Scenario Reduction in Stochastic Programming: An Approach Using Probability Metrics

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Abstract

Given a convex stochastic programming problem with a discrete initial probability distribution, the problem of optimal scenario reduction is stated as follows: Determine a scenario subset of prescribed cardinality and a probability measure based on this set that is closest to the initial distribution in terms of a natural (or canonical) probability metric. Arguments from stability analysis indicate that Fortet-Mourier type probability metrics may serve as such canonical metrics. Efficient algorithms are developed that determine optimal reduced measures approximately. Numerical experience is reported for reductions of electrical load scenario trees for power management under uncertainty. For instance, it turns out that after a 50% reduction of the scenario tree the optimal reduced tree still has about 90% of relative accuracy.

Key Words: Stochastic programming, quantitative stability, Fortet-Mourier metrics, scenario reduction, transportation problem, electrical load scenario tree
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1 Introduction

Various important real-life decision problems can be formulated as convex stochastic programs, which can mostly be written in the form

$$\min_{x \in X} \mathbb{E}_P f_0(\omega, x)\,. \tag{1}$$

Here, $X \subset \mathbb{R}^n$ is a given nonempty convex closed set, Ω a closed subset of $\mathbb{R}^s$ and $\mathcal{B}$ the Borel σ-field relative to Ω, the function f_0 from $\Omega \times \mathbb{R}^n$ to the extended reals $\overline{\mathbb{R}}$ is measurable with respect to ω and lower semicontinuous convex with respect to x, and P is a fixed probability measure on $(\Omega, \mathcal{B})$, i.e., $P \in \mathcal{P}(\Omega)$, with $\mathbb{E}_P$ denoting expectation with respect to P. This formulation covers (convex) two- and multi-stage stochastic programs with recourse. In these cases, X is the set of feasible first-stage decisions and the function values $f_0(\omega, x)$ evaluate the best possible outcomes of decisions x in case that ω is observed.

Apparently, (1) is a mathematical program. However, several important problems concerning its formulation and algorithmic approaches have to be resolved. The probability measure P does not need to be known precisely, the function f_0 – the random objective – is given implicitly (e.g. as an (iterated) optimal value), and at the same time, (1) is often large scale. To solve (1), various problem specific procedures that approximate the model and exploit its special structure have been designed. Many approximation methods exploit a discrete probability measure which, in a certain sense, approximates well not necessarily the underlying probability measure P but the optimal value $v(P)$ and the set of ε-approximate solutions $S_\varepsilon(P)$ of (1), where

$$\begin{aligned} v(P) &:= \inf\{\mathbb{E}_P f_0(\omega, x) \,:\, x \in X\} \\ S_\varepsilon(P) &:= \{x \in X : \mathbb{E}_P f_0(\omega, x) \leq v(P) + \varepsilon\} \quad (\varepsilon \geq 0). \end{aligned}$$

Consequently, $S(P) := S_0(P)$ denotes the solution set of (1).
In this context, stability properties of the model (1) with respect to perturbations (approximations) of P become important (see e.g. the surveys [5], [19] and references therein, and [9, 17, 20]). In Section 2 we present a quantitative stability result for optimal values and ε-approximate solution sets of (1) that is based on the general perturbation analysis in [1, 16], and we show that probability (semi-) metrics with *ζ-structure* (cf. [13, 22]), i.e.,

$$d_{\mathcal{F}}(P, Q) = \sup_{f \in \mathcal{F}} |\mathbb{E}_P f(\omega) - \mathbb{E}_Q f(\omega)| = \sup_{f \in \mathcal{F}} |\int_\Omega f(\omega) P(d\omega) - \int_\Omega f(\omega) Q(d\omega)| \qquad (2)$$

with $\mathcal{F}$ being a suitable class of measurable functions from Ω to $\overline{\mathbb{R}}$ and with P, Q belonging to $\mathcal{P}(\Omega)$, appear as natural and suitable distances of probability distributions for stability analysis. Furthermore, it is explained there that classes of (continuous) functions having the form

$$\mathcal{F}_c = \{f : \Omega \to \mathbb{R} : f(\omega) - f(\tilde{\omega}) \leq c(\omega, \tilde{\omega}) \text{ for all } \omega, \tilde{\omega} \in \Omega\} \qquad (3)$$

with a continuous symmetric function $c : \Omega \times \Omega \to \mathbb{R}_+$ having the property that $c(\omega, \tilde{\omega}) = 0$ iff $\omega = \tilde{\omega}$, are highly relevant in the context of convex stochastic programs of the form (1). The choice of c depends on the quantitative continuity properties of the integrand f with respect to ω and is discussed in Section 2. The following estimate is valid for the corresponding probability metric

$$\zeta_c(P, Q) := d_{\mathcal{F}_c}(P, Q) \leq \mu_c(P, Q)\,, \qquad (4)$$

where μ_c denotes the *Kantorovich-Rubinstein functional*

$$\begin{aligned} \mu_c(P, Q) \;:=\; \inf\{ &\int_{\Omega \times \Omega} c(\omega, \tilde{\omega}) \eta(d(\omega, \tilde{\omega})) \,:\, \eta \text{ is a finite Borel measure on } \mathcal{B}(\Omega \times \Omega)\,, \\ &\eta(B \times \Omega) - \eta(\Omega \times \tilde{B}) = P(B) - Q(\tilde{B}) \text{ for all } B, \tilde{B} \in \mathcal{B}\}. \quad (5) \end{aligned}$$

In case that the function c has the particular form

$$c_h(\omega, \tilde{\omega}) = \|\omega - \tilde{\omega}\| \max\{1, h(\|\omega - \omega_0\|), h(\|\tilde{\omega} - \omega_0\|)\} \qquad (6)$$

for all $\omega, \tilde{\omega} \in \Omega$, where ω_0 is some fixed element in $I\!R^s$ and $h : I\!R_+ \to I\!R_+$ is continuous and nondecreasing, the corresponding metric with ζ-structure

$$\zeta_{c_h}(P,Q) := d_{\mathcal{F}_{c_h}}(P,Q)$$

is often called a *Fortet-Mourier (type) metric* (cf. [13]).
An important instance of problem (1) is that the initial probability measure P is itself discrete with finitely many atoms (or *scenarios*) or that a good discrete approximation of P is available. Its support may be very large so that due to computational complexity and to time limitations, this probability measure is further approximated by a probability measure Q carried by a (much) smaller number of atoms. In this case the distance $\zeta_c(P,Q)$ represents the optimal value of a (finite-dimensional linear) dual transportation problem. More precisely, for $P = \sum_{i=1}^{N} p_i \delta_{\omega_i}$ and $Q = \sum_{j=1}^{M} q_j \delta_{\tilde{\omega}_j}$ we obtain

$$\zeta_c(P,Q) = \max\{\sum_{i=1}^{N} p_i u_i + \sum_{j=1}^{M} q_j v_j \ : \ u_i + v_j \leq c(\omega_i, \tilde{\omega}_j)\,,\ \forall i,j\}.$$

Here, $\delta_\omega \in \mathcal{P}(\Omega)$ denotes the Dirac measure placing unit mass at ω. In particular, the metric ζ_c can be used to evaluate distances of specific probability measures obtained during a scenario-reduction process, i.e., in case that $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_M\}$ is a subset of $\{\omega_1, \ldots, \omega_N\}$. Various reduction rules appear in the context of recent large-scale real-life applications. There are purely heuristic and ad hoc rules, e.g. [2, 3], heuristic rules inspired by the contamination technique, cf. [6], and various sampling schemes. A rule based on a uniform approximation of the random objective function f over the whole scenario space and, independently, on the decision vector is designed in [21]. A deletion rule based on maintaining first and second order moments is used in [4]. For more information on recent work for scenario generation and reduction we refer to [8].
In Section 3 we study a novel scenario reduction approach that is based on best approximations in terms of Fortet-Mourier metrics ζ_c. We show that the Fortet-Mourier distance of a discrete original probability distribution P and the optimal reduced measure Q based on a given subset of scenarios of P as well as the optimal weights of Q can be computed explicitly, i.e., without solving a transportation problem. Furthermore, we derive two heuristic algorithms for determining the optimal subset of scenarios of P with given cardinality. Here, optimality always means closeness in terms of ζ_c.
In Section 4 we report on numerical experience for the reduction of a scenario tree that represents an approximation of the electrical load process in a power management model under uncertainty. It turns out that both algorithmic approaches for determining recursively the scenario subset to be deleted (*backward reduction*) and the set of remaining scenarios (*forward selection*), respectively, work reasonably well and efficient. The reduced subtrees obtained by forward selection are slightly better, but their computation requires higher CPU times. Somewhat surprisingly, a reduction of the scenario tree by 50% of the scenarios only implies a loss of about 10% of relative accuracy. Furthermore, it is possible to determine a subtree containing less than 2% of the original number of scenarios that still carries about 50% relative accuracy.

