Level Zero Types and Hecke Algebras for Local Central Simple Algebras

Martin Grabitz, Allan J. Silberger, and Ernst-Wilhelm Zink

Abstract:

Let D be a central division algebra and $A^\times = GL_m(D)$ the unit group of a central simple algebra over a p-adic field F. The purpose of this paper is to give types (in the sense of Bushnell and Kutzko) for all level zero Bernstein components of $A^\times$ and to establish that the Hecke algebras associated to these types are isomorphic to tensor products of Iwahori Hecke algebras. The types which we consider are lifted from cuspidal representations τ of $M(k_D)$, where M is a standard Levi subgroup of GL_m and k_D is the residual field of D. Two types are equivalent if and only if the corresponding pairs $(M(k_D), \tau)$ are conjugate with respect to $A^\times$. The results are basically the same as in the split case $A^\times = GL_n(F)$ due to Bushnell and Kutzko. In the non split case there are more equivalent types and the proofs are technically more complicated.

0. Introduction

Let F be a p-adic local field, let $D := D_d$ be a central F-division algebra of index d, and let $A := M_m(D)$ be a central simple F-algebra of reduced degree $n := dm$. The purpose of this paper is to give a classification of types (see [BK2]) for all level zero Bernstein components of the unit group $A^\times$ and to establish that the Hecke algebras associated to these types are isomorphic to tensor products of Iwahori Hecke algebras, as in the split case (see [BK1] and [BK3]).

In [M1] Morris proved Hecke algebra isomorphism theorems which apply to the level zero representations of general reductive groups and in [M2] he extended this earlier work to show that cuspidal level zero representations of the finite field points of Levi factors of reductive groups inflate to types for level zero Bernstein components. Our paper, in effect, presents a special case of Morris's general theory, an example which is at the same time more general and given in greater detail than in Bushnell and Kutzko's work ([BK1] and [BK3]) for the level zero case of $M_n(F)$.

We think that the present extension of the split case is interesting enough to merit being spelled out, as we have attempted in this paper. Like Howe/Moy and Bushnell/Kutzko, we construct level zero types by inflating cuspidal representations of Levi factors with coefficients in the residue field of D to representations of unit groups of hereditary orders. Our situation is also analogous to the split case in that representations of Levi factors which are conjugate under inner automorphisms of $A^\times$ inflate to types for the same Bernstein component. However, there are more inner automorphisms acting on the set of cuspidal representations; some of these can be interpreted as a Galois action which is trivial in the split case. Although the Hecke algebra of a simple type looks like the group algebra of a semi-direct product of an infinite cyclic group normalizing a Coxeter group, the cyclic group object which serves as a part of the support of the Hecke algebra in the case of a simple type for $A^\times$ need not normalize a principal order or Iwahori subgroup-like object. The multiplication of double cosets is also more complicated in the case of general simple algebras. We prove our Hecke algebra isomorphism theorems for natural representatives of each Galois orbit of simple level zero types after arguing that all representatives of the same orbit have isomorphic Hecke algebras, that one representative is a type if and only if all are. As in other level zero situations (e.g. [M2]), we obtain our results by reducing the proofs to general arguments due to Bushnell and Kutzko ([BK2]).

We begin the paper with some background information and give statements of our main theorems (Theorems 1 and 2) in §§0.6 and 0.7. Parts 1-4 are concerned with Hecke algebras, whereas Part 5 concludes the classification of types by applying Bushnell/Kutzko's theory of covers ([BK2] 8.).

We thank Peter Schneider for a helpful remark.

§0.1 The Bernstein Spectrum and Decomposition

Let $G = G(F)$ denote the group of F-points of a connected reductive F-group. A *cuspidal pair* (M, π) for G consists of a Levi subgroup M of G and an irreducible supercuspidal representation π of M; the *Bernstein spectrum* $\Omega(G)$ is defined as the set of G-conjugacy classes of cuspidal pairs (M, π). For any irreducible smooth representation Π of G its supercuspidal support is a unique element of $\Omega(G)$.

The Bernstein spectrum has the structure of a complex locally algebraic variety. Let $X_{nr}(M)$ denote the group of unramified characters of M with its natural complex structure. Then the connected component of $\Omega(G)$ which contains the G-orbit of a cuspidal pair (M, π) is the image of the map

$$X_{nr}(M) \to \Omega(G); \quad \chi \mapsto G\text{−orbit of } (M, \chi\pi).$$

Let $\mathcal{M}(G)$ denote the category of smooth G-representations. For a connected component $\Omega \subset \Omega(G)$ let $\mathcal{M}(\Omega)$ denote the full subcategory of G-representations, all irreducible subquotients of which have supercuspidal support in Ω. The Bernstein decomposition of $\mathcal{M}(G)$ [Be] is defined as the equivalence

$$\mathcal{M}(G) = \prod_{\Omega} \mathcal{M}(\Omega), \tag{1}$$

where Ω runs over the connected components of $\Omega(G)$.

§0.2 Hecke Algebras and Intertwining Functions

Let K be an open compact subgroup of G and let (K, τ, W) be a representation τ of K in W. We call the convolution algebra consisting of all compactly supported functions $f : G \to \mathrm{End}_{\mathbb{C}}(W)$ such that

$$f(k_1 g k_2) = \tau(k_1) f(g) \tau(k_2) \tag{2}$$

for all $k_1, k_2 \in K$ the Hecke algebra of G with respect to (K, τ) and we denote it $\mathcal{H}(G, K, \tau)$.

The unit element is

$$e_\tau(x) = \begin{cases} \mu(K)^{-1}\tau(x) & \text{for } x \in K \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu(K) = \int_K 1 dy$.

We shall make use of a generalized algebra of "intertwining functions" in this paper and we include here for reference purposes a brief discussion of these functions. For any pair (τ_i, W_i, K) of irreducible representations of K acting in vector spaces W_i $(i = 1, 2)$ we call *intertwining function* any compactly supported function $f := f_{\tau_2,\tau_1}$ such that $f : G \to \mathrm{Hom}_{\mathbb{C}}(W_1, W_2)$ and $f(k'gk) = \tau_2(k')f(g)\tau_1(k)$. For functions $f_{\tau_3,\tau_2}, f_{\tau_2,\tau_1}$ we have a natural convolution product

$$f_{\tau_3,\tau_2} * f_{\tau_2,\tau_1}(g) := \int_G f_{\tau_3,\tau_2}(x) f_{\tau_2,\tau_1}(x^{-1}g)\, dx,$$

which produces a function f_{τ_3,τ_1}.

For $\varphi \in \mathrm{Hom}_K(W_2, V)$ and $w \in W_1$ we define

$$(\Pi(f_{\tau_2,\tau_1})\varphi)(w) := \int_G \Pi(g^{-1})\varphi(f_{\tau_2,\tau_1}(g)w)\, dg.$$

We obtain a mapping $\Pi(f_{\tau_2,\tau_1}) : \mathrm{Hom}_K(W_2, V) \to \mathrm{Hom}_K(W_1, V)$. We note that

$$\Pi(f_{\tau_3,\tau_2} * f_{\tau_2,\tau_1}) = \Pi(f_{\tau_2,\tau_1})\Pi(f_{\tau_3,\tau_2}),$$

which means that $\mathrm{Hom}_K(W_1, V)$ is a right module for the Hecke algebra $\mathcal{H}(G, K, \tau_1)$.

§0.3 The Concept of a Type

Bushnell and Kutzko [BK2] call a pair (K, τ), where K is an open compact subgroup and τ is an irreducible representation of K, a type for G if the category $\mathcal{M}_\tau(G)$ of all G-representations which are generated by their τ-isotypic components is closed under the formation of subquotients. In particular, they call (K, τ) a type for the connected component $\Omega \subset \Omega(G)$ if $\mathcal{M}_\tau(G) = \mathcal{M}(\Omega)$, i.e. if for every irreducible representation Π of G the restriction $\Pi|_K$ contains τ if and only if the supercuspidal support of Π belongs to Ω.

For (K, τ) a type the category $\mathcal{M}_\tau(G)$ is equivalent to a category of modules over the Hecke algebra $\mathcal{H}(G, K, \tau)$. More precisely, Bushnell and Kutzko show in [BK2](4.3) that the mapping

$$\mathcal{M}_\tau(G) \ni (\Pi, V) \mapsto \mathrm{Hom}_K(W, V) \in \mathrm{Mod}(\mathcal{H}(G, K, \tau)^{opp})$$

is an equivalence of categories if and only if the pair (K, τ) is a type. (We have to take the opposite algebra because our definition of $\mathcal{H}(G, K, \tau)$ uses τ instead of the contragredient of τ as in Bushnell and Kutzko's work.)

Henceforth we consider only the special case $G = A^\times$.

§0.4 The Connected Components of $\Omega(A^\times)$

We recall the formal set-up of Bernstein and Zelevinsky which provides a parameterization for the connected components of $\Omega(A^\times)$. Let $\mathcal{C} := \mathcal{C}(D)$ be a set of representatives for the unramified twist classes of irreducible pre-unitary supercuspidal representations of $GL_s(D)$ for all $s \geq 1$. For $\pi \in \mathcal{C}$ a representation of $GL_s(D)$ we define the degree of π to be $d(\pi) := s$. Let $\mathrm{Div}^+(\mathcal{C})$ denote the set of effective divisors over $\mathcal{C}$. To any effective divisor $\mathcal{D} = \sum_{\pi \in \mathcal{C}} m_\pi \pi$ we associate the triple:

- its degree $d(\mathcal{D}) = \sum m_\pi d(\pi)$
- the Levi subgroup $M_\mathcal{D} \subset GL_{d(\mathcal{D})}(D)$, where $M_\mathcal{D} = \prod_\pi (GL_{d(\pi)}(D))^{\times m_\pi}$ (assuming some ordering of the factors), and
- the supercuspidal representation $\pi_\mathcal{D}$ of $M_\mathcal{D}$ such that $\pi_\mathcal{D} = \otimes_\pi (\pi^{\otimes m_\pi})$.

For each $\mathcal{D} \in \mathrm{Div}^+(\mathcal{C})$ of degree m let $\Omega_\mathcal{D} \subset \Omega(A^\times)$ denote the connected component which contains the $A^\times$-orbit of $(M_\mathcal{D}, \pi_\mathcal{D})$.

1. Fact: The mapping $\mathcal{D} \mapsto \Omega_\mathcal{D}$ parameterizes the connected components of $\Omega(A^\times)$ by degree m divisors over $\mathcal{C}$.

§0.5 Standard Hereditary Orders

Let O denote the ring of integers of D and $\mathfrak{p}$ the maximal ideal of O. We fix the maximal order $\mathfrak{A}_1 = M_m(O)$, with Jacobson radical $\mathfrak{P}_1 = M_m(\mathfrak{p})$. We also fix the minimal order $\mathfrak{A}_m \subset \mathfrak{A}_1$ which consists of those elements of $\mathfrak{A}_1$ which have all matrix elements below the main diagonal in $\mathfrak{p}$. The Jacobson radical $\mathfrak{P}_m \subset \mathfrak{A}_m$ has coefficients on and below the main diagonal in $\mathfrak{p}$. A hereditary order $\mathfrak{A}$ such that $\mathfrak{A}_m \subseteq \mathfrak{A} \subseteq \mathfrak{A}_1$ will be called *standard*. Every hereditary order of A is conjugate to a unique standard hereditary order. If the standard hereditary orders satisfy $\mathfrak{A}' \subseteq \mathfrak{A}$, then the Jacobson radicals satisfy the reverse inclusions $\mathfrak{P}_1 \subseteq \mathfrak{P}_\mathfrak{A} \subseteq \mathfrak{P}_{\mathfrak{A}'} \subseteq \mathfrak{P}_m$. The mapping $\mathfrak{A} \mapsto \bar{\mathfrak{A}} = \mathfrak{A}/\mathfrak{P}_1$ sends the set of standard hereditary orders bijectively to the set of upper block triangular matrix rings in $M_m(k_D)$. In particular, to any standard hereditary order $\mathfrak{A}$ there corresponds a tuple of positive integers $s_1, \ldots, s_r$ with sum m such that the quotient ring $\mathfrak{A}/\mathfrak{P}_\mathfrak{A}$ is the semi-simple algebra

$$\mathfrak{A}/\mathfrak{P} \cong M_{s_1}(k_D) \times \cdots \times M_{s_r}(k_D), \tag{3}$$

each factor $M_{s_i}(k_D)$ being a complete matrix algebra over the residual field k_D of D. The multiplicative group of (3) is

$$\overline{\mathfrak{A}}^\times := (\mathfrak{A}/\mathfrak{P})^\times = \mathfrak{A}^\times/(1+\mathfrak{P}) \cong GL_{s_1}(k_D) \times \cdots \times GL_{s_r}(k_D). \tag{4}$$

§0.6 The Cuspidal Support of a Level Zero Representation

An irreducible smooth representation (Π, V) of $A^\times$ is called *level zero* if $V^{1+\mathfrak{P}_1} \neq (0)$, i.e. if there exists a $(1+\mathfrak{P}_1)$-fixed vector.

For (Π, V) a level zero representation we interpret $V^{1+\mathfrak{P}}$ as the Jacquet restriction of $V^{1+\mathfrak{P}_1}$ with respect to the parabolic subgroup $\mathfrak{A}^\times/(1+\mathfrak{P}_1) \subset \bar{\mathfrak{A}}_1^\times$.

It is natural to take $\mathfrak{A}$ minimal ($\mathfrak{P}_{\mathfrak{A}}$ maximal) such that $V^{1+\mathfrak{P}_{\mathfrak{A}}} \neq 0$. It follows that, as a representation of (4), all irreducible constituents $\tau = \sigma_1 \otimes \cdots \otimes \sigma_r$ occurring in $V^{1+\mathfrak{P}_{\mathfrak{A}}}$ are cuspidal, i.e. they are tensor products in which each tensor factor σ_i of $GL_{s_i}(k_D)$ is a cuspidal representation. In this case, we call $(\mathfrak{A}^\times, \tau)$ a cuspidal level zero pair and we write $\mathrm{supp}(\tau) := \{\sigma_1, \ldots, \sigma_r\}$. As we have seen, every level zero representation has a cuspidal level zero pair as a component. We will prove that each of these pairs is a type and that the connected component $\Omega \subset \Omega(A^\times)$ corresponding to the type $(\mathfrak{A}^\times, \tau)$ is determined as follows. Consider $\mathrm{supp}(\tau)$ and introduce the following equivalence relation on the set of cuspidal representations of $GL_s(k_D)$ for all $s \geq 1$: $\sigma \sim \sigma'$ if and only if $\sigma' = \sigma^\psi$ for some $\psi \in \mathrm{Gal}(k_D|k)$ acting coefficientwise on $GL_s(k_D)$. We write $[\sigma]$ for the $\mathrm{Gal}(k_D|k)$-equivalence class of σ, $r_{[\sigma]}(\tau)$ for the number of elements in $\mathrm{supp}(\tau)$ belonging to $[\sigma]$, and $d([\sigma]) := s$ for the degree of σ and $[\sigma]$. To τ we associate the effective divisor

$$\Delta(\tau) := \sum_{[\sigma]} r_{[\sigma]}(\tau)[\sigma],$$

where the sum ranges over $\mathrm{Gal}(k_D|k)$-equivalence classes of cuspidal representations occurring in $\mathrm{supp}(\tau)$.

