Stochastic Differential Algebraic Equations of Index 1 and Applications in Circuit Simulation

Renate Winkler
Humboldt-University Berlin, Institut für Mathematik

Abstract

In this paper we deal with differential-algebraic equations driven by Gaussian white noise. In a first part we use the theory of stochastic differential equations (SDEs) as well as the theory of differential-algebraic equations (DAEs) for a mathematically rigorous formulation of such problems and give the necessary analytical theory for the existence and uniqueness of strong solutions for systems of DAE-index 1. In a second part we analyze discretization methods. Due to the differential-algebraic structure implicit methods become necessary and computational errors have to be taken into account more carefully. For that purpose a result concerning the mean square numerical stability for general drift-implicit discretization schemes for SDEs is proved. Then, we apply the drift-implicit Euler scheme, the split-step backward Euler scheme, the trapezoidal rule and the drift-implicit Milstein scheme directly to the stochastic DAE, estimate the influence of errors and and prove that the convergence properties of these methods known for SDEs are preserved. We show how the theory applies to the transient noise simulation of electronic circuits and express the necessary conditions in terms of the network-topology.

1 Introduction

In this paper we deal with **S**tochastic **D**ifferential **A**lgebraic **E**quations (SDAEs) of the type

$$Ax'(t) + f(x(t),t) + G(x(t),t)\xi(t) = 0\ , \tag{1.1}$$

where f is a vector-valued function of dimension n, A is a constant singular $n \times n$ matrix with rank r, ξ stands for an m-dimensional vector of independent Gaussian white noise processes and G is an $n \times m$-dimensional matrix-function. A solution x is a vector-valued stochastic process of dimension n.
The application we have in mind is the transient noise simulation of electronic circuits. In the deterministic case the circuits are modelled by large, specially structured DAEs. Noise in the system is modelled by adding Gaussian white noise sources.

Due to the singularity of the matrix A the deterministic part of (1.1)

$$Ax'(t) + f(x(t),t) = 0 \tag{1.2}$$

forms a DAE. Solutions have to fulfil the constraints of the equation. The solution components belonging to $\ker A$ (we call them the algebraic components) do not occur differentiatedly, and the inherent dynamics live only in a lower-dimensional subspace.
DAEs are usually classified by their index. We use the concept of the tractability index (cf. [12, 21, 22]), which says that the DAE (1.2) is of index 1 if the constraints are (locally) solvable for the algebraic components. In this case (1.2) involves a coupling of an integration task and a nonlinear equation solving task. In general also the solvability of initial value problems can be expected only for initial values in a certain neighbourhood of a given solution.
If the DAE is of higher index, the constraints are not (locally) solvable for the algebraic components, and there exist solution components that are determined only by a hidden differentiation step. For a detailed analysis of DAEs we refer to [2, 5, 12, 21, 22].

SDAEs are a generalization of both, deterministic differential algebraic equations (DAEs) and stochastic differential equations (SDEs). Much research has been devoted to the numerical solution of SDEs (see [26] for a recent overview). However, only first attempts have been made towards a numerical analysis of SDAEs: In [27, 28] linear SDAEs are analyzed and the convergence of the drift-implicit Euler scheme is proved. In [25] a scheme with strong order 1 is developed for the specially structured SDAEs that arise in transient noise simulation for electronic circuits. Later we will point out its relation to the drift-implicit Milstein scheme.
In the present paper we will prove the existence and uniqueness of solutions of general nonlinear SDAEs of index 1, and develop and analyze a number of numerical schemes for SDAEs, where we will put special emphasis on estimating the influence of computational as well as truncation errors. For that purpose we firstly prove a result concerning the mean square numerical stability for general drift-implicit discretization schemes for SDEs, which might be of some independent interest.

For a mathematical treatment of the stochastic version (1.1) we understand it as a stochastic integral equation

$$Ax(s)\Big|_{t_0}^{t} + \int_{t_0}^{t} f(x(s),s)ds + \int_{t_0}^{t} G(x(s),s)dw(s) = 0\ , \tag{1.3}$$

where the Itô integral is employed, and w denotes an m-dimensional Wiener process given on the probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\geq t_0}$.