2 On stability results and probability metrics

Given the original probability measure P of (1) and an approximation Q we derive quantitative estimates of the closeness of $v(Q)$ and $S_\varepsilon(Q)$ to $v(P)$ and $S_\varepsilon(P)$ in terms of a certain probability metric. This distance of probability measures is associated with the model (1) in a natural way. Namely, with $I\!B$ denoting the closed unit ball in $I\!R^n$ we consider the following set of probability measures and distances

$$\begin{aligned}\mathcal{P}_f &:= \{Q \in \mathcal{P}(\Omega):\ -\infty < \int_\Omega \inf_{x\in X\cap\rho I\!B} f_0(\omega,x)Q(d\omega) \leq \\ &\qquad \sup_{x\in X\cap\rho I\!B}\int_\Omega f_0(\omega,x)Q(d\omega) < \infty\ ,\ \text{for each } \rho>0\}\\ d_{f,\rho}(P,Q) &:= \sup_{x\in X\cap\rho I\!B} \left|\int_\Omega f_0(\omega,x)P(d\omega) - \int_\Omega f_0(\omega,x)Q(d\omega)\right|\end{aligned}$$

for each $\rho > 0$ and P, $Q \in \mathcal{P}_f$. Note that, for any $Q\in\mathcal{P}_f$, the function $x \mapsto I\!E_Q f_0(\omega,x)$ is lower semicontinuous (by appealing to Fatou's lemma), proper (since $|I\!E_Q f_0(\omega,x)| < \infty$ holds for each $x\in X$) and convex on $I\!R^n$. Next we give a quantitative stability result for optimal values and (ε-approximate) solution sets.

Theorem 2.1 *Let $P\in\mathcal{P}_f$ and $S(P)$ be nonempty and bounded. Then there exist constants $\rho>0$ and $\bar\varepsilon>0$ such that*

$$|v(P)-v(Q)| \leq d_{f,\rho}(P,Q) \quad \textit{and} \quad \emptyset \neq S(Q) \subset S(P) + \Psi(d_{f,\rho}(P,Q))I\!B$$

whenever $Q\in\mathcal{P}_f$ with $d_{f,\rho}(P,Q)<\bar\varepsilon$, and that it holds for any $\varepsilon\in(0,\bar\varepsilon)$ that

$$\boldsymbol{d}_\infty(S_\varepsilon(P),S_\varepsilon(Q)) \leq \frac{2\rho}{\varepsilon} d_{f,\rho+\varepsilon}(P,Q) \quad \textit{whenever} \quad Q\in\mathcal{P}_f,\ d_{f,\rho+\varepsilon}(P,Q)<\varepsilon\,.$$

Here $\Psi(\eta) := \eta+\psi^{-1}(2\eta)$, $\eta\geq 0$, where $\psi(\tau) := \min\{I\!E_P f_0(\omega,x)-v(P) : d(x,S(P))\geq \tau\}$, $\tau\geq 0$, is the conditioning function of the model (1) and $\boldsymbol{d}_\infty$ is the Pompeiu-Hausdorff distance of nonempty closed subsets of $I\!R^n$.

Proof: Since the function $I\!E_P f_0(\omega,\cdot)$ is lower semicontinuous, proper and convex, we may apply Theorem 7.64 in [16]. Let $\bar\rho>0$ be chosen such that $S(P)\subset\bar\rho I\!B$ and $v(P)\geq-\bar\rho$. Let $\rho>\bar\rho$ and $\bar\varepsilon$ be chosen such that $0<\bar\varepsilon<\min\{\frac{1}{2}(\rho-\bar\rho), \frac{1}{2}\psi(\frac{1}{2}(\rho-\bar\rho))\}$. Then Theorem 7.64 in [16] says that

$$\begin{gathered}|v(P)-v(Q)| \leq \hat{\boldsymbol{d}}_\rho^+(I\!E_P f_0(\omega,\cdot), I\!E_Q f_0(\omega,\cdot)) \quad \text{and}\\ \emptyset\neq S(Q)\subset S(P)+\Psi(\hat{\boldsymbol{d}}_\rho^+(I\!E_P f_0(\omega,\cdot), I\!E_Q f_0(\omega,\cdot)))I\!B\end{gathered}$$

holds for any $Q\in\mathcal{P}_f$ with $\hat{\boldsymbol{d}}_\rho^+(I\!E_P f_0(\omega,\cdot), I\!E_Q f_0(\omega,\cdot))<\bar\varepsilon$.
Here, $\hat{\boldsymbol{d}}_\rho^+$ denotes the auxiliary epi-distance (cf. Prop. 7.61 in [16])

$$\begin{aligned}\hat{\boldsymbol{d}}_\rho^+(I\!E_P f_0(\omega,\cdot), I\!E_Q f_0(\omega,\cdot)) &= \inf\{\eta\geq 0:\ \text{for all } x\in\rho I\!B \text{ it holds that}\\ &\qquad \inf_{y\in x+\eta I\!B} I\!E_Q f_0(\omega,y) \leq \max\{I\!E_P f_0(\omega,x),-\rho\}+\eta \ \text{ and}\\ &\qquad \inf_{y\in x+\eta I\!B} I\!E_P f_0(\omega,y) \leq \max\{I\!E_Q f_0(\omega,x),-\rho\}+\eta\}.\end{aligned}$$

Hence, the first part of the result is a consequence of the estimate

$$\hat{\boldsymbol{d}}_{\rho}^{+}(\mathbb{E}_P f_0(\omega,\cdot), \mathbb{E}_Q f_0(\omega,\cdot)) \le d_{f,\rho}(P,Q)$$

(cf. Example 7.62 in [16]). Noting that the function Ψ is increasing completes the first part of the proof.
For the second part let $\varepsilon \in (0,\bar{\varepsilon})$ and $Q \in \mathcal{P}_f$ be such that $d_{f,\rho+\varepsilon}(P,Q) < \varepsilon$. Then

$$\emptyset \neq S(Q) \subset (\bar{\rho} + \Psi(\bar{\varepsilon}))\mathbb{B} \quad \text{and} \quad v(Q) \ge -(\bar{\rho} + \bar{\varepsilon}).$$

With $\rho > \hat{\rho} = \min\{\bar{\rho} + \Psi(\bar{\varepsilon}), \bar{\rho} + \bar{\varepsilon}\}$ and $\bar{\varepsilon} \le \rho - \hat{\rho}$ it follows from Theorem 7.69 in [16] that

$$\hat{\boldsymbol{d}}_{\rho}(S_\varepsilon(P), S_\varepsilon(Q)) \le \hat{\boldsymbol{d}}_{\rho+\varepsilon}^{+}(\mathbb{E}_P f_0(\omega,\cdot), \mathbb{E}_Q f_0(\omega,\cdot)),$$

where $\hat{\boldsymbol{d}}_{\rho}$ is the set distance $\hat{\boldsymbol{d}}_{\rho}(C,D) := \inf\{\eta \ge 0 : C \cap \rho\mathbb{B} \subset D + \eta\mathbb{B}, D \cap \rho\mathbb{B} \subset C + \eta\mathbb{B}\}$ for nonempty subsets C and D of $\mathbb{R}^n$. Using the same argument as above, we may estimate the auxiliary epi-distance $\hat{\boldsymbol{d}}_{\rho+\varepsilon}^{+}(\mathbb{E}_P f_0(\omega,\cdot), \mathbb{E}_Q f_0(\omega,\cdot))$ from above by $d_{f,\rho+\varepsilon}(P,Q)$. Moreover, since the functions $\mathbb{E}_P f_0(\omega,\cdot)$ and $\mathbb{E}_Q f_0(\omega,\cdot)$ are lower semicontinuous and convex, their level sets $S_{\bar{\varepsilon}}(P)$ and $S_{\bar{\varepsilon}}(Q)$ are also bounded. Hence, we may choose the constant ρ such that

$$\hat{\boldsymbol{d}}_{\rho}(S_\varepsilon(P), S_\varepsilon(Q)) = \boldsymbol{d}_{\infty}(S_\varepsilon(P), S_\varepsilon(Q)).$$