In 5.1 and 5.2 we show that $[\sigma]$ determines an unramified twist class of irreducible supercuspidal representations of $GL_{d([\sigma])}(D)$, hence a unique $\pi_{[\sigma]} \in \mathcal{C}$. Therefore, $\Delta(\tau)$ determines $\mathcal{D}(\tau) := \sum_{[\sigma]} r_{[\sigma]}(\tau)\pi_{[\sigma]} \in \mathrm{Div}^+(\mathcal{C})$, and we prove:

Theorem 1: *Let $(\mathfrak{A}^\times, \tau)$ be any cuspidal level zero pair. Then $(\mathfrak{A}^\times, \tau)$ is a type for the connected component $\Omega_{\mathcal{D}} \subset \Omega(A^\times)$, where $\mathcal{D} = \mathcal{D}(\tau)$ is determined by $\Delta(\tau)$.*

We call two cuspidal level zero pairs $(\mathfrak{A}^\times, \tau)$ and $(\mathfrak{A}'^\times, \tau')$ equivalent if $\Delta(\tau) = \Delta(\tau')$ and we obtain a bijection between equivalence classes of such pairs and level zero connected components of $\Omega(A^\times)$. In the case of $M_n(F)$ there is no Galois action and this means that $\Delta(\tau) = \Delta(\tau')$ if and only if $\mathrm{supp}(\tau)$=$\mathrm{supp}(\tau')$ as multisets.

§0.7 The Hecke Algebra of a Level Zero Type

Let $(\mathfrak{A}^\times, \tau)$ be a cuspidal level zero pair. In verifying that $(\mathfrak{A}^\times, \tau)$ is a type we study the structure of the Hecke algebra $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$. We prove:

Theorem 2: *If $(\mathfrak{A}^\times, \tau)$ has the divisor $\Delta(\tau)$, then*

$$\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau) \cong \bigotimes_{[\sigma]} \mathcal{H}(r_{[\sigma]}(\tau), q^{d \cdot d([\sigma])}),$$

a tensor product of affine Hecke algebras (see Part 4 for the notation $\mathcal{H}(r, z)$).

In Part 1 we determine the support of $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$ and, applying [BK2](7.2)(ii), we show that $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$ is a tensor product of Hecke algebras such that $\mathfrak{A} = \mathfrak{A}_r$ is a principal order and such that all tensor factors of τ are $\mathrm{Gal}(k_D|k)$-equivalent representations. In (1.10), the final result of the part, we show that in this particular case $\mathcal{H}(A^\times, \mathfrak{A}_r^\times, \tau)$ is isomorphic to $\mathcal{H}(A^\times, \mathfrak{A}_r^\times, \sigma^r)$, where $\sigma \in \mathrm{supp}(\tau)$ and $\sigma^r := \sigma^{\otimes r}$. Thus, we conclude that $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$ depends only upon the divisor $\Delta(\tau)$. In Part 4 we show that $\mathcal{H}(A^\times, \mathfrak{a}_r^\times, \sigma^r) \cong \mathcal{H}(r, q^{d\cdot d(\sigma)})$ (see Theorem 4.2), i.e. is isomorphic to an affine Hecke algebra of type A (see [BK1], Chapter 5 for the case $M_n(F)$). As a preparation in Parts 2 and 3 we study the multiplication of double cosets, the main results being Propositions 2.6, 2.7, and 3.1. Here differences in the proofs between the general and the split case become visible. As we have noted, the final results do not reflect these differences.

§0.8 Generalized Tits Systems for Unit Groups of Simple Algebras

This section is included for reference purposes and to establish notation.

Let o_F denote the ring of integers of the p-adic local field F, $\mathfrak{p}_F$ the prime ideal of o_F, and $k = o_F/p_F$ the residual field of F. In D we fix a pair (F_d, ϖ) consisting of a maximal unramified extension of F and a prime element of D which normalizes F_d; in this case $\varpi^d = \varpi_F$ is a prime element of F. We may also identify k_D with the residual field of F_d and note that $k_D|k$ is a degree d extension.

Let ${}^0A$ denote the subgroup of $A^\times$ consisting of all elements x such that $\mathrm{Nrd}_{A|F}(x) \in o_F^\times$. Note that all compact subgroups of $A^\times$ are in ${}^0A$.

Let $\tilde{W}_A$ denote the subgroup of $A^\times$ consisting of all monomial matrices with non-zero entries which are powers of ϖ. Each element $w \in \tilde{W}_A$ has a unique product representation of the form

$$w = \varpi^v p, \tag{5}$$

where $v = (v_1, \ldots, v_m) \in \mathbb{Z}^m$, $\varpi^v = \mathrm{diag}(\varpi^{v_1}, \ldots, \varpi^{v_m})$, and $p = (\delta_{i,\sigma(j)})_{i,j=1}^m$ is the matrix of a permutation $\sigma \in \mathfrak{S}_m$, i.e. $w = \varpi^v p$ is the matrix obtained by permuting the columns of ϖ^v by σ. The subgroup $W_A = \tilde{W}_A \cap {}^0A$ consists of all $w = \varpi^v \cdot p$ such that $v_1 + \cdots + v_m = 0$, and we have the semi-direct product

$$\tilde{W}_A = W_A \rtimes \langle h_A \rangle.$$

Let $\mathcal{I}_A$ be the minimal standard hereditary order in A, and let $N_A \subset A^\times$ and ${}^0N_A = N_A \cap {}^0A$ denote the subgroups of monomial matrices.

Set

$$h_A = \begin{pmatrix} & I_{m-1} \\ \varpi & \end{pmatrix}.$$

For $i = 1, \ldots, m-1$ let $s_{i,A}$ denote the matrix of the transposition $i \leftrightarrow i+1$ and set $s_{0,A} := h_A s_{1,A} h_A^{-1}$.

2. Fact(see [I]): The triple $(A^\times, \mathcal{I}_A^\times, N_A)$ is a generalized Tits system with $\tilde{W}_A \cong N_A/(\mathcal{I}_A^\times \cap N_A)$ as generalized Weyl group. The set $S_A = \{s_{0,A}, \ldots, s_{m-1,A}\}$ is a Coxeter system of type $\tilde{A}_{m-1}$; it generates the group $W_A \cong {}^0N_A/({}^0N_A \cap \mathcal{I}_A^\times)$ and $(\mathcal{I}_A^\times, {}^0N_A)$ is an affine BN-pair of the group ${}^0A$.

We have the Bruhat decomposition

$$\tilde{W}_A \longleftrightarrow \mathcal{I}_A^\times \backslash A^\times / \mathcal{I}_A^\times,$$

and, more generally:

3. Fact: Let $\mathfrak{A}_i$ be a standard hereditary order for $i = 1, 2$. Then there is a natural bijective correspondence

$$\mathfrak{A}_1^\times \backslash A^\times / \mathfrak{A}_2^\times \longleftrightarrow (\mathfrak{A}_1^\times \cap \tilde{W}_A) \backslash \tilde{W}_A / (\mathfrak{A}_2^\times \cap \tilde{W}_A).$$

If $M = GL_{s_1} \times \cdots \times GL_{s_r}$, where $s_1 + \cdots + s_r = m$, and $\mathfrak{A}^\times = M(O) \cdot (1 + \mathfrak{P}_{\mathfrak{A}})$, then

$$\mathfrak{A}^\times \cap \tilde{W}_A = M(O) \cap W_A \cong \mathfrak{S}_{s_1} \times \cdots \times \mathfrak{S}_{s_r}.$$

Write $l_A(w)$ for the length function on W_A corresponding to the system S_A. For $w \in W_A$ and $s \in S_A$ such that $l_A(ws) > l_A(w)$

$$\mathcal{I}_A^\times w \mathcal{I}_A^\times s \mathcal{I}_A^\times = \mathcal{I}_A^\times ws \mathcal{I}_A^\times \tag{6}$$

and

$$(\mathcal{I}_A^\times : \mathcal{I}_A^\times \cap w \mathcal{I}_A^\times w^{-1}) = q^{d \cdot l_A(w)}, \tag{7}$$

where $q^d = |k_D|$.

Notations:

$F,\ o_F,\ \varpi_F,\ \mathfrak{p}_F$	p-adic local field, integers, prime element, maximal ideal
$k = o_F/\mathfrak{p}_F,\ q$	residual field of F, $\lvert k\rvert$
$D \vert F$	central division F-algebra of index d
$O, \mathfrak{p}$	valuation ring and valuation ideal of D
$k_D := O/\mathfrak{p},\ q^d$	residual field of D, $\lvert k_D\rvert$
F_d, ϖ	a maximal unramified extension of F in D, a prime element of D which normalizes F_d; $(\varpi^d = \varpi_F)$
$A := M_m(D)$	central simple algebra over F
$n := dm$	reduced degree of $A \vert F$
$\mathfrak{A}, \mathfrak{P} := \mathfrak{P}_{\mathfrak{A}}$	standard hereditary order in A, its Jacobson radical
$\mathfrak{A}_r, \mathfrak{P}_r$	standard principal order of period $r \vert m$, its Jacobson radical; in (3) we have $s_1 = \cdots = s_r = s = \frac{m}{r}$.
$\mathcal{I}_A := \mathfrak{A}_m$	minimal standard hereditary order in A.

1. The Support of the Hecke Algebra

Let $\mathfrak{A} \subset A$ be a standard hereditary order and let (τ, W) be an irreducible cuspidal representation of the group $\bar{\mathfrak{A}}^\times$. By inflation we regard τ as a representation of $\mathfrak{A}^\times$ (see (4)). In this Part we want to construct a vector space basis for the Hecke algebra $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$ (see §0.2).

For any $x \in A^\times$ define

$$\tau^x(y) := \tau(xyx^{-1}) \qquad (y \in \mathfrak{A}^\times).$$

1.1 **Lemma**: *For $x \in A^\times$ there exists $f \in \mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$ such that $f(x) \neq 0$ if and only if $\mathrm{Hom}_{\mathfrak{A}^\times \cap x^{-1}\mathfrak{A}^\times x}(\tau, \tau^x) \neq (0)$, in which case setting $f(x) := J \in \mathrm{Hom}_{\mathfrak{A}^\times \cap x^{-1}\mathfrak{A}^\times x}(\tau, \tau^x)$ uniquely determines f with support in $\mathfrak{A}^\times x \mathfrak{A}^\times$. If $\tau|_{\mathfrak{A}^\times \cap x^{-1}\mathfrak{A}^\times x}$ is irreducible, then this space of functions is one-dimensional.*

Proof: For $y \in \mathfrak{A}^\times \cap x^{-1}\mathfrak{A}^\times x$ we have

$$\begin{aligned} \tau^x(y)f(x) &= \tau(xyx^{-1})f(x) = f(xy) = f(x)\tau(y) \\ \text{i.e. } f(x) &= J \in \mathrm{Hom}_{\mathfrak{A}^\times \cap x^{-1}\mathfrak{A}^\times x}(\tau, \tau^x). \end{aligned}$$

Conversely, if $0 \neq J \in \mathrm{Hom}_{\mathfrak{A}^\times \cap x^{-1}\mathfrak{A}^\times x}(\tau, \tau^x)$ exists, then $f(y_1 x y_2) := \tau(y_1) J \tau(y_2)$ for $y_1, y_2 \in \mathfrak{A}^\times$ defines a function with support $\mathfrak{A}^\times x \mathfrak{A}^\times$. Since $y_1 x y_2 = x$ implies that $f(y_1 x y_2) = f(x)$, the function f is well defined. That the space of functions with support in $\mathfrak{A}^\times x \mathfrak{A}^\times$ is at most one-dimensional follows from Schur's Lemma, when $\tau|_{\mathfrak{A}^\times \cap x^{-1}\mathfrak{A}^\times x}$ is irreducible. □

We consider $\mathfrak{A}^\times = U^-(\mathfrak{P}) \cdot M(O) \cdot U^+(O)$, the Iwahori factorization of $\mathfrak{A}^\times$, and write $\mathcal{N}_{A^\times}(M(O))$ for the normalizer of $M(O)$ in $A^\times$. We shall also consider $\bar{\mathfrak{A}}^\times = M(k_D) = \prod_{i=1}^{r} GL_{s_i}(k_D)$ as a block-diagonal group and $\tau = \sigma_1 \otimes \cdots \otimes \sigma_r$, where σ_i is a cuspidal representation of $GL_{s_i}(k_D)$.

1.2 **Proposition:** *The support of $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$ consists of the set of double cosets $\mathfrak{A}^\times w \mathfrak{A}^\times$, where $w \in \tilde{W}_A$ and, in addition:*

(i) *$w \in \mathcal{N}_{A^\times}(M(O))$;*

(ii) *conjugation by w fixes the class of the representation τ of $M(O)$.*

Proof: During the proof we write $\mathcal{H} := \mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$. We proceed in several steps.

