde{P}}_1\hat{\tilde{G}}_2^{-1}B\tilde{D}^-DD^- = D\hat{\tilde{P}}_1\hat{\tilde{G}}_2^{-1}B\tilde{P}_0D^-\\
&= D\hat{\tilde{P}}_1\hat{\tilde{G}}_2^{-1}BD^- = D\hat{P}_1\hat{G}_2^{-1}BD^-.
\end{aligned}
$$

Thus, the inherent regular ODE does not depend on the choice of P_0.

Let P_0 be fixed. It turns out that expression $Q_1G_2^{-1}$ is independent of the choice of P_1 (Q_1). To show that we use formula (3.15) in Remark 3.2 which clearly remains valid if each index is increased by 1. Combining with the above, $DQ_1G_2^{-1}$ becomes independent of the specific choices of both P_0 and P_1 and so does the function space $C^1_{DQ_1G_2^{-1}}$.

This remark also has connection to the question about hidden constraint below and the solvability theorem next to it.

Namely, how does the hidden constraint look like? Using the reflexive generalized inverse A^- (see Section 2), we obtain from (1.8)

$$R(Dx)' + A^-Bx = A^-q,$$

hence

$$D\hat{Q}_1D^-R(Dx)' + D\hat{Q}_1D^-A^-Bx = D\hat{Q}_1D^-A^-q, \tag{3.28}$$

$$(D\hat{Q}_1x)' - (D\hat{Q}_1D^-)'Dx + D\hat{Q}_1D^-A^-Bx = D\hat{Q}_1D^-A^-q. \tag{3.29}$$

Taking into account (3.22) we obtain the hidden constraint equation

$$(D\hat{Q}_1G_2^{-1}q)' - (D\hat{Q}_1D^-)'Dx + D\hat{Q}_1D^-A^-Bx = D\hat{Q}_1D^-A^-q. \tag{3.30}$$

Note that equation (3.30) is independent of the specific choice of projectors P_0 (Q_0) and W_0, since, due to (3.22) and (3.23), so is $Dx = D\hat{P}_1x + D\hat{Q}_1x$ and $A^-(Bx-q)=$ $R(Dx)' = \tilde{A}^-(Bx-q)$ holds if $A\tilde{A}^- = I - \tilde{W}_0$.

Turning back to (3.25), let us stress that

$$\Pi_{\text{can }2} := KD^-D\hat{P}_1 = KP_0\hat{P}_1 \tag{3.31}$$

is again a projector. Namely, it holds that

$$\Pi^2_{\text{can }2} = KP_0\hat{P}_1KP_0\hat{P}_1 = KP_0\hat{P}_1P_0KP_0\hat{P}_1 = KP_0\hat{P}_1$$

and

$$\ker \Pi_{\text{can }2} = \ker D^-D\hat{P}_1 = \boldsymbol{N}_1 \oplus \boldsymbol{N}_0 = P_0\boldsymbol{N}_1 \oplus \boldsymbol{N}_0, \tag{3.32}$$

$$D\Pi_{\text{can }2} = D\hat{P}_1, \quad \operatorname{im} D\Pi_{\text{can }2} = D\boldsymbol{S}_1. \tag{3.33}$$

In the next Theorem we prove that geometric space $\mathbb{S}_{\text{ind }2}$ that contains all solutions of the homogeneous equation is just the image of $\Pi_{\text{can }2}$, i. e.

$$\mathbb{S}_{\text{ind }2} = \operatorname{im} \Pi_{\text{can }2}. \tag{3.34}$$

For the index-1 tractable DAEs we proved that

$$\mathbb{S}_{\text{ind }1} := \boldsymbol{S}_0$$

is the respective geometric solution space and $\Pi_{\mathrm{can}\ 1} := P_{0c} = (I - Q_0 G_1^{-1} B P_0) P_0$ represents the projector onto $\mathbb{S}_{\mathrm{ind}\ 1}$. For an index-2 DAE we will have a lower dimensional geometric solution space

$$\mathbb{S}_{\mathrm{ind}\ 2} \subset \mathbb{S}_{\mathrm{ind}\ 1}.$$

THEOREM 3.4. *Let an index-2 tractable DAE (1.8) be given and condition C2 be fulfilled.*

1. *For each* $q \in C^1_{DQ_1G_2^{-1}}$, $d \in D(t_0)\boldsymbol{S}_1(t_0)$, $t_0 \in \mathcal{I}$, *the initial value problem*

$$A(Dx)' + Bx = q, \quad D(t_0)\hat{P}_1(t_0)x(t_0) = d \tag{3.35}$$

is uniquely solvable on C^1_D.

2. *(1.8) has perturbation index 2.*

3. *Through each* $t_0 \in \mathcal{I}$, $x_0 \in \mathbb{S}_{\mathrm{ind}\ 2}(t_0)$ *passes exactly one solution of the homogeneous equation.*

Proof.

1. First we solve the IVP for the inherent regular ODE (3.27) with the initial condition $u(t_0) = d$. Since $D\boldsymbol{S}_1$ is an invariant subspace for (3.27), it holds that $u = D\hat{P}_1 D^- u$. Then we construct the continuous function

$$x := KD^-u + D^-D\hat{Q}_1\hat{G}_2^{-1}q + Q_0\hat{P}_1\hat{G}_2^{-1}q + Q_0\hat{Q}_1D^-(D\hat{Q}_1\hat{G}_2^{-1}q)'.$$

Further we obtain

$$Dx = Ru + DQ_1G_2^{-1}q \in C^1.$$

Clearly, function x constructed now satisfies (1.8) and

$$\begin{aligned} D(t_0)\hat{P}_1(t_0)x(t_0) &= D(t_0)\hat{P}_1(t_0)[D(t_0)]^-D(t_0)x(t_0) \\ &= D(t_0)\hat{P}_1(t_0)[D(t_0)]^-u(t_0) = d. \end{aligned}$$

Assume that $\tilde{x} \in C^1_D$ also is a solution of (3.35) different from the one constructed above. Then $\hat{x} = \tilde{x} - x$ satisfies (3.35) with $q = 0$ and $d = 0$. (3.25) yields $\hat{x} = KD^-D\hat{P}_1\hat{x}$, but the solution $D\hat{P}_1\hat{x}$ of the regular ODE vanishes identically, i.e. $\hat{x} = 0$, in contrast to the assumption.

Note that in problem formulation (3.35) the sign $\hat{}$ cannot be removed.

2. Let $\mathcal{I}$ be a compact interval. Let us compare the solutions x and x_q of the homogeneous equation and the inhomogeneous one supplied with the same initial condition, i.e.

$$D(t_0)P_1(t_0)(x(t_0) - x_q(t_0)) = 0.$$

We derive

$$\begin{aligned} \|u_q - u\|_\infty &\le K_1\|D\hat{P}_1\hat{G}_2^{-1}q + (D\hat{P}_1D^-)'D\hat{Q}_1\hat{G}_2^{-1}q\|_\infty \\ &\le K_2\|q\|_\infty, \end{aligned}$$

further

$$\|x_q - x\|_\infty \le K_3\left(\|q\|_\infty + \|(DQ_1G_2^{-1}q)'\|_\infty\right).$$

3. For $x_0 \in \mathbb{S}_{\text{ind }2}(t_0)$ it holds that $x_0 = \Pi_{\text{can }2}(t_0)x_0$.
The IVP

$$A(Dx)' + Bx = 0, \quad D(t_0)\hat{P}_1(t_0)x(t_0) = D(t_0)\hat{P}_1(t_0)x_0,$$

has the solution

$$x = KD^- u = KD^- D\hat{P}_1 D^- u,$$

thus

$$\begin{aligned} x(t_0) &= \Pi_{\text{can }2}(t_0)[D(t_0)]^- u(t_0) \\ &= \Pi_{\text{can }2}(t_0)[D(t_0)]^- D(t_0)\hat{P}_1(t_0)x_0 = \Pi_{\text{can }2}(t_0)x_0 = x_0 \end{aligned}$$

Note that part 3 of Theorem 3.4 (as well as 3 of Theorem 3.2) itself provides the uniqueness claim in part 1) of Theorem 3.4 (in 1 of Theorem 3.2) by setting $x_0 := \hat{x}(0) = 0$. □

REMARK 3.4. The initial condition $D(t_0)\hat{P}_1(t_0)x(t_0) = d$, $d \in D(t_0)\boldsymbol{S}_1(t_0)$, can be replaced by

$$D(t_0)\hat{P}_1(t_0)x(t_0) = D(t_0)\hat{P}_1(t_0)x^0, \quad x^0 \in \mathbb{C}^m. \tag{3.36}$$

Then, the variational equation for x'_{x_0} is simply

$$A(DX)' + BX = 0, \quad D(t_0)\hat{P}_1(t_0)(X(t_0) - I) = 0. \tag{3.37}$$

REMARK 3.5. Formally, the case of index-1 tractability might be considered as index-2 tractability with $\dim \boldsymbol{N}_0 \cap \boldsymbol{S}_0 = \{0\}$, thus $\boldsymbol{N}_1 = \{0\}$, $P_1 = I$ $G_2 = G_1$. In this sense, statements (1) and (3) of Theorem 3.4 confirm once more the results and expressions of Theorem 3.2. In particular, the inherent ODEs (3.27) and (3.8) coincide and so do the geometric solution spaces.