This completes the proof. $\square$

Theorem 2.1 is taken from the paper [18], which also contains more general results (e.g. allowing for unbounded solution sets $S(P)$). Its proof is included for convenience of the reader. The theorem illuminates the role of the distances $d_{f,\rho}(P,Q)$ with properly chosen $\rho > 0$ as *minimal* probability metrics implying stability of optimal values and (approximate) solutions to (1). Here, *minimal* means that the distance $d_{f,\rho}(P,Q)$ processes the minimal information of problem (1) implying stability.
Clearly, the result remains valid when bounding $d_{f,\rho}(P,Q)$ from above by another distance $d(P,Q)$ and reducing $\mathcal{P}_f$ to a subset of $\mathcal{P}(\Omega)$ on which d is well defined. Such a distance d will be called a *canonical* probability metric associated with (1) if it has the form (2) with a class $\mathcal{F}$ of functions from Ω to $\overline{\mathbb{R}}$ that contains the functions $C \cdot f_0(\cdot,x)$ for each $x \in X \cap \rho\mathbb{B}$ and some constants $C > 0$ and $\rho > 0$, as well as further functions having important analytical properties of $f_0(\cdot,x)$ without becoming too large. Typical analytical properties defining relevant classes $\mathcal{F}$ in the theory of probability metrics are: Hölder or Lipschitz continuity and m-th order differentiability together with Hölder or Lipschitz continuity of the m-th derivative (see [13]). Hence, the problem arises to explore analytical properties of integrands f_0 in stochastic programming.
Typical integrands $f_0(\cdot,x)$, $x \in X$, in convex stochastic programming models (1) are nondifferentiable but locally Lipschitz continuous on Ω. More precisely, they often satisfy the following property: There exists a continuous symmetric function $c : \Omega \times \Omega \to \mathbb{R}_+$ having the property that $c(\omega,\tilde{\omega}) = 0$ holds iff $\omega = \tilde{\omega}$, and a nondecreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ such that for each $x \in X$ and $\omega, \tilde{\omega} \in \Omega$,

$$|f_0(\omega,x) - f_0(\tilde{\omega},x)| \le g(\|x\|)c(\omega,\tilde{\omega}). \tag{7}$$

From now on we require that the function c is measurable and satisfies the properties
(C1) $c(\omega,\tilde{\omega}) = 0$ iff $\omega = \tilde{\omega}$;
(C2) $c(\omega,\tilde{\omega}) = c(\tilde{\omega},\omega)\ \forall \omega,\tilde{\omega} \in \Omega$;
(C3) $\sup\{c(\omega,\tilde{\omega}) : \omega,\tilde{\omega} \in B, \|\omega-\tilde{\omega}\| \leq \delta\} \to 0$ as $\delta \to 0$ for each bounded $B \subseteq \Omega$;
(C4) there exists a measurable function $\lambda : \Omega \to I\!R_+$ that maps bounded sets to bounded sets and has the property $c(\omega,\tilde{\omega}) \leq \lambda(\omega) + \lambda(\tilde{\omega})\ \forall \omega,\tilde{\omega} \in \Omega$.
If Ω is compact, continuity of c implies (C3) and (C4). An important example of a function c satisfying the conditions (C1)–(C4) is a function of the form (6). It clearly satisfies (C1)–(C3) and also (C4) by considering the function $\lambda_h(\omega) := \|\omega - \omega_0\| \max\{1, h(\|\omega-\omega_0\|)\}$. A typical choice for ω_0 is $\omega_0 = I\!E_P\omega$. In the special case of $h(r) = r^{p-1}$ for $r \in I\!R_+$ and $\omega_0 = 0$ in (6) we use the notations c_p and ζ_p for the corresponding Fortet-Mourier metric. It is worth noting that the following estimates

$$\zeta_1(P,Q) \leq \zeta_{c_h}(P,Q) \quad \text{and} \quad \zeta_p(P,Q) \leq \zeta_q(P,Q)$$

$$\Big|\int_\Omega \|\omega\|^p (P-Q)(d\omega)\Big| \leq p \cdot \mu_{c_p}(P,Q) = p \cdot \zeta_p(P,Q)$$

are valid for each continuous nondecreasing function $h : I\!R_+ \to I\!R_+$ and $1 \leq p \leq q < \infty$ (cf. [13] and, in particular, Theorem 6.2.5). ζ_1 is also called *Kantorovich* or L_1-*Wasserstein* metric. Since convergence with respect to the Kantorovich metric implies weak convergence of probability measures, the estimates show that the closeness of two measures P and Q in terms of ζ_p implies the closeness of their p-th order moments and with respect to the topology of weak convergence. For further information on Fortet-Mourier type metrics the reader is referred to the monographs [13, 15].
Condition (7) motivates to consider the class (3) and the metric ζ_c on the set $\mathcal{P}_c := \{Q \in \mathcal{P}(\Omega) : \int_\Omega \lambda(\omega) Q(d\omega) < \infty\}$ with λ from (C4),

$$\zeta_c(P,Q) = d_{\mathcal{F}_c}(P,Q)$$

as a *canonical* probability metric for convex stochastic programs of the form (1).
Then we have for all $x \in X \cap \rho I\!B$,

$$d_{f,\rho}(P,Q) \leq g(\rho)\zeta_c(P,Q) \quad \text{if} \quad (g(\rho))^{-1} f_0(\cdot,x) \in \mathcal{F}_c\,.$$

Remark 2.2 *(Choice of c for multi-stage stochastic programs)*
In [14, 18] it is shown that linear two-stage stochastic programs with fixed recourse enjoy quantitative stability properties with respect to the Fortet-Mourier metric ζ_2 (i.e., the corresponding integrand f_0 satisfies condition (7) when setting $c = c_2$, where $c_2(\omega,\tilde{\omega}) := \|\omega-\tilde{\omega}\| \max\{1, \|\omega\|, \|\tilde{\omega}\|\}, \forall \omega,\tilde{\omega} \in \Omega$). In [18] this result is extended to the case of linear multi-stage stochastic programming models with fixed recourse in all stages under additional assumptions for the dependence structure of the underlying discrete-time stochastic process. The corresponding result asserts quantitative stability with respect to the metric ζ_K, where K denotes the number of stages of the model. The result also says that such models are even quantitatively stable with respect to ζ_1 if either costs or technology matrices and right-hand sides in all stages are random. In all these cases Fortet-Mourier metrics serve as canonical distances. In case Ω is compact and

condition (7) is satisfied for some function c_h, but h is not known explicitly, an estimate of the form $c_1(\omega,\tilde{\omega}) \le c_h(\omega,\tilde{\omega}) \le C \cdot c_1(\omega,\tilde{\omega})$ with some constant $C > 0$ is valid and, hence, c_1 can be used. However, any available knowledge on h should be incorporated !

There is a well developed theory for dual representations of the metric ζ_c in terms of Kantorovich-Rubinstein functionals μ_c (5) (cf. [15]). As an example we quote the following result (see Sect. 5.3 in [13]), which will be used in Section 3.

Theorem 2.3 *Let Ω be a closed subset of $I\!R^s$ and $c : \Omega \times \Omega \to I\!R_+$ satisfy the assumptions (C1)–(C4). Then it holds that*

$$\zeta_c(P,Q) = \mu_c(P,Q)$$

for all probability measures $P, Q \in \mathcal{P}_c(\Omega) = \{Q \in \mathcal{P}(\Omega) : \int_\Omega \lambda(\omega) Q(d\omega) < \infty\}$.