1.3 **Lemma**: *If $w = \varpi^v p \in \tilde{W}_A$ and w is in the support of $\mathcal{H}$, then ϖ^v normalizes $M(O)$.*

Proof: Consider $w = \varpi^v p$ and write the exponent vector $v = (v_1, \ldots, v_m) \in \mathbb{Z}^m$ (see (§0.8)) as a vector of vectors $v = (v^{(1)}, \ldots, v^{(r)})$, where, for $1 \le i \le r$, the vector $v^{(i)}$ is an s_i-vector. Replacing w by

$$p'\, w = p' \varpi^v p'^{-1} \cdot p'p \quad \text{for} \quad p' \in \mathfrak{A}^\times \cap \tilde{W}_A,$$

we may assume that each of the subvectors $v^{(i)}$ is a non-increasing sequences of integers. Assume that there is a j such that in some subvector $v^{(i)}$ we have $v_j > v_{j+1}$. Then

$$M' = (GL_j \times GL_{m-j}) \cap M$$

is a proper Levi subgroup of M; we write $P' = M'U' \subset M$ for the corresponding lower parabolic subgroup of M. Thus,

$$U'(D) \cap \mathfrak{A}^\times \subseteq M(D) \cap \mathfrak{A}^\times = M(O),$$

$$\varpi^{-v}(U'(D) \cap \mathfrak{A}^\times)\varpi^v \subseteq 1 + \mathfrak{P}_1,$$

and, therefore,

$$f(u'\varpi^v p) = f(\varpi^v p(p^{-1}\varpi^{-v}u'\varpi^v p)) = f(\varpi^v p)$$

for any $f \in \mathcal{H}$ and $u' \in U'(D) \cap \mathfrak{A}^\times$, since the conjugation by ϖ^{-v} maps $U'(D) \cap \mathfrak{A}^\times$ into $1 + \mathfrak{P}_1$ and the permutation matrix p normalizes $1 + \mathfrak{P}_1$. Noting that $U'(D) \cap \mathfrak{A}^\times$ modulo $1 + \mathfrak{P}$ is the unipotent radical of a proper parabolic subgroup of $M(k_D)$, we conclude that τ cuspidal implies that

$$0 = \int\limits_{U'(D)\cap\mathfrak{A}^\times} \tau(u')f(\varpi^v\, p)du' = \int\limits_{U'(D)\cap\mathfrak{A}^\times} f(\varpi^v\, p)du',$$

so $f(\varpi^v p) = 0$. Therefore, all the subvectors $v^{(i)}$ of $v \in \mathbb{Z}^m$ have to be "scalar" for $\varpi^v p$ to be in the support of $\mathcal{H}$; in other words, it is necessary that $\varpi^v \in \mathcal{N}_{A^\times}(M(O))$. □

1.4 **Lemma**: *Assume that ϖ^v normalizes $M(O)$ but that the permutation matrix p does not normalize M. Then $w = \varpi^v p$ is not in the support of $\mathcal{H}$.*

Proof: Let $P = MU$ be the upper block triangular parabolic subgroup of $A^\times$ which has M as its Levi subgroup. If $p \in A^\times$ is a permutation matrix which does not normalize M, then $P' = pPp^{-1} \cap M$ is a proper parabolic subgroup of M with the Levi decomposition

$$P' = M'U' = (pMp^{-1} \cap M)(pUp^{-1} \cap M).$$

In $\mathfrak{A}^\times$ we have $P' \cap \mathfrak{A}^\times = (M' \cap \mathfrak{A}^\times)(U' \cap \mathfrak{A}^\times)$ with $U' \cap \mathfrak{A}^\times = pUp^{-1} \cap M(O)$. Since τ is a cuspidal representation of $M(k_D)$ and since the reduction of $U' \cap \mathfrak{A}^\times$ is the unipotent radical of a proper parabolic subgroup of $M(k_D)$, it follows that

$$0 = \int\limits_{U'\cap\mathfrak{A}^\times} \tau(u')f(\varpi^v p)du' = \int\limits_{U'\cap\mathfrak{A}^\times} f(\varpi^v p(p^{-1}\varpi^{-v}u'\varpi^v p))du'$$

and our assertion follows from the fact that $p^{-1}\varpi^{-v}u'\varpi^v p \in 1 + \mathfrak{P}$. To see this observe that the diagonal matrices lie in $M \cap pMp^{-1}$; hence ϖ^v normalizes pUp^{-1} and, by hypothesis, also $M(O)$. Therefore ϖ^v normalizes $U' \cap \mathfrak{A}^\times = pUp^{-1} \cap M(O)$ and we may conclude that $\varpi^{-v}u'\varpi^v \in pUp^{-1} \cap M(O)$, i.e. that $p^{-1}\varpi^{-v}u'\varpi^v p \in U(O) \subseteq 1 + \mathfrak{P}$. □

1.5 **Lemma**: *If $w \in \tilde{W}_A \cap \mathcal{N}_{A^\times}(M(O))$, then*

$$\mathrm{Hom}_{\mathfrak{A}^\times \cap w^{-1}\mathfrak{A}^\times w}(\tau, \tau^w) = \mathrm{Hom}_{M(k_D)}(\tau, \tau^w).$$

Proof: Note that $\mathfrak{A}^\times = M(O) \cdot (1 + \mathfrak{P})$, where the normal subgroup $1 + \mathfrak{P}$ is in the kernel of τ. The hypothesis $w \in \tilde{W}_A \cap \mathcal{N}_{A^\times}(M(O))$ implies that

$$\mathfrak{A}^\times \cap w^{-1}\mathfrak{A}^\times w = M(O) \cdot ((1 + \mathfrak{P}) \cap w^{-1}(1 + \mathfrak{P})w)$$

and that τ and τ^w are both trivial on $(1+\mathfrak{P})\cap w^{-1}(1+\mathfrak{P})w$, i.e. $\tau^w(x) = \tau(wxw^{-1})$ and conjugation by w maps $(1+\mathfrak{P})\cap w^{-1}(1+\mathfrak{P})w$ into $1+\mathfrak{P}$. Thus the intertwining map factors through the projection of $\mathfrak{A}^\times \cap w^{-1}\mathfrak{A}^\times w$ upon $\bar{\mathfrak{A}}^\times = M(k_D)$. □

Since τ is irreducible, 1.5 implies that $w \in \tilde{W}_A \cap \mathcal{N}_{A^\times}(M(O))$ is in the support of $\mathcal{H}$ if and only if $\tau^w = \tau$. This completes the proof of 1.2. □

From Part 0, Fact 3 and the observation that $\tilde{W}_A \cap M(O)$ is a normal subgroup of $\tilde{W}_A \cap \mathcal{N}_{A^\times}(M(O))$, we have the injective mapping

$$\tilde{W}_A \cap \mathcal{N}_{A^\times}(M(O))/(\tilde{W}_A \cap M(O)) \hookrightarrow \mathfrak{A}^\times\backslash A^\times/\mathfrak{A}^\times.$$

Let $\mathrm{Stab}(M(O),\tau)$ denote the subgroup of $\mathcal{N}_{A^\times}(M(O))$ consisting of those elements which fix the class of τ and note that

$$\tilde{W}_A \cap M(O) \subseteq \tilde{W}_A \cap \mathrm{Stab}(M(O),\tau) \subseteq \tilde{W}_A \cap \mathcal{N}_{A^\times}(M(O)).$$

Let $\tilde{W}_A(\tau)$ be a set of representatives for $(\tilde{W}_A \cap \mathrm{Stab}(M(O),\tau))/(\tilde{W}_A \cap M(O))$. From 1.2 we obtain:

1.6 **Corollary**: *The mapping $w \mapsto \mathfrak{A}^\times w \mathfrak{A}^\times$ defines a bijection from $\tilde{W}_A(\tau)$ to the set of double cosets in the support of $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$.*

In §0.6 we introduced for any cuspidal level zero pair $(\mathfrak{A}^\times, \tau)$ the divisor $\Delta(\tau)$. At this moment we do not yet know that this data determines a single type. Lemma 1.7 and Proposition 1.10 show that any two interpretations $(\mathfrak{A}_1^\times, \tau_1)$ and $(\mathfrak{A}_2^\times, \tau_2)$ of this data lead to isomorphic Hecke algebras and representations τ_1 and τ_2 which occur as components of the same representations with the same multiplicities.

1.7 **Lemma**: *For $i = 1,2$ let $\mathfrak{A}_i$ be a standard hereditary order and τ_i a cuspidal representation of $M_i(k_D) = \bar{\mathfrak{A}}_i^\times$ such that $\Delta(\tau_1) = \Delta(\tau_2)$. Assume that the same representatives with the same multiplicities are chosen from each class $[\sigma]$ so that τ_1 and τ_2 are tensor products in possibly permuted order of equivalent representations. Then the Hecke algebras $\mathcal{H}(A^\times, \mathfrak{A}_1^\times, \tau_1)$ and $\mathcal{H}(A^\times, \mathfrak{A}_2^\times, \tau_2)$ are isomorphic and, moreover, for any irreducible smooth representation Π of $A^\times$ the multiplicity of τ_1 in $\Pi|_{\mathfrak{A}_1^\times}$ equals the multiplicity of τ_2 in $\Pi|_{\mathfrak{A}_2^\times}$.*

Proof: Since the symmetric group $\mathfrak{S}_r$ is generated by transpositions r_i which switch the pair $(i, i+1)$ for $1 \le i < r$, it is sufficient to consider pairs $(\mathfrak{A}_1^\times, \tau_1)$ and $(\mathfrak{A}_2^\times, \tau_2)$ in which σ_i and σ_{i+1} are inequivalent and such that τ_1 and τ_2 differ by a transposition of the i-th and $i+1$-th tensor factors. This means that M_1 and M_2 can differ as block diagonal groups in at most their i-th and $i+1$-th blocks. Let M' denote the Levi factor which contains M_1 and M_2, in which the two blocks $GL_{s_i} \times GL_{s_{i+1}}$ are replaced by the single block $GL_{s_i+s_{i+1}}$. Let $\mathfrak{A}'$ denote the standard hereditary order such that $\bar{\mathfrak{A}}'^\times \cong M'(k_D)$. Let

$$\tau' := \mathrm{Ind}_{\mathfrak{A}_1^\times}^{\mathfrak{A}'^\times} \tau_1 \cong \mathrm{Ind}_{\mathfrak{A}_2^\times}^{\mathfrak{A}'^\times} \tau_2,$$

τ' being an irreducible representation since $\sigma_i \not\sim \sigma_{i+1}$. It is well known both that τ' is irreducible and that the class of τ' does not depend upon the order of the tensor factors σ_i, σ_{i+1}. Therefore, if $\Pi_{\mathfrak{A}'^\times}$ contains τ' if and only if

$\Pi_{\mathfrak{A}_i^\times}$ contains τ_i for $i = 1, 2$, and the multiplicities are the same.

To see that the commuting algebras $\mathcal{H}(A^\times, \mathfrak{A}_1^\times, \tau_1)$ and $\mathcal{H}(A^\times, \mathfrak{A}_2^\times, \tau_2)$ are isomorphic we use [BK1](4.1.3) to deduce that for each $i = 1, 2$ the algebras $\mathcal{H}(A^\times, \mathfrak{A}_i^\times, \tau_i)$ and $\mathcal{H}(A^\times, \mathfrak{A}'^\times, \mathrm{Ind}_{\mathfrak{A}_i^\times}^{\mathfrak{A}'^\times}(\tau_i))$ are isomorphic. □

Using 1.7 we may assume without loss of generality that the tensor factors of τ are ordered such that the set $\{1, \ldots, r\}$ is partitioned into subsets such that if σ_i and $\sigma_{i'}$ belong to the same $\mathrm{Gal}(k_D|k)$-orbit, then the same is true for all i'' between i and i'. In other respects we may assume that the representative τ of the divisor Δ is arbitrary.

Next we write $\widetilde{M}$ for the smallest Levi subgroup of $A^\times$ such that $M \subseteq \widetilde{M}$ and such that $\tilde{W}_A \cap \mathrm{Stab}(M(O), \tau) \subset \widetilde{M}(D)$. To specify $\widetilde{M}$ we first use (4) to define the partition of $\{1, \ldots, m\}$ such that

$$(8) \qquad \mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_r, \qquad \mathcal{P}_i = \{h_{i-1} + 1, \ldots, h_i\}, \qquad h_i = \sum_{v=1}^{i} s_v.$$

We set $\mathcal{P}_i \sim \mathcal{P}_j$ if σ_i and σ_j are $\mathrm{Gal}(k_D|k)$-equivalent. The union of the $\mathcal{P}_i$ over $\mathrm{Gal}(k_D|k)$-equivalent σ_i gives a coarser partition $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}_1 \cup \cdots \cup \widetilde{\mathcal{P}}_t$ of $\{1, \ldots, m\}$. Let $l, \tilde{l}$ denote the functions on $\{1, \ldots, m\}$ such that $l(x) = i$ if $x \in \mathcal{P}_i$ and $\tilde{l}(x) = i$ if $x \in \widetilde{\mathcal{P}}_i$. The Levi subgroup $M \subset A^\times$ corresponds to $\mathcal{P}, l$ and has the representation

$$M = \{(a_{ij}) \in A^\times \mid a_{ij} = 0, \quad l(i) \neq l(j)\};$$

the larger Levi subgroup $\widetilde{M}$ corresponds to $\widetilde{\mathcal{P}}, \tilde{l}$ and has the similar representation

$$\widetilde{M} = \{(a_{ij}) \in A^\times \mid a_{ij} = 0, \quad \tilde{l}(i) \neq \tilde{l}(j)\}.$$

As usual, we write $d(\sigma) := s$ if the cuspidal representation σ is a representation of $GL_s(k_D)$. Associated to the divisor $\Delta(\tau)$, we have

$$\widetilde{M} = \prod_{[\sigma]} GL_{r_{[\sigma]}(\tau) d(\sigma)}.$$

Our assumption that $i \leq i'' \leq i'$ and $\sigma_i \sim \sigma_{i'}$ implies $\sigma_i \sim \sigma_{i''}$ implies that $\widetilde{M}$ is block diagonal. We may therefore represent the upper and lower block triangular parabolic subgroups which have $\widetilde{M}$ as their Levi factors in the form

$$\begin{aligned} \widetilde{P} = \widetilde{M} \ltimes \widetilde{U} &= \{(a_{ij}) \in A^\times \mid \quad a_{ij} = 0, \qquad \tilde{l}(i) > \tilde{l}(j)\} \\ \widetilde{P}^- = \widetilde{M} \ltimes \widetilde{U}^- &= \{(a_{ij}) \in A^\times \mid \quad a_{ij} = 0, \qquad \tilde{l}(i) < \tilde{l}(j)\}. \end{aligned}$$

The following Lemma is now obvious:

1.8 **Lemma**:

(i) $\mathfrak{A}^\times = (\mathfrak{A}^\times \cap \widetilde{U}^-)(\mathfrak{A}^\times \cap \widetilde{M})(\mathfrak{A}^\times \cap \widetilde{U})$;
(ii) $\mathfrak{A}^\times \cap \widetilde{U}^-$ *and* $\mathfrak{A}^\times \cap \widetilde{U}$ *are contained in* $1 + \mathfrak{P}$.

We now have the following important consequence of [BK2](7.2):

1.9 **Proposition:** *Let* $(\mathfrak{A}^\times, \tau, \widetilde{M})$ *as before. There is a canonical isomorphism*

$$t_{\widetilde{P}} : \mathcal{H}(\widetilde{M}, \widetilde{M} \cap \mathfrak{A}^\times, \tau) \xrightarrow{\sim} \mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$$

such that $\mathrm{supp}(t_{\widetilde{P}} f) = \mathfrak{A}^\times \cdot \mathrm{supp}(f) \cdot \mathfrak{A}^\times$ *for all* $f \in \mathcal{H}(\widetilde{M}, \widetilde{M} \cap \mathfrak{A}^\times, \tau)$.