In Chapter 2 we start by defining SDAEs of index 1 and present an analysis of such systems. We formulate initial value problems and prove the existence and uniqueness of strong solutions as well as results concerning their growth using corresponding results for SDEs. This generalizes results in [27, 28] to nonlinear systems.
To facilitate an analogous approach for discretization methods we first analyze drift-implicit discretization methods for SDEs in Chapter 3. We provide a proof of mean square numerical stability, which allows to estimate the influence of computational as well as truncation errors in the discrete systems.
In Chapter 4 we present and discuss discretization schemes suitable for SDAEs, in particu-

lar the drift-implicit Euler scheme, the split-step backward Euler scheme, the trapezoidal rule, and the drift-implicit Milstein scheme.
Finally, in Chapter 5 we describe the SDAEs arising in transient noise simulation for electrical circuits and give sufficient conditions for these systems to meet the assumptions stated in the previous chapters .

2 SDAEs of index 1

We consider the SDAE

$$Ax(s)\Big|_{t_0}^{t} + \int_{t_0}^{t} f(x(s),s)ds + \int_{t_0}^{t} G(x(s),s)dw(s) = 0 \,, \quad t \in \mathcal{J} \,, \tag{2.1}$$

abbreviated as

$$Ax'(t) + f(x(t),t) + G(x(t),t)\xi(t) = 0 \quad , \quad t \in \mathcal{J} \,, \tag{2.2}$$

where A is a constant nonsingular matrix in $I\!R^{n\times n}$, $\mathcal{J} = [t_0, T]$, $f : I\!R^n \times \mathcal{J} \to I\!R^n$, and $G : I\!R^n \times \mathcal{J} \to I\!R^{n\times m}$ are continuous functions, and, moreover, f possesses continuous derivatives with respect to x.
w denotes an m-dimensional Wiener process given on the probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\geq t_0}$, ξ stands for the corresponding white noise.

To avoid a solution process that is directly affected by white noise we have to assume that the noise sources do not appear in the constraints. This means that

$$\operatorname{im} G(x,t) \subseteq \operatorname{im} A \quad \forall (x,t) \in I\!R^n \times \mathcal{J} \,.$$

In [28, 27] SDAEs with this property are called SDAEs without direct noise. Further, we assume here that the deterministic part

$$Ax'(t) + f(x(t),t) = 0 \quad , \quad t \in \mathcal{J} \,, \tag{2.3}$$

is globally an index 1 DAE in the sense that the constraints are regularly and globally uniquely solvable for the algebraic variables, the components of x belonging to the kernel of the matrix A. This is stronger than the usual index 1 condition, which requires only the non-singularity of the corresponding Jacobian and guarantees only local solvability of the constraints for the algebraic variables. Summarizing both assumptions we define:

Definition: The SDAE (2.1) is said to be an **index 1 SDAE** if

- the noise sources do not appear in the constraints, and
- the deterministic part is an index 1 DAE,
  moreover, the constraints are globally uniquely solvable for the algebraic variables.

To be more precise we will distinguish the differential and algebraic solution components as well as the constraints by means of the special projectors

$$\begin{array}{llll} Q \quad \text{onto} \quad \ker A & , & P := I - Q \quad \text{along} \quad \ker A \, , \\ R \quad \text{along} \quad \operatorname{im} A \, . \end{array}$$

(A matrix Q is a projector iff $Q^2 = Q$, it projects onto its image and along its kernel.) Now we split the solution components into differential and algebraic components

$$x(t) \;=\; Px(t) + Qx(t) \;=:\; u(t) + v(t) \;,$$

and the equations of the SDAE (2.2) into differential ones and constraints:

$$Ax'(t) + (I-R)f(x(t),t) + G(x(t),t)\xi(t) \;=\; 0 \tag{2.4}$$

$$Rf(x(t),t) \;=\; 0 \,. \tag{2.5}$$

Solving the constraints for the algebraic solution components means solving $Rf(u+v,t) = 0$, where $Av = 0$ for v , or, equivalently, solving