3.3. Canonical projectors. For the projector onto the geometrical solution space $\mathbb{S}_{\text{ind }2}$ of an index–2 we had a rather complicated expression (3.31). The next Lemma demonstrates how to simplify by a proper construction.

LEMMA 3.5. *For an index-2 tractable DAE satisfying condition C2 there is a "canonical" projector Q_{0c} onto N_0 such that $\Pi_{\text{can}_2} = P_{0c}P_{1c}$ holds true.*

Proof. We begin constructing the chain (2.6) starting with an arbitrary Q_0 and the corresponding reflexive generalized inverse D^- of D. Next we check that for

$$Q_{0c} := Q_0\hat{P}_1\hat{G}_2^{-1}B + Q_0\hat{Q}_1D^-(D\hat{Q}_1D^-)'D \tag{3.38}$$

$Q_{0c}^2 = Q_{0c}$ and $\operatorname{im} Q_{0c} = \boldsymbol{N}_0$. In this verification one makes use of the relation $Q_0\hat{P}_1\hat{G}_2^{-1}BQ_0 = Q_0\hat{P}_1\hat{G}_2^{-1}G_1Q_0 = Q_0\hat{P}_1Q_0 = Q_0$. Since Q_{0c} turned to be a projector onto $\boldsymbol{N}_0$ we may build up the chain (2.6) also starting with Q_{0c}. Let us mark the respective matrices and subspaces by an index c. Note immediately that matrix K introduced before Theorem 3.4 can be expressed in the new terms as $K = P_{0c} + Q_0$ and thus $KD^- = P_{0c}D^-$.

Now we intend to show that Q_{0c} is the canonical projector claimed in the statement, i. e. we have to show that the expression

$$\mathcal{A} := \left\{Q_{0c}P_{1c}G_{2c}^{-1}BP_{0c} + Q_{0c}Q_{1c}D_c^-(DQ_{1c}D_c^-)'D\right\}P_{0c}P_{1c}$$

is well-defined and that it vanishes. Here in the braces the formal expression $I - K_c$ is inserted. We have $\boldsymbol{S}_{1c} = \boldsymbol{S}_1$, $G_{1c} = G_1(P_0 + Q_{0c})$, $\boldsymbol{N}_{1c} = (P_{0c} + Q_0)\boldsymbol{N}_1$. Further, $Q_{1c} := (P_0 + Q_{0c})^{-1}\hat{Q}_1(P_0 + Q_{0c}) = (P_{0c} + Q_0)\hat{Q}_1$, thus, Q_{1c} is the projector onto $\boldsymbol{N}_{1c}$ along $\boldsymbol{S}_{1c}$. Consequently, $\hat{Q}_{1c} = Q_{1c} = Q_{1c}G_{2c}^{-1}BP_{1c}$. From $DD^- = R = DD_c^-$ and $P_0 = D^-D$, $P_{0c} = D_c^-D$ we derive

$$
\begin{aligned}
&D^- = D^-DD_c^-, \quad D_c^- = D_c^-DD^- \\
&DQ_{1c}D_c^- = D(P_{0c} + Q_0)\hat{Q}_1D_c^-DD^- = D\hat{Q}_1D^-, \\
&DQ_{1c} = D\hat{Q}_1 \quad DP_{1c} = D\hat{P}_1.
\end{aligned}
$$

In particular the derivative $(DQ_{1c}D_c^-)' = (DQ_1D^-)'$ in $\mathcal{A}$ exists.

Additionally, we compute

$$
\begin{aligned}
G_{2c} &= \hat{G}_2(P_0 + Q_{0c} + Q_0P_{0c}\hat{Q}_1), \\
G_{2c}^{-1} &= (P_{0c} + Q_0 - Q_0P_{0c}\hat{Q}_1)\hat{G}_2^{-1} \\
Q_{0c}Q_{1c} &= Q_{0c}(P_{0c} + Q_0)\hat{Q}_1 = Q_0\hat{Q}_1 \\
Q_{0c}Q_{1c}D_c^- &= Q_0\hat{Q}_1D_c^-DD^- = Q_0\hat{Q}_1D^-.
\end{aligned}
$$

Thus, the second term of $\mathcal{A}$ is

$$
Q_{0c}Q_{1c}D_c^-(DQ_{1c}D_c^-)'DP_{0c}P_{1c} = Q_0\hat{Q}_1D^-(D\hat{Q}_1D^-)'D\hat{P}_1.
$$

Let us turn to the first term of $\mathcal{A}$ and compute

$$
\begin{aligned}
Q_{0c}P_{1c}G_2^{-1}BP_{0c}P_{1c} &= Q_{0c}G_{2c}^{-1}BP_{0c}P_{1c} - Q_{0c}Q_{1c}G_{2c}^{-1}BP_{0c}P_{1c} \\
&= Q_{0c}G_{2c}^{-1}BP_{0c}P_{1c} \\
&= (Q_0 - Q_0P_{0c}\hat{Q}_1)\hat{G}_2^{-1}BP_{0c}P_{1c} \\
&= Q_0\hat{G}_2^{-1}BP_{0c}P_{1c} - Q_0P_{0c}\hat{Q}_1\hat{G}_2^{-1}BP_{0c}P_{1c} \\
&= Q_0\hat{G}_2^{-1}B(P_0 + Q_0)P_{0c}P_{1c} - Q_0P_{0c}\hat{Q}_1P_{0c}P_{1c} \\
&= Q_0\hat{G}_2^{-1}B(P_0 + Q_0)P_{0c}P_{1c} \\
&= Q_0\hat{G}_2^{-1}BP_0\hat{P}_1 + Q_0P_{0c}\hat{P}_1.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathcal{A} &= Q_0\hat{G}_2^{-1}BP_0\hat{P}_1 + Q_0\hat{P}_1 - Q_{0c}\hat{P}_1 + Q_0\hat{Q}_1D^-(D\hat{Q}_1D^-)'D\hat{P}_1 \\
&= Q_0\hat{G}_2^{-1}BP_0\hat{P}_1 + Q_0\hat{P}_1 - (Q_0\hat{P}_1\hat{G}_2^{-1}B + Q_0Q_0D^-(D\hat{Q}_1D^-)'D)\hat{P}_1 \\
&\quad + Q_0\hat{Q}_1D^-(D\hat{Q}_1D^-)'D\hat{P}_1 \\
&= Q_0\hat{G}_2^{-1}BP_0\hat{P}_1 + Q_0\hat{P}_1 - Q_0\hat{P}_1\hat{G}_2^{-1}B\hat{P}_1 \\
&= Q_0(\hat{G}_2^{-1}BP_0 + I - \hat{P}_1\hat{G}_2^{-1}B)\hat{P}_1 \\
&= Q_0(\hat{G}_2^{-1}BP_0 + I - \hat{G}_2^{-1}B + \hat{Q}_1\hat{G}_2^{-1}B)\hat{P}_1 \\
&= Q_0(I - \hat{G}_2^{-1}BQ_0 + \hat{Q}_1)\hat{P}_1 = Q_0(I - Q_0 + \hat{Q}_1)\hat{P}_1 = 0.
\end{aligned}
$$

□

While for an index–1 DAE P_{0c} just defined coincides with (3.9) and thus it is a projector onto $\boldsymbol{S}_0$, this is not the case with the index–2 equations. Namely, one cannot claim in general that $\operatorname{im} P_{0c} = \boldsymbol{S}_0$.

Due to Lemma 3.5, with $D^- := D_c^-$, we may obtain formally nicer solution representation. In particular, we arrive at the theorem below.

THEOREM 3.6. *For an index-μ-tractable DAE, $\mu = 1, 2$, under condition C2, there is a canonical projector Q_{0c} onto $\boldsymbol{N}_0$ so that*

$$\begin{aligned} \Pi_{\text{can }1} &= P_{0c} && \textit{projects onto} \quad \mathbb{S}_{\text{ind }1}, \\ \Pi_{\text{can }2} &= P_{0c}P_{1c} && \textit{projects onto} \quad \mathbb{S}_{\text{ind }2}. \end{aligned} \tag{3.39}$$

In both cases, the reflexive generalized inverses D^- of D can be chosen so that $D^-D = P_{0c}$ hold and the solutions of homogeneous equation be

$$x = D^- u, \tag{3.40}$$

where the functions u are the solutions of the homogeneous inherent regular ODEs.

4. Fundamental matrices. Throughout this section when speaking about index–1 or index–2 equations, we assume that conditions C1 or C2, respectively, are fulfilled. When not marked additionally, D^- and A^- denote generalized inverses associated with arbitrarily fixed projectors P_0 and W_0.

In parallel to the homogeneous equation (1.8) ($q = 0$), one may consider matrix equations

$$A(DX)' + BX = 0 \tag{4.1}$$

with respect to matrices $X : \mathcal{I} \to L(\mathbb{C}^k, \mathbb{C}^m)$ for arbitrary k, $1 \le k \le m$, with coefficients as fixed in Section 2.