Note that the theorem applies to discrete probability measures with finite support in Ω since they always belong to $\mathcal{P}_c(\Omega)$.
Let us now consider a stochastic program (1), and assume that the integrand f_0 satisfies the condition (7) for some function c having the properties (C1)–(C4) and that the original probability measure belongs to $\mathcal{P}_c(\Omega)$. If the solution set $S(P)$ of (1) is nonempty and bounded, Theorem 2.1 applies and we may conclude Lipschitz stability properties of the optimal value v and the ε-approximate solution set S_ε at P with respect to the Fortet-Mourier metric ζ_c.
This motivates to take the probability distance ζ_c as a basis for approximating the original measure P. For instance, the principle of *optimal scenario generation* or *selection* for (1) may be formulated in the following way: Determine a discrete probability measure Q^* having M scenarios in Ω such that:

$$\zeta_c(P,Q^*) = \min\{\zeta_c(P, \sum_{j=1}^{M} q_j \delta_{\omega_j}) \ : \ \sum_{j=1}^{M} q_j = 1\,,\ q_j \ge 0\,,\ \omega_j \in \Omega\,,\ j = 1,\ldots,M\}\,. \quad (8)$$

Further constraints can be incorporated into (8), e.g., implying that the scenarios exhibit a certain prescribed tree structure.
Similarly, the principle of *optimal scenario reduction* of a given discrete approximation to P may be written as: Determine an index set $J_* \subset \{1,\ldots,M\}$ of given cardinality $\#J_* = k$ and a probability measure $Q_* = \sum_{\substack{j=1\\ j\notin J_*}}^{M} q_j^* \delta_{\omega_j}$ such that:

$$\zeta_c(P,Q_*) = \min\{\zeta_c(P, \sum_{\substack{j=1\\ j\notin J}}^{M} q_j \delta_{\omega_j}) \ : J \subset \{1,\ldots,M\}\,, \#J = k\,, \sum_{j\notin J} q_j = 1\,,\ q_j \ge 0\}. \quad (9)$$

We note that problem (8) represents a nonconvex optimization problem that is nondifferentiable and large scale in most practical situations. Its algorithmic solution appears to be hopeless for general measures P, supports Ω, functions c, and "large" numbers M of scenarios. An attempt for solving (8) is made in [12] in case of $c = c_1$, $\Omega = I\!R^s$, and of tree-structured scenarios $\{\omega_j : j = 1,\ldots,M\}$. The authors of [12] develop a

deterministic iteration scheme in case P is completely known and a stochastic approximation method based on empirical approximations of P. So far numerical experience is available for low dimensional test problems.
When assuming that P is discrete with finite support or replaced by a good discrete approximation, the situation becomes quite different. Then the Fortet-Mourier distance $\zeta_c(P, \sum_{\substack{j=1 \\ j \notin J}}^{M} q_j \delta_{\omega_j})$ is the optimal value of a (large scale) linear program (cf. (10) below), and the weights q_j and scenarios ω_j enter the right-hand sides of linear constraints and the cost function, respectively. With regard to problem (9) the situation becomes quite comfortable as will be shown in the next section.

3 Scenario reduction

Assume that the initial probability distribution P is discrete and carried by finitely many scenarios $\omega_i \in \Omega$ with weights $p_i > 0, i = 1, \ldots, N$, and $\sum_{i=1}^{N} p_i = 1$, i.e., $P = \sum_{i=1}^{N} p_i \delta_{\omega_i}$. Let $J \subset \{1, \ldots, N\}$ and consider the probability measure Q having scenarios ω_j with probabilities q_j, $j \in \{1, \ldots, N\} \setminus J$, i.e., compared to P the measure $Q = \sum_{j \notin J} q_j \delta_{\omega_j}$ is *reduced* by deleting all scenarios ω_j , $j \in J$, and by assigning new probabilistic weights q_j to each scenario ω_j, $j \notin J$. The optimal reduction concept described in Section 2 (see (9)) recommends to consider the probability distance

$$D(J;q) := \zeta_c(\sum_{i=1}^{N} p_i \delta_{\omega_i}, \sum_{j \notin J} q_j \delta_{\omega_j}) = \max\{\sum_{i=1}^{N} p_i u_i + \sum_{j \notin J} q_j v_j \ : \ u_i + v_j \leq c(\omega_i, \omega_j)\} \quad (10)$$

where the constraints of the linear program are satisfied for all $i \in \{1, \ldots, N\}$ and $j \notin J$, and the function c is chosen such that the underlying stochastic program behaves stable with respect to the Fortet-Mourier metric ζ_c. We assume throughout this Section that c satisfies (C1)–(C4). The reduction concept (9) says that the index set J is selected such that the distance $D(J;q)$ of the original and the reduced measure is optimal subject to all index sets with given cardinality. We distinguish two cases: *optimal* or *prescribed* weights q_j, $j \notin J$. Our first result provides an explicit representation of $D(J;q)$ in case of optimal weights q.

Theorem 3.1 *(optimal weights)*
Given $J \subset \{1, \ldots, N\}$ we have

$$D_J = \min\{D(J;q) \ : \ q_j \geq 0 \, , \sum_{j \notin J} q_j = 1\} = \sum_{i \in J} p_i \min_{j \notin J} c(\omega_i, \omega_j). \quad (11)$$

Moreover, the minimum is attained at $\bar{q}_j = p_j + \sum_{\substack{i \in J \\ j(i) = j}} p_i$, for each $j \notin J$, where $j(i) \in \arg\min_{j \notin J} c(\omega_i, \omega_j)$ for $i \in J$ (optimal redistribution rule).

Proof: We set $c_{ij} := c(\omega_i, \omega_j)$ for $i, j \in I := \{1, \ldots, N\}$ and make use of the primal as well as the dual representation of $D(J; q)$ for given J (cf. Theorem 2.3), i.e.,

$$D(J;q) = \min\{\sum_{i,j} c_{ij}\eta_{ij} : \eta_{ij} \geq 0\,, \sum_{i,j} \eta_{ij} = 1\,, \sum_{j\notin J} \eta_{ij} - \sum_{i=1}^{N} \eta_{ij} = p_i - q_j\,,\ i \in I\,,\ j \notin J\}$$

$$= \max\{\sum_{i=1}^{N} p_i u_i + \sum_{j\notin J} q_j v_j \,:\, u_i + v_j \leq c_{ij}\,,\ i \in I\,,\ j \notin J\}.$$

First, we show that the expression $\sum_{i\in J} p_i \min_{j\notin J} c_{ij}$ provides a lower bound of $D(J; q)$ for any feasible q. We set $u_i := \min_{k\notin J} c_{ik}$ for each $i \in I$ and $v_j := 0$ for each $j \notin J$. We show that this choice of u_i and v_j is feasible for the dual representation of $D(J; q)$. Noting that $u_i = 0$ for any $i \notin J$ we obtain $u_i + v_j = u_i \leq c_{ij}$ for all $i \in J$ and $j \notin J$, and $u_i + v_j = 0 \leq c_{ij}$ for all $i, j \notin J$. Hence, it holds

$$\sum_{i=1}^{N} p_i u_i + \sum_{j\notin J} q_j v_j = \sum_{i\in J} p_i \min_{k\notin J} c_{ik} \leq D(J;q) \quad \text{ for any feasible } q.$$

Next we show that this lower bound for $D(J; q)$ valid for all feasible q is attained for some $\bar{q}$. To this end, we select $j(i) \in \arg\min_{j\notin J} c_{ij}$ for each $i \in J$ and define

$$\eta_{ij} := \begin{cases} p_i & , j = j(i)\,,\ i \in J, \\ p_i & , i = j \notin J, \\ 0 & , \text{otherwise.} \end{cases}$$

Setting $\bar{q}_j = \sum_{i=1}^{N} \eta_{ij}$ for each $j \notin J$ we obtain that η_{ij} is feasible for the primal representation of $D(J; \bar{q})$. Hence, it also holds that

$$D(J;\bar{q}) \leq \sum_{i,j} c_{ij}\eta_{ij} = \sum_{i\in J} p_i \min_{j\notin J} c_{ij}.$$

We conclude that $\bar{q}$ is optimal, so that $D(J; \bar{q}) = \sum_{i\in J} p_i \min_{j\notin J} c_{ij}$, where $\bar{q}_j = p_j + \sum_{\substack{i\in J \\ j(i)=j}} p_i$ for each $j \notin J$. $\square$

The theorem provides an explicit formula for the optimal weights when the index set J of deleted scenarios is given. Its interpretation is that the new probability of a kept scenario is equal to the sum of its former probability and of all probabilities of deleted scenarios that are closest to it with respect to c.
When fixing the redistribution of the deleted weight $p_J = \sum_{i\in J} p_i$ of P by a rule of the form

$$q_j = p_j + \lambda_j p_J\,, \quad \text{for each } j \notin J \tag{12}$$

where the redistribution weights $\lambda_j \geq 0$, $j \notin J$, with $\sum_{j\notin J} \lambda_j = 1$, are given, the following upper bound for $D(J; q)$ is valid.