$$Av + Rf(u+v,t) = 0 \tag{2.6}$$

for v. We denote the solution by

$$v \;=\; \hat{v}(u,t) \;. \tag{2.7}$$

Summarizing we see : (2.1) is an index 1 SDAE iff the constraints can be described by the deterministic equation $Rf(x,t) = 0$, the Jacobian $J(x,t) := A + Rf'_x(x,t)$ is nonsingular and the implicitly defined function $\hat{v}$ exists globally and uniquely.
The latter condition is guaranteed if the inverse $(A+Rf'_x(x,t))^{-1}$ is uniformly bounded. Furthermore, then also perturbed constraints

$$Rf(u+v,t) = Rd \quad , \; d \in I\!R^n \;,$$

are uniquely solvable for the algebraic variables. We will denote the solution by

$$v = \hat{v}(u,t,d) \quad \text{with the convention that} \quad \hat{v}(u,t,0) = \hat{v}(u,t) \;.$$

Now, let us consider initial conditions to (2.1). A solution process has to fulfil (2.1) and, therefore, also the constraints (2.5) almost surely. To find a solution to a given $\mathcal{F}_{t_0}$-measurable initial value x_0 , this value has to fulfil the constraints at the initial time-point almost surely, i.e.,

$$Rf(x_0,t_0) = 0 \quad \text{a.s.} \;. \tag{2.8}$$

x_0 is said to be a consistent initial value for the index 1 SDAE iff (2.8) is fulfilled. (2.8) represents $n-\operatorname{rank} A$ independent scalar conditions. So there are only $\operatorname{rank} A$ free initial parameters. One way to determine a consistent initial value x_0 with given $Ax_0 := Ax^0$ for any $\mathcal{F}_{t_0}$-measurable $I\!R^n$-valued random variable x^0 is to solve the system

$$A(x_0 - x^0) = 0 \quad , \quad Rf(x_0,t_0) = 0 \;,$$

or, equivalently,

$$A(x_0 - x^0) + Rf(x_0, t_0) = 0 \ .$$

Using the function (2.7) the solution of this system can be represented as

$$x_0 := Px_0 + Qx_0 := Px^0 + \hat{v}(Px^0, t_0) \ .$$

In general, unless $Qx^0 = \hat{v}(Px^0, t_0)$, the consistent initial value x_0 will differ from the given value x^0. According to this setting initial value problems can be formulated as

$$A(x(t) - x^0) + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} G(x(s), s)dw(s) = 0 \quad , t \in \mathcal{J} \ , \tag{2.9}$$

or, abbreviated, as

$$Ax'(t) + f(x(t), t) + G(x(t), t)\xi(t) = 0 \quad , t \in \mathcal{J} \quad , A(x(t_0) - x^0) = 0 \ . \tag{2.10}$$

Definition A process $x(\cdot) = (x(t))_{t\in\mathcal{J}}$ with continuous sample paths is called a **(strong) solution** of (2.9) iff

- $x(\cdot)$ is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathcal{J}}$,
- $\int_{t_0}^{t} |f_i(x(s), s)|ds < \infty$ a.s. , $\forall i = 1, \ldots, n$, $\forall t \in \mathcal{J}$,
  $\int_{t_0}^{t} g_{ij}^2(x(s), s)dw(s) < \infty$ a.s. , $\forall i = 1, \ldots, n$, $\forall j = 1, \ldots, m$, $\forall t \in \mathcal{J}$,
- (2.9) holds a.s. .

For the theoretical understanding we now perform a decoupling procedure. We aim to show that (2.9) is equivalent to a composition of the solution in dependence on the differential solution components and a certain ordinary SDE in these parts of the solutions. Therefore, we follow the steps:

- Multiplying by the projectors $(I - R)$ and R onto and along im A we split the equations into differential ones and constraints:

$$\begin{aligned} Ax'(t) + (I-R)f(x(t), t) + G(x(t), t)\xi(t) &= 0 \ , \quad A(x(t_0) - x^0) = 0 \\ Rf(x(t), t) &= 0 \ . \end{aligned}$$