DEFINITION 4.1. *A matrix function $X : \mathcal{I} \to L(\mathbb{C}^k, \mathbb{C}^m)$ will be called solution of (4.1), if its each column is a solution of (1.8) with $q = 0$.*

Thus, a continuous matrix function $X : \mathcal{I} \to L(\mathbb{C}^k, \mathbb{C}^m)$ for which the product DX is differentiable and satisfies (4.1) pointwisely, is a solution of (4.1). The relevant functionspace will be denoted by $\mathcal{C}_D^{1,k}$,

$$\mathcal{C}_D^{1,k} := \{X \in C(\mathcal{I}, L(\mathbb{C}^k, \mathbb{C}^m)) : DX \in C^1(\mathcal{I}, L(\mathbb{C}^k, \mathbb{C}^m))\}.$$

DEFINITION 4.2. *A solution $X \in \mathcal{C}_D^{1,k}$ will be called fundamental for an index–i DAE if* $\operatorname{im} X = \mathbb{S}_{\text{ind }i}$, *$i = 1$ or 2, respectively.*

In the index-1 case, due to statement 3 in Theorem 3.2 and decomposition (2.7), one has $m_1 := \dim \mathbb{S}_{\text{ind }1} = r$. Hence, no fundamental matrix can exist with $k < m_1$.

Similarly, in the index-2 case, statement 3 in Theorem 3.4 and decomposition (2.10) yields $m_2 := \dim \mathbb{S}_{\text{ind }2} = r_1 - (m - r)$. Hence, no fundamental matrix can exist with $k < m_2$.

For $k \ge m_1$ and $k \ge m_2$, respectively, fundamental matrices do exist. We show this by direct construction of samples for each k.

Let U_i be the fundamental matrices for the inherent regular (homogeneous) ODEs (3.8) and (3.30) of homogeneous DAE (1.8) of index i, $i = 1, 2$, with $U_i(t_0) = I$, respectively.

THEOREM 4.3. *For arbitrary $k \ge m_i$ and $\overline{P_i} \in L(\mathbb{C}^k, \mathbb{C}^m)$ such that*

$$\operatorname{im} \overline{P_i} = \begin{cases} \operatorname{im} P_0(t_0) & \textit{if} \quad i = 1, \\ \operatorname{im} \hat{P}_1(t_0) & \textit{if} \quad i = 2, \end{cases} \tag{4.2}$$

function $X_i : \mathcal{I} \to L(\mathbb{C}^k, \mathbb{C}^m)$ defined by

$$X_i(t) := \Pi_{\text{can }i}(t) D^-(t) U_i(t) D(t_0) \overline{P_i} \tag{4.3}$$

is a fundamental matrix for (1.8) of index i, $i = 1, 2$.

Proof. X_i is continuous by construction. For $i = 1$ we have $\Pi_{\mathrm{can}\ 1} = P_{0c}$, $DP_{0c} = D$, hence $D\Pi_{\mathrm{can}\ 1}D^- = DD^- = R \in C^1$. For $i = 2$, we use Lemma 3.5 and Theorem 3.6 to obtain

$$D\Pi_{\mathrm{can}\ 2}D^- = DP_{0c}P_{1c}D^- = DP_{1c}D^- = R - DQ_{1c}D^- = R - DQ_1D^- \in C^1.$$

Since $U_i \in \mathbb{C}^1$, we obtain $DX_i \in \mathbb{C}^1$, i.e. $X_i \in C_D^{1,k}$.

By its definition, $\mathrm{im}\ X_i \subseteq \mathrm{im}\ \Pi_{\mathrm{can}\ i} = \mathbb{S}_{\mathrm{ind}\ i}$.

On the other hand, in the index-1 case $\ker \Pi_{\mathrm{can}\ 1} = \ker P_{0c} = \boldsymbol{N}_0 = \mathrm{im}\ Q_0$ and $\mathrm{im}\ D^- = \mathrm{im}\ P_0$, thus $\boldsymbol{N}_0 \cap \mathrm{im}\ D^- = \{0\}$. Hence,

$$\begin{aligned}\dim \mathrm{im}\ X_1 &= \dim \mathrm{im}\ D^-U_1D(t_0)\overline{P_1} = \dim \mathrm{im}\ RU_1D(t_0)\overline{P_1}\\ &= \dim \mathrm{im}\ U_1D(t_0)\overline{P_1} = \dim \mathrm{im}\ D(t_0)\overline{P_1} = \dim \mathrm{im}\ D(t_0) = r = m_1.\end{aligned}$$

Therefore, $\mathrm{im}\ X_1 = \mathbb{S}_{\mathrm{ind}\ 1}$ as required.

In the index–2 case, one has $U_2D(t_0)\overline{P_2} = D\hat{P}_1D^-U_2D(t_0)\overline{P_2}$. A consequence is that

$$X_2 = \Pi_{\mathrm{can}\ 2}D^-U_2D(t_0)\overline{P_2} = \Pi_{\mathrm{can}\ 2}P_0\hat{P}_1D^-U_2D(t_0)\overline{P_2} = \Pi_{\mathrm{can}\ 2}D^-U_2D(t_0)\overline{P_2}.$$

Next, $\ker \Pi_{\mathrm{can}\ 2}D^- \cap \mathrm{im}\ D\hat{P}_1 = \{0\}$, i.e.

$$\begin{aligned}\dim \mathrm{im}\ X_2 &= \dim \mathrm{im}\ D\hat{P}_1D^-U_2D(t_0)\overline{P_2} = \dim \mathrm{im}\ U_2D(t_0)\overline{P_2}\\ &= \dim \mathrm{im}\ D(t_0)\overline{P_2} = m_2.\end{aligned}$$

Hence, $\mathrm{im}\ X_2 = \mathbb{S}_{\mathrm{ind}\ 2}$. ◻

In the proof we used only the image $D\overline{P_i}$. Noting that

$$\begin{aligned}D(t_0)\Pi_{\mathrm{can}\ 1}(t_0) &= (D(t_0)P_0(t_0))P_{0c}(t_0) = D(t_0)(P_0(t_0)P_{0c}(t_0))\\ &= D(t_0)P_0(t_0) = D(t_0),\end{aligned}$$

and again using Theorem 3.6

$$\begin{aligned}D(t_0)\Pi_{\mathrm{can}\ 2}(t_0) &= D(t_0)(P_{0c}(t_0)P_{1c}(t_0)) = (D(t_0)P_{0c}(t_0))P_{1c}(t_0)\\ &= D(t_0)P_{1c}(t_0) = D(t_0)\hat{P}_1(t_0),\end{aligned}$$

one concludes

COROLLARY 4.4. *For arbitrary $k \geq m_i$ and $\overline{\Pi_i} \in L(\mathbb{C}^k, \mathbb{C}^m)$ such that*

$$\mathrm{im}\ \overline{\Pi_i} = \mathrm{im}\ \Pi_{\mathrm{can}\ i}(t_0) = \mathbb{S}_{\mathrm{ind}\ i}(t_0), \tag{4.4}$$

function $X_i : \mathcal{I} \to L(\mathbb{C}^k, \mathbb{C}^m)$ defined by

$$X_i(t) := \Pi_{\mathrm{can}\ i}(t)D^-(t)U_iD(t_0)\overline{\Pi_i} \tag{4.5}$$

is a fundamental matrix for (1.8) of index i, $i = 1, 2$.

REMARK 4.1. Although Theorem 4.3 allows arbitrary $k \geq m_i$ and the minimal value $k = m_i$ would be sufficient, the reasonable choice is $k = m$. From one hand, it is much easier to construct a projector ($k = m$) than either a basis ($k = m_i$) in $\mathbb{S}_i$ or vectors spanning $\mathbb{S}_i$. Frequently, the projectors are given explicitly by the form of the equation. Obviously, statements claimed in [1] concerning transformations between different fundamental matrices including those of different sizes remain valid. We show below that a kind of normalization may be applied to fundamental matrices of "maximal size" m. Further, a generalized inverse will be introduced for fundamental matrices of maximal size. Then, the group properties will be shown to take place. As in [1], we use the following definition.

DEFINITION 4.5. *A fundamental matrix X_i of maximal size $k = m$ for the homogeneous DAE 1.8 of index i will be called* normalized at $t = t_0$ *if*

$$\Pi_{\mathrm{can}\ i}(t_0)(X_i(t_0) - I) = 0. \tag{4.6}$$

THEOREM 4.6. *For an index-i DAE with $i = 1, 2$ and arbitrary $t_0 \in \mathcal{I}$ there exists a unique fundamental matrix X_i normalized at $t = t_0$ and for it $X_i(t_0) = \Pi_{\mathrm{can}\ i}(t_0)$ holds.*

Proof. The normalization means $\Pi_{\mathrm{can}\ i}(t_0)X_i(t_0) = \Pi_{\mathrm{can}\ i}(t_0)$. The existence is ensured by Corollary 4.4. Indeed, set $k = m$ and $\overline{\Pi}_i = \Pi_{\mathrm{can}\ i}(t_0)$. Then

$$X_i(t) = \Pi_{\mathrm{can}\ i}(t)D^-(t)U_i(t)D(t_0)\Pi_{\mathrm{can}\ i}(t_0) \tag{4.7}$$

and $\Pi_{\mathrm{can}\ i}(t_0)X_i(t_0) = X_i(t_0) = \Pi_{\mathrm{can}\ i}(t_0)P_0(t_0)\Pi_{\mathrm{can}\ i}(t_0)$.