Proof: In the terminology of [BK2](6.1), Lemma 1.8 implies that the pair $(\mathfrak{A}^\times, \tau)$ is decomposed with respect to $(\widetilde{M}, \widetilde{P})$. Moreover, in the terminology of [BK2](6.2), we have

$$\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau) = \mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)_{\widetilde{M}},$$

since $\tilde{W}_A \cap \mathrm{Stab}(M(O), \tau) \subseteq \widetilde{M}$, i.e. the support of our Hecke algebra is in $\mathfrak{A}^\times \cdot \widetilde{M} \cdot \mathfrak{A}^\times$. It follows that [BK2](7.2)(ii) implies the present Proposition. □

Since $\widetilde{M}$ is the direct product of subgroups

$$\widetilde{M}_v = \{a \in \widetilde{M};\ a_{ij} = \delta_{ij} \ \text{ if } \ \tilde{l}(i) = \tilde{l}(j) \neq v\}$$

and $\widetilde{M}_v \cap \mathfrak{A}^\times$ supports all constituents of τ which are in a single $\mathrm{Gal}(k_D|k)$-equivalence class $[\sigma] = [\sigma]_v$, we have the isomorphism

$$\bigotimes_v \mathcal{H}(\widetilde{M}_v, \widetilde{M}_v \cap \mathfrak{A}^\times, \tau^{(v)}) \xrightarrow{\sim} \mathcal{H}(\widetilde{M}, \widetilde{M} \cap \mathfrak{A}^\times, \tau). \tag{9}$$

As a consequence 1.9 reduces the study of the structure of level zero Hecke algebras to the case in which the cuspidal pair $(\mathfrak{A}^\times, \tau)$ corresponds to a principal order $\mathfrak{A}$ and τ with only $\mathrm{Gal}(k_D|k)$-equivalent tensor factors. The Hecke algebra isomorphism assertion of the next Proposition could have been given a simpler proof by using 1.9. However, the following Proposition also completes the proof that one representative of a divisor Δ is a type if and only if the same is true for all representatives of Δ. Thus, after 1.10, it will be sufficient to prove that τ is a type when all $\mathrm{Gal}(k_D|k)$-equivalent tensor factors of τ are equivalent.

1.10 **Proposition**: *The Hecke algebra* $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$ *depends, up to isomorphism, only on the divisor* $\Delta(\tau)$. *Moreover, if for* $i = 1, 2$ *the pair* $(\mathfrak{A}_i^\times, \tau_i)$ *consists of a standard hereditary order and cuspidal representation of* $\bar{\mathfrak{A}}_i^\times$ *such that* $\Delta(\tau_1) = \Delta(\tau_2)$, *then, for any irreducible smooth representation* Π *of* $A^\times$, *the representations* τ_1 *and* τ_2 *occur in restrictions of* Π *with the same multiplicities. Thus* τ_1 *is a type if and only if* τ_2 *is a type and for the same Bernstein components.*

Proof: In view of 1.7 it is enough to prove 1.10 for $\tau_1 = \sigma_1 \otimes \cdots \otimes \sigma_r$, where $\sigma_1, \ldots, \sigma_\ell$ are $\mathrm{Gal}(k_D|k)$-conjugate and no other tensor factor of τ_1 belongs to the $\mathrm{Gal}(k_D|k)$-orbit of σ_1, and $\tau_2 = {}^{\phi}\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r$. In this case, $\mathfrak{A} := \mathfrak{A}_1 = \mathfrak{A}_2$; clearly, $s := s_1 = \cdots = s_\ell$. In fact, 1.7 implies that the order of the tensor factors in τ_2 can be arbitrary, so we redefine

$$\tau_2 := \sigma_2 \otimes \cdots \sigma_\ell \otimes {}^{\phi}\sigma_1 \otimes \sigma_{\ell+1} \otimes \cdots \otimes \sigma_r.$$

With these definitions of τ_1 and τ_2 we want to construct two compactly supported intertwining functions f_{τ_2,τ_1} and f_{τ_1,τ_2} (see §0.2) such that

(10) $f_{\tau_2,\tau_1}(a'xa) = \tau_2(a')f_{\tau_2,\tau_1}(x)\tau_1(a)$ and $f_{\tau_1,\tau_2}(a'xa) = \tau_1(a')f_{\tau_1,\tau_2}(x)\tau_2(a)$

for all $a, a' \in \mathfrak{A}^\times$ and $x \in A^\times$ and

$$f_{\tau_1,\tau_2} * f_{\tau_2,\tau_1} = e_{\tau_1} \quad \text{and} \quad f_{\tau_2,\tau_1} * f_{\tau_1,\tau_2} = e_{\tau_2}. \tag{11}$$

For functions with the properties (10) and (11) it is easy to see that we have the isomorphisms

$$f_{\tau_1,\tau_2} * \mathcal{H}(A^\times, \mathfrak{A}^\times, \tau_2) * f_{\tau_2,\tau_1} = \mathcal{H}(A^\times, \mathfrak{A}^\times, \tau_1)$$

and

$$f_{\tau_2,\tau_1} * \mathcal{H}(A^\times, \mathfrak{A}^\times, \tau_1) * f_{\tau_1,\tau_2} = \mathcal{H}(A^\times, \mathfrak{A}^\times, \tau_2).$$

Moreover, let Π be a smooth representation of $A^\times$ and consider the space $\mathrm{Hom}_{\mathfrak{A}^\times}(\tau_1, \Pi)$. As remarked in §0.3 this space is a module for the Hecke algebra $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau_1)$. For any function $\varphi \in \mathrm{Hom}_{\mathfrak{A}^\times}(\tau_1, \Pi)$ we have $\Pi(f_{\tau_1,\tau_2})\varphi \in \mathrm{Hom}_{\mathfrak{A}^\times}(\tau_2, \Pi)$. If $\varphi \neq 0$, then (11) implies that $\Pi(f_{\tau_1,\tau_2})\varphi \neq 0$ too. Similarly one sees that $\Pi(f_{\tau_2,\tau_1})$ defines a monomorphism of the $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau_2)$-module $\mathrm{Hom}_{\mathfrak{A}^\times}(\tau_2, \Pi)$ to $\mathrm{Hom}_{\mathfrak{A}^\times}(\tau_1, \Pi)$. Since these convolutions are inverses of each other, the morphisms are surjective as well, so isomorphisms, allowing each module to be regarded as a module over either Hecke algebra. The assertions in 1.10 to the effect that in any irreducible smooth representation the multiplicities of τ_1 and τ_2 as components are the same follow from these observations. Thus τ_1 is a type if and only if τ_2 is and for the same Bernstein components.

It remains to construct the functions f_{τ_2,τ_1} and f_{τ_1,τ_2}. We consider the matrix

$$h_0 = \left(\begin{array}{c|c|c} & I_{(\ell-1)s} & \\ \hline \varpi I_s & & \\ \hline & & I_{m-\ell s} \end{array}\right)$$

and claim that the double cosets $\mathfrak{A}^\times h_0 \mathfrak{A}^\times$ and $\mathfrak{A}^\times h_0^{-1} \mathfrak{A}^\times$ support the respective functions f_{τ_2,τ_1} and f_{τ_1,τ_2}. First, we want to check (10) for the function f_{τ_2,τ_1}; we shall leave the analogous verification for f_{τ_1,τ_2} to the reader.

To check that a function satisfying (10) can be defined with support $\mathfrak{A}^\times h_0 \mathfrak{A}^\times$ we must verify, as in 1.1, that

$$f_{\tau_2,\tau_1}(h_0)\tau_1(y) = f_{\tau_2,\tau_1}(h_0 y h_0^{-1} \cdot h_0) = \tau_2({}^{h_0}y) f_{\tau_2,\tau_1}(h_0) \tag{12}$$

for all $y \in \mathfrak{A}^\times \cap h_0^{-1}\mathfrak{A}^\times h_0$. We can argue (see (1.5)) that $(1+\mathfrak{P}) \cap h_0^{-1}(1+\mathfrak{P})h_0$ is in the kernel of τ_1 and its h_0-conjugate $h_0(1+\mathfrak{P})h_0^{-1} \cap (1+\mathfrak{P})$ lies in the kernel of τ_2, so it is enough to check (12) for $y \in M(k_D)$, since h_0 normalizes $M(O)$, each of the representations of $M(O)$ factors through $M(k_D)$, and h_0 acts on $M(k_D)$.

Proceeding with the verification, let $\mathcal{S}$ denote the permutation of the tensor factors

$$v := v_1 \otimes \cdots \otimes v_r \mapsto v_2 \otimes \cdots \otimes v_\ell \otimes v_1 \otimes v_{\ell+1} \otimes \cdots \otimes v_r.$$

Note that, with $m = \mathrm{diag}(m_1, \ldots, m_r) \in M(k_D)$, we have

$$\tau_1(m)v = \sigma_1(m_1)v_1 \otimes \cdots \otimes \sigma_r(m_r)v_r.$$

Since, by definition, $\tau_1(m) = {}^{h_0}\tau_1({}^{h_0}m)$, we have

$$\tau_1(m)v = {}^{h_0}\tau_1(\mathrm{diag}(m_2, \ldots, m_\ell, {}^{\phi}m_1, m_{\ell+1}, \ldots, m_r))v$$

$$= \mathcal{S}^{-1}\sigma_2(m_2) \otimes \cdots \otimes \sigma_\ell(m_\ell) \otimes {}^{\phi}\sigma_1({}^{\phi}m_1) \otimes \sigma_{\ell+1}(m_{\ell+1}) \otimes \cdots \otimes \sigma_r(m_r)\mathcal{S}v$$

or

$$\mathcal{S}\tau_1(m)v = \tau_2({}^{h_0}m)\mathcal{S}v.$$

Thus we see that (12) holds with $f_{\tau_2,\tau_1}(h_0) = \mathcal{S}$. This implies that with this definition we have f_{τ_2,τ_1} satisfying (10) with support $\mathfrak{A}^\times h_0 \mathfrak{A}^\times$. A similar verification, left to the reader, shows that we have a function f_{τ_1,τ_2} satisfying (10) with support the double coset $\mathfrak{A}^\times h_0^{-1}\mathfrak{A}^\times$. We may also choose $f_{\tau_1,\tau_2}(h_0^{-1}) = \mathcal{S}^{-1}$.

Now let us verify (11). First, up to a constant factor,

$$f_{\tau_1,\tau_2} * f_{\tau_2,\tau_1}(1) = \mathcal{S}^{-1}\mathcal{S} = e_{\tau_1}(1)$$

and

$$f_{\tau_2,\tau_1} * f_{\tau_1,\tau_2}(1) = \mathcal{S}\mathcal{S}^{-1} = e_{\tau_2}(1).$$

We will verify only one of these relations:

$$f_{\tau_1,\tau_2} * f_{\tau_2,\tau_1}(1) = \int_{\mathfrak{A}^\times h_0^{-1}\mathfrak{A}^\times} f_{\tau_1,\tau_2}(x) f_{\tau_2,\tau_1}(x^{-1})dx = \mu(\mathfrak{A}^\times)[\mathfrak{A}^\times : \mathfrak{A}^\times \cap h_0^{-1}\mathfrak{A}^\times h_0]\mathcal{S}^{-1}\mathcal{S}.$$

Adjusting each of the functions by a factor of $\mu(\mathfrak{A}^\times)^{-1}[\mathfrak{A}^\times : \mathfrak{A}^\times \cap h_0^{-1}\mathfrak{A}^\times h_0]^{-1/2}$ we have

$$f_{\tau_1,\tau_2} * f_{\tau_2,\tau_1}(1) = \mu(\mathfrak{A}^\times)^{-1} I_{\tau_1} = e_{\tau_1}(1).$$

We have finally to show that these convolution products are functions with support only on the identity double coset. Assume that $w \in \tilde{W}_A - \mathfrak{A}^\times$ and that $f_{\tau_1,\tau_2} * f_{\tau_2,\tau_1}(w) \neq 0$. Then $f_{\tau_1,\tau_2} * f_{\tau_2,\tau_1} \in \mathcal{H}(A^\times, \mathfrak{A}^\times, \tau_1)$ implies that $w \in \widetilde{M}$. On the other hand, up to a constant, for any $w \in \tilde{W}$, we have

$$\int_{\mathfrak{A}^\times h_0^{-1}\mathfrak{A}^\times} f_{\tau_1,\tau_2}(x) f_{\tau_2,\tau_1}(x^{-1}w)dx = \sum_{u \in \mathfrak{A}^\times/(\mathfrak{A}^\times \cap h_0^{-1}\mathfrak{A}^\times h_0)} \mathcal{S}^{-1} f_{\tau_2,\tau_1}(h_0 u w),$$

so for this to be non-zero we have to show that there exists u such that $h_0uw \in \mathfrak{A}^\times h_0\mathfrak{A}^\times$, i.e. that $\mathfrak{A}^\times w\mathfrak{A}^\times \cap h_0^{-1}\mathfrak{A}^\times h_0 \neq \emptyset$. We obtain a contradiction if we show that $w \in \tilde{W}_A - \mathfrak{A}^\times$ and $\mathfrak{A}^\times w\mathfrak{A}^\times \cap h_0^{-1}\mathfrak{A}^\times h_0 \neq \emptyset$ implies $w \notin \widetilde{M}$. We give the argument in detail for the case $\ell = 1$; the general case is more complicated but depends on the same ideas. Write $B := h_0^{-1}Ah_0$, where $A \in \mathfrak{A}^\times$. Thus,

$$B = \left(\begin{array}{c|c} \varpi^{-1}A_{11}\varpi & \varpi^{-1}A_{12} \\ \hline \varpi A_{21} & A_{22}\end{array}\right) = \left(\begin{array}{c|c} \varpi^{-1}I_s & \\ \hline & I_{m-s}\end{array}\right)\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22}\end{array}\right)\left(\begin{array}{c|c} \varpi I_s & \\ \hline & I_{m-s}\end{array}\right).$$

Clearly, $B \notin \mathfrak{A}^\times$ if and only if $\varpi^{-1}A_{12}$ has an entry which is not in O. In this case, the algorithm for constructing Bruhat representatives (see the proof of (3.1)) implies that the Bruhat representative $w(B)$ for B has support at a position corresponding to a position in A_{12} and this implies that $w(B) \notin \widetilde{M}$. (Some entry in the Bruhat representative of a matrix occurs in a position where the matrix had an entry with minimum ordinal.) $\square$

2. Multiplying Double Cosets in the Simple Type Case

Let $\mathfrak{A} := \mathfrak{A}_r$ be a standard principal order ($rs = m$ and $\overline{\mathfrak{A}}_r^\times = M(k_D) = [GL_s(k_D)]^r$) and let $\tau = \sigma^r = \sigma^{\otimes r}$ be a tensor power of an irreducible cuspidal representation of $GL_s(k_D)$. In this Part we want to study the Hecke algebra $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$.

Since conjugation by $\varpi \in D$ induces a generator ϕ of the Galois group $\mathrm{Gal}(k_D|k)$ and a corresponding automorphism ϕ of $GL_s(k_D)$, we may set $\sigma^{\phi^i} := \sigma \circ \phi^i$ and let $l \geq 1$ be minimum such that σ^{ϕ^l} is equivalent to σ. Then $l|d$, where d is the index of $D|F$.

