PDAEs and Further Mixed Systems as Abstract Differential Algebraic Systems

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Abstract

Abstract differential algebraic systems (ADASs), i.e., differential algebraic systems with operators acting in real Hilbert spaces are introduced for a systematical treatment of coupled systems of PDEs, DAEs and integral equations. Using the finite-dimensional decoupling theory for DAEs as motivation, this paper will examine what one appropriate analogue is for infinite-dimensional systems. This leads to an index definition for ADASs. Thereby, instead of the inherent regular ODE one obtains an explicit (abstract) differential equation. In particular, when discussing PDAEs, the inherent regular differential equation is actually a parabolic PDE. The decoupling procedure provides, additionally, appropriate initial and boundary conditions for unique solvability of the coupled systems. The concept to handle ADASs is explained in different case studies.

Keywords: differential-algebraic equations; partial differential equations, integral equations, partial differential algebraic equations, implicit differential equation, coupled systems, index

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1 Introduction

Nowadays, we notice a remarkable interest in partial differential algebraic equations (PDAEs), which consist of coupled partial differential equations (PDEs) and standard differential algebraic equations (DAEs) (e.g. [CM99], [Lan00], [LSEL99], [GR99], [Sim00], [MB00]). Furthermore, there is also an actual interest in coupling PDEs and integral equations (e.g. [Bru00]). To study all these problems systematically we are led to differential algebraic systems

$$A(x,t)(d(x,t))_t + b(x,t) = 0 \tag{1.1}$$

in an abstract setting. The unknown function $x(t)$ is now a path in an appropriate (infinite-dimensional) Hilbert space. The operator $A(x,t)$ has a somehow uniform nontrivial nullspace.

At this place, let us mention the considerable work done in studying (abstract) descriptor systems (e.g. [Kur93])

$$Ax_t + B(t)x = C(t)u(t), \tag{1.2}$$

where the coefficient operators $A, B(t)$ and $C(t)$ are bounded maps acting in real Hilbert spaces.
When dealing with bounded maps, the setting, in particular all function spaces including certain boundary conditions, should be given at the beginning. However, formulating appropriate boundary and initial conditions is one of the tasks to be solved during the modelling procedure.
In the finite-dimensional case, each solution of a linear DAE is shown to consist of a solution of the inherent regular ordinary differential equation (ODE) and certain packaging material, which is difficult to describe. In particular, if the DAE

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t) \tag{1.3}$$

has index $\mu = 2$, it can be decoupled (cf. [BM00]) into

$$\begin{aligned} x(t) \quad = \quad & D(t)^{-}u(t) + (P_0(t)Q_1(t) + Q_0(t)P_1(t))G_2(t)^{-1}q(t) \\ & + (D(t)Q_1(t)G_2(t)^{-1}q(t))', \end{aligned} \tag{1.4}$$

where the function $u(t)$ is a solution of the inherent regular ODE

$$\begin{aligned} & u'(t) - (DP_1D^{-})'(t)u(t) + (DP_1G_2^{-1}BD^{-})(t)u(t) \\ = \quad & (DP_1G_2^{-1}q)(t) + (DP_1D^{-})'(t)(DQ_1G_2^{-1}q)(t), \end{aligned} \tag{1.5}$$

which satisfies $u(t) \in \operatorname{im} D(t)P_1(t)$. For the matrix functions used in these expressions we refer to Section 2 below and [BM00].
In our case studies concerning a setting in infinite-dimensional spaces, we apply analogous decoupling procedures. Thereby, instead of the inherent regular ODE we obtain an explicit (abstract) differential equation. In particular, when discussing PDAEs, the inherent regular differential equation is actually a parabolic PDE.
To obtain uniquely solvable abstract differential algebraic systems we have to make sure that the inherent abstract regular differential equation is uniquely solvable. This will be realized by equipping the problem with appropriate initial and boundary conditions. Notice that, in the finite-dimensional case, just initial conditions for (1.5) have to be given. Additionally, in the infinite-dimensional case, when constructing the terms for describing the packaging material (cf. (1.4)), it may happen that we have to put certain homogeneous boundary conditions into the related function spaces. A clear answer to what is actually needed is obtained during the decoupling procedure.
The paper is organized as follows. In Section 2, basic material on linear operators in Hilbert spaces as well as the basic notions concerning abstract DAEs are collected. For more transparency, Section 3 repeats the semi-explicit finite-dimensional case. Section 4 is devoted to classical PDEs. Here, the physical laws implying the wave equation and those leading to the heat equation are treated in their original form as abstract DAEs. It turns out that the classical PDEs represent nothing else than the inherent regular differential equation of these abstract DAEs. In Section 5 and 6, systems with components of different type (a PDE combined with a Fredholm integral equation and a PDE combined

with a DAE) are considered. Systems of the last type are of interest when coupling circuit and device simulation. Finally, Section 7 deals with different linear PDAEs which are under discussion in current literature.

2 Fundamentals

In this section we collect the necessary background (cf. [Mär00]). Consider the abstract differential algebraic system (ADAS)

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{2.1}$$

where $A(t) : Z \to Y$, $B(t) : X \to Y$, $D(t) : X \to Z$ are linear operators acting in the real Hilbert spaces X, Y, Z. Before starting the investigation of equation (2.1) we collect some basic facts concerning linear operators in Hilbert spaces (e.g. [Kat95], [AG81]).

A linear operator $\mathfrak{A} : H_1 \to H_2$ acting in the Hilbert spaces H_1 and H_2 is said to be

(i) densely defined if its definition domain $\mathcal{D}_{\mathfrak{A}}$ is dense in H_1,

(ii) normally solvable if its range $\operatorname{im}\mathfrak{A}$ is closed in H_2,

(iii) densely solvable if its range $\operatorname{im}\mathfrak{A}$ is dense in H_2.

For a given set $M \subseteq H$ of a Hilbert space H, we denote its closure in H by $\operatorname{cl} M$. A subspace L of the Hilbert space H is a closed linear manifold, i.e., $\operatorname{cl} L = L \subseteq H$.

$L_b(H_1, H_2)$ denotes the set of the bounded linear maps defined on H_1. Recall that, for each bounded linear map defined on a subset of H_1, there is a bounded linear extension defined on H_1. We assume this extension to be realized. Denote further $L_b(H) = L_b(H, H)$.

For $\mathfrak{A} \in L_b(H_1, H_2)$, the kernel $\ker\mathfrak{A} = \{z \in H_1 : \mathfrak{A}z = 0\}$ is a closed linear manifold, which is a subspace. This justifies to call $\ker\mathfrak{A}$ the nullspace of $\mathfrak{A}$.

Each idempotent map $\mathcal{P} \in L_b(H)$ is said to be a projector. Due to $\mathcal{P}^2 = \mathcal{P}$ it holds that $\ker(I-\mathcal{P}) = \operatorname{im}\mathcal{P}$, thus $(I-\mathcal{P})^2 = I-\mathcal{P} \in L_b(H)$, $\ker\mathcal{P} \oplus \operatorname{im}\mathcal{P} = H$, and both $\ker\mathcal{P}$ and $\operatorname{im}\mathcal{P}$ are subspaces.

For a densely defined map $\mathfrak{A} : \mathcal{D}_{\mathfrak{A}} \subseteq H_1 \to H_2$ with closed range $\operatorname{im}\mathfrak{A} \subseteq H_2$, an outer or generalized inverse $\mathfrak{A}^-$ is defined to be a linear map $\mathfrak{A}^- : H_2 \to H_1$ such that $\mathfrak{A}^-\mathfrak{A}\mathfrak{A}^- = \mathfrak{A}^-$ is satisfied. The generalized inverse is said to be reflexive if $\mathfrak{A}\mathfrak{A}^-\mathfrak{A} = \mathfrak{A}$ is fulfilled additionally.