In the index–1 case, this latter expression is trivially equal to $\Pi_{\mathrm{can}\ 1}(t_0)$, i.e. the fundamental matrix (4.7) is normalized.

In the index–2 case, we recall that $P_0\hat{P}_1Q_0 = 0$. Thus, $\Pi_{\mathrm{can}\ 2}P_0 = (KP_0\hat{P}_1)P_0 = (KP_0\hat{P}_1)(P_0 + Q_0) = (KP_0\hat{P}_1) = \Pi_{\mathrm{can}\ 2}$, and by this the normalization is verified.

Meanwhile we got that for X_i defined by (4.7) the second part of the statement holds.

Let $\widehat{X}_i$ be another maximal fundamental solution different from (4.7) and also normalized at the same $t = t_0$. Columns of $\widehat{X}_i$ are solutions of the homogeneous DAE (1.8) and X_i is fundamental, so $\widehat{X}_i = X_iC$ with some $C \in L(\mathbb{C}^m)$. Due to (4.7) with no restriction of generality we may assume $C = \Pi_{\mathrm{can}\ i}(t_0)C$. Since both $\widehat{X}_i$ and X_i are normalized, $\Pi_{\mathrm{can}\ i}(t_0)\widehat{X}_i(t_0) = \Pi_{\mathrm{can}\ i}(t_0) = \Pi_{\mathrm{can}\ i}(t_0)X_i(t_0)$, i.e. $\Pi_{\mathrm{can}\ i}(t_0)(\widehat{X}_i(t_0) - X_i(t_0)) = \Pi_{\mathrm{can}\ i}(t_0)X_i(t_0)(C - I) = 0$. On the other hand, by construction $(I - \Pi_{\mathrm{can}\ i}(t_0))X_i(t_0) = 0$ and hence $(I - \Pi_{\mathrm{can}\ i}(t_0))\widehat{X}_i(t_0) = (I - \Pi_{\mathrm{can}\ i}(t_0))X_i(t_0)C = 0$, $(I - \Pi_{\mathrm{can}\ i}(t_0))X_i(t_0)(C - I) = 0$, i.e. $X_i(t_0)(C - I) = 0$. But $X_i(t_0) = \Pi_{\mathrm{can}\ i}(t_0)$, hence $\Pi_{\mathrm{can}\ i}(t_0)(C - I) = 0$. Now with the assumption on C, $C - \Pi_{\mathrm{can}\ i}(t_0) = 0$ holds. The substitution of C yields $\widehat{X}_i = X_iC = X_i\Pi_{\mathrm{can}\ i}(t_0) = X_i$ in contrast to the assumption. This completes the proof of uniqueness. □

For brevity, let us denote the fundamental matrix normalized at $t = \widehat{t}$ by $X_i(., \widehat{t})$, while the fundamental matrix U_i of the regular homogeneous ODE (3.8) or (3.30) with $U_i(\widehat{t}) = I$ be denoted by $U_i(., \widehat{t})$. For the fundamental matrix $X_i(., \widehat{t})$ let us introduce a generalized reflexive inverse $X_i^-(., \widehat{t})$ by

$$X_i(t, \widehat{t})X_i^-(t, \widehat{t}) = \Pi_{\mathrm{can}\ i}(t) \quad \text{and} \quad X_i^-(t, \widehat{t})X_i(t, \widehat{t}) = \Pi_{\mathrm{can}\ i}(\widehat{t}) \tag{4.8}$$

for all $t \in \mathcal{I}$ and some fixed $\widehat{t} \in \mathcal{I}$.

THEOREM 4.7. *For normalized fundamental solutions the identities*

$$X_i^-(t_2, t_1) = X_i(t_1, t_2) \tag{4.9}$$

$$\textit{and} \quad X_i(t_1, t_3) = X_i(t_1, t_2)X_i(t_2, t_3) \tag{4.10}$$

hold for all $t_1, t_2, t_3 \in \mathcal{I}$ and both indices $i = 1, 2$.

Proof. From (4.7) we have the representations

$$X_i(t_k,t_l) = \Pi_{\mathrm{can}\ i}(t_k)D^-(t_k)U_i(t_k,t_l)D(t_l)\Pi_{\mathrm{can}\ i}(t_l),$$
$$(k,l) = (1,2),\ (2,1),\ (1,3),\ (2,3).$$

In the index–1 case for $(k,l) = (1,2)$ and $(2,1)$ we have

$$\begin{aligned}
&X_1(t_k,t_l)X_1(t_l,t_k) = \\
&\Pi_{\mathrm{can}\ 1}(t_k)D^-(t_k)U_1(t_k,t_l)D(t_l)\Pi_{\mathrm{can}\ 1}(t_l)D^-(t_l)U_1(t_l,t_k)D(t_k)\Pi_{\mathrm{can}\ 1}(t_k) = \\
&\Pi_{\mathrm{can}\ 1}(t_k)D^-(t_k)U_1(t_k,t_l)R(t_l)U_1(t_l,t_k)D(t_k)\Pi_{\mathrm{can}\ 1}(t_k) = \\
&\Pi_{\mathrm{can}\ 1}(t_k)D^-(t_k)U_1(t_k,t_l)U_1(t_l,t_k)D(t_k)\Pi_{\mathrm{can}\ 1}(t_k) = \\
&\Pi_{\mathrm{can}\ 1}(t_k)D^-(t_k)D(t_k)\Pi_{\mathrm{can}\ 1}(t_k) = \Pi_{\mathrm{can}\ 1}(t_k)P_0(t_k)\Pi_{\mathrm{can}\ 1}(t_k) = \Pi_{\mathrm{can}\ 1}(t_k).
\end{aligned}$$

Since the reflexive inverse is defined uniquely by (4.8), the above chain confirms the identity (4.9). Further,

$$\begin{aligned}
&X_1(t_1,t_2)X_1(t_2,t_3) = \\
&\Pi_{\mathrm{can}\ 1}(t_1)D^-(t_1)U_1(t_1,t_2)D(t_2)\Pi_{\mathrm{can}\ 1}(t_2)D^-(t_2)U_1(t_2,t_3)D(t_3)\Pi_{\mathrm{can}\ 1}(t_3) = \\
&\Pi_{\mathrm{can}\ 1}(t_1)D^-(t_1)U_1(t_1,t_2)R(t_2)U_1(t_2,t_3)D(t_3)\Pi_{\mathrm{can}\ 1}(t_3) = \\
&\Pi_{\mathrm{can}\ 1}(t_1)D^-(t_1)U_1(t_1,t_2)U_1(t_2,t_3)D(t_3)\Pi_{\mathrm{can}\ 1}(t_3) = \\
&\Pi_{\mathrm{can}\ 1}(t_1)D^-(t_1)U_1(t_1,t_3)D(t_3)\Pi_{\mathrm{can}\ 1}(t_3) = X_1(t_1,t_3).
\end{aligned}$$

For both identities we used the invariant subspace of (3.8) and the group properties of U_1.

In the index–2 case one makes use of the group property of U_2 and the invariant subspace of (3.27). The next chain verifies the identity (4.9).

$$\begin{aligned}
&X_2(t_k,t_l)X_2(t_l,t_k) = \\
&\Pi_{\mathrm{can}\ 2}(t_k)D^-(t_k)U_2(t_k,t_l)D(t_l)\Pi_{\mathrm{can}\ 2}(t_l)D^-(t_l)U_2(t_l,t_k)D(t_k)\Pi_{\mathrm{can}\ 2}(t_k) = \\
&\Pi_{\mathrm{can}\ 2}(t_k)D^-(t_k)U_2(t_k,t_l)D(t_l)\hat{P}_1(t_l)D^-(t_l)U_2(t_l,t_k)D(t_k)\Pi_{\mathrm{can}\ 2}(t_k) = \\
&\Pi_{\mathrm{can}\ 2}(t_k)D^-(t_k)U_2(t_k,t_l)U_2(t_l,t_k)D(t_k)\Pi_{\mathrm{can}\ 2}(t_k) = \\
&\Pi_{\mathrm{can}\ 2}(t_k)P_0(t_k)\Pi_{\mathrm{can}\ 2}(t_k) = \Pi_{can2}(t_k).
\end{aligned}$$

The next chain verifies the identity (4.10).