- Multiplying by the projectors P and Q along and onto ker A we split the solution vector into differential and algebraic parts:

$$x(t) = Px(t) + Qx(t) \ =: \ u(t) + v(t) \ , \ (u = Pu \ , \ v = Qv) \ .$$

- We solve the constraints for the “algebraic components”

  $$Pv = 0 \ , \ Rf(u + v, t) = 0 \Longleftrightarrow v = \hat{v}(u, t) \ ,$$

  and reassemble the solution as

  $$x(t) = Px(t) + Qx(t) := u(t) + \hat{v}(u(t), t) \ . \tag{2.11}$$

- We insert $v(t) = \hat{v}(u(t), t)$ into the stochastic differential equations and use $Ax'(t) = Au'(t)$ to obtain :

  $$Au' + (I - R)f(u + \hat{v}(u, t), t) + G(u + \hat{v}(u, t), t)\xi(t) = 0 \ , \ u(t_0) = Px^0 \ .$$

- We scale the system by a suitable non-singular matrix D such that $DA = P$. Using $Pu'(t) = u'(t)$ we obtain:

  $$u' + \underbrace{D(I - R)f(u + \hat{v}(u, t), t)}_{:= \hat{f}(u,t)} + \underbrace{DG(u + \hat{v}(u, t), t)}_{:= \hat{G}(u,t)}\xi(t) \;=\; 0 \ , \tag{2.12}$$

  $$u(t_0) \;=\; Px^0 \ . \tag{2.13}$$

(2.12) is a regular SDE in the differential part u of the solution with $\operatorname{im} P$ as an invariant subspace. This can be seen as follows:

> Since $DA = P$, it holds that $QDA = QP = 0$ and thus $QD(I - R) = 0$. Multiplying (2.12) by Q we obtain
>
> $$Qu'(t) = (Qu)'(t) = 0 \quad , \quad t \in \mathcal{J} \ .$$
>
> Hence, $Qu(t_0) = 0$ implies that $Qu(t) = 0$ for all $t \in \mathcal{J}$, or, in other words, $u(t_0) \in \operatorname{im} P$ implies that $u(t) \in \operatorname{im} P$ for all $t \in \mathcal{J}$.

Definition: (2.12) is called an **inherent regular SDE** of the SDAE (2.2).

The inherent regular SDE (2.12) together with the initial condition (2.13) and the assembling of the solution (2.11) is equivalent to the original initial value problem for the SDAE (2.9). Based on this fact we are now able to prove our main theorem on existence and uniqueness of strong solutions of index 1 SDAEs:

Theorem 2.1 *Suppose that (2.1) is an index 1 SDAE and that the Jacobian* $J(x, t) := A + Rf'_x(x, t)$ *of (2.6) possesses a globally bounded inverse.*
Suppose that f *and* G *are globally Lipschitz-continuous with respect to* x *with a Lipschitz-constant* L*, continuous with respect to* t*, and that* Ax^0 *is* $\mathcal{F}_{t_0}$ *-measurable, independent of the Wiener process* w*, and with finite second moments.*
Then there exists a solution $x(\cdot)$ *of (2.9) that is pathwise unique, i.e., if* $\tilde{x}(\cdot)$ *is another*

solution of (2.9), then $\sup_{t\in\mathcal{J}} |x(t)-\tilde{x}(t)| = 0$ *holds almost surely. Moreover, the solution process* $x(\cdot)$ *is square-integrable and it holds*

$$I\!\!E|x(t)|^2 \leq c_0(t) + c_1(1{+}I\!\!E|Px_0|^2)\cdot e^{c_2L(t-t_0)}$$

with a continuous function $c_0(\cdot)$ *(resulting from the inhomogeneity in the constraints), and constants* c_1, c_2*.*
If, additionally, the function Rf *is Lipschitz continuous with respect to* t*, then there exist constants* c_3, c_4 *such that*

$$I\!\!E|x(t)-x_0|^2 \leq c_3(t-t_0)^2 + c_4(1{+}I\!\!E|Px_0|^2)\cdot(t-t_0)\cdot e^{c_2L(t-t_0)} \quad .$$

Proof: First, we note that for continuous functions and compact time-intervals the global Lipschitz-continuity with respect to x implies the usual growth condition: It holds that

$$|f(x,t)| \leq \max(\|f(0,\cdot)\|_\infty, L_f)\ (1+|x|) \quad \forall x\in I\!\!R^n\ , \quad \forall t\in\mathcal{J}\ ,$$

where $\|y(\cdot)\|_\infty := \max_{t\in\mathcal{J}}|y(t)|$ denotes the Chebychev-norm and L_f denotes the Lipschitz-constant of f with respect to x.
An analogous argument applies to the matrix-valued function G.