Let $F_l \subset F_d$ be the unramified extension of F of degree l and let D' be the centralizer of F_l in D. Since ϖ normalizes F_d, it also normalizes F_l, which implies that $\varpi_{D'} := \varpi^l$ is a prime element of D' and $\varpi_{D'}^{d/l} = \varpi_F$; D' is a central division algebra over F_l of index d/l. Since D and D' have the same residual fields we have

$$|O_D/\mathfrak{p}_D| = |O_{D'}/\mathfrak{p}_{D'}| = q^d. \tag{13}$$

Let us write $A_{r,l} := M_r(D')$ and regard $A = (A|F, \varpi) = A_{m,1}$. We write $B := A_{m,l}$ and $C := A_{r,l}$ and fix the embedding

$$\bigotimes I_s : C \hookrightarrow B = M_r(M_s(D')) \qquad (c_{ij})_{i,j=1}^r = c \mapsto c \otimes I_s = (c_{ij}I_s)_{i,j=1}^r, \tag{14}$$

the matrix $c \otimes I_s$ being a block matrix with scalar blocks.

We have natural analogues of the various structures and translations of notations relative to A for B and C. In particular, for the algebra C we have the Coxeter system (W_C, S_C) with $S_C = \{s_{0,C}, \ldots, s_{r-1,C}\}$ and the generalized Weyl group $\tilde{W}_C$ (see §0.8). From (14) we obtain the explicit image

$$\begin{array}{rcl} W & := & W_C \otimes I_s \quad S = \{s_0, \ldots, s_{r-1}\} := \{s_{0,C} \otimes I_s, \ldots, s_{r-1,C} \otimes I_s\} \\ \tilde{W} & := & \tilde{W}_C \otimes I_s = W \rtimes \langle h \rangle, \quad h := h_C \otimes I_s \end{array} \tag{15}$$

Since $\tilde{W}_C \subset C^\times = A_{r,l}^\times$ consists of all monomial matrices with non-zero entries which are powers of ϖ^l, 1.2 and 1.10 imply:

2.1 **Proposition**: *For $\mathfrak{A} = \mathfrak{A}_r$, $\tau = \sigma^r$, and l the length of the* $\mathrm{Gal}(k_D|k)$*-orbit of σ the group $\tilde{W} = \tilde{W}_C \otimes I_s$ is a set of representatives for*

$$(\tilde{W}_A \cap \mathrm{Stab}(M(O), \tau))/(\tilde{W}_A \cap M(O)).$$

The correspondence

$$w \mapsto \mathfrak{A}_r^\times w \mathfrak{A}_r^\times$$

defines a bijection between $\tilde{W}$ and the set of double cosets in the support of $\mathcal{H}(A^\times, \mathfrak{A}_r^\times, \tau)$. Concretely, in the notation of (5), $\varpi^v p \otimes I_s \in \tilde{W}$ if and only if $v = lv'$ ($v' \in \mathbb{Z}^r$) and p is an $r \times r$ permutation matrix.

In order to exploit 2.1 for multiplying double cosets we consider the above embeddings in more detail. Consider the Coxeter system (W_A, S_A) (§0.8) with the length function l_A. Since $\tilde{W}_A = W_A \rtimes \langle h_A \rangle$ (semi-direct product) and since h_A, acting by conjugation, stabilizes the Coxeter system (W_A, S_A), we can extend l_A from W_A to $\tilde{W}_A$ simply by assigning h_A the length 0, $l_A(w)$ for any $w \in \tilde{W}_A$ being the minimum number of elements of S_A occurring in any representation of w as a word in the alphabet $S_A \cup \{h_A\}$. It follows that $l_A(w)$ can be computed by multiplying w by whatever power of h_A maps w to W_A and computing the length in W_A.

From the Bruhat decomposition theory we know that

$$(\mathcal{I}_A^\times : \mathcal{I}_A^\times \cap w\mathcal{I}_A^\times w^{-1}) = q^{d \cdot l_A(w)}. \tag{16}$$

For $B = A_{m,l}$ and $\varpi_{D'} = \varpi^l$ instead of ϖ we see that $\mathcal{I}_B, W_B, \tilde{W}_B$ arise from $\mathcal{I}_A, W_A, \tilde{W}_A$ by taking intersections with $B \subset A$. Moreover $s_{i,A} = s_{i,B}$ for $i = 1, \ldots, m-1$, whereas $s_{0,A} \neq s_{0,B}$ and $h_A \neq h_B$.

Define the standard principal order $\mathfrak{B}_r := \mathfrak{A}_r \cap B$ of B and the length functions l_B and l_C, which correspond, respectively, to (W_B, S_B) and $(W, S) = (W_C, S_C)$. We also extend these length functions to $\tilde{W}_B$ and $\tilde{W}_C$ as above.

2.2 **Proposition**: *For $w \in \tilde{W} \subset \tilde{W}_B$*

(i) $(\mathfrak{B}_r^\times : \mathfrak{B}_r^\times \cap w\mathfrak{B}_r^\times w^{-1}) = Q^{l_C(w)}$, *where* $Q = |M_s(k_{D'})| = q^{d\cdot s^2}$.

(ii) $l_B(w) = s^2 \cdot l_C(w)$.

Proof: To prove (i) we first change (16), replacing A by B; (13) implies that this is meaningful. Observing that $\tilde{W}$ normalizes $M(O)$ and stabilizes the set of positive roots in $M(O)$, we obtain assertion (i) immediately from the Bruhat decomposition of $GL_r(M_s(D'))$ with respect to $\tilde{W}$. (ii) is immediate from (i) and (16) applied to B. □

2.3 **Lemma**: *Let $w \in \tilde{W}$, $s \in S = S_C \otimes I_s$ such that $l_C(ws) > l_C(w)$. Then:*

$$\mathfrak{B}_r^\times w\mathfrak{B}_r^\times s\mathfrak{B}_r^\times = \mathfrak{B}_r^\times ws\mathfrak{B}_r^\times.$$

Proof: From $l_C(ws) > l_C(w)$ and $l_C(s) = 1$ it follows that $l_C(ws) = l_C(w) + l_C(s)$. Therefore, by 2.2(ii), we have $l_B(ws) = l_B(w) + l_B(s)$. But $(B^\times, \mathfrak{B}_m^\times, N_B)$ is a generalized Tits system with the generalized Weyl group $\tilde{W}_B$. Therefore the last equation implies (see [Bou], Lie IV, no. 2.4. Cor.1) that

$$\mathfrak{B}_m^\times w\mathfrak{B}_m^\times s\mathfrak{B}_m^\times = \mathfrak{B}_m^\times ws\mathfrak{B}_m^\times. \tag{17}$$

Since $\mathfrak{B}_r^\times = M(O_{D'})\mathfrak{B}_m^\times$ and since w normalizes $M(O_{D'})$, we obtain from (17) that

$$w\mathfrak{B}_r^\times s = wM(O_{D'})\mathfrak{B}_m^\times s = M(O_{D'})w\mathfrak{B}_m^\times s \subseteq M(O_{D'})\mathfrak{B}_m^\times ws\mathfrak{B}_m^\times \subseteq \mathfrak{B}_r^\times ws\mathfrak{B}_r^\times.$$

Therefore $\mathfrak{B}_r^\times w\mathfrak{B}_r^\times s\mathfrak{B}_r^\times \subseteq \mathfrak{B}_r^\times ws\mathfrak{B}_r^\times$ and the opposite inclusion is obvious. □

We recall the natural bijective correspondence between standard hereditary orders of B and A such that, for every standard hereditary order $\mathfrak{A} \subset A$, $\mathfrak{B} = \mathfrak{A} \cap B$ is a standard hereditary order of B and conversely. This and (13) imply that

$$\mathfrak{B}^\times/(1+\mathfrak{P}_\mathfrak{B}) = \mathfrak{A}^\times/(1+\mathfrak{P}_\mathfrak{A}) = M(k_D);$$

hence $\mathfrak{B}^\times \cap \tilde{W}_B = \mathfrak{A}^\times \cap \tilde{W}_A \cong \mathfrak{S}_{s_1} \times \cdots \times \mathfrak{S}_{s_r}$, i.e. (see (4))

$$\bar{\mathfrak{B}}^\times = \bar{\mathfrak{A}}^\times = GL_{s_1}(k_D) \times \cdots \times GL_{s_r}(k_D).$$

2.4 **Lemma**: $\mathfrak{A}^\times x\mathfrak{A}^\times \cap B = \mathfrak{B}^\times x\mathfrak{B}^\times$ *for any $x \in B^\times$.*

Proof: The inclusion $\supset$ is obvious. To see the opposite inclusion let $w \in \tilde{W}_B$ be such that $\mathfrak{B}^\times x\mathfrak{B}^\times = \mathfrak{B}^\times w\mathfrak{B}^\times$, hence $\mathfrak{A}^\times x\mathfrak{A}^\times = \mathfrak{A}^\times w\mathfrak{A}^\times$. Since $\mathfrak{A}^\times w\mathfrak{A}^\times \cap B$ is a union of $\mathfrak{B}^\times$ double cosets, we may consider $w' \in \tilde{W}_B$ such that $\mathfrak{B}^\times w'\mathfrak{B}^\times \subset \mathfrak{A}^\times w\mathfrak{A}^\times \cap B$. In this case, $\mathfrak{A}^\times w'\mathfrak{A}^\times = \mathfrak{A}^\times w\mathfrak{A}^\times$, so §0.8, Fact 3 implies that $w' \in (\mathfrak{A}^\times \cap \tilde{W}_A)w(\mathfrak{A}^\times \cap \tilde{W}_A)$. Therefore, $(\mathfrak{A}^\times \cap \tilde{W}_A) = (\mathfrak{B}^\times \cap \tilde{W}_B)$ implies that $\mathfrak{B}^\times w'\mathfrak{B}^\times = \mathfrak{B}^\times w\mathfrak{B}^\times$. □

2.5 **Lemma:** $\mathfrak{A}_r^\times x\mathfrak{A}_r^\times s\mathfrak{A}_r^\times \cap B = \mathfrak{B}_r^\times x\mathfrak{B}_r^\times s\mathfrak{B}_r^\times$ *for $x \in B^\times$ and $s_0 \neq s \in S \subset S_B$.*

Proof: By hypothesis, $s = s_i$ for $i = 1, \ldots, r-1$. Regard $\mathfrak{A}_r$ as a set of matrices in $M_r(M_s(O))$ and for $r \geq 1$ let $E_{i,i+1} \subset M_r$ be the matrix with its only non-zero coefficient at $(i,i+1)$ and that coefficient 1. Then we have the natural bijections

$$\begin{aligned}\mathfrak{B}_r^\times/(\mathfrak{B}_r^\times\cap s_i\mathfrak{B}_r^\times s_i) &\cong I_m+E_{i,i+1}\otimes M_s(k_D)\\ &\cong \mathfrak{A}_r^\times/(\mathfrak{A}_r^\times\cap s_i\mathfrak{A}_r^\times s_i)\end{aligned}$$

for all $i=1,\ldots,r-1$, since D and D' have the same residual fields. Therefore,

$$\mathfrak{A}_r^\times=\mathfrak{B}_r^\times(\mathfrak{A}_r^\times\cap s_i\mathfrak{A}_r^\times s_i)=\mathfrak{A}_r^\times\cap\mathfrak{B}_r^\times s_i\mathfrak{A}_r^\times s_i,$$

which implies that

$$\mathfrak{A}_r^\times x\mathfrak{A}_r^\times s_i\mathfrak{A}_r^\times=\mathfrak{A}_r^\times x(\mathfrak{A}_r^\times\cap\mathfrak{B}_r^\times s_i\mathfrak{A}_r^\times s_i)s_i\mathfrak{A}_r^\times=(\mathfrak{A}_r^\times x\mathfrak{A}_r^\times s_i\mathfrak{A}_r^\times)\cap(\mathfrak{A}_r^\times x\mathfrak{B}_r^\times s_i\mathfrak{A}_r^\times).$$

Thus, $\mathfrak{A}_r^\times x\mathfrak{A}_r^\times s_i\mathfrak{A}_r^\times=\mathfrak{A}_r^\times x\mathfrak{B}_r^\times s_i\mathfrak{A}_r^\times$ for all $i=1,\ldots,r-1$. Intersecting with B, we obtain our Lemma by applying 2.4. □

2.6 **Lemma:** *Let $w\in\tilde{W}$ and $s_0\neq s\in S$ and assume that $l_C(ws)>l_C(w)$. Then there exists $I\subset\tilde{W}_A-\tilde{W}_B$ such that*

$$\mathfrak{A}_r^\times w\mathfrak{A}_r^\times s\mathfrak{A}_r^\times=\mathfrak{A}_r^\times ws\mathfrak{A}_r^\times\cup(\bigcup_{w'\in I}\mathfrak{A}_r^\times w'\mathfrak{A}_r^\times).$$

Proof: Lemmas 2.3 and 2.5 imply that

$$\mathfrak{A}_r^\times w\mathfrak{A}_r^\times s\mathfrak{A}_r^\times\cap B=\mathfrak{B}_r^\times w\mathfrak{B}_r^\times s\mathfrak{B}_r^\times=\mathfrak{B}_r^\times ws\mathfrak{B}_r^\times.$$

Applying 2.4, we see that

$$\mathfrak{A}_r^\times w\mathfrak{A}_r^\times s\mathfrak{A}_r^\times\cap B=\mathfrak{A}_r^\times ws\mathfrak{A}_r^\times\cap B,$$

i.e.

$$\mathfrak{A}_r^\times w\mathfrak{A}_r^\times s\mathfrak{A}_r^\times=\mathfrak{A}_r^\times ws\mathfrak{A}_r^\times\cup(\bigcup_{w'\in I}\mathfrak{A}_r^\times w'\mathfrak{A}_r^\times),$$

where for $w'\in I\subset\tilde{W}_A$ we have $\mathfrak{A}_r^\times w'\mathfrak{A}_r^\times\cap B=\emptyset$. Thus, $I\subset\tilde{W}_A-\tilde{W}_B$. □

Finally we need the following analogue of 2.6 in which s is replaced by the generator h of $\tilde{W}/W$.

2.7 **Lemma**: *For $w\in\tilde{W}$ and $h'\in\{h,h^{-1}\}$ there exists $I=I(w,h')\subset\tilde{W}_A-\tilde{W}_B$ such that*

$$\begin{aligned}\mathfrak{A}_r^\times h'\mathfrak{A}_r^\times w\mathfrak{A}_r^\times &= \mathfrak{A}_r^\times h'w\mathfrak{A}_r^\times\cup(\bigcup_{w'\in I}\mathfrak{A}_r^\times w'\mathfrak{A}_r^\times)\\ \mathfrak{A}_r^\times w\mathfrak{A}_r^\times h'\mathfrak{A}_r^\times &= \mathfrak{A}_r^\times wh'\mathfrak{A}_r^\times\cup(\bigcup_{w'\in I}\mathfrak{A}_r^\times w'\mathfrak{A}_r^\times).\end{aligned}$$

Proof: We apply the Main Lemma 3.1, to be stated and proved below. Write $\mathfrak{A}_r^\times h'\mathfrak{A}_r^\times w\mathfrak{A}_r^\times=\mathfrak{A}_r^\times h'\mathfrak{A}_r^\times h'^{-1}(h'w)\mathfrak{A}_r^\times$, and apply the Main Lemma for $b\in h'\mathfrak{A}_r^\times(h')^{-1}$ and $h'w\in\tilde{W}$. □

Remark: If $l=1$, then 2.6 and 2.7 become simpler, since $l=1$ implies that $A=B$, so $I=\emptyset$. Obviously $l=1$ in the split case $D=F$, since $l|d$ and $D=F$ implies that $d=1$.