$$\begin{aligned}
&X_2(t_1,t_2)X_2(t_2,t_3) = \\
&\Pi_{\mathrm{can}\ 2}(t_1)D^-(t_1)U_2(t_1,t_2)D(t_2)\Pi_{\mathrm{can}\ 2}(t_2)D^-(t_2)U_2(t_2,t_3)D(t_3)\Pi_{\mathrm{can}\ 2}(t_3) = \\
&\Pi_{\mathrm{can}\ 2}(t_1)D^-(t_1)U_2(t_1,t_2)D(t_2)P_1(t_2)D^-(t_2)U_2(t_2,t_3)D(t_3)\Pi_{\mathrm{can}\ 2}(t_3) = \\
&\Pi_{\mathrm{can}\ 2}(t_1)D^-(t_1)U_2(t_1,t_2)U_2(t_2,t_3)D(t_3)\Pi_{\mathrm{can}\ 2}(t_3) = \\
&\Pi_{\mathrm{can}\ 2}(t_1)D^-(t_1)U_2(t_1,t_3)D(t_3)\Pi_{\mathrm{can}\ 2}(t_3) = X_2(t_1,t_3).
\end{aligned}$$

□

5. Adjoint equation. In section 2 we proved Lemma 2.2 ensuring that in the equation

$$A_*(D_*y)' + B_*y = p \tag{5.1}$$

where $A_* := D^*$, $D_* := A^*$ and $B_* := -B^*$, the ordered matrix pair (A_*, D_*) gives to rise a smooth $\mathbb{C}^m$ decomposition provided the pair (A, D) does so. The projector

R_* satisfies relation $R_* = R^*$. For equation (5.1) let us introduce the subspaces and the matrices as it was given for (1.8) by (2.6):

$$\left.\begin{array}{l} G_{*0} := A_*D_*, \quad B_{*0} := B_* = -B^*, \\ Q_{*i} \text{ is a projector onto } \ker G_{*i}, \quad P_{*i} := I - Q_{*i}, \\ W_{*i} \text{ is a projector}, \ker W_{*i} = \text{im } G_{*i}, \\ G_{*i+1} := G_{*i} + B_{*i}Q_{*i}, \qquad B_{*i+1} = B_{*i}P_{*i}, \\ \boldsymbol{N}_{*i} := \ker G_{*i} = \text{im } Q_{*i}, \\ \boldsymbol{S}_{*i} := \{z \in \mathbb{C}^m : B_{*i}z \in \text{ im } G_{*i}\} = \ker W_{*i}B_{*i}, Cc \end{array}\right\} \quad i = 0, 1. \tag{5.2}$$

Additionally, we have the reflexive inverses $D_*^- = A^{*-}$ and $A_*^- = D^{*-}$ with products $A^*A^{*-} = R_* = R^*$, $A^{*-}A^* = P_{*0}$ and $D^*D^{*-} = I - W_{*0}$, $D^{*-}D^* = R_* = R^*$. For the auxiliary matrices we have $\mathcal{G}_{*i+1} := G_{*i} + W_{*i}B_{*i}Q_{*i}$, $i = 0, 1$.

Remark 5.1. In general, one cannot claim $A^{*-} = A^{-*}$ or $D^{*-} = D^{-*}$. Instead, we have the following identities:

$$\begin{array}{ll} D^{-*} = D^{*-}P_0^*, & D^{*-} = D^{-*}(I - W_{*0}), \\ A^{-*} = (I - W_0^*)A^{*-}, & A^{*-} = P_{*0}A^{-*}. \end{array} \tag{5.3}$$

When, however, $W_0 = Q_{*0}^*$ and $W_{*0} = Q_0^*$ is set, then $A^{*-} = A^{-*}$ or $D^{*-} = D^{-*}$ holds. In this case, we have $A^-A = R$ and $AA^- = I - Q_{*0}^*$, i.e. $A^*A^{-*} = R^*$ and $A^{-*}A^* = I - Q_{*0} = P_{*0}$, while A^{*-} is defined by $A^*A^{*-} = R_* = R^*$ and $A^{*-}A^* = P_{*0}$. The reflexive generalized inverse is uniquely defined by the given projectors P_{*0} and R^*, therefore $A^{-*} = A^{*-}$ takes place. Coincidence of D^{-*} with D^{*-} can be shown using similar arguments.

Equation (5.1), or equivalently, equation (1.9) is very closely related to equation (1.8) as the following theorem claims.

Theorem 5.1. *Equation (1.9) is of index 1 or of index 2 if and only if so is equation (1.8).* To prove the theorem first we verify the following five lemmas.

Lemma 5.2. $\boldsymbol{N}_{*0} \cap \boldsymbol{S}_{*0} = \text{im } G_1{}^\perp$ *and* $\boldsymbol{N}_0 \cap \boldsymbol{S}_0 = \text{im } G_{*1}{}^\perp$.

Proof. Let $y \in \boldsymbol{N}_{*0} \bigcap S_{*0}$. Then, $A_*D_*y = 0$ and there exists $z \in \mathbb{C}^m$ such that $B_*y = A_*D_*z$. Hence, $Q_0^*B_*y = Q_0^*A_*D_*z = Q_0^*D^*A^*z = (DQ_0)^*A^*z = 0$. Therefore, $G_1^*y = (AD + BQ_0)^*y = (A_*D_* - Q_0^*B_*)y = A_*D_*y - Q_0^*B_*y = 0$, $y \in \ker G_1^*$, $\ker G_1^* = \text{im } G_1{}^\perp$.

In the opposite direction: Let $y \in \ker G_1^*$, i.e. $A_*D_*y = Q_0^*B_*y$. Premultiplication by Q_0^* gives $0 = Q_0^*B_*y$, hence $A_*D_*y = 0$, i.e. $y \in \boldsymbol{N}_{*0}$. Further, $B_*y = (Q_0^* + P_0^*)B_*y = P_0^*B_*y$. Due to $P_0^* = (D^-D)^*$, $B_*y \in \text{im } D^*$. By Lemma 2.2, one may apply Lemma 2.3 to $H = A_*$, $J = D_*$ and thus 3 of Lemma 2.3 gives $\text{im } D^* = \text{im } D^*A^*$, i.e. $B_*y \in \boldsymbol{S}_{*0}$. The second statement can be checked similarly. ☐

Lemma 5.3. $\boldsymbol{N}_i \cap \boldsymbol{S}_i = \ker \mathcal{G}_{i+1}$ *and* $\boldsymbol{N}_{*i} \cap \boldsymbol{S}_{*i} = \ker \mathcal{G}_{*i+1}$, $i = 0, 1$.

Proof. Let $i = 0$ or $i = 1$ and $y \in \boldsymbol{N}_i \cap \boldsymbol{S}_i$. Then, $G_iy = 0$ and $W_iB_iy = 0$. Hence, $Q_iy = y$ and $W_iB_iQ_iy = W_iB_iy = 0$, i.e. $\mathcal{G}_{i+1}y = 0$.

In the opposite direction: Let $\mathcal{G}_{i+1}y = 0$. Premultiplication by W_i yields $0 = W_i\mathcal{G}_{i+1}y = W_iG_iy + W_iB_iQ_iy = W_iB_iQ_iy$. Thus, $G_iy = \mathcal{G}_{i+1}y - W_iB_iQ_iy = 0$, too, or, equivalently, $y \in \boldsymbol{N}_i$, $Q_iy = y$. Hence, $W_iB_iy = W_iB_iQ_iy = 0$, i.e. $y \in \boldsymbol{S}_i$.

The second part of the statement can be obtained by adding subscript $*$ everywhere in the above proof. ☐

Lemma 5.4.

$$\begin{array}{lcl} \ker \mathcal{G}_1 = \ker \mathcal{G}_{*1}^* & \textit{and} & \text{im } \mathcal{G}_1^* = \text{im } \mathcal{G}_{*1}, \\ \ker \mathcal{G}_{*1} = \ker \mathcal{G}_1^* & \textit{and} & \text{im } \mathcal{G}_{*1}^* = \text{im } \mathcal{G}_1. \end{array}$$

Proof. We have

$$\ker W_0 = \operatorname{im} AD = \ker D^*A^{*\perp} = \operatorname{im} Q_{*0}{}^{\perp} = \ker Q_{*0}^* \tag{5.4}$$

$$\ker W_{*0} = \operatorname{im} D^*A^* = \ker AD^{\perp} = \operatorname{im} Q_0{}^{\perp} = \ker Q_0^*. \tag{5.5}$$

Let

$$x \in \ker \mathcal{G}_1, \quad \text{i.e.} \quad \mathcal{G}_1 x = 0. \tag{5.6}$$

Then $0 = W_0\mathcal{G}_1 x = W_0ADx + W_0BQ_0x = W_0BQ_0x$. (5.4) involves $Q_{*0}^*BQ_0x = 0$. Further, $0 = (I - W_0)\mathcal{G}_1 x = ADx - W_0ADx = ADx$, i.e. $Dx = 0$, $P_0x = D^-Dx = 0$. Hence $Q_{*0}^*Bx = Q_{*0}^*B(Q_0x + P_0x) = Q_{*0}^*BQ_0x$. Finally, $Q_{*0}^*B(I - W_{*0}^*)x = Q_{*0}^*B(D^*D^{*-})^*x = Q_{*0}^*BD^{*-*}Dx = 0$, i.e. $Q_{*0}^*BW_{*0}^*x = Q_{*0}^*Bx = 0$ and so $\mathcal{G}_{*1}^*x = ADx - Q_{*0}^*BW_{*0}^*x = 0$.