Theorem 3.2 *(prescribed redistribution)*
If q is redistributed by formula (12), we have $D(J;q) \leq \sum_{i\in J} p_i \sum_{j\notin J} \lambda_j c(\omega_i, \omega_j)$.
Moreover, equality holds if $\#J = 1$ and c satisfies the triangular inequality.

Proof:
We use the primal representation of $D(J;q)$ and set $\eta_{ij} = \begin{cases} p_i\lambda_j & , i \in J\,,\, j \notin J, \\ p_i & , i = j \notin J, \\ 0 & , \text{otherwise.} \end{cases}$

Then we have $\sum_{j\notin J} \eta_{ij} = p_i$, for $i = 1, \ldots, N$, and $\sum_{i=1}^{N} \eta_{ij} = p_j + \lambda_j p_J = q_j$ for $j \notin J$. Hence, we obtain with $c_{ij} := c(\omega_i, \omega_j)$

$$D(J;q) \leq \sum_{i,j} c_{ij}\eta_{ij} = \sum_{i\in J} p_i \sum_{j\notin J} \lambda_j c_{ij}\,.$$

Finally, let $J = \{l\}$ and assume that c satisfies the triangular inequality. We set $u_i = -c_{il} = -v_i\,,\, i = 1, \ldots, N$, and note that $u_i + v_j = c_{jl} - c_{il} \leq c_{ij}$ holds for all $i, j \in \{1, \ldots, N\}\,,\, j \neq l$. Hence, we obtain from the dual representation of $D(J;q)$ that

$$D(J;q) \geq \sum_{i=1}^{N} (q_i - p_i) c_{il} = p_l \sum_{\substack{i=1 \\ i\neq l}}^{N} \lambda_i c_{il}\,. \quad \square$$

Simple examples show that equality gets lost in Theorem 3.2 in general if $\#J \geq 2$ and also if $\#J = 1$ and c does not satisfy the triangle inequality. We stress here that the latter property of c is not needed for the optimal redistribution in Theorem 3.1.
Next we discuss the optimal choice of an index set J for scenario reduction with optimal weights and fixed cardinality $\#J$ of J, i.e., the solution of the problem

$$\min\{D_J = \sum_{i\in J} p_i \min_{j\notin J} c(\omega_i, \omega_j) \;:\; J \subset \{1, ..., N\}\,,\, \#J = k\} \tag{13}$$

for given $k \in I\!N\,,\, 1 \leq k < N$. First we consider the extremal cases of problem (13): deleting a single scenario and all but one scenario.

Example 3.3 *(single scenario deletion)*
In case $\#J = 1$, the optimal deletion problem (13) is of the form

$$\min_{l\in\{1,\ldots,N\}} p_l \min_{j\neq l} c(\omega_l, \omega_j). \tag{14}$$

If the minimum is attained at $l_* \in \{1, ..., N\}$, i.e., the scenario ω_{l_*} is deleted, the optimal redistribution rule is $\bar{q}_l = p_l$ for each $l \notin \{l_*, j(l_*)\}$ and $\bar{q}_{j(l_*)} = p_{j(l_*)} + p_{l_*}$, where $j(l_*) \in \arg\min_{j\neq l_*} c(\omega_{l_*}, \omega_j)$. Of course, the optimal deletion of a single scenario may be repeated recursively until a prescribed number k of scenarios is deleted (as in Algorithm 3.6).

Example 3.4 *(keeping only one scenario)*
In case $\#J = N-1$, the problem (13) is of the form

$$\min_{u\in\{1,\ldots,N\}} \sum_{i=1}^{N} p_i c(\omega_i, \omega_u)\,. \tag{15}$$

If the minimum is attained at $u^* \in \{1, ..., N\}$, the scenario ω_{u^*} is kept and the optimal redistribution rule provides $\bar{q}_{u^*} = p_{u^*} + \sum_{i\neq u^*} p_i = 1$.

Since (13) represents a special combinatorial optimization problem, efficient solution algorithms like those for the cases $k = 1$ and $k = N-1$ are hardly available in general. However, the next result provides lower and upper bounds for the optimal value of (13) that correspond to recursive extensions of the strategies (14) for $\#J = 1$ and (15) for $\#J = N-1$. It also turns out that the lower bound is attained under additional assumptions.

Theorem 3.5 *(bounds)*

$$\sum_{i=1}^{k} p_{l_i} \min_{j\notin\{l_1,\ldots,l_i\}} c(\omega_{l_i}, \omega_j) \leq \min\{D_J : J \subset \{1, ..., N\}\,,\, \#J = k\} \leq \sum_{i\in J_u} p_i \min_{j\notin J_u} c(\omega_i, \omega_j)\,,$$

where $J_u = \{1, \ldots, N\} \setminus \{u_1, \ldots, u_{N-k}\}$ and the indices l_i and u_j are chosen recursively such that

$$l_i \in \arg \min_{l\in\{1,\ldots,N\}\setminus\{l_1,\ldots,l_{i-1}\}} p_l \min_{j\notin\{l_1,\ldots,l_{i-1},l\}} c(\omega_j, \omega_l)\ ,\ i = 1, \ldots, k\ , \quad \textit{and} \tag{16}$$

$$u_j \in \arg \min_{l\notin\{u_1,\ldots,u_{j-1}\}} \sum_{\substack{i=1\\ i\notin\{u_1,\ldots,u_{j-1},l\}}}^{N} p_i \min_{u\in\{u_1,\ldots,u_{j-1},l\}} c(\omega_u, \omega_i)\ ,\ j = 1, \ldots, N-k\,. \tag{17}$$

Moreover, the index set $\{l_1, \ldots, l_k\}$ is a solution of (13) if for each $i = 1, \ldots, k-1$ the set $\arg\min_{j\notin\{l_1,\ldots,l_i\}} c(\omega_j, \omega_{l_i}) \setminus \{l_{i+1}, \ldots, l_k\}$ is nonempty.

Proof: For any index set $J = \{j_1, \ldots, j_k\} \subset \{1, \ldots, N\}$ with $\#J = k$ we have, by Theorem 3.1, that

$$\begin{aligned} D_J &= \sum_{i=1}^{k} p_{j_i} \min_{j\notin\{j_1,\ldots,j_k\}} c(\omega_{j_i}, \omega_j) \geq \sum_{i=1}^{k} p_{j_i} \min_{j\notin\{j_1,\ldots,j_i\}} c(\omega_{j_i}, \omega_j) \\ &\geq \sum_{i=1}^{k} p_{l_i} \min_{j\notin\{l_1,\ldots,l_i\}} c(\omega_{l_i}, \omega_j)\,, \end{aligned}$$

where the latter estimate is a consequence of the definition of the numbers l_i, $i = 1, \ldots, k$. If $\arg\min_{j\notin\{l_1,\ldots,l_i\}} c(\omega_j, \omega_{l_i}) \setminus \{l_{i+1}, \ldots, l_k\} \neq \emptyset$ holds for each $i = 1, \ldots, k-1$, we obtain

$$\min_{j\notin\{l_1,\ldots,l_i\}} c(\omega_{l_i}, \omega_j) = \min_{j\notin\{l_1,\ldots,l_k\}} c(\omega_{l_i}, \omega_j)$$

for each $i = 1, \ldots, k$. Hence, the above estimate may be continued to

$$D_J \geq \sum_{i=1}^{k} p_{l_i} \min_{j \notin \{l_1,\ldots,l_k\}} c(\omega_{l_i}, \omega_j) = D_{\{l_1,\ldots,l_k\}} \,.$$

Thus, the index set $\{l_1, \ldots, l_k\}$ is a solution of (13). Finally, the estimate

$$\min\{D_J : J \subset \{1, ..., N\}\,,\, \#J = k\} \leq D_{J_u} = \sum_{i \in J_u} p_i \min_{i \notin J_u} c(\omega_i, \omega_j)$$

is valid, which completes the proof. □

Theorem 3.5 suggests two different *heuristic algorithms* for the optimal reduction of the N original scenarios to $N-k$ scenarios. The first one determines the k scenarios that have to be deleted by exploiting the lower bound technique (16). It will be called *backward reduction*, while the second one is called *forward selection*, because it selects the remaining $N-k$ scenarios by using the upper bound technique (17).