3. **The Main Lemma**.

Recall the generator of $\tilde{W}/W$:

$$h = \left(\begin{array}{c|c} & I_{m-s} \\ \hline \varpi^l I_s & \end{array}\right) = \left(\begin{array}{c|c} & I_{r-1} \\ \hline \varpi^l & \end{array}\right) \otimes I_s.$$

In the following we also write $\mathfrak{A}_{r,1}^\times := h\mathfrak{A}_r^\times h^{-1}$ and $\mathfrak{A}_{r,2}^\times := h^{-1}\mathfrak{A}_r^\times h$.

3.1 **Main Lemma**: *Let $w \in \tilde{W}$ and assume that $b \in \mathfrak{A}_{r,1}^\times \cup \mathfrak{A}_{r,2}^\times$. (i) If $\mathfrak{A}_r^\times wb\mathfrak{A}_r^\times \cap \tilde{W}_B \neq \emptyset$, then $\mathfrak{A}_r^\times wb\mathfrak{A}_r^\times = \mathfrak{A}_r^\times w\mathfrak{A}_r^\times$, and (ii) if $\mathfrak{A}_r^\times bw\mathfrak{A}_r^\times \cap \tilde{W}_B \neq \emptyset$, then $\mathfrak{A}_r^\times bw\mathfrak{A}_r^\times = \mathfrak{A}_r^\times w\mathfrak{A}_r^\times$.*

Proof: It is sufficient to prove (i), since (ii) follows from (i) by taking inverses. In fact, we we shall prove (i) only in the case that $b \in \mathfrak{A}_{r,1}^\times$, because a simple modification of the argument for this case proves (i) in the case that $b \in \mathfrak{A}_{r,2}^\times$. Thus we shall only sketch the argument for the case $b \in \mathfrak{A}_{r,2}^\times$.

Assume that $b \in \mathfrak{A}_{r,1}^\times$; we shall prove (i). Consider the standard parabolic subgroup $P = M \ltimes U$ of G, where $M = G_{m-s} \times G_s$, and the opposite parabolic subgroup $P^- = M \ltimes U^-$ of G. Thus U is an upper triangular unipotent subgroup of G which is complementary to and normalized by M and U^- is lower triangular unipotent with the same properties. The group $\mathfrak{A}_{r,1}^\times$ has the Iwahori decomposition $\mathfrak{A}_{r,1}^\times = (\mathfrak{A}_{r,1}^\times \cap U)(\mathfrak{A}_{r,1}^\times \cap M)(\mathfrak{A}_{r,1}^\times \cap U^-)$, where the second and third factors are subgroups of $\mathfrak{A}_r^\times$, and the first factor consists of matrices

$$\mathfrak{A}_{r,1}^\times \cap U = \left(\begin{array}{c|c} I_{m-s} & M_s(\mathfrak{P}^{1-l}) \\ & \vdots \\ & M_s(\mathfrak{P}^{1-l}) \\ \hline & I_s \end{array}\right) \tag{18}$$

with the last column blocks as indicated. Therefore, given $b \in \mathfrak{A}_{r,1}^\times$, we can choose $c \in \mathfrak{A}_r^\times$ such that $b' = bc \in \mathfrak{A}_{r,1}^\times \cap U$. From this we see that $\mathfrak{A}_r^\times wb\mathfrak{A}_r^\times = \mathfrak{A}_r^\times wb'\mathfrak{A}_r^\times$. From the Bruhat decomposition of $A^\times$ we know that there is a unique $w' \in \tilde{W}_A$ such that $\mathfrak{A}_m^\times wb'\mathfrak{A}_m^\times = \mathfrak{A}_m^\times w'\mathfrak{A}_m^\times$.

We want to show that, if $w' \in \tilde{W}_B$, then $w' = w$. The argument for this is to compute the Bruhat representative for wb' and to observe that this representative can belong to $\tilde{W}_B$ only if it equals w.

We briefly recall the algorithm for computing Bruhat representatives for elements of $A^\times$. Let $g = (g_{ij})_{i,j=1}^m \in A^\times$ and let $v_0 = \min v_D(g_{ij})$ be the minimum exponent of the entries of g. Choose (i_0, j_0) such that $v_D(g_{i_0,j_0}) = v_0$ and such that j_0 is smallest and i_0 largest with this property. By elementary row and column operations we zero all the entries in row i_0 and column j_0, except the entry (i_0, j_0) itself which is transformed to ϖ^{v_0}. Write $g' = (g'_{ij})$ for this matrix and observe that the row and column operations which construct g' can be represented by left and right multiplications by elements of $\mathfrak{A}_m^\times$. By induction applied to the submatrix of g' obtained by omitting the i_0-th row and j_0-th column we arrive via left and right multiplications by elements of $\mathfrak{A}_m^\times$ at a monomial representative w' for g in the double coset $\mathfrak{A}_m^\times g \mathfrak{A}_m^\times$ such that each non-zero entry of w' is a power of ϖ.

Keeping in mind the observation encapsulated in the following lemma, let us compute the Bruhat representative of wb'.

3.2 **Lemma**: *Let w be a monomial element and b' an upper triangular unipotent element of $A^\times$. Let $\sigma = \sigma_w \in \mathfrak{S}_m$ denote the permutation such that $\sigma(i) = j$ if and only if (i,j) is in the support of w. Then:*

$$(wb')_{ij} = \begin{cases} w_{ij}, \text{ if } j = \sigma(i) \\ w_{i,\sigma(i)} b'_{\sigma(i),j}, \text{ otherwise.} \end{cases}$$

Proof: The second equation follows from matrix multiplication, from the hypothesis that w is monomial, and the first from the second, from the assumption that b' is triangular unipotent. $\square$

From 3.2 it follows that, if the distinguished positions (i_0, j_0) which we choose at the various stages of applying the Bruhat representative algorithm to wb' all lie in the support of w, then $w' = w$. Otherwise, after perhaps several left and right multiplications by elements of $\mathfrak{A}_m^\times$, we have constructed out of wb' a matrix g ($aga' = wb'$ with $a, a' \in \mathfrak{A}_m^\times$) and in the next step of applying our algorithm there is a first position (i_0, j_0) such that $j_0 \neq \sigma(i_0)$. In this case, it follows from (18) that $g_{i_0,j_0} = (wb')_{i_0,j_0}$, since each preceding step has involved only the entries residing in the i_0-th row or j_0-th column of wb' for each (i_0, j_0) with $j_0 = \sigma(i_0)$ in a sequence of such elements. Thus we have, at this step,

$$g_{i_0,j_0} = (wb')_{i_0,j_0} = w_{i_0,\sigma(i_0)} b'_{\sigma(i_0),j_0}, \tag{19}$$

since $\sigma(i_0) \neq j_0$. From (18) and the fact that at this step $v_0 = v_D(g_{i_0,j_0}) = v_D((wb')_{\sigma(i_0),j_0})$ it follows that

$$\sigma(i_0) \leq m - s < j_0. \tag{20}$$

In order that (i_0, j_0) be the distinguished position for this step of the algorithm it is necessary that $v_0 = v_D(g_{i_0,j_0}) < v_D(g_{i_0,\sigma(i_0)})$. Using 3.2 we have

$$g_{i_0,\sigma(i_0)} = (wb')_{i_0,\sigma(i_0)} = w_{i_0,\sigma(i_0)} = \varpi^{v'},$$

where $l \mid v'$, and therefore, by (18) and (19),

$$1 - l \leq v_D(b'_{\sigma(i_0),j_0}) = v_0 - v' < 0,$$

which implies that l cannnot divide v_0. Thus $w' \notin \tilde{W}_B$, since ϖ^{v_0} occurs as an entry of w'.

Now we use the fact that $\mathfrak{A}_r^\times w' \mathfrak{A}_r^\times \cap \tilde{W}_A = (\mathfrak{A}_r^\times \cap W_A) w' (\mathfrak{A}_r^\times \cap W_A)$ (see §0.8, Fact 3). Therefore, since w' has an exponent not divisible by l, the same is true for all elements of $\mathfrak{A}_r^\times w' \mathfrak{A}_r^\times \cap \tilde{W}_A$. Hence, $\mathfrak{A}_r^\times w' \mathfrak{A}_r^\times \cap \tilde{W}_B = \emptyset$. This proves (i) of the Main Lemma in the case that $b \in \mathfrak{A}_{r,1}^\times$.

Now assume that $b \in \mathfrak{A}_{r,2}^\times$ and let us sketch the modifications in the preceding argument which we need in order to prove the Main Lemma in this case.

Take $M = G_s \times G_{m-s}$ as block groups in the reverse order from before and let U and U^- be the upper and lower triangular unipotent groups such that $M \ltimes U$ is a standard parabolic subgroup and $M \ltimes U^-$ its opposite. Now $\mathfrak{A}_{r,2}^\times = (\mathfrak{A}_{r,2}^\times \cap U)(\mathfrak{A}_{r,2}^\times \cap M)(\mathfrak{A}_{r,2}^\times \cap U^-)$, where the second and third factors are in $\mathfrak{A}_r^\times$ and the first factor is

$$\mathfrak{A}_{r,2}^\times \cap U = \left(\begin{array}{c|c} I_s & M_s(\mathfrak{P}^{1-l}), \cdots, M_s(\mathfrak{P}^{1-l}) \\ \hline & \\ & I_{m-s} \end{array}\right). \tag{21}$$

For $b \in \mathfrak{A}_{r,2}^\times$ there exists $b' \in \mathfrak{A}_{r,2}^\times \cap U$ such that $\mathfrak{A}_r^\times wb\mathfrak{A}_r^\times = \mathfrak{A}_r^\times wb'\mathfrak{A}_r^\times$. As before, we construct the Bruhat representative w' for wb' and see that $w' = w$ if and only if $w' \in \tilde{W}_B$. We conclude by arguing as before that if $w' \notin \tilde{W}_B$, then $\mathfrak{A}_r^\times wb\mathfrak{A}_r^\times \cap \tilde{W}_B = \emptyset$. □

4. The Hecke Algebra of a Simple Level Zero Type

As in Part 2, we want to study the Hecke algebra $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$, where $(\mathfrak{A}^\times, \tau)$ is a cuspidal pair in which $\mathfrak{A} = \mathfrak{A}_r$ is a principal order and $\tau = \sigma^r$ is the r-th tensor power of a cuspidal representation σ of $GL_s(k_D)$ $(rs = m)$. We want to prove that, in this case, the Hecke algebra $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$ is isomorphic to an affine Hecke algebra of type $\tilde{A}_{r-1}$. We refer here to [BK1](5.4): For a positive integer r and $z \in \mathbb{C}^\times$ we have the finite and affine Hecke algebras of type A_{r-1} which we denote by $\mathcal{H}_0(r,z) \subset \mathcal{H}(r,z)$, respectively. These are complex algebras which are given by generators and relations as follows:

I. The algebra $\mathcal{H}_0(r,z)$ has the $r-1$ generators $\rho_1, \ldots, \rho_{r-1}$ with the relations:

(i) $(\rho_i + 1)(\rho_i - z) = 0$ for all $i = 1, \ldots, r-1$;
(ii) $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$ for $i = 1, \ldots, r-2$;
(iii) $\rho_i\rho_j = \rho_j\rho_i$ if $|i-j| \geq 2$.

II. The affine algebra $\mathcal{H}(r,z)$ contains $\mathcal{H}_0(r,z)$ and has the two additional generators ξ and ξ' with the three additional relations:

(iv) $\xi\xi' = \xi'\xi = 1$ (hence we write $\xi^{-1} := \xi'$);
(v) $\xi^2\rho_1\xi^{-2} = \rho_{r-1}$; and
(vi) $\xi\rho_i\xi^{-1} = \rho_{i-1}$ for $i \geq 2$.

To construct an isomorphism $\Phi : \mathcal{H}(r, q^{ds}) \xrightarrow{\sim} \mathcal{H}(A^\times, \mathfrak{A}_r^\times, \sigma^r)$ $(q^d = |k_D|)$ we consider the subalgebra $\mathcal{H}(\mathfrak{A}_1^\times, \mathfrak{A}_r^\times, \tau)$. Reducing mod $\mathfrak{P}_1$, we may apply the finite field results of [HM] 1.5, Theorem 5.1:

4.1 **FACT**: *There is a uniquely determined isomorphism*

$$\Phi_0 : \mathcal{H}_0(r, q^{ds}) \xrightarrow{\sim} \mathcal{H}(\mathfrak{A}_1^\times, \mathfrak{A}_r^\times, \tau)$$

which sends the generator ρ_i $(i = 1, \ldots, r-1)$ to an operator valued function with support on the single double coset $\mathfrak{A}_r^\times s_i\ \mathfrak{A}_r^\times$, where $s_i = s_{i,C} \otimes I_s$ for $i = 1, \ldots, r-1$ (see Part 2 for notations).

Now in view of 2.1 we can state:

4.2 **Theorem**: *Let $h = h_C \otimes I_s \in \tilde{W}$ and let $\varphi_h \in \mathcal{H}(A^\times, \mathfrak{A}_r^\times, \tau)$ be any non-zero function with the support $\mathfrak{A}_r^\times h \mathfrak{A}_r^\times$. Then there is a unique isomorphism*

$$\Phi : \mathcal{H}(r, q^{ds}) \xrightarrow{\sim} \mathcal{H}(A^\times, \mathfrak{A}_r^\times, \tau)$$

such that $\Phi(\xi) = \varphi_h$ and such that Φ extends Φ_0.

Notation: During the proof we will write $\mathcal{H} := \mathcal{H}(A^\times, \mathfrak{A}_r^\times, \tau)$.

Proof: Proposition 2.1 implies that there are non-zero functions $\varphi_h, \varphi_{h^{-1}} \in \mathcal{H}$ with supports $\mathfrak{A}_r^\times h \mathfrak{A}_r^\times$ and $\mathfrak{A}_r^\times h^{-1} \mathfrak{A}_r^\times$, respectively. In particular, by Lemma 1.1, $\varphi_h(h) = J \in \mathrm{Hom}_{\mathfrak{A}^\times \cap h^{-1}\mathfrak{A}^\times h}(\tau, \tau^h)$, i.e. $J\tau(y) = \tau^h(y)J$ for all $y \in \mathfrak{A}_r^\times \cap h^{-1}\mathfrak{A}_r^\times h$. Replacing y by $h^{-1}yh$, we see that $\tau^{h^{-1}}(y)J^{-1} = J^{-1}\tau(y)$ for $y \in \mathfrak{A}^\times \cap h\mathfrak{A}^\times h^{-1}$. Therefore, we find that $J^{-1} \in \mathrm{Hom}_{\mathfrak{A}^\times \cap h\mathfrak{A}^\times h^{-1}}(\tau, \tau^{h^{-1}})$, so, again by 1.1, we obtain a function $\tilde{\varphi}_{h^{-1}} \in \mathcal{H}$ which is supported on $\mathfrak{A}_r^\times h^{-1} \mathfrak{A}_r^\times$ and has the value J^{-1} at h^{-1}.