For the proof in the opposite direction, let

$$x \in \ker \mathcal{G}_{*1}^*, \quad \text{i.e.} \quad \mathcal{G}_{*1}^* x = 0. \tag{5.7}$$

Then $ADx = 0$ and $Q_{*0}^*BW_{*0}^*x = 0$. Due to the first equality $P_0x = 0$, i.e. $x = Q_0x$, while the second one combined with (5.4) yields $W_0BW_{*0}^*x = 0$. Since

$$\operatorname{im} W_{*0}^* = \ker W_{*0}{}^{\perp} = \ker Q_0^{*\perp} = \operatorname{im} Q_0, \tag{5.8}$$

we have $W_{*0}^*Q_0 = Q_0$ and thus, finally, $\mathcal{G}_1x = ADx + W_0BQ_0x = 0$. For the second identity of the lemma, take the orthogonal complement of both sides in the first equality. The second pair holds by symmetry. ◻

The next two lemmas help in showing other connections between G_i, G_{*i} and $\mathcal{G}_i, \mathcal{G}_{*i}$. They allow to formulate the statements of Lemma 5.3 in terms of G_i and G_{*i}. Let $F_0 = I + D^-A^-BQ_0$, $F_{*0} = I - A^{*-}D^{*-}B^*Q_{*0}$. By direct substitution of $F_0^{-1} := I - D^-A^-BQ_0$ and $F_{*0}^{-1} := I + A^{*-}D^{*-}B^*Q_{*0}$, one can verify that F_0 and F_{*0} are nonsingular.

Further, let the generalized reflexive inverses G_1^- and G_{*1}^- be introduced by $G_1^- = P_1G_2^{-1}(I - W_1)$ and $G_{*1}^- = P_{*1}G_{*2}^{-1}(I - W_{*1})$. Then,

$$G_1G_1^- = I - W_1, \qquad G_1^-G_1 = P_1,$$
$$G_{*1}G_{*1}^- = I - W_{*1}, \qquad G_{*1}^-G_{*1} = P_{*1}.$$

Denote $F_1 = I + G_1^-BP_0Q_1$ and $F_{*1} = I - G_{*1}^-B^*P_{*0}Q_{*1}$.

LEMMA 5.5. $\operatorname{im} G_i = \operatorname{im} \mathcal{G}_i$ *and* $\operatorname{im} G_{*i} = \operatorname{im} \mathcal{G}_{*i}$, $i = 1, 2$.

Proof. We shall check that $G_i = \mathcal{G}_iF_{i-1}$ and $G_{*i} = \mathcal{G}_{*i}F_{*i-1}$ holds.

$$G_1 = \mathcal{G}_1 + (I - W_0)BQ_0 = \mathcal{G}_1 + AA^-BQ_0 = \mathcal{G}_1 + \mathcal{G}_1D^-A^-BQ_0$$
$$= \mathcal{G}_1(I + D^-A^-BQ_0) = \mathcal{G}_1F_0.$$

Since $Q_1G_1^- = 0$ holds, we derive

$$G_2 = G_1 + BP_0Q_1 = G_1 + W_1BP_0Q_1 + (I - W_1)BP_0Q_1$$
$$= \mathcal{G}_2 + G_1G_1^-BP_0Q_1 = \mathcal{G}_2(I + G_1^-BP_0Q_1) = \mathcal{G}_2F_1,$$

This completes the proof of the first part. The second part can be proven similarly by the use of F_{*0} and F_{*1}. ◻

LEMMA 5.6. *$F_0 \ker \mathcal{G}_2 = \ker G_{*2}^*$ and $F_{*0} \ker \mathcal{G}_{*2} = \ker G_2^*$.*

Proof. Let $y \in \ker G_{*2}^*$. Then $y \in \ker G_{*1}^*$ and $Q_{*1}^* P_{*0}^* By = 0$. From the first inclusion $Q_{*0}^* By = 0$ and $ADy = 0$ follows. Therefore $y = Q_0 y$ and $W_0 BQ_0 y = 0$ holds. We have $G_1 y = By = P_{*0}^* By$. Let $x = F_0^{-1} y$. Then $G_1 x = (AD + BQ_0)(I - D^- A^- BQ_0)y = [AD + BQ_0 - (I - W_0)BQ_0 - BQ_0 D^- A^- BQ_0]y = ADy - W_0 BQ_0 y - B(Q_0 D^-)A^- BQ_0 = 0$, i.e. $x = Q_1 x$.

Due to $Q_{*1}^* P_{*0}^* By = 0$, there exists a v such that $P_{*0}^* By = G_{*1}^* v$, and $0 = Q_{*0}^* Bv$ holds. By the use of $P_{*0}^* By = BQ_0 y$, we have $BQ_0 y = ADv$, i.e. $P_0 v = D^- A^- BQ_0 y$. Now,

$$\begin{aligned}
W_1 BP_0 Q_1 x &= W_1 BP_0 x = W_1 BP_0 F_0^{-1} y = W_1 B(P_0 - D^- A^- BQ_0)y \\
&= -W_1 BD^- A^- BQ_0 y = -W_1 BP_0 v = -W_1 Bv + W_1 BQ_0 v \\
&= -W_1 P_{*0}^* Bv + W_1(AD + BQ_0)Q_0 v \\
&= -W_1 ADP_0 D^{*-*} A^{*-*} Bv + W_1(AD + BQ_0)Q_0 v \\
&= -W_1(AD + BQ_0)P_0 D^{*-*} A^{*-*} Bv + W_1(AD + BQ_0)Q_0 v \\
&= (W_1 G_1)(Q_0 - P_0 D^{*-*} A^{*-*} B)v = 0.
\end{aligned}$$

By this, the proof is completed in one direction.

Let $\mathcal{G}_2 x = 0$. Then $G_1 x = 0$, i.e. $x = Q_1 x$ and $W_1 BP_0 Q_1 x = 0$, i.e. $W_1 BP_0 x = 0$. From $G_1 x = 0$ we get $ADx = -BQ_0 x$, i.e. $P_0 x = -D^- A^- BQ_0 x$ and $Q_{*0}^* BQ_0 x = 0$. Let $y = F_0 x = (I + D^- A^- BQ_0)x = (I - P_0)x = Q_0 x$. Then $G_{*1}^* y = (AD - Q_{*0}^* B)y = ADQ_0 x - Q_{*0}^* BQ_0 x = 0$.

Due to $\ker W_1 = \operatorname{im} G_1 = \operatorname{im} \mathcal{G}_1 = \operatorname{im} \mathcal{G}_{*1}^* = \operatorname{im} F_{*0}^{-*} G_{*1}^* = F_{*0}^{-*} \operatorname{im} G_{*1}^*$, the relation $W_1 BP_0 x = 0$ involves $Q_{*1}^* F_{*0}^* BP_0 x = 0$. Using the identity

$$\begin{aligned}
F_{*0}^* &= Q_{*0}^* + P_{*0}^* - Q_{*0}^* BD^{*-*} A^{*-*} = Q_{*0}^* + ADD^{*-*} A^{*-*} - Q_{*0}^* BD^{*-*} A^{*-*} \\
&= Q_{*0}^* + G_{*1}^* D^{*-*} A^{*-*}
\end{aligned}$$

we obtain

$$0 = Q_{*1}^* F_{*0}^* BP_0 x = Q_{*1}^* Q_{*0}^* BP_0 x + Q_{*1}^* G_{*1}^* D^{*-*} A^{*-*} BP_0 x = Q_{*1}^* Q_{*0}^* BP_0 x.$$

Finally,

$$\begin{aligned}
Q_{*1}^* P_{*0}^* By &= Q_{*1}^* P_{*0}^* BQ_0 x = Q_{*1}^* BQ_0 x - Q_{*1}^*(Q_{*0}^* BQ_0 x) \\
&= Q_{*1}^* BQ_0 x = -Q_{*1}^* ADx = -Q_{*1}^* ADP_0 x \\
&= -Q_{*1}^* ADP_0 x + Q_{*1}^* Q_{*0}^* BP_0 x = -Q_{*1}^* G_{*1} P_0 x = 0.
\end{aligned}$$

The second part can be obtained in a similar way. ∎

Now we can prove Theorem 5.1.

Proof. Let (1.8) be index-1 tractable. Then, by Lemma 2.2, C1 holds for (A_*, D_*), too. Further, G_1 is nonsingular. Then, by Lemma 5.2 $\boldsymbol{N}_{*0} \cap \boldsymbol{S}_{*0} = \{0\}$. Therefore, 1.9 is index-1 tractable.

Let (1.8) be index-2 tractable. Then, by Lemma 5.3 $\dim \ker \mathcal{G}_1 = \text{const}_1$. Lemma 5.4 results in $\dim \ker \mathcal{G}_{*1} = \dim \ker \mathcal{G}_{*1}^* = \dim \ker \mathcal{G}_1$. Now, Lemma 5.3 provides $\dim \boldsymbol{N}_{*0} \cap \boldsymbol{S}_{*0} = \text{const}_1$. By Lemmas 5.5 and 5.6, $\dim \ker \mathcal{G}_{*2} = \dim \ker G_2 = \dim \ker \mathcal{G}_2$. Then, again due to lemma 5.3, $\dim \boldsymbol{N}_{*1} \cap \boldsymbol{S}_{*1} = \{0\}$. In the opposite direction the proof is similar. Note that this proof replaces the above one for the index-1 case. ∎

Having checked the coincidence of indices of equation (1.8) and (1.9), we formulate a basic statement concerning this pair of equations.