Algorithm 3.6 *(backward reduction)*
In the first step, an index k_1 with $1 \leq k_1 \leq k$ is determined such that $J_1 = \{l_1^{(1)}, \ldots, l_{k_1}^{(1)}\}$ is a solution of (13) for $k = k_1$ by using the strategy described in Theorem 3.5. Next, the optimal redistribution rule of Theorem 3.1 is used and the probability measure P_1 is considered, where the scenarios indexed by J_1 are deleted. If $k_1 < k$, then the measure P_1 is reduced in a second step by deleting all scenarios in some index set J_2 with $1 \leq k_2 = \#J_2 \leq k - k_1$ which is obtained in the same way using Theorem 3.5. This procedure is continued until, in step r, we have $\sum_{i=1}^{r} k_i = k$ and $J = \cup_{i=1}^{r} J_i$. Finally, the optimal redistribution rule is used again for the index set J.

A particular variant of Algorithm 3.6 consists in the case that $k_i = 1$ for $i = 1, \ldots, k$ (*backward reduction of single scenarios*). This variant (without the final optimal redistribution) was already announced in [7, 10].

Algorithm 3.7 *(forward selection)*
The indices u_j, $j = 1, \ldots, N-k$, are determined recursively as in (17). Finally, set $J := \{1, \ldots, N\} \setminus \{u_1, \ldots, u_{N-k}\}$ and redistribute for the index set J according to Theorem 3.1.

Both algorithms provide an approximate solution of problem (13) and, hence, an upper bound for its optimal value. Note that these bounds, which represent Fortet-Mourier distances of the original P and the optimal reduced distribution, can be computed by using the explicit formula in Theorem 3.1. The performance of both algorithms is evaluated for a real-life test example in Section 4.
According to Theorem 3.2 the optimal choice of an index set J for scenario deletion with prescribed redistribution and fixed cardinality $\#J$ may be formulated as follows:

$$\min\{\sum_{i \in J} p_i \sum_{j \notin J} \lambda_j c(\omega_i, \omega_j) \;:\; J \subset \{1, ..., N\}\,,\, \#J = k\} \tag{18}$$

for given $k \in I\!N$, $1 \leq k < N$ and weights λ_j, $j \notin J$. Let us consider the special case of $k = 1$.

Example 3.8 *(deleting a single scenario with prescribed redistribution)*
In case $\#J = 1$, problem (18) takes the form

$$\min_{i \in \{1,\dots,N\}} p_i \sum_{j \neq i} \lambda_j c(\omega_i, \omega_j).$$

If P is a uniform discrete distribution, i.e., $p_i = \frac{1}{N}$ for each i, it might be desirable that the reduced measure Q has uniform weights too, i.e., $q_j = \frac{1}{N-1}$ for $j \notin J$. This corresponds to the choice of $\lambda_j = \frac{1}{N-1}$ for $j \notin J$.

Although a prescription of the weights for the reduced distribution might sometimes be useful, our theoretical results tend to preferring the optimal redistribution rule. For the latter rule no additional assumptions on c are needed, setwise reduction algorithms are available and Fortet-Mourier distances can be computed without solving transportation problems.

4 Numerical results for electrical load scenario trees

The optimization of electric power production in hydro-thermal generation systems for time horizons of one week or longer is inherently a stochastic decision problem. Indeed, forecasts on electrical load, flows to hydro reservoirs, and on fuel and electricity prices cannot be exact. For this reason, the weekly cost-optimal production of electric power in a hydro-thermal generation system is modelled in [10, 11] as a multistage (mixed-integer) stochastic program. The optimal scheduling decisions for all power units minimize the expected production costs subject to the operational requirements of the generation system. The scheduling decisions for thermal units are: which units to commit in each period, and at what generating capacity. The decision variables for pumped storage hydro units are the generation and pumping levels for each period. Power contracts for delivery and purchase are regarded as special thermal units. The basic system requirement is to meet the electrical load and the spinning reserve capacity. Further operating constraints are capacity limits for thermal and hydro units, minimum up/down-time requirements for thermal units and operating ranges and dynamics of hydro storage units. The scheduling horizon of one week is discretized into hourly intervals. Accordingly, the stochastic data process is approximated by a discrete-time stochastic process (with hourly time steps). The numerical tests of the stochastic model are performed in [10, 11] for a real-life hydro-thermal generation system consisting of 25 (coal-fired or gas-burning) thermal units and 7 pumped hydro units, and for stochastic electrical load (i.e., the remaining data were deterministic). The stochastic load process is approximated by a scenario tree. The resulting problem to be solved is a large-scale mixed-integer linear program with a special sparsity structure. The mixed-integer model is large even for a relatively small number of nodes in the tree. The single scenario model (i.e., 168 nodes) already contains 4200 binary and 6652 continuous variables, 13441 constraints and 19657 nonzeros in the constraint matrix.
In [10], an initial load scenario tree was constructed according to the following steps:

1. Calibration of a time series model for the load, generation of a large number of load scenarios.

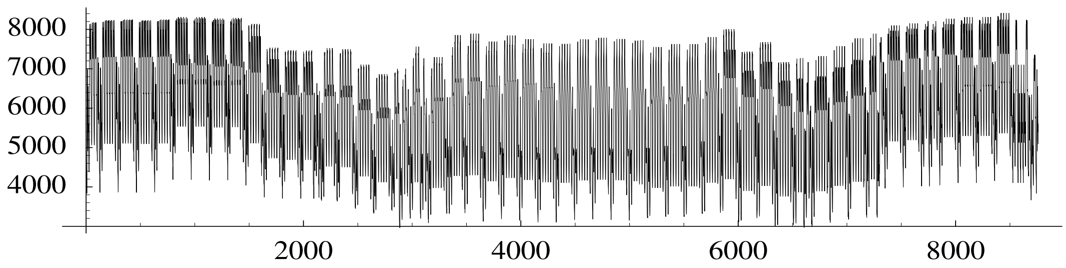

Figure 1: Time plot of the load profile for one year

2. Construction of a (dense) initial load scenario tree using the sample means and standard deviations of the simulated load scenarios.

The time series model for the load d_t in period t was calibrated from a historical load profile of one year (cf. Figure 1). The time series model for the load process $\{d_t\}_{t\in\boldsymbol{Z}}$ is the SARIMA$(7,0,9)\times(0,1,0)_{168}$ model (cf. [10])

$$\begin{aligned} d_t &= \hat{\phi}_1 d_{t-1} + \ldots + \hat{\phi}_7 d_{t-7} - d_{t-168} - \hat{\phi}_1 d_{t-169} - \ldots - \hat{\phi}_7 d_{t-175} \\ &\quad + Z_t + \hat{\theta}_1 Z_{t-1} + \ldots + \hat{\theta}_9 Z_{t-9}, \quad t \in \boldsymbol{Z}. \end{aligned} \tag{19}$$

The estimated model coefficients are

$$\begin{aligned} (\hat{\phi}_1,\ldots,\hat{\phi}_7) &= (2.79, -4.35, 5.16, -4.88, 3.67, -1.92, 0.50), \\ (\hat{\theta}_1,\ldots,\hat{\theta}_9) &= (-1.27, 1.53, -1.35, 0.88, -0.31, -0.06, 0.18, 0.11, 0.07) \end{aligned}$$

and Z_t, $t\in\boldsymbol{Z}$, are independent, normally distributed random variables with mean 0 and standard deviation 108.3.