4.3 **Lemma**: *$\varphi_h * \tilde{\varphi}_{h^{-1}} = \lambda e$ is a scalar multiple of the unit $e \in \mathcal{H}$.*

Proof: The support of $\varphi_h * \tilde{\varphi}_{h^{-1}}$ is contained in $\mathfrak{A}_r^\times h \mathfrak{A}_r^\times h^{-1} \mathfrak{A}_r^\times$. Using 2.7 we find $\mathfrak{A}_r^\times h \mathfrak{A}_r^\times h^{-1} \mathfrak{A}_r^\times = \mathfrak{A}_r^\times \cup (\bigcup_{w' \in I} \mathfrak{A}_r^\times w' \mathfrak{A}_r^\times)$ where $I \subset \tilde{W}_A - \tilde{W}_B$. On the other hand by 2.1 the support of $\varphi_h * \tilde{\varphi}_{h^{-1}}$ is contained in $\bigcup_{w \in \tilde{W}} \mathfrak{A}_r^\times w \mathfrak{A}_r^\times$, hence it is in $\mathfrak{A}_r^\times$. Moreover for $y \in \mathfrak{A}_r^\times$ we find

$$(\varphi_h * \tilde{\varphi}_{h^{-1}})(y) = \int_{x \in \mathfrak{A}_r^\times h \mathfrak{A}_r^\times} \varphi_h(x)\tilde{\varphi}_{h^{-1}}(x^{-1}y)dx$$

Putting $x = y_1 h y_2$, we obtain

$$\varphi_h(x)\tilde{\varphi}_{h^{-1}}(x^{-1}y) = \tau(y_1)J\tau(y_2)\tau(y_2)^{-1}J^{-1}\tau(y_1^{-1}y) = \tau(y),$$

hence

$$(\varphi_h * \tilde{\varphi}_{h^{-1}})(y) = \mu(\mathfrak{A}_r^\times h \mathfrak{A}_r^\times) \cdot \tau(y).$$

From the definition of $e_\tau \in \mathcal{H}$ (§0.2) we see that

$$\varphi_h * \tilde{\varphi}_{h^{-1}} = \lambda \cdot e,$$

where $\lambda = \mu(\mathfrak{A}_r^\times)\mu(\mathfrak{A}_r^\times h \mathfrak{A}_r^\times)$. □

Using 4.3 we put $\varphi_{h^{-1}} := \lambda^{-1}\tilde{\varphi}_{h^{-1}}$ and

$$(\varphi_h)^k := \begin{cases} \varphi_h * \cdots * \varphi_h & k \text{ times if } k > 0 \\ \varphi_{h^{-1}} * \cdots * \varphi_{h^{-1}} & -k \text{ times if } k < 0 \\ e & \text{if } k = 0. \end{cases}$$

Again using 2.7 we see that the support of $(\varphi_h)^k$ is the double coset $\mathfrak{A}_r^\times h^k \mathfrak{A}_r^\times$. The function $(\varphi_h)^k$ is non-trivial since it is a unit in $\mathcal{H}$.
For $1, \varrho_1, \ldots, \varrho_{r-1}, \xi, \xi' \in \mathcal{H}(r, q^{ds})$ we put

$$\begin{array}{rrcllcl} (22) \quad & \Phi(1) & = & e, & \Phi(\varrho_i) & = & \Phi_0(\varrho_i) =: \varphi_{s_i} \\ & \Phi(\xi) & = & \varphi_h, & \Phi(\xi') & = & \varphi_{h^{-1}}. \end{array}$$

Since $\Phi_0(\varrho_i)$ is supported on $\mathfrak{A}_r^\times s_i \mathfrak{A}_r^\times$, the notation $\varphi_{s_i} = \Phi_0(\varrho_i)$ is not misleading.

4.4 **Lemma**: *The assignments of (22) extend to an algebra homomorphism*

$$\Phi : \mathcal{H}(r, q^{ds}) \longrightarrow \mathcal{H}.$$

Proof: We have described $\mathcal{H}(r, q^{ds})$ in terms of the generators $\varrho_1, \ldots, \varrho_{r-1}, \xi, \xi'$ and six relations. We must verify that the same relations hold for the elements of $\mathcal{H}$ given in (22). From 4.1, 4.3, and the definition of $\varphi_{h^{-1}}$ given above, we know already that the relations (i)-(iv) are fulfilled. We have to show that $\varphi_h^2 * \varphi_{s_1} * \varphi_h^{-2} = \varphi_{s_{r-1}}$ and that $\varphi_h * \varphi_{s_i} * \varphi_h^{-1} = \varphi_{s_{i-1}}$ for $i = 2, \ldots, r-1$. From 4.1 and relation (i) we know that φ_{s_i} has the support $\mathfrak{A}_r^\times s_i \mathfrak{A}_r^\times$ and that $(\varphi_{s_i} + e) * (\varphi_{s_i} - q^{ds} e) = 0$. Obviously the function $\varphi_{s_i} \in \mathcal{H}$ is characterized by these two properties. It is sufficient to show that the left sides of the above equations have the same characterizing properties. Since the relation is preserved when conjugating with φ_h, it is enough to show that both sides of each equation have the same support. But we know the support of $(\varphi_h)^k$ and of φ_{s_i}. Therefore, using 2.6, 2.7 and 2.1, we see that the supports of $\varphi_h^2 * \varphi_{s_1} * \varphi_h^{-2}$ and $\varphi_h * \varphi_{s_i} * \varphi_h^{-1}$ are, respectively, $\mathfrak{A}_r^\times h^2 s_1 h^{-2} \mathfrak{A}_r^\times = \mathfrak{A}_r^\times s_{r-1} \mathfrak{A}_r^\times$ and $\mathfrak{A}_r^\times h s_i h^{-1} \mathfrak{A}_r^\times = \mathfrak{A}_r^\times s_{i-1} \mathfrak{A}_r^\times$. □

Finally we want to prove that our homomorphism $\Phi : \mathcal{H}(r, q^{ds}) \longrightarrow \mathcal{H}$ is an isomorphism. For this we need some additional facts concerning the affine Hecke algebras $\mathcal{H}(r, z)$. Observe that (W, S) is a Coxeter system and that $\tilde{W} = W \rtimes \langle h \rangle$ is an extended Coxeter group, since h normalizes S.

Therefore:

4.5 **FACT** (see [BK1] p. 179, 180): *There is a well defined map*

$$\tilde{W} \ni w \longmapsto [w] \in \mathcal{H}(r, z)$$

such that:

$$\begin{array}{rcll} [s_i] & = & \varrho_i & \text{for} \quad i = 1, \cdots, r-1 \\ [s_0] & = & \xi \varrho_1 \xi^{-1} & \\ [h] & = & \xi, & \end{array}$$

and if $w \in \tilde{W}$ has the minimal expression $w = h^k e_1 \cdots e_l$, where $e_1, \ldots, e_l \in S = \{s_0, \ldots, s_{r-1}\}$ and $l = l_C(w)$ is the length with respect to S, then:

$$[w] = \xi^k [e_1] \cdots [e_l]. \tag{23}$$

The relations in $\mathcal{H}(r, z)$ imply that $[w]$ is well defined independent of the reduced expression for w and that

$$[h^k s h^{-k}] = [h]^k [s] [h]^{-k} \tag{24}$$

for all $k \in \mathbb{Z}$ and $s \in S$. Moreover, all elements $\{[w] : w \in \tilde{W}\}$ are units in $\mathcal{H}(r, z)$ (since $\xi, \varrho_1, \ldots, \varrho_{r-1}$ are units) and form a basis of $\mathcal{H}(r, z)$ as a $\mathbb{C}$-vector space.

From 2.1, 1.1, and 4.5 we see that it is now enough to prove:

4.6 **Lemma**: *$\Phi : \mathcal{H}(r, q^{ds}) \longrightarrow \mathcal{H}$ sends $[w]$ to a function $\Phi([w])$ which has support on $\mathfrak{A}_r^\times w\, \mathfrak{A}_r^\times$.*

Proof: We use the expression (23) and proceed by induction on l. For $l = 0$, i.e. $w = h^k$ the assertion is already known. For the induction step we use the fact that Φ is a ring homomorphism, hence

$$\Phi([w]) = \Phi(\xi^k [e_1] \cdots [e_{l-1}]) * \Phi([e_l]).$$

By the induction hypothesis the first factor has the support $\mathfrak{A}_r^\times w' \mathfrak{A}_r^\times$, where $w' = h^k e_1 \cdots e_{l-1} \in \tilde{W}$. If $e_l = s_i \neq s_0$, then $[e_l] = \varrho_i$, and $\Phi([e_l]) = \varphi_{s_i}$ has the support $\mathfrak{A}_r^\times s_i \mathfrak{A}_r^\times$ (see 4.1 and (22)). From 2.1 and 2.6 we conclude that $\Phi([w])$ has the support $\mathfrak{A}_r^\times w \mathfrak{A}_r^\times$. If $e_l = s_0$, take $h^{-1}wh = h^{-1}w'h \cdot h^{-1}s_0 h$ and use the fact that $h^{-1}s_0 h = s_1$. (As we have seen, conjugation by h does not change the length.) By the induction hypothesis, $\Phi([h^{-1}w'h])$ has the support $\mathfrak{A}_r^\times w' \mathfrak{A}_r^\times$ and, therefore, $\Phi([h^{-1}wh]) = \Phi([h^{-1}w'h]) * \varphi_{s_1}$ has the support $\mathfrak{A}_r^\times h^{-1}wh \mathfrak{A}_r^\times$, as we see again from 2.1 and 2.6. Finally from (23) and (24) we see that $[w] = [h][h^{-1}wh][h^{-1}]$ implies that

$$\Phi([w]) = \varphi_h * \Phi([h^{-1}wh]) * \varphi_{h^{-1}}.$$

Now we apply 2.7 to conclude that $\Phi([w])$ has support $\mathfrak{A}_r^\times w \mathfrak{A}_r^\times$. □

This completes the proof of Theorem 4.2. □

For later use we note:

4.7 **Corollary**: *Every function $f \in \mathcal{H}(A^\times, \mathfrak{A}_r^\times, \tau)$ with support on a single double coset $\mathfrak{A}_r^\times w \mathfrak{A}_r^\times$ ($w \in \tilde{W}$) belongs to $\mathcal{H}(A^\times, \mathfrak{A}_r^\times, \tau)^\times$.*

Proof: Let $f \in \mathcal{H}(A^\times, \mathfrak{A}_r^\times, \tau)$ have support $\mathfrak{A}_r^\times w \mathfrak{A}_r^\times$ for $w \in \tilde{W}$. Then by construction there is a scalar $\lambda = \lambda(f) \neq 0$ such that $f = \Phi(\lambda \cdot [w])$. □

5. Level Zero Types.

We consider pairs $(\mathfrak{A}^\times, \tau)$, where $\mathfrak{A} \subset A$ is a standard hereditary order and τ is the inflation of an irreducible cuspidal representation of $M(k_D) = \bar{\mathfrak{A}}^\times$. Using the results of Parts 1 and 4 we want to show that these pairs are types in the sense of Bushnell and Kutzko [BK2]. We also want to show that, if τ, τ' as representations of $\bar{\mathfrak{A}}^\times, \bar{\mathfrak{A}}'^\times$ are $A^\times$-conjugate, then τ and τ' are types for the same component of the Bernstein spectrum.

First we consider the case in which $\mathfrak{A} = \mathfrak{A}_1 = M_m(O)$ is the standard maximal order of A.

5.1 **Proposition**: *If σ is a cuspidal representation of $\bar{\mathfrak{A}}_1^\times = GL_m(k_D)$, then the set Ω of equivalence classes of irreducible smooth representations Π of $A^\times$ such that $\sigma \subset \Pi|_{\mathfrak{A}_1^\times}$ comprises a single unramified twist class of level zero supercuspidal representations of $A^\times$. Hence Ω is a connected component of the Bernstein spectrum $\Omega(A^\times)$ and $(\mathfrak{A}_1^\times, \sigma)$ is a type for that component. Moreover, for every $\Pi \in \Omega$ the type σ occurs in $\Pi|_{\mathfrak{A}_1^\times}$ simply.*

Proof: We apply Proposition 2.1 to the case $r=1, s=m$. We see that a double coset $\mathfrak{A}_1^\times w \mathfrak{A}_1^\times$ supports an element of the Hecke algebra $\mathcal{H}(A^\times, \mathfrak{A}_1^\times, \sigma)$ if and only if $w = \varpi^{lv} I_m$ with $v \in \mathbb{Z}$. In particular, we have an intertwining operator J such that $\sigma(\varpi^l u \varpi^{-l}) = J\sigma(u)J^{-1}$ for all $u \in \mathfrak{A}_1^\times$. Putting $\widetilde{\sigma}(u\varpi^{lv}) := \sigma(u)J^v$ for $u \in \mathfrak{A}_1^\times$ and $v \in \mathbb{Z}$, we obtain an extension $\widetilde{\sigma}$ of σ to the group $H = \mathfrak{A}_1^\times \rtimes \langle \varpi^l \rangle$. If $g \in A^\times$ intertwines $\widetilde{\sigma}$, then g intertwines σ, which implies, by 2.1, that $g \in H$. Therefore, the compact mod center induction $\mathrm{cInd}_H^{A^\times}(\widetilde{\sigma})$ is irreducible and supercuspidal, since H is compact mod center and $\mathcal{H}(A^\times, H, \widetilde{\sigma})$ is one-dimensional. Furthermore, if $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ are two extensions of σ to H which induce to equivalent irreducible representations of $A^\times$, then there exists $g \in A^\times$ such that g intertwines $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$. Again this implies that g intertwines σ, hence that $g \in H$. It follows that $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ are equivalent. Fixing a character χ of the subgroup $\langle \varpi_F \rangle$ in the center of $A^\times$, we have

$$\mathrm{cInd}_{\mathfrak{A}_1^\times \cdot \langle \varpi_F \rangle}^{A^\times}(\sigma \otimes \chi) = \bigoplus_{\widetilde{\sigma}} \mathrm{cInd}_H^{A^\times}(\widetilde{\sigma}), \tag{25}$$

a direct sum over the d/l different extensions of $\sigma \otimes \chi$ to H. Since

$$\mathrm{Hom}_{\mathfrak{A}_1^\times \cdot \langle \varpi_F \rangle}(\sigma \otimes \chi, \Pi) = \mathrm{Hom}_{A^\times}(\mathrm{cInd}_{\mathfrak{A}_1^\times \cdot \langle \varpi_F \rangle}^{A^\times}(\sigma \otimes \chi), \Pi),$$

we see that an irreducible representation Π of $A^\times$ contains σ as a component if and only if Π occurs in (25) for some (not necessarily unitary) character χ, in which case the multiplicity is one. Finally, we have an injection

$$\mathrm{Nrd}_{A|F} : H/\mathfrak{A}_1^\times \hookrightarrow F^\times / o_F^\times,$$

which implies that all extensions $\widetilde{\sigma}$ from σ to H are covered if we twist $\mathrm{cInd}_H^{A^\times}(\widetilde{\sigma})$ by unramified characters of $A^\times$. Therefore, $\Pi|_{\mathfrak{A}_1^\times} \supset \sigma$ if and only if Π is an unramified twist of $\mathrm{cInd}_H^{A^\times}(\widetilde{\sigma})$. □