THEOREM 5.7. *Let equation (1.8) be of index 2. Then, Condition C2 holds or it is violated simultaneously for equation (1.8) and (1.9).*

REMARK 5.2. *For the pair of equations (1.8) and (1.9), C1 holds simultaneously as stated in Lemma 2.2.*

For the proof of the Theorem 5.7 we need the next auxiliary propositions.

LEMMA 5.8. $D\boldsymbol{S}_1 = R(A^*\boldsymbol{N}_{*1})^\perp$ *and* $A^*\boldsymbol{S}_{*1} = R^*(D\boldsymbol{N}_1)^\perp$.

Proof. Let $x \in D\boldsymbol{S}_1$. Then, there exists a $z \in \boldsymbol{S}_1$ such that $x = Dz$. Thus, $0 = W_1BP_0z = W_1BD^-Dz = W_1BD^-x = W_1BD^-DD^-x = W_1BP_0D^-x$. It means that $D^-x \in \boldsymbol{S}_1$. Therefore, there exists v such that $BP_0D^-x = G_1v$, that is $BD^-x = G_1v$. A consequence of the latter formula is $Q^*_{*0}BD^-x - Q^*_{*0}G_1v = 0$. Noting that $Q^*_{*0}G_1 = Q^*_{*0}(AD + BQ_0) = (D^*A^*Q_{*0})^* + Q^*_{*0}BQ_0 = Q^*_{*0}BQ_0$, we obtain $Q^*_{*0}B(D^-x - Q_0v) = 0$. On the other hand, $AD(D^-x - Q_0v) = ARx - ADQ_0v = Ax$. From this one gets $G^*_{*1}(D^-x - Q_0v) = (AD - Q^*_{*0}B)(D^-x - Q_0v) = Ax$. Due to $Q^*_{*1}G^*_{*1} = 0$, we arrive at $Q^*_{*1}Ax = 0$ and $Q^*_{*1}ARx = 0$. Frim this, we conclude that $x, Rx \in (A^*\boldsymbol{N}_{*1})^\perp$. Finally, noting that $x \in \mathrm{im}\ D = \mathrm{im}\ R$, i.e. there exists w such that $x = Rw$, we find that $(I - R)x = (I - R)Rw = Rw - Rw = 0$, i.e. $x = Rx$. This accomplishes the proof in one direction.

Let $x \in R(A^*\boldsymbol{N}_{*1})^\perp$. Then, $x \in \mathrm{im}\ R$, i.e. there exists z such that $x = Rz$ and $z \in (A^*\boldsymbol{N}_{*1})^\perp$. Since $\boldsymbol{N}_{*1} = \mathrm{im}\ Q_{*1}$, $A^*\boldsymbol{N}_{*1} = \mathrm{im}\ A^*Q_{*1}$, i.e. $z \in \ker Q^*_{*1}A$. We have $0 = Q^*_{*1}Az = Q^*_{*1}(AR)z = Q^*_{*1}A(Rz) = Q^*_{*1}Ax$, i.e. $x \in (A^*\boldsymbol{N}_{*1})^\perp$. Further, $Ax \in \ker Q^*_{*1}$ means that there exists w such that $Ax = G^*_{*1}w$. Premultiplication by Q^*_{*0} yields $Q^*_{*0}Bw = 0$ and $Ax = ADw$, $P_0w = D^-A^-ADw = D^-A^-Ax = D^-x$. $Bw = P^*_{*0}Bw = AA^{*-*}Bw$, i.e. $Bw \in \mathrm{im}\ A = \mathrm{im}\ AD$, i.e. there exists u such that $Bw = ADu$ and $Q_0u = 0$, i.e. $Bw = ADu = G_1u$. Let $v = Q_0w$. Then $P_0v = 0$ and $G_1v = ADv + BQ_0v = BQ_0v = BQ_0w$. Thus, $BP_0D^-x = BP_0w = Bw - BQ_0w = G_1u - G_1v = G_1(u - v)$, i.e. $BP_0D^-x \in \mathrm{im}\ G_1$. It gives $D^-x \in \boldsymbol{S}_1$, i.e. $Rx \in DS_1$. Now $(I - R)x = x - Rx = Rz - R^2z = 0$ leads to the final result $x \in D\boldsymbol{S}_1$.

The second statement can be obtained in the same way, changing A, D, B and the associated matrices, projectors, inverses for $D^*, A^*, -B^*$ and their associates. □

LEMMA 5.9. *For the pair of index-2 tractable DAEs (1.8) and (1.9), it holds that*

$$(D\hat{P}_1D^-)^* = A^*\hat{P}_{*1}A^{*-} \tag{5.9}$$

Proof. Both $(D\hat{P}_1D^-)^*$ and $A^*\hat{P}_{*1}A^{*-}$ are projectors. They are uniquely determined by their kernels and images. By construction, $(D\hat{P}_1D^-)^*$ projects onto $(D\boldsymbol{N}_1 \oplus \ker R)^\perp$ along $(D\boldsymbol{S}_1)^\perp$ while $A^*\hat{P}_{*1}A^{*-}$ projects onto $A^*\boldsymbol{S}_{*1}$ along $A^*\boldsymbol{N}_{*1} \oplus \ker R^*$.

If we succeed to show that

$$(D\boldsymbol{N}_1 \oplus \ker R)^\perp = R^*(D\boldsymbol{N}_1)^\perp, \tag{5.10}$$

$$(A^*\boldsymbol{N}_{*1} \oplus \ker R^*)^\perp = R(A^*\boldsymbol{N}_{*1})^\perp \tag{5.11}$$

hold true, then, due to Lemma 5.8, we are done.

Observe that $D\boldsymbol{N}_1 \oplus \ker R = \mathrm{im}\ D\hat{Q}_1D^- \oplus \mathrm{im}\ (I - R) = \mathrm{im}\ (I - D\hat{P}_1D^-)$, hence $(D\boldsymbol{N}_1 \oplus \ker R)^\perp = \ker((I - D\hat{P}_1D^-)^*) = \ker[(D\hat{Q}_1D^-)^* + (I - R^*)]$. For $z \in (D\boldsymbol{N}_1 \oplus \ker R)^\perp$, $[(D\hat{Q}_1D^-)^* + (I - R^*)]z = 0$, i.e. $D^{-*}\hat{Q}_1^*D^*z = 0$ and $(I - R^*)z = 0$. Hence, $z = R^*z$ and $z \in \ker(D\hat{Q}_1D^-)^* = (\mathrm{im}\ D\hat{Q}_1D^-)^\perp = (D\boldsymbol{N}_1)^\perp$ and finally $z \in R^*(D\boldsymbol{N}_1)^\perp$.

On the other hand, $z \in R^*(D\boldsymbol{N}_1)^\perp = R^*(\mathrm{im}\, D\hat{Q}_1D^-)^\perp = R^*\ker(D\hat{Q}_1D^-)^*$ implies the representation $z = R^*w$ with $(DQ_1D^-)^*w = 0$. Further, we have

$$\begin{aligned}[(D\hat{Q}_1D^-)^* + (I - R^*)]z &= (D\hat{Q}_1D^-)^*z = D^{-*}\hat{Q}_1^*D^*z \\ &= D^{-*}\hat{Q}_1^*D^*D^{*-}D^*w = D^{-*}\hat{Q}_1^*D^*w \\ &= (D\hat{Q}_1D^-)^*w = 0,\end{aligned}$$

i.e. $z \in (D\boldsymbol{N}_1 \oplus \ker R)^\perp$. By this, (5.10) is verified. Relation (5.11) can be proven analogously. □

Now we turn to the proof of the Theorem 5.7.