As a special case, the map

$$\begin{aligned} \mathfrak{A} &: \quad L_2(\Omega) \to L_2(\Omega), \quad \mathcal{D}_{\mathfrak{A}} = H^1(\Omega), \\ \mathfrak{A}u &:= \quad -\Delta u + cu, \quad u \in \mathcal{D}_{\mathfrak{A}}, \end{aligned}$$

is considered below. Here, Δ denotes the energetic extension of the Laplace operator onto $H^1(\Omega)$. Provided $c \geq 0$, the Poisson equation completed by homogeneous Dirichlet conditions has a uniquely determined weak solution, i.e.,

for each given $f \in L_2(\Omega)$, there is a unique $u \in H_0^1(\Omega)$ such that $\mathfrak{A}u = f$ is satisfied. If $\mathfrak{A}^-$ denotes the corresponding solution operator defined on $\operatorname{im}\mathfrak{A} = L_2(\Omega)$, it holds that $\operatorname{im}\mathfrak{A}^- = H_0^1(\Omega)$, $\mathfrak{A}\mathfrak{A}^- = I$, $\mathfrak{A}\mathfrak{A}^-\mathfrak{A} = \mathfrak{A}$, $\mathfrak{A}^-\mathfrak{A}\mathfrak{A}^- = \mathfrak{A}^-$. However, the relation $\mathfrak{A}^-\mathfrak{A}u = u$ holds only if $u \in H_0^1(\Omega)$.
Next, let a normally solvable operator $\mathfrak{A} \in L_b(H_1, H_2)$ and projectors $\mathcal{Q} \in L_b(H_1)$, $\mathcal{R} \in L_b(H_2)$ be given such that $\operatorname{im}\mathcal{Q} = \ker\mathfrak{A}$, $\operatorname{im}\mathcal{R} = \operatorname{im}\mathfrak{A}$. Now, a reflexive generalized inverse $\mathfrak{A}^-$ can be constructed so that $\mathfrak{A}^- \in L_b(H_2, H_1)$, $\mathfrak{A}^-\mathfrak{A}\mathfrak{A}^- = \mathfrak{A}^-$, $\mathfrak{A}\mathfrak{A}^-\mathfrak{A} = \mathfrak{A}$, $\mathfrak{A}\mathfrak{A}^- = \mathcal{R}$, $\mathfrak{A}^-\mathfrak{A} = I - \mathcal{Q} =: \mathcal{P}$ becomes true. Namely, taking into account the decompositions $H_1 = \ker\mathfrak{A} \oplus \operatorname{im}\mathcal{P}$, $H_2 = \ker\mathcal{R} \oplus \operatorname{im}\mathfrak{A}$, we know $\mathfrak{A}$ to be a bijection between the subspaces $\operatorname{im}\mathcal{P}$ and $\operatorname{im}\mathfrak{A}$. Define $\mathfrak{A}^-$ on $\operatorname{im}\mathfrak{A} \subseteq H_2$ to be the corresponding inverse. For arbitrary $z \in H_2$ we set $\mathfrak{A}^-z = \mathfrak{A}^-(\mathcal{R}z + (I - \mathcal{R})z) = \mathfrak{A}^-\mathcal{R}z$. Obviously, $\mathfrak{A}^-$ is bounded, and the relations $\mathfrak{A}^-\mathfrak{A}x = \mathfrak{A}^-\mathfrak{A}\mathcal{P}x = \mathcal{P}x$, $\mathfrak{A}\mathfrak{A}^-z = \mathfrak{A}\mathfrak{A}^-\mathcal{R}z = \mathcal{R}z$ for all $x \in H_1$, $z \in H_2$ are satisfied, hence the above four properties are satisfied.

Now, we want to consider equation (2.1). For all $t \in \mathcal{I}$ let $A(t)$ and $D(t)$ be bounded and normally solvable. Moreover, let $A(\cdot)$ and $D(\cdot)$ depend continuously (in the norm sense) on t. As bounded maps, $A(t)$ and $D(t)$ have nullspaces $\ker A(t)$ and $\ker D(t)$, respectively, which are closed linear manifolds, i.e., subspaces. Due to the normal solvability, $\operatorname{im} A(t)$ and $\operatorname{im} D(t)$ are subspaces, too. For all $t \in \mathcal{I}$, let the operator $B(t)$ be defined on a dense subset $\mathcal{D}_B \subseteq X$, and let $B(\cdot)x \in C(\mathcal{I}, Y)$ for all $x \in \mathcal{D}_B$. A continuous path $x \in C(\mathcal{I}, X)$ with $x(t) \in \mathcal{D}_B$, $t \in \mathcal{I}$, is called a solution of (2.1) if $Dx \in C^1(\mathcal{I}, Z)$ and equation (2.1) is satisfied pointwise.

Definition 2.1 *The leading term of (2.1) is properly stated if the operators $A(t)$ and $D(t)$, $t \in \mathcal{I}$, are well matched in the following sense:*

(i) $\operatorname{im} A(t)D(t) = \operatorname{im} A(t)$, $\ker A(t)D(t) = \ker D(t)$, $\ker A(t) \oplus \operatorname{im} D(t) = Z$, $\forall t \in \mathcal{I}$.

(ii) The projector $R(t) \in L_b(Z)$ realizing this decomposition of Z (i.e., $R(t)^2 = R(t)$, $\operatorname{im} D(t) = \operatorname{im} R(t)$, $\ker R(t) = \ker A(t)$) depends continuously differentially on t.

Remark 2.2 *Since $R(t)$ depends smoothly on t, the subspaces* $\operatorname{im} D(t)$ *are isomorphic for different values of t. For the same reason, the nullspaces* $\ker A(t)$ *are also isomorphic.*

Denote by $Q_0(t) \in L_b(X)$ an arbitrary projector such that $\operatorname{im} Q_0(t) = N_0(t) := \ker A(t)D(t) = \ker D(t)$, $t \in \mathcal{I}$. $Q_0(t)$ is assumed to be continuous in t. Let $P_0(t) = I - Q_0(t)$. $D(t)^- : Z \to X$ denotes the reflexive generalized inverse of $D(t)$ such that $D(t)D(t)^- = R(t)$, $D(t)^-D(t) = P_0(t)$, $t \in \mathcal{I}$.
Next we introduce further linear maps, linear manifolds and subspaces for all $t \in \mathcal{I}$:

$$G_0(t) = A(t)D(t), \quad B_0(t) = B(t),$$

for $i \geq 0$:

$$\begin{aligned}
Q_i(t) &\in L_b(X),\ Q_i(t)^2 = Q_i(t),\ \operatorname{im} Q_i(t) = N_i(t),\ P_i(t) := I - Q_i(t),\\
N_i(t) &= \operatorname{cl}\,\ker G_i(t),\\
W_i(t) &\in L_b(Y),\ W_i(t)^2 = W_i(t),\ \ker W_i(t) = \operatorname{cl}\operatorname{im} G_i(t),\\
S_i(t) &= \ker(W_i(t)B_i(t)) = \ker(W_i(t)B(t)),\\
G_{i+1}(t) &= G_i(t) + B_i(t)Q_i(t),\\
B_{i+1}(t) &= B_i(t)P_i(t) - C_{i+1}(t)P_0(t)P_1(t)\cdots P_i(t),\\
C_{i+1}(t) &= G_{i+1}(t)D(t)^-(D(t)P_1(t)\cdots P_{i+1}(t)D(t)^-)'D(t).
\end{aligned}$$

By construction, the operators $G_i(t)$ are at least densely defined in X. As operator products (in particular those with projectors), they are often defined on larger regions by trivial reasons. We will always use their maximal trivial extensions.
In particular, we have

$$\begin{aligned}
G_1 &= AD + BQ_0,\\
G_2 &= AD + BQ_0 + BP_0Q_1 - C_1P_0Q_1,\ C_1 = G_1D^-(DP_1D^-)'D.
\end{aligned}$$

Obviously, we may form these maps and subspaces provided that the derivatives used for $C_i(t)$ do exist. This is closely related to a further smooth decomposition of the smooth subspace $\operatorname{im} D(t) = \operatorname{im} R(t)$ (cf. [BM00], [Mär01] for the finite dimensional case). In the following case studies we will always have constant maps D and constant projectors P_i such that the terms C_i disappear.
Using the introduced operators and projectors we may generalize the tractability index to abstract DAEs as follows.