5.2 **Proposition**: *Two cuspidal types $(\mathfrak{A}_1^\times, \sigma)$, $(\mathfrak{A}_1^\times, \sigma')$ as in 5.1 are types for the same connected component Ω if and only if σ and σ' belong to the same $\mathrm{Gal}(k_D|k)$-orbit, if and only σ and σ' are conjugate under the action of the normalizer of $\mathfrak{A}_1^\times$ on representations of $\mathfrak{A}_1^\times$.*

Proof: Let $\widetilde{\sigma}$ be an extension of σ to H as in the proof of 5.1. Then the restriction of the representation $\mathrm{cInd}_H^{\mathfrak{A}_1^\times \rtimes \langle \varpi \rangle}(\widetilde{\sigma})$ contains every $\mathrm{Gal}(k_D|k)$-conjugate of σ; these conjugates are also conjugates by the normalizer of $\mathfrak{A}_1^\times$, since $N_{A^\times}(\mathfrak{A}_1^\times) = \mathfrak{A}_1^\times \rtimes \langle \varpi \rangle$. Thus the two notions of conjugation amount to the same thing. Thus we have proved that if σ and σ' are conjugate, in either sense, then they are types for the same connected component Ω. Conversely, let us show that if (σ, W) is contained in $\Pi|_{\mathfrak{A}_1^\times}$ and if the irreducible level zero representation σ' of $\mathfrak{A}_1^\times$ is also a component of $\Pi|_{\mathfrak{A}_1^\times}$, where Π is an irreducible smooth representation of $A^\times$, then σ' is a $\mathrm{Gal}(k_D|k)$-conjugate of σ. By 5.1 we can choose an extension $\widetilde{\sigma}$ of σ such that Π is equivalent to $\mathrm{cInd}_H^{\mathfrak{A}_1^\times \rtimes \langle \varpi \rangle}(\widetilde{\sigma})$. We realize Π as right translation on the space V of W-valued functions f on $A^\times$ such that f is compactly supported mod $\langle \varpi_F \rangle$ and $f(hx) = \widetilde{\sigma}(h)f(x)$ for all

$h \in H$ and $x \in A^\times$. For any level zero component of $\Pi|_{\mathfrak{A}_1^\times}$ we have an $x \in A^\times$ and $f \in V$ such that $0 \neq f(x)$ and $f(xu) = f(x)$ for all $u \in (1+\mathfrak{P}_1)$. Using the Cartan decomposition for $A^\times$, we take $x = \varpi^v$, where $v = (v_1, \ldots, v_m)$ and $v_1 \geq v_2 \geq \cdots \geq v_m$. If $v_i > v_{i+1}$ for some $i < m$, we may consider the Levi group $M' = GL_i \times GL_{m-i}$ together with the lower standard parabolic subgroup $P' = M' \ltimes U'$ in GL_m. We have

$$\varpi^{-v}(U' \cap \mathfrak{A}_1^\times)\varpi^v \subseteq 1 + \mathfrak{P}_1.$$

Therefore,

$$f(u'\varpi^v) = f(\varpi^v(\varpi^{-v}u'\varpi^v)) = f(\varpi^v) \tag{26}$$

for all $u' \in U' \cap \mathfrak{A}_1^\times$. But $U' \cap \mathfrak{A}^\times / U' \cap (1+\mathfrak{P}_1)$ is a unipotent subgroup of $GL_m(k_D)$, and $\tilde{\sigma}$ is a cuspidal representation, so, by (26),

$$0 = \int\limits_{U' \cap \mathfrak{A}_1^\times} \tilde{\sigma}(u') f(\varpi^v) du' = \int\limits_{U' \cap \mathfrak{A}_1^\times} f(\varpi^v) du',$$

which contradicts the assumption that $f(\varpi^v) \neq 0$ unless $v_1 = v_2 = \cdots = v_m$, i.e. $\varpi^v = \varpi^{v_1} I_m \in H$. Thus every irreducible level zero component σ' of $\Pi|_{\mathfrak{A}_1^\times}$ belongs to the $\mathrm{Gal}(k_D|k)$-orbit of σ. □

Now we consider $\mathcal{C}(k_D)$ the set of classes of irreducible cuspidal representations of $GL_s(k_D)$ for all $s \geq 1$. If $\sigma \in \mathcal{C}(k_D)$ is a representation of $GL_s(k_D)$, we define its degree to be $d(\sigma) := s$. With respect to the natural action of $\mathrm{Gal}(k_D|k)$ on $\mathcal{C}(k_D)$ we consider the set $\overline{\mathcal{C}}(k_D)$ of Galois orbits. From 5.1 and 5.2 we obtain the natural injection

$$[\sigma] \in \overline{\mathcal{C}}(k_D) \longmapsto \pi_{[\sigma]} \in \mathcal{C}(D)_{\text{level zero}}, \tag{27}$$

which preserves degrees. We also consider the map between effective divisors

$$\mathrm{Div}^+(\overline{\mathcal{C}}(k_D)) \longrightarrow \mathrm{Div}^+(\mathcal{C}(D)_{\text{level zero}}) \tag{28}$$

which is induced by (27). The degree m divisors $\mathcal{D} \in \mathrm{Div}^+(\mathcal{C}(D)_{\text{level zero}})$ naturally parameterize the level zero supported connected components of the Bernstein spectrum $\Omega(A^\times)$.

5.3 **Proposition**: *Let $(\mathfrak{A}^\times, \tau)$ be a pair with divisor $\Delta(\tau)$ such that all* $\mathrm{Gal}(k_D|k)$*-equivalent constituents of τ are isomorphic. Then $(\mathfrak{A}^\times, \tau)$ is a type for the connected component $\Omega_{\mathcal{D}} \subset \Omega(A^\times)$, where $\mathcal{D}$ is the image of $\Delta(\tau)$ under (28).*

Proof: Let $M = GL_{s_1} \times \cdots \times GL_{s_r}$ and $M(k_D) = \bar{\mathfrak{A}}^\times$. We use (4) to regard τ as a representation of either $M(k_D)$ or $M(O)$. From 5.1 it follows that $(M(O), \tau)$ is a cuspidal type for the group $M(D)$. If we can prove that $(\mathfrak{A}^\times, \tau)$ is a "cover" of $(M(O), \tau)$, then [BK2](8.3) implies our assertion. To prove that $(M(O), \tau)$ is a cover, we have to verify the properties of a cover which are given in [BK2](8.1): For each parabolic subgroup $P = M \ltimes U$ of $A^\times$ with Levi subgroup M and opposite group $P^- = M \ltimes U^-$ we must show:

(i) $\mathfrak{A}^\times = (U^- \cap \mathfrak{A}^\times)(M \cap \mathfrak{A}^\times)(U \cap \mathfrak{A}^\times)$,
and $U^- \cap \mathfrak{A}^\times$, $U \cap \mathfrak{A}^\times$ are both in the kernel of τ.

(ii) $\qquad (\mathfrak{A}^\times \cap M, \tau|_{\mathfrak{A}^\times\cap M}) = (M(O), \tau).$

(iii) There exists an invertible element of $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$ supported on a double coset $\mathfrak{A}^\times z_P \mathfrak{A}^\times$, where z_P lies in the center of M and satisfies the conditions:

$$\begin{aligned} z_P(\mathfrak{A}^\times \cap U)z_P^{-1} &\subseteq \mathfrak{A}^\times \cap U \\ z_P^{-1}(\mathfrak{A}^\times \cap U^-)z_P &\subseteq \mathfrak{A}^\times \cap U^-, \end{aligned}$$

and for any compact open subgroups $H_1, H_2 \subset U$ and $K_1, K_2 \subseteq U^-$ we have $z_P^m H_1 z_P^{-m} \subset H_2$ and $z_P^{-m} K_1 z_P^m \subset K_2$ for all sufficiently large positive integers m.

As usual we assume without loss of generality that M is block diagonal. The conditions (i), (ii), and (iii) are then certainly satisfied in the case that P contains the upper triangular group. Referring to M as below (8) we take $z_P = \varpi_F^v$, $v = (v_1, \dots, v_m)$, $v_i = v_j$ if l(i) = l(j), $v_i > v_j$ if l(i) < l(j).

In every case (ii) is clear. Noting that $\mathfrak{A}^\times = M(O)\cdot(1+\mathfrak{P}_\mathfrak{A})$ and that $(1+\mathfrak{P}_\mathfrak{A}) = \prod_{1\le i,j\le m} H_{i,j}$, a product of abelian pro-p-groups in which we have uniqueness of representation independent of the order of factors; in fact, the same is true for the groups $(1+\mathfrak{P}_\mathfrak{A})\cap U$, $(1+\mathfrak{P}_\mathfrak{A})\cap M$, and $(1+\mathfrak{P}_\mathfrak{A})\cap U^-$ for any $P = M \ltimes U$. (i) is therefore also clear for all $P = M \ltimes U$. Finally for any $P = M \ltimes U$ there is a Weyl chamber which is positive with respect to a minimal parabolic subgroup contained in P. By choosing an appropriate wall of this chamber and letting it correspond to the positive elements on the central torus of M we satisfy the third condition too with elements z_P in this torus. We note that $z_P \in \tilde{W}_A \cap \mathrm{Stab}(M(O), \tau)$ lies in the support of $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)$ (see 1.2). We are left to show that there is an element of $\mathcal{H}(A^\times, \mathfrak{A}^\times, \tau)^\times$ with support the double coset $\mathfrak{A}^\times z_P \mathfrak{A}^\times$. However, from 4.7, 1.9, and 1.(9) it is a general property of our Hecke algebras that functions which have support on a single double coset are units of the Hecke algebra. $\square$

5.4 **Theorem**: *Let $(\mathfrak{A}^\times, \tau)$ be any cuspidal level zero pair. Then $(\mathfrak{A}^\times, \tau)$ is a type for the connected component $\Omega_\mathcal{D} \subset \Omega(A^\times)$, where $\mathcal{D} = \mathcal{D}(\tau)$ is the image of $\Delta(\tau)$ under (28).*

Proof: We observe that there exists a cuspidal level zero pair fulfilling the conditions of 5.3 with the divisor $\Delta(\tau)$ and we apply 1.10. $\square$

Now we summarize our results:

5.5 **Theorem**: *Let (Π, V) be an irreducible representation of $A^\times$ which is of level zero. Let $\mathcal{D} = \mathcal{D}(\Pi)$ be the divisor (see §0.4) such that the supercuspidal support of Π belongs to $\Omega_\mathcal{D}$. Let $\mathfrak{A}$ be a standard hereditary order which is minimal such that $V^{1+\mathfrak{P}} \neq (0)$ for $\mathfrak{P} = \mathfrak{P}_\mathfrak{A}$. Then:*

(i) *The space $V^{1+\mathfrak{P}}$ decomposes as the direct sum of all irreducible cuspidal representations τ of $\overline{\mathfrak{A}}^\times$ such that $\mathcal{D}(\tau) = \mathcal{D}(\Pi)$, each occurring with the same multiplicity. In particular, the support of $\mathcal{D}(\Pi)$ consists of level zero supercuspidal representations.*

(ii) *If* (Π, V) *is supercuspidal and level zero, then* $\mathcal{D}(\Pi) \in \mathcal{C}(D)$*, which implies that* $\mathfrak{A} = \mathfrak{A}_1$ *and* $\tau = \sigma = \sigma_1$ *and* $V^{1+\mathfrak{P}_1}$ *is the direct sum of the* $\mathrm{Gal}(k_D|k)$*-conjugates of* σ*, each occurring with multiplicity one. Thus,* $\mathcal{D}(\Pi) = \pi_{[\sigma]}$.

(iii) *If the supercuspidal support of an irreducible representation* (Π, V) *of* $A^\times$ *is level zero, then* (Π, V) *is also of level zero.*

Proof: (i): We know that if $V^{1+\mathfrak{P}_\mathfrak{A}}$ satisfies the above minimality condition, then it is a finite direct sum of cuspidal types $(\mathfrak{A}^\times, \tau)$. From 5.4 it follows that if τ occurs in $V^{1+\mathfrak{P}_\mathfrak{A}}$, then $\mathcal{D}(\Pi) = \mathcal{D}(\tau)$. From 1.7 and 1.10 it follows that the multiplicities are the same for all τ' such that $\Delta(\tau') = \Delta(\tau)$; in particular, all τ' occur and all occur with the same multiplicities. Since $\mathcal{D}(\Pi) = \mathcal{D}(\tau)$, the support of $\mathcal{D}(\Pi)$ consists of level zero representations.

(ii) follows from (i).

(iii): Up to $A^\times$-conjugation the supercuspidal support of Π is (M, π), where $M = GL_{s_1} \times \cdots \times GL_{s_r}$ is a standard Levi subgroup of GL_m. By assumption the supercuspidal representation π has a fixed vector for $\Pi|_{(1+\mathfrak{P}_1)\cap M}$. This means that $\pi = \otimes_{i=1}^r \pi_i$ is a tensor product of irreducible supercuspidal level zero representations π_i of $GL_{s_i}(D)$. Applying (ii) to each of the π_i, we see that the supercuspidal support of Π lies in $\Omega_\mathcal{D}$ for some divisor $\mathcal{D} = \sum_{[\sigma]} r_{[\sigma]}\pi_{[\sigma]}$ as described above. Therefore, if $(\overline{\mathfrak{A}}^\times, \tau)$ is a pair such that $\mathcal{D}(\tau) = \mathcal{D}$, then, by 5.4, $(\mathfrak{A}^\times, \tau)$ is a type for $\Omega_\mathcal{D}$, hence is contained in V and this implies that (Π, V) is level zero. $\square$

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Humboldt-Universität Berlin, FB Reine Mathematik, Unter den Linden 6, 10099 Berlin, Germany

E-mail address: grabitz@mathematik.hu-berlin

Cleveland State University, Cleveland, Ohio 44115, USA

E-mail address: silberger@math.csuohio.edu

Humboldt-Universität Berlin, FB Reine Mathematik, Unter den Linden 6, 10099 Berlin, Germany

E-mail address: zink@mathematik.hu-berlin