Next, we have a global Lipschitz-property for the implicit function $\hat{v}$ from (2.7), too. $\hat{v}$ solves

$$h(v;u,t) \ :=\ Av + Rf(u+v,t) \ =\ 0\ .$$

Since the function h is continuously differentiable and Lipschitz continuous with respect to v and u, also $\hat{v}$ is continuously differentiable with respect to u and it holds that

$$\begin{aligned}\hat{v}'_u(u,t) &= -h'_v(\hat{v}(u,t),u,t)^{-1}h'_u(\hat{v}(u,t),u,t)\\ &= -(A+Rf'_x(u+\hat{v}(u,t),t))^{-1}Rf'_x(u+\hat{v}(u,t),t)\ .\end{aligned}$$

Since the Jacobian $h'_v(v,u,t) = J(u+v,t)$ is supposed to have a uniformly bounded inverse, i.e., $\|J(x,t)^{-1}\| \leq M$ with a uniform constant M, $\|\hat{v}'_u(u,t)\|$ is bounded by $L_{\hat{v}} := M\cdot\|R\|\cdot L_f$. Hence, $\hat{v}$ is Lipschitz continuous with respect to u with this constant. If, additionally, the function Rf is Lipschitz continuous with respect to t, then $\hat{v}$ is Lipschitz continuous with respect to t with a constant $L_{\hat{v},t}$. Considering the dependence of the function $\hat{v} = \hat{v}(u,t,d)$ on perturbations of the constraints Rd, $d\in I\!\!R^n$, we note that $\hat{v}$ is Lipschitz continuous with respect to d with a constant $L_{\hat{v},d} = M\cdot\|R\|$.

Now, we have the conditions of the usual existence and uniqueness theorem for SDEs (see e.g. [1, 17]) for the inherent regular SDE (2.12),(2.13):

- *Lipschitz-condition:* For $\hat{f}$ with $\hat{f}(u,t) := f(u+\hat{v}(u,t),t)$ it holds:

$$\begin{aligned}|\hat{f}(u,t)-\hat{f}(\tilde{u},t)| &\leq L_f|u+\hat{v}(u,t)-\tilde{u}-\hat{v}(\tilde{u},t)|\\ &\leq L_f(1+L_{\hat{v}})|u-\tilde{u}| \ =:\ L_{\hat{f}}|u-\tilde{u}| \qquad \forall u,\tilde{u}\in I\!\!R^n,\ \forall t\in\mathcal{J}\ .\end{aligned}$$

  An analogous argument applies to $\hat{G}$.

- *Growth condition:* Since f and $\hat{v}$ depend continuously on t, also $\hat{f}$ depends continuously on t. Hence, the growth condition follows from the global Lipschitz condition. Again, an analogous argument applies to $\hat{G}$.
- *Initial condition:* Since $Px^0 = DAx^0$, where D is a constant nonsingular matrix, with Ax^0 also the initial value for the inherent regular SDE Px^0 is $\mathcal{F}_{t_0}$-measurable, independent of the Wiener process w, and with finite second moments.

Applying the usual existence and uniqueness theorem for SDEs to the inherent regular SDE we obtain: (2.12),(2.13) has a pathwise unique continuous solution process u, which is square-integrable and fulfils

$$\mathbb{E}|u(t)|^2 \leq (1+\mathbb{E}|u_0|^2) \cdot e^{c_5 \hat{L}(t-t_0)} \quad \forall t \in \mathcal{J} \tag{2.14}$$
$$\mathbb{E}|u(t)-u_0|^2 \leq c_6(1+\mathbb{E}|u_0|^2) \cdot (t-t_0) \cdot e^{c_5 \hat{L}(t-t_0)} \quad \forall t \in \mathcal{J} \tag{2.15}$$