Definition 2.3 *Equation (2.1) with a properly stated leading term is said to be an abstract DAE with index μ if $\dim(\operatorname{im} W_i(t)) > 0$, $W_i(t)$ depends continuously on t, $i = 0, \ldots, \mu - 1$, and $G_\mu(t)$ is injective and densely solvable.*

If $B(t) : X \to Y$ is a bounded map that is continuous with respect to t in the norm sense, then the assertions obtained for the finite dimensional case (cf. [BM00], [Mär00]) can be modified in a straightforward way to hold true for ADAS. The more challenging problem is an unbounded map $B(t)$.

3 Semi-Explicit Finite-Dimensional Constant Coefficient Case

For more transparency, we describe the new defined spaces and operators for the special case of standard semi-explicit DAEs

$$\begin{aligned}
x_1'(t) + B_{11}x_1(t) + B_{12}x_2(t) &= q_1(t),\\
B_{21}x_1(t) + B_{22}x_2(t) &= q_2(t)
\end{aligned} \tag{3.1}$$

with $B_{21}B_{12}$ nonsingular. Put $B_{22} = \mathfrak{B} - \lambda I$. Choosing $X = Y = \mathbb{R}^{n_1+n_2}$, $Z = \mathbb{R}^{n_1}$, $A = \begin{pmatrix} I \\ 0 \end{pmatrix}$, $D = \begin{pmatrix} I & 0 \end{pmatrix}$, $D^- = \begin{pmatrix} I \\ 0 \end{pmatrix}$ we may rewrite (3.1) as (2.1) and apply the index criteria. We have

$$\begin{aligned} G_0 &= AD = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad N_0 = 0 \times \mathbb{R}^{n_2}, \\ W_0 &= \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad S_0 \cap N_0 = 0 \times \ker(\mathfrak{B} - \lambda I), \quad G_1 = \begin{pmatrix} I & B_{12} \\ 0 & \mathfrak{B} - \lambda I \end{pmatrix}. \end{aligned}$$

The system (3.1) has index $\mu = 1$ if and only if λ does not belong to the spectrum $\sigma(\mathfrak{B})$ of $\mathfrak{B}$, i.e., if $\mathfrak{B} - \lambda I$ is nonsingular. Clearly, provided that λ is an eigenvalue of $\mathfrak{B}$, G_1 is no longer injective. Then we compute

$$\begin{aligned} N_1 &= \{z \in \mathbb{R}^m : z_1 = -B_{12}z_2, \ z_2 \in \ker(\mathfrak{B} - \lambda I)\}, \\ S_1 &= \{z \in \mathbb{R}^m : B_{21}z_1 \in \operatorname{im}(\mathfrak{B} - \lambda I)\}, \\ N_1 \cap S_1 &= \{z \in \mathbb{R}^m : z_1 = -B_{12}z_2, \quad z_2 \in \ker(\mathfrak{B} - \lambda I), \\ &\qquad - B_{21}B_{12}z_2 \in \operatorname{im}(\mathfrak{B} - \lambda I)\}. \end{aligned}$$

Since $B_{21}B_{12}$ is nonsingular, we know that $N_1 \cap S_1 = 0$ if and only if

$$\ker(\mathfrak{B} - \lambda I) \cap \operatorname{im}(B_{21}B_{12})^{-1}(\mathfrak{B} - \lambda I) = 0.$$

In the finite-dimensional case we always have $\dim N_2 = \dim(N_1 \cap S_1)$ (cf. [BM00]).

4 Classical Partial Differential Equations

In this section we treat abstract DAEs which are the origin of classical partial differential equations.

4.1 Wave equation

The flow in an ideal gas is determined by three laws (cf. e.g. [Bra92]). As usually, we denote the velocity by v, the density by ρ, and the pressure by p.

1. Continuity equation
$$\frac{\partial \rho}{\partial t} = -\rho_0 \operatorname{div} v. \tag{4.1}$$
Due to the conservation of mass, the variation of the mass in a (sub)volume V is equal to the flow over the surface, i.e., $\int_{\partial V} \rho v \cdot \mathfrak{n} \, dO$. The Gaussian integral theorem implies the above equation, where ρ is approximated by the fixed density ρ_0.

2. Newton's Theorem
$$\rho_0 \frac{\partial v}{\partial t} = -\operatorname{grad} p. \tag{4.2}$$

The pressure gradient induces a force field causing the acceleration of particles.

3. State equation
$$p = c^2 \rho. \tag{4.3}$$
In ideal gases the pressure is proportional to the density under constant temperature.

From the three laws above the wave equation

$$\frac{\partial^2}{\partial t^2} p = c^2 \frac{\partial^2 \rho}{\partial t^2} = -c^2 \frac{\partial}{\partial t} \rho_0 \operatorname{div} v = c^2 \operatorname{div} \rho_0 \frac{\partial v}{\partial t} = c^2 \operatorname{div} \operatorname{grad} p = c^2 \Delta p$$

is deduced. Considering the equations (4.1) - (4.3) to form a system of equations we know that this is an abstract DAE of index 1. Namely, choosing

$$x(t) = \begin{pmatrix} \rho(t,\cdot) \\ v(t,\cdot) \\ p(t,\cdot) \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & \rho_0 I \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \end{pmatrix},$$

$$D^- = \begin{pmatrix} 1 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \rho_0 \operatorname{div} & 0 \\ 0 & 0 & \operatorname{grad} \\ -c^2 & 0 & 1 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we find, independently of the special function spaces,

$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho_0 I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_1 = G_0 + BQ_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho_0 I & \operatorname{grad} \\ 0 & 0 & 1 \end{pmatrix}.$$

Obviously, G_1 is always nonsingular and its inverse is given by

$$G_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho_0} I & -\frac{1}{\rho_0} \operatorname{grad} \\ 0 & 0 & 1 \end{pmatrix}.$$

Correspondingly to the decoupling procedure for the finite dimensional case (see [BM00, Mär01]) we compute

$$DG_1^{-1} BD = \begin{pmatrix} 0 & \rho_0 \operatorname{div} \\ \frac{c^2}{\rho_0} \operatorname{grad} & 0 \end{pmatrix}, \quad Dx = \begin{pmatrix} \rho \\ v \end{pmatrix}.$$

Hence, not surprisingly, the inherent regular differential equation reads

$$\begin{aligned} \rho' + \rho_0 \operatorname{div} v &= 0 \\ v' + \frac{c^2}{\rho_0} \operatorname{grad} \rho &= 0 \end{aligned} \tag{4.4}$$

while

$$\begin{pmatrix}0\\0\\p\end{pmatrix} = Q_0 x = Q_0 G_1^{-1} B D^- D x = \begin{pmatrix}0\\0\\c^2\rho\end{pmatrix}$$

represents the constraint. The initial condition for the resulting DAE (2.1) consists of the initial condition for the inherent regular equation (4.4). Boundary conditions for (4.4) should be incorporated by specifying the function space X and the definition domain $\mathcal{D}_B \subset X$.