For the generation system under consideration, the load forecast is reliable for the time span $t=1,\ldots,24$. A large number M of *simulated load scenarios* (sample paths) $\tilde{d}^\ell = (\tilde{d}^\ell_t)_{t=25}^{168}$, $\ell = 1,\ldots,M$, is generated from the SARIMA equation (19) using M i.i.d. realizations of Z_t, $t=16,\ldots,168$, and starting values d_t, $t=-150,\ldots,24$. The *empirical means* $\bar{d}_t$ and *standard deviations* $\bar{\sigma}_t$ of the simulated load scenarios form the basis of the scenario tree building scheme. Since there exists a normally fixed daily time when already observable meteorological and load data provide the opportunity to readjust forecasts, $t_k = 24k$, $k=1,\ldots,6$, is a reasonable choice for the branching points. A balanced *ternary tree* with branching points t_k allows to distinguish the events with the verbal description "low load", "medium load" and "high load" in the time period $t = t_k+1,\ldots,t_{k+1}$, $k=1,\ldots,6$. (For convenience of notation set $t_7 := 168$.) Thus, the stages of the stochastic program correspond to one day (or 24 hours) and the tree consists of $N := 3^6 = 729$ scenarios $d^i = (d^i_t)_{t=1}^{168}$, $i=1,\ldots,N$.

The scenarios of the initial load scenario tree and their probabilities can be assigned in various ways. The predicted load for the (first stage) planning period $t=1,\ldots,24$ yields the first 24 components for all scenarios. To each scenario i, $i=1,\ldots,N$, we assign a vector $\omega^i = (\omega^i_k)_{k=2}^7$ with $\omega^i_k \in \{-1,0,1\}$ for $k=2,\ldots,7$. It provides a unique description of the path in the ternary tree that corresponds to scenario i. In particular, set $\omega^i_k := -1$ if the values of the scenario i for $t = t_k+1,\ldots,t_{k+1}$ are realizations of the event with the verbal description "low load" for this time span.

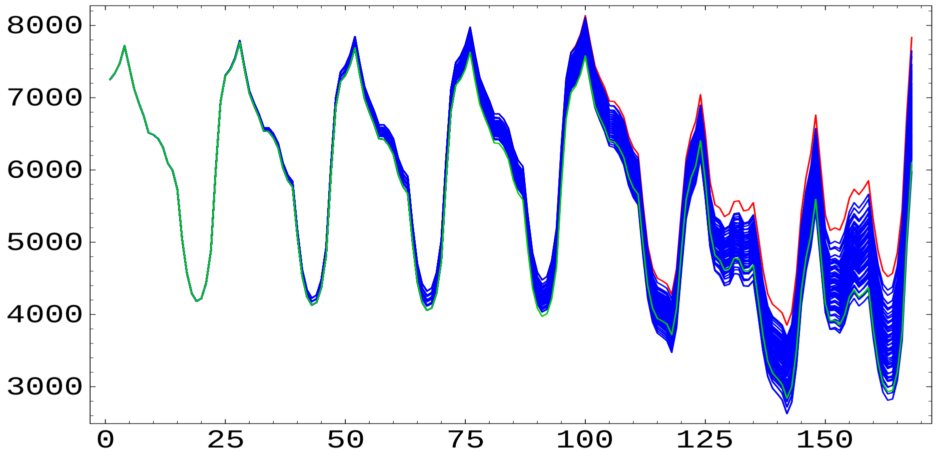

Figure 2: Ternary load scenario tree for one week

Accordingly, we set $\omega_k^i := 0$ ($\omega_k^i := 1$) to describe the event "medium load" ("high load") for $t = t_k + 1, \ldots, t_{k+1}$. Modifying the binary tree building formula from [10] we define the value of the scenario i at t as

$$d_t^i := \bar{d}_t + \sum_{j=1}^{k-1} \omega_j^i \frac{\sqrt{3}\bar{\sigma}_{t_{j+1}}}{2^{(8-j)/2}} + \omega_k^i \frac{\sqrt{3}\bar{\sigma}_{t_{k+1}}}{2^{(8-k)/2}} \frac{t - t_k}{t_{k+1} - t_k} \tag{20}$$

for $t = t_k + 1, \ldots, t_{k+1}$, $k = 1, \ldots, 6$. We let all scenarios have equal probabilities $N^{-1} = 3^{-6}$. (Alternative scenario probabilities might be computed from histograms of the simulated scenarios.)

The ternary scenario tree (20) has the following properties. First, for $t = 25, \ldots, 168$, the mean of the random vector represented by the ternary tree coincides with the empirical mean $\bar{d}_t$. Secondly, the symmetry of the load tree is consistent with the normality assumptions imposed on the time series model for the load process. Third, for $k = 1, \ldots, 6$, the events "low load" ("medium load", "high load") for $t = t_k + 1, \ldots, t_{k+1}$ are expressed in terms of scaled empirical standard deviations $\bar{\sigma}_{t_{k+1}}$. To model increasing load uncertainty, the variances $\mathrm{var}(d_t)$ of scenario values are strictly increasing with t. The variances at the branching points are

$$\mathrm{var}(d_{t_k}) = 2^{-6}\bar{\sigma}_{t_2}^2 + \cdots + 2^{-(8-k)}\bar{\sigma}_{t_k}^2, \quad k = 2, \ldots, 7.$$

Hence, in case $\bar{\sigma}_{t_k} \approx \bar{\sigma}$, $k = 2, \ldots, 7$, the variance in the final time period is $\mathrm{var}(d_{t_7}) \approx \bar{\sigma}^2(\frac{1}{2^6} + \cdots + \frac{1}{2}) \approx \bar{\sigma}^2$. Furthermore, it holds for the extremal scenario s with $\omega_k^s = 1$ for all k that

$$d_{t_7}^s = \bar{d}_{t_7} + \sqrt{3} \cdot 2^{-6/2}\bar{\sigma}_{t_2} + \cdots + \sqrt{3} \cdot 2^{-1/2}\bar{\sigma}_{t_7}\,.$$

Thus, unrealistic ("too large") load values are avoided. Finally, we note that the scenarios are linearly interpolated between the t_k's in order to save time required for computing $\bar{\sigma}_t^2$ for all $t = 25, \ldots, 168$.

The Figures 2 and 3 show the ternary load scenario tree $\{d_t^i\}_{t=1}^{168}$ and the corresponding mean shifted tree $\{d_t^i - \bar{d}_t\}_{t=1}^{168}$, respectively, where d_t^i is generated by the scheme (20), with $N = 3^6 = 729$ scenarios and branching points $t_k = 24k$, $k = 1, \ldots, 6$. This tree is

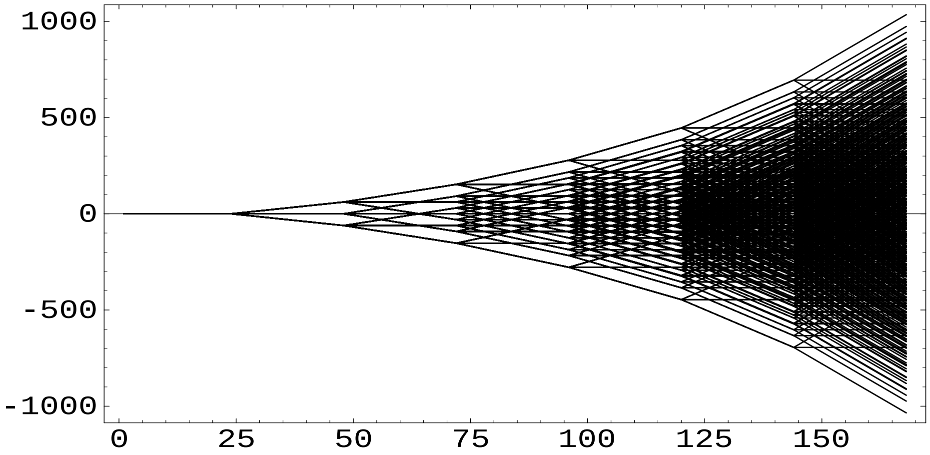

Figure 3: Mean shifted ternary load scenario tree for one week

Scenarios in the reduced tree	**Backward reduction of scenario sets** time [s]	ζ_1-distance abs.	rel.[%]	**Backward reduction of single scenarios** time [s]	ζ_1-distance abs.	rel.[%]	**Forward selection of scenarios** time [s]	ζ_1-distance abs.	rel.[%]
600	8	66.63	3.37	8	66.63	3.37	8149	66.64	3.36
500	8	118.34	5.99	8	118.34	5.99	7637	118.32	5.99
400	8	176.24	8.92	8	176.25	8.92	6146	170.48	8.63
300	8	260.66	13.19	8	262.17	13.3	4280	235.76	11.93
200	8	348.77	17.65	8	357.19	18.08	2319	331.34	16.76
100	8	502.90	25.45	8	505.70	25.59	721	483.94	24.49
81	8	546.74	27.67	8	559.89	28.33	498	530.04	26.84
50	8	645.07	32.64	8	664.34	33.62	212	628.31	31.80
27	8	759.88	38.45	8	782.79	39.61	73	749.09	37.91
10	8	989.12	50.05	8	1019.73	51.60	19	951.02	48.13
9	8	1019.90	51.61	8	1049.02	53.09	17	970.23	49.10
8	8	1045.78	52.92	8	1071.23	54.21	15	1010.92	51.16
7	8	1073.14	54.31	8	1122.04	56.78	13	1051.64	53.22
6	8	1107.82	56.06	8	1147.86	58.09	12	1097.48	55.54
5	8	1153.44	58.37	8	1189.47	60.19	11	1143.42	57.86
4	8	1218.29	61.65	8	1290.15	65.29	10	1201.11	60.78
3	8	1303.74	65.98	8	1360.97	68.87	9	1259.25	63.73
2	8	1506.35	76.23	8	1666.22	84.20	9	1618.16	81.89
1	8	1976.07	100.0	8	2027.32	102.59	8	1976.07	100.0