with constants c_5, c_6, where $\hat{L}$ is a Lipschitz-constant for the functions $\hat{f}, \hat{G}$.
Assembling the solution x of the SDAE (2.1) as in (2.11)

$$x(t) := u(t) + \hat{v}(u(t), t) \quad , \quad t \in \mathcal{J} \ ,$$

gives us a pathwise unique continuous solution process of the original SDAE.
Due to the Lipschitz and continuity properties of the implicit function $\hat{v}$ also the solution $x(t)$ is square integrable for all $t \in \mathcal{J}$: It holds that

$$\begin{aligned} \mathbb{E}|x(t)|^2 &= \mathbb{E}|u(t)+\hat{v}(u(t),t)|^2 = \mathbb{E}|\hat{v}(0,t)+(u(t)+\hat{v}(u(t),t)-\hat{v}(0,t))|^2 \\ &\leq 2|\hat{v}(0,t)|^2 + 2\mathbb{E}|u(t)+\hat{v}(u(t),t)-\hat{v}(0,t)|^2 \\ &\leq 2|\hat{v}(0,t)|^2 + 2(1+L_{\hat{v}})^2\mathbb{E}|u(t)|^2 \\ &\leq 2|\hat{v}(0,t)|^2 + 2(1+L_{\hat{v}})^2(1+\mathbb{E}|u_0|^2) \cdot e^{c_5\hat{L}(t-t_0)} \\ &=: c_0(t) + c_1(1+\mathbb{E}|Px_0|^2) \cdot e^{c_2 L(t-t_0)} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}|x(t)-x_0|^2 &= \mathbb{E}|u(t)+\hat{v}(u(t),t)-(u_0+\hat{v}(u_0,t_0))|^2 \\ &\leq 2\mathbb{E}|\hat{v}(u_0,t)-\hat{v}(u_0,t_0)|^2 + 2(1+L_{\hat{v}})^2\mathbb{E}|u(t)-u_0|^2 \\ &\leq 2L_{\hat{v},t}(t-t_0)^2 + (1+L_{\hat{v}})^2 c_6(1+\mathbb{E}|u_0|^2)\cdot(t-t_0)\cdot e^{c_5\hat{L}(t-t_0)} \\ &=: c_3(t-t_0)^2 + c_4(1+\mathbb{E}|Px_0|^2)\cdot(t-t_0)\cdot e^{c_2 L(t-t_0)} \end{aligned}$$

#

3 Numerical stability, consistency and mean-square convergence for discretization methods for SDEs

In numerical computations one always has to be aware of errors and has to choose stable algorithms in the sense that the effect of errors on the computed solutions is controlled. The numerical treatment of SDAEs incorporates not only truncation errors and roundoff

errors, but also defects in solving the constraints or in solving the nonlinear equations in drift-implicit methods. It is not appropriate to assume that these errors tends to zero if the stepsizes do. Another reason for looking carefully at the effect of errors is the ill-posedness of higher index DAEs due to the differentiation task involved causing a weak instability of numerical methods.
Analogously to the analysis of SDAEs in Chapter 2 we will trace back the properties of certain discretization schemes for SDAEs to those for SDEs. Though there is a well-developed convergence analysis for discretization schemes for SDEs, less emphasis has been put on a numerical stability analysis to estimate the effect of errors. Therefore, we supplement the known convergence results by a theorem concerning the numerical stability of discretization schemes for SDEs.

Several different stability properties of numerical methods are discussed in the literature (see e.g. [6, 14, 19]). Numerical stability for a discretization scheme means that the influence of errors occurring in the right-hand side of the discrete system on the discrete solution is bounded by the maximum of these errors multiplied by a so-called stability constant that is independent of the grid. Numerical stability allows to conclude convergence from consistency. In order to distinguish this stability concept from others, it is sometimes called zero-stability. It should not be mistaken for properties like asymptotic stability, which guarantee that for fixed stepsizes (and long or unbounded time-intervals) qualitative properties of the exact solutions like damping behaviour in dissipative systems are preserved by the discrete approximations.