4.2 Heat equation

Let $T(x,t)$ be the distribution of temperature in a body. Then the heat flow reads

$$F = -\kappa \operatorname{grad} T, \tag{4.5}$$

where the diffusion constant κ represents a material constant. The change of energy in a volume element is composed due to the conservation of energy, of the heat flow over the surface and the applied heat Q. Now it follows that

$$\begin{aligned}\frac{\partial E}{\partial t} &= -\operatorname{div} F + Q\\ &= \operatorname{div} \kappa \operatorname{grad} T + Q\\ &= \kappa \Delta u + Q,\end{aligned} \tag{4.6}$$

if κ is assumed to be constant. With the specific heat $a = \partial E/\partial T$ (a further material constant) we finally obtain the heat equation

$$\frac{\partial T}{\partial t} = \frac{\kappa}{a}\Delta T + \frac{1}{a} Q. \tag{4.7}$$

On the other hand, we may consider the original equations

$$\begin{aligned} a\frac{\partial T}{\partial t} &= -\operatorname{div} F + Q,\\ F &= -\kappa \operatorname{grad} T\end{aligned} \tag{4.8}$$

to form an abstract index-1 DAE (2.1) for $x(t) = \begin{pmatrix}T(t,\cdot)\\F(t,\cdot)\end{pmatrix}$ with

$$\begin{aligned} A &= \begin{pmatrix}a\\0\end{pmatrix}, \quad D = \begin{pmatrix}1 & 0\end{pmatrix}, \quad D^- = \begin{pmatrix}1\\0\end{pmatrix}, \quad Q_0 = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix},\\ B &= \begin{pmatrix}0 & \operatorname{div}\\ \kappa \operatorname{grad} & 1\end{pmatrix}, \quad G_1 = \begin{pmatrix}a & \operatorname{div}\\0 & 1\end{pmatrix}.\end{aligned}$$

Also in this case, G_1 is always nonsingular. Following again the decoupling procedure for the description of the inherent regular differential equation and the constraints, we obtain

$$G_1^{-1} = \begin{pmatrix}\frac{1}{a} & -\frac{1}{a}\operatorname{div}\\ 0 & 1\end{pmatrix}, \quad DG_1^{-1}BD^- = -\frac{\kappa}{a}\operatorname{div}\operatorname{grad}, \quad Dx = T,$$

which implies the inherent regular differential equation to be

$$T' - \frac{\kappa}{a}\,\mathrm{div}\,\mathrm{grad}\,T = \frac{1}{a}Q$$

and

$$\begin{pmatrix}0\\F\end{pmatrix} = Q_0x = -Q_0G_1^{-1}BD^-Dx + Q_0G_1^{-1}\begin{pmatrix}Q\\0\end{pmatrix} = \begin{pmatrix}0\\-\kappa\,\mathrm{grad}\,T\end{pmatrix}$$

to represent the constraint. Again, the function spaces and incorporated boundary conditions as well as an initial condition for the inherent regular differential equation should enable us to obtain unique solvability.

5 A Coupled System of a PDE and Fredholm Integral Equations

Given a linear Fredholm integral operator $F : L_2(\Omega)^s \to L_2(\Omega)^s$, $\|F\| < 1$, a linear differential operator

$$K : C_0^2(\Omega) \to L_2(\Omega), \quad Kw := -\Delta w + cw \text{ for } w \in C_0^2(\Omega), \quad c \geq 0,$$

with linear bounded coupling operators $L : L_2(\Omega)^s \to L_2(\Omega)$, $E : L_2(\Omega) \to L_2(\Omega)^s$, the system to be considered is ([Bru00])

$$\begin{array}{rcl} x_1'(t) + Kx_1(t) + Lx_2(t) &=& q_1(t), \\ Ex_1(t) + (I-F)x_2(t) &=& q_2(t), \quad t \in [0,1]. \end{array} \tag{5.1}$$

Using the corresponding matrix representations for A, D, B, with $X := L_2(\Omega) \times L_2(\Omega)^s$, $Y := X$, $Z := L_2(\Omega)$, we rewrite (5.1) in the form (2.1). Namely, we have

$$A = \begin{pmatrix}1\\0\end{pmatrix}, \quad D = \begin{pmatrix}1 & 0\end{pmatrix}, \quad P_0 = AD = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \quad D^- = \begin{pmatrix}1\\0\end{pmatrix}, \quad R = 1$$

and $B = \begin{pmatrix}K & L\\E & I-F\end{pmatrix}$. By construction B is defined on $\mathcal{D}_B := C_0^2(\Omega) \times L_2(\Omega)^s$. Clearly, it holds that $N_0 = 0 \times L_2(\Omega)^s$. Choosing

$$Q_0 = \begin{pmatrix}0 & 0\\0 & I\end{pmatrix}, \quad W_0 = \begin{pmatrix}0 & 0\\0 & I\end{pmatrix}$$

we obtain

$$W_0B = \begin{pmatrix}0 & 0\\E & I-F\end{pmatrix}, \quad G_1 = \begin{pmatrix}1 & L\\0 & I-F\end{pmatrix}.$$

defined on X (as trivial extensions of bounded maps). G_1 is a bijection such that this abstract DAE has the index 1. This implies

$$G_1^{-1} = \begin{pmatrix}1 & -L(I-F)^{-1}\\0 & (I-F)^{-1}\end{pmatrix}, \quad DG_1^{-1}BD^- = K - L(I-F)^{-1}E.$$

Each solution of the DAE is given by the expression

$$\begin{aligned} x(t) &= D^- u(t) + Q_0 x(t), \\ Q_0 x(t) &= Q_0 G_1^{-1} q(t) - Q_0 G_1^{-1} B D^- u(t), \end{aligned}$$

where $u(t)$ is a solution of the abstract regular differential equation

$$u'(t) + DG_1^{-1} BD^- u(t) = DG_1^{-1} q(t),$$

which corresponds to

$$x_1'(t) + (K - L(I-F)^{-1}E)x_1(t) = q_1(t) - L(I-F)^{-1}Eq_2(t). \tag{5.2}$$

Obviously, one has to state an appropriate initial condition for (5.2), i.e., $x_1(0) = x_1^0 \in C_0^2(\Omega)$. Better solvability will be obtained by defining B (resp. K) on $H_0^1(\Omega)$ instead of $C_0^2(\Omega)$ and using weak solutions.

6 A PDE and a DAE Coupled by a Restriction Operator

In this section we consider systems where PDE and DAE model equations are coupled via certain boundary conditions. Such systems arise, for instance, by coupling device and circuit simulation. Consider the system

$$\tilde{u}_t(y,t) - \tilde{u}_{yy}(y,t) + c\tilde{u}(y,t) = f(y,t),\ y \in \Omega,\ t \geq 0,\ \tilde{u}|_{\partial\Omega} = 0, \tag{6.1}$$
$$\tilde{A}(t)(\tilde{D}(t)\tilde{x}(t))' + \tilde{B}(t)\tilde{x}(t) + r(t)\tilde{u}(\cdot,t) = g(t), \quad t \geq 0. \tag{6.2}$$

Assume the linear restriction map $r(t) : H^1(\Omega) \to \mathbb{R}^m$ (usually describing boundary conditions for $\tilde{u}$) to be bounded and to depend continuously on t. Rewrite the system (6.1)-6.2) as an abstract DAE for $x(t) = \begin{pmatrix} \tilde{u}(\cdot,t) \\ \tilde{x}(t) \end{pmatrix}$ and choose $X = Y = L_2(\Omega) \times \mathbb{R}^m$, $Z = L_2(\Omega) \times \mathbb{R}^n$,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{D} \end{pmatrix}, \quad AD = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A}\tilde{D} \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{Q}_0 \end{pmatrix},$$
$$D^- = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{D}^- \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{R} \end{pmatrix}, \quad B = \begin{pmatrix} -\Delta + c & 0 \\ r & \tilde{B} \end{pmatrix},$$

where $\mathcal{D}_B := H_0^1(\Omega) \times \mathbb{R}^m$. $G_1 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{G}_1 \end{pmatrix}$ is defined on X, $N_1 = 0 \times \tilde{N}_1$, $\operatorname{im} G_1 = L_2(\Omega) \times \operatorname{im} \tilde{G}_1$, $S_1 = \{x \in X : rx_1 + \tilde{B}\tilde{P}_0 x_2 \in \operatorname{im} \tilde{G}_1\}$. Supposing that the operator $r(t)$ maps into $\operatorname{im} \tilde{G}_1(t)$ it holds that

$$S_1 = \{x \in X : \tilde{B}\tilde{P}_0 x_2 \in \operatorname{im} \tilde{G}_1\} = L_2(\Omega) \times \tilde{S}_1.$$

Then, $Q_1 = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{Q}_1 \end{pmatrix}$ is the projector onto N_1 along S_1, $G_2 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{G}_2 \end{pmatrix}$.