Proof. Due to Theorem 5.1, equation (1.9) also is of index 2. Then, by Lemma 2.5 decompositions (2.10) and

$$A^*\boldsymbol{S}_{*1} \oplus A^*\boldsymbol{N}_{*1} \oplus \ker D^* = \mathbb{C}^m, \quad t \in \mathcal{I} \tag{5.12}$$

take place. Let C2 holds for (1.8). Since there exists a differentiable basis in $D\boldsymbol{N}_1$, the orthogonal projector onto $D\boldsymbol{N}_1$ is differentiable, and so is the orthogonal projector Δ onto $(D\boldsymbol{N}_1)^\perp$, $(D\boldsymbol{N}_1)^\perp = \mathrm{im}\ \Delta$. By the help of the Lemma, $A^*\boldsymbol{S}_{*1} = R^*(D\boldsymbol{N}_1)^\perp = \mathrm{im}\ R^*\Delta$. Thus the orthogonal projector $(R^*\Delta)(R^*\Delta)^+$ onto $A^*\boldsymbol{S}_{*1}$ is differentiable, i.e. there exists a differentiable basis in $A^*\boldsymbol{S}_{*1}$. In $\ker D^*$, the differentiable basis exists due to C1. Together with Lemma 5.9, it yields immediately that the subspace $A^*\boldsymbol{N}_{*1}$ complementary to $A^*\boldsymbol{S}_{*1} \oplus \ker D^*$ has a differentiable basis. □

COROLLARY 5.10. *Given an index-2 tractable DAE (1.8) and let condition C2 be fulfilled. Then,*

1. *for each $p \in C^1_{A^*Q_{*1}G^{-1}_{*2}}$, $a \in A^*(t_0)\boldsymbol{S}_{*1}(t_0)$, the initial value problem for equation (1.9) with initial value $A^*(t_0)\hat{P}_{*1}(t_0)y(t_0) = a$ is uniquely solvable in $C^1_{A^*}$, where $\hat{P}_{*1}$ is the special projector satisfying $\hat{P}_{*1} = I - Q_{*1}G^{-1}_{*2}B^*P_{*0}$,*
2. *(1.9) has perturbation index 2,*
3. *through each $t_0 \in \mathcal{I}$, $y \in \mathbb{S}_{*\mathrm{ind}\ 2}$ passes exactly one solution of the homogeneous equation.*

COROLLARY 5.11. *Given an index-i matrix DAE (4.1), $i = 1$ or $i = 2$, and let condition Ci hold. Then, for the adjoint matrix equation*

$$D^*(A^*X_*)' - B^*X_* = 0 \tag{5.13}$$

and arbitrary $t_0 \in \mathcal{I}$ there exists a unique maximal fundamental matrix X_{*i} normalized at $t = t_0$ and it has the form

$$X_{*i}(t,t_0) = \Pi_{*\mathrm{can}\ i}(t)A^{*-}(t)U_{*i}(t)A^*(t_0)\Pi_{*\mathrm{can}\ i}(t_0) \tag{5.14}$$

where U_{*i} is the normalized fundamental matrix of the inherent regular ODE for the adjoint equation (1.9). In the above statements, $\Pi_{*\mathrm{can}\ i}(t)$ is the canonical projector onto $\mathbb{S}_{*\mathrm{ind}\ i}$, they are constructed by the same scheme as described in Section 3 for (1.8).

In Section 2 we stated formula (2.1). If applied to fundamental matrices of (4.1) and (5.13) the Lagrange identity

$$X^*_{*i}ADX_i = const, \tag{5.15}$$

is valid for arbitrary pair of fundamental matrices X_i, X_{*i} even of different dimensions since $(X^*_{*i}ADX_i)' = (X^*_{*i}A)DX_i + X^*_{*i}A(DX_i)' = X^*_{*i}BX_i - X^*_{*i}BX_i = 0$.

When the fundamental matrices are normalized at the same point $t = t_0$, the value of the constant can be computed. Indeed,

$$\begin{aligned} X^*_{*i}(t,t_0)A(t)D(t)X_i(t,t_0) &\equiv X^*_{*i}(t_0,t_0)A(t_0)D(t_0)X_i(t_0,t_0) \\ &= \Pi^*_{*can\ i}(t_0)A(t_0)D(t_0)\Pi_{can\ i}(t_0). \end{aligned} \tag{5.16}$$

This identity allows to state relationships between the normalized fundamental matrices of equation (1.8) and (1.9). For this purpose we use the reflexive inverses defined by (4.8), $\widehat{t} = t_0$ and similarly by

$$X_{*i}(t,t_0)X^-_{*i}(t,t_0) = \Pi_{*\mathrm{can}\ i}(t) \quad \text{and} \quad X^-_{*i}(t,t_0)X_{*i}(t,t_0) = \Pi_{*\mathrm{can}\ i}(t_0). \tag{5.17}$$

Let us multiply (5.16) from the right by $X_i^-(t,t_0)$ and obtain

$$X^*_{*i}(t,t_0)A(t)D(t)\Pi_{\mathrm{can}\ i}(t) = \Pi^*_{*\mathrm{can}\ i}(t_0)A(t_0)D(t_0)X_i^-(t,t_0). \tag{5.18}$$

Taking the adjoint of (5.18) we get

$$\Pi^*_{\mathrm{can}\ i}D^*A^*X_{*i} = X_i^{-*}D^*(t_0)A^*(t_0)\Pi_{*\mathrm{can}\ i}(t_0) \tag{5.19}$$

Here and further, the argument t and the pair (t,t_0) are omitted when no confusion may arise.

When $i = 1$, then $\Pi_{can\ 1} = P_{0c}$ and $\Pi_{*can\ 1} = P_{*0c}$. Thus, $D^*A^*X_{*1} = X_1^{-*}D^*(t_0)A^*(t_0)$, or, by multiplication by D^{-*} from the left and by $A^{*-}(t_0)$ from the right, a more transparent form appears. Namely, we have

$$A^*(t)X_{*1}(t)A^{*-}(t_0) = (D(t_0)X_1^-(t)D^-(t))^*.$$

Multiplying from the left by A_c^{*-} and by $A^*(t_0)$ from the right and taking into account (5.17) we obtain an explicit formula for X_{*1}. It is worth noting that $D^{-*}X_1^{-*} = D^{*-}X_1^{-*}$. Similarly, proceeding from the adjoint of (5.16), one obtains $ADX_1 = X_{*1}^{-*}A(t_0)D(t_0)$ and finally,

$$D(t)X_1(t)D^-(t_0) = (A^*(t_0)X^-_{*1}(t)A^{*-}(t))^*.$$

Again, one can get an explicit formula, correctly reflecting the geometrical solution space. Both formulae will be given in Theorem 5.12 below. We wish, however, formulate it after we have verified that it is valid in the index–2 case, too. For that purpose, we recall Lemma 5.9. By its help we are ready to find the connection between the fundamental matrices for the pair of index–2 equations as we treated the index–1 pairs.

Recall that $\Pi_{\mathrm{can}\ 2} = P_{0c}P_{1c}$, $\Pi_{*\mathrm{can}\ 2} = P_{*0c}P_{*1c}$ and $P_0 = P_0P_{0c}$, $P_0P_{0c}P_{1c} = P_0\hat{P}_1$, $P_{*0} = P_{*0}P_{*0c}$, $P_{*0}P_{*0c}P_{*1c} = P_{*0}\hat{P}_{*1}$. Multiplication of (5.19) by D^{-*} from the left yields

$$D^{-*}\hat{P}_1^*D^*A^*X_{*2} = D^{-*}X_2^{-*}D^*(t_0)A^*(t_0)\hat{P}_{*1}(t_0). \tag{5.20}$$

By Lemma 5.9 we find

$$D^{-*}\hat{P}_1^*D^*A^*X_{*2} = A^*\hat{P}_{*1}A^{*-}A^*X_{*2} = A^*X_{*2} \tag{5.21}$$

and further

$$\begin{aligned} &X_2^{-*}D^*(t_0)A^*(t_0)\hat{P}_{*1}(t_0) = X_2^{-*}D^*(t_0)A^*(t_0)\hat{P}_{*1}(t_0)A^{*-}(t_0)A^*(t_0) \\ &= X_2^{-*}D^*(t_0)[D(t_0)\hat{P}_1(t_0)D^-(t_0))]^*A^*(t_0) = X_2^{-*}D^*(t_0)A^*(t_0). \end{aligned}$$

Consequently, (5.20) reads now

$$A^* X_{*2} = D^{-*} X_2^{-*} D^*(t_0) A^*(t_0).$$

It remains to multiply by A_c^{*-} from the left and to take into account that $A_c^{*-} A^* = P_{*0c}$ and $P_{*0c} X_{*2} = X_{*2}$ and by this, we arrive at the explicit expression for X_{*2}. Observe that $D^{-*} X_i^{-*} = D^{*-} X_i^{-*}$ holds. Now the proof of the first formula in the next theorem is completed for both indices 1 and 2. The second formula comes by symmetry and by the use of $A^{*-*} X_i^{-*} = A^- X_i^{-*}$. The possibility of the changes A^{*-*} and D^{-*} for A^- and D^{*-} makes transparent that the final statement is independent of special choices of P_0, P_{*0} and W_0, W_{*0}, too.

THEOREM 5.12. *The fundamental matrices of index-i DAE (1.8) and (1.9), $i = 1, 2$ normalized at the same point $t = t_0$ are connected by the formulae*

$$\begin{aligned} X_{*i}(t) &= A_c^{*-}(t) D^{*-}(t) X_i^{-*}(t) D^*(t_0) A^*(t_0), \\ X_i(t) &= D_c^-(t) A^-(t) X_{*i}^{-*}(t) A(t_0) D(t_0), \end{aligned}$$

provided the proper reflexive inverses (4.8) and (5.17) are taken and D_c^- and A_c^{*-} are constructed by the help of P_{0c} and P_{*0c}.

6. Final remarks. In the paper our aim was to give a precise and detailed analysis of low index DAEs with not very smooth data as they arise in most of applications rather then a general DAE theory supposing highly smooth data. As a first complement to the present paper, [8] deals with nonlinear equations

$$A(x(t), t)(D(t)x(t))' + b(x(t), t) = 0$$

and

$$A(x(t), t)(d(x(t), t)' + b(x(t), t) = 0.$$

In particular, different approaches of integration methods ([7, 6]) are now unified and understood anew.

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