Table 1: Results of load scenario tree reduction

used as a (dense) initial scenario tree in order to test the performance of the reduction algorithms of Section 3. The test runs were performed on an HP 9000 (780/J280) Compute-Server with 180 MHz frequency and 768 MByte main memory under HP-UX 10.20. We compared the three algorithms *backward reduction of scenario sets* (Algorithm 3.6), *backward reduction of single scenarios* (variant of Algorithm 3.6) and *forward selection of scenarios* (Algorithm 3.7), where all of them were implemented in C. For computing the distances of the initial probability measure $P = 3^{-6} \sum_{i=1}^{3^6} \delta_{d^i}$ (i.e., given by the initial tree with identical scenario probabilities) and the reduced probability measure Q (reduced tree with optimal redistribution) we used the ζ_1- or Kantorovich metric and the explicit formula of Theorem 3.1. This choice of the metric is justified by Remark 2.2 when considering the stochastic power management model with fixed binary decisions and with electrical load appearing as stochastic right-hand side.

Table 1 displays computing times (in seconds) and (absolute and relative) ζ_1-distances of initial and reduced scenario trees for different numbers of scenarios. Here, the relative ζ_1-distance of P and Q is defined as the quotient

$$\frac{\zeta_1(P, Q)}{\zeta_1(P, \delta_{d^{i_*}})}$$

where $i_* \in \{1, \ldots, N\}$ denotes the first index obtained by forward selection (see also Example 3.4) and $\delta_{d^{i_*}}$ is the corresponding Dirac measure. Hence, the distance $\zeta_1(P, \delta_{d^{i_*}})$ corresponds to the best possible ζ_1-distance of P to one of its scenarios endowed with unit mass. The test runs have shown that the "distances" $c_1(d^i, d^j) = \|d^i - d^j\|$ ($\|\cdot\|$ denoting the Euclidean norm) for each pair (d^i, d^j) of scenarios of the initial load tree are computed within 6s CPU time. These "distances" are needed in all algorithms.

Table 1 shows that all algorithms work reasonably good, and that backward reduction algorithms are (much) faster than forward selection ones. Furthermore, besides the favorable simple algorithmic structure, backward reduction of single scenarios is also competitive to backward reduction of scenario sets with respect to computing times and accuracy (at least for reduced trees with scenario numbers ≥ 100). For reduced trees with small scenario numbers ≤ 50 forward selection performs fast and produces better trees than backward reduction. Table 1 also reflects the surprising effect that a reduction of the scenario tree by 50% of the original scenarios only implies a loss of about 10% of relative accuracy. Furthermore, it is possible to determine a subtree containing less than 2% of the original number of scenarios that still carries about 50% of relative accuracy. The Figures 4, 5 and 6 show (mean) shifted reduced load trees with 10 and 50 scenarios that are obtained by all three algorithms. The figures display the scenarios within the extremal paths of the initial scenario tree indicated by dashed lines and with grey levels proportional to scenario probabilities.

The different performances of backward reduction and forward selection with respect to running times are due to the following algorithmic detail. The inner minima and optimal indices in (14) and (16) are efficiently evaluated by the preprocessing step that sorts the "distances" $c(\cdot, \cdot)$ from each scenario to the remaining scenarios and stores the corresponding permutation vectors of scenario indices (requiring 1s CPU-time). After selecting the next scenario that enters the index set J the optimal indices of the inner

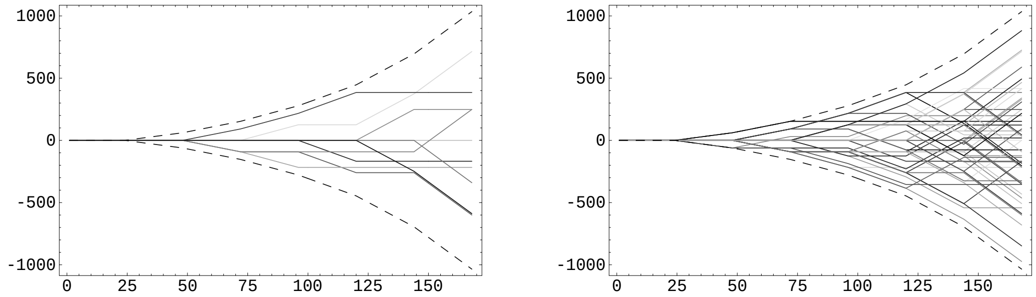

Figure 4: Backward reduction of scenario sets: Reduced trees with 10 and 50 scenarios

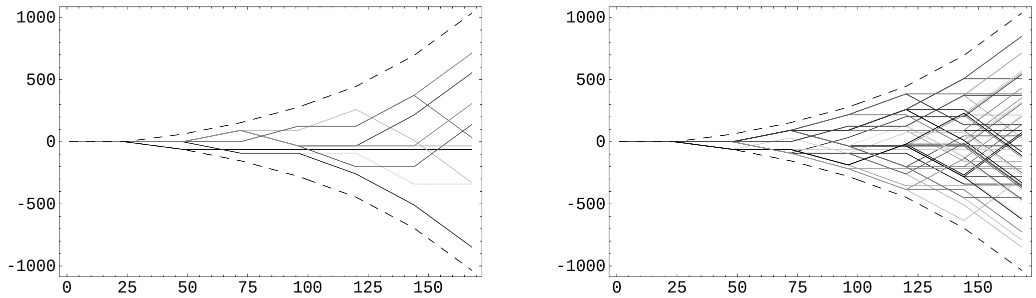

Figure 5: Backward reduction of single scenarios: Reduced trees with 10 and 50 scenarios

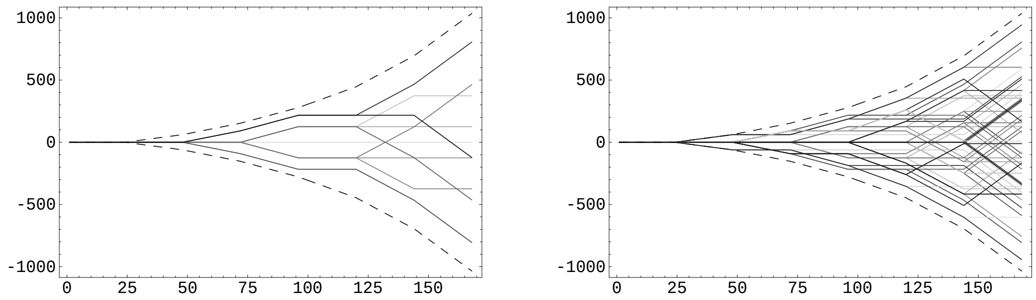

Figure 6: Forward selection of scenarios: Reduced trees with 10 and 50 scenarios

minimization are updated by shifting pointers along the permutation vectors. Then the outer minimization for selecting the next scenario for J according to (14) and (16) can be easily performed. This explains the identical running times for both backward reduction algorithms for different reduced trees. The algorithm for fast evaluating the inner minima was adapted for forward selection methods. However, the computing times indicate that the adaptation has not been successful since updating the pointers to the permutation vectors is more costly.

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