In the general case the projector Q_1 onto N_1 along S_1 is more difficult to construct. Obviously, we have $N_1 \cap S_1 = 0 \times (\tilde{N}_1 \cap \tilde{S}_1)$. G_2 is a bijection if $\tilde{G}_2$ is so. Consequently, the coupled system (6.1)-(6.2) interpreted as an abstract DAE has the same index as the DAE (6.2).

7 Linear PDAEs

Linear systems of the form

$$\mathcal{A}u_t(t,y) + \mathcal{B}u_{yy}(t,y) + \mathcal{C}u(t,y) = q(t,y),\ y \in \Omega,\ t \in \mathcal{I} \tag{7.1}$$

where at least one of the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C} \in L(\mathbb{R}^n)$ is singular, are discussed as partial differential algebraic equations (PDAEs) or constraint PDEs e.g. in [CM99] and [LSEL99]. These systems may combine parabolic, elliptic and ordinary differential equations, but also algebraic equations.
In [CM99] and [LSEL99], Laplace transformations and Fourier analysis are applied to obtain sequences of (ordinary) DAEs, which are investigated in detail then. In [MB00], a differentiation index for general PDAEs of first order (containing first order derivatives) is proposed.
Here we choose a completely different approach. We treat the PDAE as an abstract DAE. The resulting index of the abstract DAE is, roughly speaking, the uniform differential time index in [LSEL99]. However, there are various index notions and all of them are based on special conditions that would require extensive explanations.
It should be stressed that the index notion given here, that is, the index of the PDAE understood as an abstract DAE, is not restricted to time-invariant linear systems of the form (7.1). Additionally, it does not matter at all whether the region Ω is one-dimensional or not.
We are exclusively interested in systems (7.1) with a singular matrix $\mathcal{A}$. We interprete the unknown function to be an abstract function in time $x(t) := u(t,.) \in X$, where we have typically $X = L_2(\Omega)^s$. Then we describe (7.1) as an ADAS

$$A(Dx(t))' + Bx(t) = q(t). \tag{7.2}$$

Thereby, roughly speaking, the leading term $A(Dx(t))'$ is determined by the leading term $\mathcal{A}u_t(t,.)$ of (7.1) in such a way that $\mathcal{A} = AD$ and A, D are well matched. The second term in (7.2) is determined by the second and third term of (7.1) so that $B = \mathcal{B} \otimes \Delta + \mathcal{C}$, where Δ denotes the Laplacian.
Having obtained the abstract DAE, we may check the index and try to find the inherent regular differential equation as we used to do in the finite-dimensional case (e.g. [BM00]). In this way we realize which boundary and initial value conditions have to be given.
Sometimes it is reasonable to shift the variable x in (7.2) by a given function $\xi \in C(\mathcal{I}, X)$, $D\xi \in C^1(\mathcal{I}, Z)$, $\xi(t) \in \mathcal{D}_B$, $t \in \mathcal{I}$. Then, one solves the equation resulting for $\tilde{x} = x - \xi$, namely

$$A(D\tilde{x}(t))' + B\tilde{x}(t) = q(t) - A(D\xi(t))' - B\xi(t) =: \tilde{q}(t). \tag{7.3}$$

Those shifts are usually made in order to obtain problems with homogeneous boundary conditions from those with certain inhomogeneous boundary conditions. Naturally, if $A(D\xi(t))'$ is a nontrivial term, the solution of (7.3) depends on this term, i.e., on a derivative of certain problem data. However, we do not say that, for fixed operators A, D, B, equation (7.3) with $\xi \neq 0$ has a higher index than the same equation with $\xi = 0$. In particular, for $A = 1$, $D = 1$, $B = -\Delta + 1$, (7.3) represents a homogenized standard scalar parabolic PDE, but it would not make sense to call it a problem of index $\mu = 1$. Actually, the solution depends only on the boundary values itself. In the following, the index of an equation results from the case when homogeneous boundary conditions are given and put into the corresponding function spaces. Possible homogenizations are assumed to be done before.

7.1 A special example considered in [LSEL99]

Consider the PDAE

$$\begin{aligned} u_t + v_t - u_{yy} - v_{yy} + c_1 u + c_2 v &= r, \\ -u_{yy} + u &= s, \quad y \in \Omega,\ t \geq 0, \end{aligned} \tag{7.4}$$

with $\Omega = (-l, l)$, $c_1, c_2 \geq 0$. Choose $X = Y = L_2(\Omega) \times L_2(\Omega)$, $Z = L_2(\Omega)$ and the operators in matrix representation

$$\begin{aligned} A &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in L_b(Z, Y), \quad D = \begin{pmatrix} 1 & 1 \end{pmatrix} \in L_b(X, Z), \quad R = 1 \\ P_0 &= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in L_b(X), \quad D^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in L_b(Z, X). \end{aligned}$$

For the first time we choose the definition region for $B \in L(X, Y)$,

$$B := \begin{pmatrix} -\Delta + c_1 & -\Delta + c_2 \\ -\Delta + 1 & 0, \end{pmatrix}$$

to be $\mathcal{D}_B^{(1)} = C^2(\Omega) \times C^2(\Omega)$. Compute

$$G_1 = \begin{pmatrix} 1 + c_1 - c_2 & 1 \\ -\Delta + 1 & 0 \end{pmatrix}$$

and ask whether G_1 is densely solvable and injective. For a positive answer, boundary conditions for the first component are needed. This is why we replace $\mathcal{D}_B^{(1)}$ by $\mathcal{D}_B^{(2)} = C_0^2(\Omega) \times C^2(\Omega)$, which is also dense in X. Now G_1 is injective and densely solvable. Thus, this system has index $\mu = 1$. Compute further $(\Delta + 1)^-$ on $L_2(\Omega)$ (cf. Section 2 for the use of generalized inverses)

$$\begin{aligned} G_1^{-1} &= \begin{pmatrix} 0 & (-\Delta + 1)^- \\ 1 & -(1 + c_1 - c_2)(-\Delta + 1)^- \end{pmatrix}, \quad Q_0 G_1^{-1} B = Q_0, \\ D G_1^{-1} &= (1 \ \ -(c_1 - c_2)(-\Delta + 1)^-), \quad D G_1^{-1} B D^- = -\Delta + c_2. \end{aligned}$$

Each solution of the ADAS (7.2) may be expressed by

$$\begin{aligned} x(t) &= \begin{pmatrix} u(t,\cdot) \\ v(t,\cdot) \end{pmatrix} = D^- Dx(t) + Q_0 G_1^{-1} \begin{pmatrix} r(t,\cdot) \\ s(t,\cdot) \end{pmatrix} \\ &= D^- Dx(t) + \begin{pmatrix} (-\Delta+1)^- s(t,\cdot) \\ -(-\Delta+1)^- s(t,\cdot) \end{pmatrix} \end{aligned} \tag{7.5}$$

while $Dx(t) = u(t,\cdot) + v(t,\cdot)$ satisfies the inherent regular differential equation

$$(Dx)' + DG_1^{-1} BD^- Dx = DG_1^{-1} q,$$

i.e., the parabolic equation

$$(u+v)' - \Delta(u+v) + c_2(u+v) = r - (c_1 - c_2)(-\Delta+1)^- s. \tag{7.6}$$

For the sake of uniqueness, we have to equip this parabolic equation with initial and boundary conditions. Using homogeneous Dirichlet boundary conditions, we finally choose $\mathcal{D}_B = C_0^2(\Omega) \times C_0^2(\Omega)$ instead of $\mathcal{D}_B^{(2)}$ or, considering weak solutions, $\mathcal{D}_B = H_0^1(\Omega) \times H_0^1(\Omega)$.
The initial condition now reads $u(0) + v(0) \in D\mathcal{D}_B$. In particular, in this weak setting, for each given $u(0) + v(0) \in H_0^1(\Omega)$ and $v, s \in L_b(\Omega)$, the inherent regular differential equation (7.6) has a unique solution $u + v \in C^1(\mathcal{I}, H_0^1(\Omega))$, and by (7.5) the corresponding solution of the original problem (7.4) is given.

7.2 The previous example with inhomogeneous Dirichlet conditions

To consider (7.1) with inhomogeneous boundary conditions, say

$$u(t)|_{\partial\Omega} = \tilde{\alpha}(t),\ t \in \mathcal{I},$$

we apply the usual homogenization, introducing a continuously differentiable function $\alpha(t) \in H^1(\Omega)$ such that $\alpha(t)|_{\partial\Omega} = \tilde{\alpha}(t)$, $t \in \mathcal{I}$, and then shifting the variable $u - \alpha =: \tilde{u}$. The resulting system for $x = \begin{pmatrix} \tilde{u} \\ v \end{pmatrix}$ has the new-right hand side

$$q = \begin{pmatrix} r - \alpha_t + \Delta\alpha - c_1\alpha \\ s + \Delta\alpha - \alpha \end{pmatrix}.$$

Again, we are looking for solutions satisfying homogeneous boundary conditions as in Section 7.1.
If $\tilde{\alpha}(t)$ actually depends on t, then also $\alpha(t)$ does so, and the solution x formally depends via q on α_t. However, we do not interpret this fact by a higher index value.

7.3 A general semi-explicit system with two unknown functions

Consider the PDAE

$$\begin{aligned} u_t + b_1 u_{yy} + b_2 v_{yy} + c_1 u + c_2 v &= r, \\ b_3 u_{yy} + b_4 v_{yy} + c_3 u + c_4 v &= s, \quad y \in \Omega, \quad t \geq 0, \end{aligned} \tag{7.7}$$

where

$$b_1 b_4 - b_2 b_3 \neq 0. \tag{7.8}$$

Choose $X = Y = L_2(\Omega) \times L_2(\Omega)$, $Z = L_2(\Omega)$ and

$$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

further we obtain $B = \begin{pmatrix} b_1\Delta + c_1 & b_2\Delta + c_2 \\ b_3\Delta + c_3 & b_4\Delta + c_4 \end{pmatrix}$. Put, for the first time, $\mathcal{D}_B^{(1)} = H^1(\Omega) \times H^1(\Omega)$. Let $\sigma(\Delta)$ denote the spectrum of the Laplacian. It should be stressed that $\sigma(\Delta)$ naturally depends on the region Ω. In particular, if there is only one spatial dimension and $\Omega = (0, L)$, then $\sigma(\Delta)$ heavily depends on L. To obtain injectivity for

$$G_1 = \begin{pmatrix} 1 & b_2\Delta + c_2 \\ 0 & b_4\Delta + c_4 \end{pmatrix}$$

we turn to $\mathcal{D}_B^{(2)} = H^1(\Omega) \times H_0^1(\Omega)$ provided that $b_4 \neq 0$. More precisely, G_1 is injective on $\mathcal{D}_B^{(2)}$ provided that

$$b_4\gamma + c_4 \neq 0, \text{ for all } \gamma \in \sigma(\Delta). \tag{7.9}$$

Condition (7.9) implies the ADAS (7.2) representing our PDAE (7.7) to have index $\mu = 1$. Note that in case of $b_4 = 0$, $c_4 \neq 0$, we can proceed with $\mathcal{D}_B^{(1)}$. The resulting ADAS has also index $\mu = 1$. In both index-1 cases, the corresponding inherent regular differential equation is related to the component $Dx = u$. In order to obtain unique solvability, boundary and initial conditions for this component have to be added. Hence, we turn to $\mathcal{D}_B = H_0^1(\Omega) \times H_0^1(\Omega)$ for $b_4 \neq 0$ and $\mathcal{D}_B = H_0^1(\Omega) \times H^1(\Omega)$ for $b_4 = 0$, $c_4 \neq 0$. The initial condition is stated to be

$$Dx(0) = u(0) \in D\mathcal{D}_B = H_0^1(\Omega).$$

Next, suppose that $b_4 = 0$, $c_4 = 0$, which implies (by (7.8)) that $b_2 \neq 0$, $b_3 \neq 0$ have to be given. Obviously,

$$G_1 = \begin{pmatrix} 1 & b_2\Delta + c_2 \\ 0 & 0 \end{pmatrix}$$

is no longer injective on $\mathcal{D}_B^{(2)}$. We form the next subspace as described in Section 2. We have

$$\begin{aligned} \ker G_1 &= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}_B^{(2)} : u + (b_2\Delta + c_2)v = 0 \right\} \\ &= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}_B^{(2)} : v = -(b_2\Delta + c_2)^- u \right\} \end{aligned}$$

provided that

$$b_2\gamma + c_2 \neq 0, \ \text{ for all } \ \gamma \in \sigma(\Delta), \tag{7.10}$$

is given. The operator

$$Q_1 = \begin{pmatrix} 1 & 0 \\ -(b_2\Delta + c_2)^- & 0 \end{pmatrix}$$

is a bounded idempotent map acting on $X = L_2(\Omega) \times L_2(\Omega)$,

$$\operatorname{im} Q_1 = N_1 = \operatorname{cl}\, \ker G_1 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in X : u \in L_2(\Omega), \ v = -(b_2\Delta + c_2)^- u \right\}.$$

Note that $Q_1 Q_0 = 0$ is satisfied. Compute

$$G_2 = G_1 + BP_0Q_1 = \begin{pmatrix} 1 + c_1 + b_1\Delta & c_2 + b_2\Delta \\ c_3 + b_3\Delta & 0 \end{pmatrix}$$

and turn to $\mathcal{D}_B = H_0^1(\Omega) \times H_0^1(\Omega)$. Now, provided that

$$c_3 + b_3\gamma \neq 0, \ \text{ for all } \ \gamma \in \sigma(\Delta), \tag{7.11}$$

G_2 is injective and densely solvable. Hence, if $b_4 = c_4 = 0$ and the conditions (7.10), (7.11) are satisfied, then the resulting ADAS has index $\mu = 2$. The corresponding inherent regular differential equation is trivially $0 = 0$, since the component $DP_1x = 0$ disappears. Consequently, no additional initial condition is allowed.
Finally, we deal with the case if $b_4 \neq 0$, $c_4 \neq 0$, but there is a $\gamma_* \in \sigma(\Delta)$ such that

$$b_4\gamma_* + c_4 = 0.$$

Then, with $\mathcal{D}_B^{(1)}$, we have that

$$\ker G_1 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}_B^{(1)} : u + (b_2\Delta + c_2)v = 0, v = \Pi_* v \right\},$$

where $\Pi_* \in L_b(L_2(\Omega))$ denotes the spectral projection onto the eigenspace of Δ corresponding to the eigenvalue γ_*, thus

$$\Delta\Pi_* = \gamma_*\Pi_*, \quad \Pi_*\Delta(I - \Pi_*) = 0.$$

Now, if $b_2\gamma_* + c_2 = 0$, we find $N_1 = 0 \times \operatorname{im}\Pi_*$ and

$$Q_1 = \begin{pmatrix} 0 & 0 \\ 0 & \Pi_* \end{pmatrix}, \quad G_2 = G_1 + BP_0Q_1 = G_1, \quad N_2 = N_1$$

and so on. There is no injective G_j in the resulting sequence. Therefore, the system (7.7) is no longer a regular abstract DAE. The operator pair $\{AD, B\}$ behaves like a singular matrix pencil. Characteristic of this situation is the particular case of $b_1 = -1$, $b_2 = 1$, $b_3 = 0$, $b_4 = 1$, $c_1 = 0$, $c_2 = c_4 = -\gamma_*$, $c_3 = 0$, i.e., the system (7.7)

$$\begin{aligned} u_t - \Delta u \quad + \quad (\Delta - \gamma_*)v &= r, \\ (\Delta - \gamma_*)v &= s, \end{aligned}$$

which is no longer solvable for all sufficiently smooth functions $s : \mathcal{I} \to L_2(\Omega)$ and which does not determine the component $\Pi_* v$. Notice that in [LSEL99] this case is forbidden by means of the condition

$$(b_1 + \gamma c_1)(b_4 + \gamma c_4) - (b_2 + \gamma c_2)(b_3 + \gamma c_3) \neq 0, \ \gamma \in \sigma(\Delta). \tag{7.12}$$

Next we assume that

$$b_4\gamma_* + c_4 = 0, \ \ b_2\gamma_* + c_2 =: \alpha \neq 0. \tag{7.13}$$

We derive

$$\begin{aligned} \ker G_1 &= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}_B^{(1)} : v = -\frac{1}{\alpha}u, \ u = \Pi_* u \right\}, \\ N_1 &= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in X : u = \Pi_* u, \ v = -\frac{1}{\alpha}u \right\}, \\ Q_1 &= \begin{pmatrix} \Pi_* & 0 \\ -\frac{1}{\alpha}\Pi_* & 0 \end{pmatrix}, \quad Q_1Q_0 = 0, \end{aligned}$$

further

$$G_2 = \begin{pmatrix} 1 + (b_1\Delta + c_1)\Pi_* & b_2\Delta + c_2 \\ (b_3\Delta + c_3)\Pi_* & b_4\Delta + c_4 \end{pmatrix} = \begin{pmatrix} 1 + (b_1\gamma_* + c_1)\Pi_* & b_2\Delta + c_2 \\ (b_3\gamma_* + c_3)\Pi_* & b_4\Delta + c_4 \end{pmatrix}.$$

The homogeneous equation $G_2 \begin{pmatrix} u \\ v \end{pmatrix} = 0$ reads in detail

$$u + (b_1\gamma_* + c_1)\Pi_* u + (b_2\Delta + c_2)v = 0, \tag{7.14}$$

$$(b_4\Delta + c_4)v = -(b_3\gamma_* + c_3)\Pi_* u. \tag{7.15}$$

If $b_3\gamma_* + c_3 = 0$, then (7.14), (7.15) yield

$$v = \Pi_* v, \quad u = \Pi_* u, \quad v = -\frac{1}{\alpha}(1 + b_1\gamma_* + c_1)u,$$

but $\Pi_* u$ is not determined and G_2 is not injective,

$$\ker G_2 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}_B^{(1)} : u = \Pi_* u,\ v = -\frac{1}{\alpha}(1 + b_1\gamma_* + c_1)u \right\},$$

$$Q_2 = \begin{pmatrix} \Pi_* & 0 \\ -\frac{1}{\alpha}(1 + b_1\gamma_* + c_1)\Pi_* & 0 \end{pmatrix}, \quad Q_2Q_1 = Q_2, \quad Q_2Q_0 = 0,$$

$G_3 = G_2$ and so on. Again, we arrive at a singular problem. This situation, too, is excluded by condition (7.12) (cf. [LSEL99]).
Provided that

$$b_4\gamma_* + c_4 = 0, \quad b_2\gamma_* + c_2 =: \alpha \neq 0, \quad b_3\gamma_* + c_3 =: \beta \neq 0, \tag{7.16}$$

equation (7.15) leads to $\Pi_* u = 0$, $v = \Pi_* v$ since the Laplacian, as a symmetric operator on $H_0^1(\Omega)$, does not have generalized eigenfunctions. Then, from (7.14), we obtain that $u = \Pi_* u$ and $v = -\frac{1}{\alpha}(1 + b_1\gamma_* + c_1)u$, hence $u = 0$, $v = 0$, i.e., G_2 is injective. G_2 is defined on $\mathcal{D}_B^{(1)}$ and we have $\operatorname{im} G_2 = Y$,

$$G_2^{-1} = \begin{pmatrix} I - \Pi_* & \frac{1}{\beta}\Pi_* - (b_2\Delta + c_2)(b_4\Delta + c_4)_*^-(I - \Pi_*) \\ \frac{1}{\alpha}\Pi_* & (b_4\Delta + c_4)_*^-(I - \Pi_*) - \frac{1}{\alpha\beta}(1 + b_1\gamma_* + c_1)\Pi_* \end{pmatrix},$$

where $(b_4\Delta + c_4)_*^-$ denotes the solution operator for $(b_4\Delta + c_4)z = y$, $y \in \operatorname{im}(I - \Pi_*)$, $z \in H_0^1(\Omega)$, $\Pi_* z = 0$. Now we are able to formulate the corresponding inherent regular differential equation that is related to the component $DP_1 \begin{pmatrix} u \\ v \end{pmatrix} = (I - \Pi_*)u$. We have $DP_1D^- = I - \Pi_*$, $Q_1G_2^{-1}BP_0 = Q_1$,

$$\begin{aligned} &DP_1G_2^{-1}BD^- = \\ &\quad (I - \Pi_*)(b_1\Delta + c_1) - (I - \Pi_*)(b_2\Delta + c_2)(b_4\Delta + c_4)_*^-(I - \Pi_*)(b_3\Delta + c_3) \end{aligned}$$

and

$$DP_1G_2^{-1}\begin{pmatrix} r \\ s \end{pmatrix} = (I - \Pi_*)r - (I - \Pi_*)(b_2\Delta + c_2)(b_4\Delta + c_4)_*^-(I - \Pi_*)s,$$

therefore, the inherent regular equation is

$$\begin{aligned} ((I - \Pi_*)u)_t + \{(b_1\Delta + c_1) - (b_2\Delta + c_2)(b_4\Delta + c_4)_*^-(b_3\Delta + c_3)\}(I - \Pi_*)u \\ = (I - \Pi_*)r - (b_2\Delta + c_2)(b_4\Delta + c_4)_*^-(I - \Pi_*)s. \end{aligned}$$

This equation has to be completed by boundary conditions via choosing $\mathcal{D}_B^{(2)}$, but also by an initial condition $(I - \Pi_*)u(0) \in H_0^1(\Omega)$. The part to be differentiated is

$$DQ_1\begin{pmatrix} u \\ v \end{pmatrix} = \Pi_* u = DQ_1G_2^{-1}\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\beta}\Pi_* \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} = \frac{1}{\beta}\Pi_* s.$$

Finally, the solutions of (7.7) can be expressed as

$$\begin{aligned}\begin{pmatrix}u\\v\end{pmatrix} &= \begin{pmatrix}1\\0\end{pmatrix}((I-\Pi_*)u+\frac{1}{\beta}\Pi_* s)+\begin{pmatrix}0\\v\end{pmatrix},\\ \begin{pmatrix}0\\v\end{pmatrix} &= Q_0P_1G_2^{-1}\begin{pmatrix}r\\s\end{pmatrix}-Q_0P_1G_2^{-1}BD^-(I-\Pi_*)u+\begin{pmatrix}0\\ \frac{1}{\alpha\beta}(\Pi_* s)_t\end{pmatrix}.\end{aligned}$$

Consequently, provided condition (7.16) is satisfied, the abstract formulation of the PDAE (7.7) is an ADAS with index $\mu = 2$.

7.4 A superconducting coil

This example was studied in [MT95, CM99]. There, a superconducting coil is simulated by the hyperbolic PDAE

$$\frac{\partial^2}{\partial x^2}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}0 & 0\\ -\frac{LC}{l^2} & \frac{L}{D}\end{pmatrix}\frac{\partial^2}{\partial t^2}\begin{pmatrix}u_1\\u_2\end{pmatrix}+\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} \tag{7.17}$$

with $x \in \Omega = (0,l)$, $t \geq 0$, $C, L, D > 0$. The initial and boundary conditions are prescribed as follows

$$\begin{aligned}u(0,x) &= 0, \quad \frac{\partial}{\partial t}u(0,x)=0,\\ u(t,0) &= 0, \quad u_1(t,l)=E, \quad u_2(t,l)=\gamma^2 E,\end{aligned}$$

for $u = \begin{pmatrix}u_1\\u_2\end{pmatrix}$, $E > 0$ and $\gamma = (CD)^{1/2}$. For details of the physical meaning of the parameters see [MT95]. Transforming (7.17) into the form (2.1) with $v_1 := u$, $v_2 := u_t$ and $v = \begin{pmatrix}v_1\\v_2\end{pmatrix}$, we obtain (see also [CM99])

$$\begin{pmatrix}0&0&0&0\\0&0&\frac{LC}{l^2}&-\frac{L}{D}\\1&0&0&0\\0&1&0&0\end{pmatrix}v_t+\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}v_{xx}+\begin{pmatrix}0&-1&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}v=0$$

with the initial and boundary conditions

$$v(0,x)=0,\;\; v_1(t,0)=0,\;\; v_{11}(t,l)=E,\;\; v_{12}(t,l)=\gamma^2 E.$$

Rewriting this system as abstract DAE of the form (2.1), the corresponding operators are given by

$$A=\begin{pmatrix}0&0&0\\0&0&1\\1&0&0\\0&1&0\end{pmatrix},\quad D=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{LC}{l^2}&-\frac{L}{D}\end{pmatrix},\quad B=\begin{pmatrix}\Delta&-1&0&0\\0&\Delta&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}.$$

We may choose

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{l^2}{CD} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ which implies } D^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{l^2}{LC} \\ 0 & 0 & 0 \end{pmatrix}, \quad R = I.$$

Applying the common homogenization technique (see Section 7.2) we are looking for a function $\bar{v}_1 \in H^1(\Omega) \times H^1(\Omega)$ fulfilling the boundary conditions. Then, the corresponding PDAE with homogeneous boundary conditions reads

$$A\tilde{v}_t + B\tilde{v} = q, \quad q = \begin{pmatrix} -\Delta\bar{v}_1 + \begin{pmatrix} \bar{v}_{12} \\ 0 \end{pmatrix} \\ -\bar{v}_{1t} \end{pmatrix}$$

for $\tilde{v} := \begin{pmatrix} v_1 - \bar{v}_1 \\ v_2 \end{pmatrix}$. We obtain

$$G_1 = A + BQ_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{CL}{l^2} & -\frac{L}{D} \\ 1 & 0 & 0 & -\frac{l^2}{CD} \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

and it is obvious that G_1 does not remain injective. Consequently, the index is greater than 1. Continuing the matrix chain, we construct a projector onto the nullspace of G_1 satisfying $Q_1Q_0 = 0$ as follows

$$Q_1 = \begin{pmatrix} 0 & \frac{l^2}{CD} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{l^2}{CD} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The next matrix chain element is then given by

$$G_2 = G_1 + BP_0Q_1 = \begin{pmatrix} 0 & \frac{l^2}{CD}\Delta - 1 & 0 & 0 \\ 0 & \Delta & \frac{CL}{l^2} & -\frac{L}{D} \\ 1 & 0 & 0 & -\frac{l^2}{CD} \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Obviously, G_2 is injective and densely solvable on $D_B = H^1(\Omega) \times H_0^1(\Omega) \times L_2(\Omega) \times L_2(\Omega)$. Using $L_0^- = (-\Delta + \alpha)^-$ defined on $L_2(\Omega)$ (see Section 2) and $\alpha = \frac{CD}{l^2}$, the inverse operator G_2^{-1} is given by

$$G_2^{-1} = \begin{pmatrix} -L_0^- & 0 & 1 & -\alpha^{-1} \\ -\alpha L_0^- & 0 & 0 & 0 \\ (\frac{D}{L}\Delta - 1)L_0^- & (\frac{\alpha L}{D})^{-1} & 0 & -\alpha^{-1} \\ -\alpha L_0^- & 0 & 0 & -1 \end{pmatrix}.$$

Furthermore, the operator projecting onto $\ker G_1$ along S_1 reads

$$Q_{1s} = L_0^- \otimes \begin{pmatrix} -\Delta & +1 & 0 & 0 \\ -\alpha\Delta & +\alpha & 0 & 0 \\ -\Delta & +1 & 0 & 0 \\ -\alpha\Delta & +\alpha & 0 & 0 \end{pmatrix}.$$

Consequently, the abstract DAE (7.17) has the index 2. This does not contradict to the result obtained in [CM99] (perturbation index $\nu_P^\infty \geq 3$) since our index definition for abstract DAEs orientates on the perturbations with respect to time only. However, we obtain also informations about the influence of perturbations with respect to the space variables considering the inherent explicit differential equation

$$\tilde{u}' + DP_{1s}G_2^{-1}BD^-\tilde{u} = DP_{1s}G_2^{-1}q,$$

for which

$$\tilde{u} = DP_{1s}\tilde{v} = \begin{pmatrix} L_0^-(\alpha\tilde{v}_{11} - \tilde{v}_{12}) \\ L_0^-\Delta(\alpha\tilde{v}_{11} - \tilde{v}_{12}) \\ \frac{L}{D}(\alpha\tilde{v}_{21} - \tilde{v}_{22}) \end{pmatrix}, \quad DP_{1s}G_2^{-1}q = \begin{pmatrix} -L_0^-(\alpha\bar{v}_{11} - \bar{v}_{12})_t \\ -L_0^-\Delta(\alpha\bar{v}_{11} - \bar{v}_{12})_t \\ -\Delta L_0^-\Delta(\alpha\bar{v}_{11} - \bar{v}_{12}) \end{pmatrix}$$

and

$$DP_1G_2^{-1}BD^- = \begin{pmatrix} 0 & 0 & -\frac{D}{L\alpha}(1 + L_0^-\Delta) \\ 0 & 0 & -\frac{D}{L}L_0^-\Delta \\ \alpha\Delta L_0^-\Delta & \Delta - \alpha\Delta L_0^- & 0 \end{pmatrix}.$$

The latter component contains the term $-\alpha\Delta L_0^-\Delta(\alpha\bar{v}_{11} - \bar{v}_{12})$. This means that an index defined by the perturbation or differentiation index [CM99, MB00] including the space variables has at least a value equal to four.
The obvious constraints are given by $DQ_1\tilde{v} = DQ_1G_2^{-1}q$ which is equivalent to

$$\Delta\tilde{v}_{11} - \tilde{v}_{12} = \Delta\bar{v}_{11} - \bar{v}_{12} + \Delta(\bar{v}_{11} - \bar{v}_{12})_t.$$

Finally the hidden constraints read

$$Q_0\tilde{v} - Q_0Q_1D^-(DQ_1\tilde{v})_t + Q_0P_1G_2^{-1}BD^-DP_1\tilde{v} = QP_1G_2^{-1}q$$

or, equivalently,

$$\tilde{v}_{22} - L_0^-\Delta(\tilde{v}_{21} - \tilde{v}_{22}) = \alpha L_0^-\Delta(\bar{v}_{11} - \bar{v}_{12})_t.$$

8 Conclusions

Coupled systems of differential algebraic equations with partial differential equations or integral equations may be considered as abstract DAEs with coefficient matrices operating in real Hilbert spaces. The decoupling concept based on the tractability index for (ordinary) DAEs can be extended to ADASs if the arising operators are normally solvable.

The different case studies make clear that this extended concept provides a strategy to handle such coupled systems. This way, we find consistent initial and boundary conditions as well as appropriate solution spaces. Furthermore, this concept seems to provide a way to extend existence and uniqueness results from ODE and PDE theory to coupled systems of partial differential equations and differential algebraic equations.

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