We aim at a numerical stability inequality for discretization schemes for SDEs concerning the mean-square norm of errors of the discrete solution. To avoid rough estimates (losing a factor $h^{1/2}$) we will estimate them by the mean-square norm as well as the conditional mean of the errors perturbing the right-hand sides. This phenomenon is already known from the convergence proofs e.g. in [3, 24]. In [11, 18] it occurs only implicitly due to the comparison with the truncated Itô-Taylor-expansions.

We denote the mean-square norm of a vector-valued square-integrable random variable $z \in L_2(\Omega, I\!R^n)$ by

$$\|z\|_{L_2} := (I\!E|z|^2)^{1/2} .$$

Let us consider the SDE

$$x(s)\Big|_{t_0}^{t} + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} G(x(s), s)dw(s) = 0 , \quad t \in \mathcal{J} , \quad x(t_0) = x_0 , \tag{3.1}$$

where $\mathcal{J} = [t_0, T]$, $f : I\!R^n \times \mathcal{J} \to I\!R^n$, $G : I\!R^n \times \mathcal{J} \to I\!R^{n\times m}$, w is an m-dimensional Wiener process on the given probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\in\mathcal{J}}$, and x_0 is a given $\mathcal{F}_{t_0}$-measurable initial value, independent of the Wiener process and with finite second moments. Assume that there exists a pathwise unique strong solution $x(\cdot)$.

Moreover, let us consider a generally drift-implicit numerical scheme in the form

$$x_\ell = x_{\ell-1} + \underbrace{\varphi(x_{\ell-1}, x_\ell; t_{\ell-1}, h_\ell)}_{=:\varphi_\ell(x_{\ell-1},x_\ell)} + \underbrace{\psi(x_{\ell-1}; t_{\ell-1}, h_\ell, I_\ell)}_{=:\psi_\ell(x_{\ell-1})} , \quad \ell = 1, \ldots, N , \quad x_0 \in I\!R^n \quad (3.2)$$

on the deterministic grid $t_0 < t_1 < \ldots < t_N = T$ with stepsizes $h_\ell := t_\ell - t_{\ell-1}$, $\ell = 1, \ldots, N$. I_ℓ denotes a collection of multiple stochastic integrals

$$I_{i_1,\ldots,i_k;\ell} = \int_{t_{\ell-1}}^{t_\ell} dw_{i_k}(s_1) \int_{t_{\ell-1}}^{s_1} dw_{i_{k-1}}(s_2) \cdots \int_{t_{\ell-1}}^{s_{k-1}} dw_{i_1}(s_k) \quad ,$$

where $i_1, \ldots, i_k$ take values in $\{0, 1, \ldots, m\}$ ($dw_0(s)$ is understood to mean ds) up to a certain finite order k_{max}.
For example, for the drift-implicit Euler scheme

$$x_\ell := x_{\ell-1} + h_\ell f(x_\ell, t_\ell) + G(x_{\ell-1}, t_{\ell-1})\Delta w_\ell \quad , \quad \ell = 1, \ldots, N \quad ,$$

we have $k_{max} = 1$ and

$$\varphi(z, x; t, h) := hf(x, t + h) \quad , \quad \psi(z; t, h, \Delta w) := G(z, t)\Delta w \quad .$$

A similar setting for explicit schemes is used in [3]. We now formulate and prove our main theorem on numerical stability that estimates the effects of errors.

Theorem 3.1 *Suppose that the discretization scheme (3.2) for the SDE (3.1) fulfils the following properties:*

- *for all* $z, \tilde z, x, \tilde x \in I\!R^n$, $t \in \mathcal{J}$, $h \le h^1$ *it holds:*
  (A1) $|\varphi(z, x; t, h) - \varphi(\tilde z, \tilde x; t, h)| \le hL_1|z - \tilde z| + hL_2|x - \tilde x|$.
- *for all* $\ell = 1, \ldots, N$, *for all* $h_\ell \le h^1$, *and for all* $\mathcal{F}_{t_{\ell-1}}$*-measurable random variables* $x_{\ell-1}, \tilde x_{\ell-1}$ *it holds:*
  (A2) $I\!E(\psi_\ell(x_{\ell-1}) - \psi_\ell(\tilde x_{\ell-1})|\mathcal{F}_{t_{\ell-1}}) = 0$,
  (A3) $I\!E(|\psi_\ell(x_{\ell-1}) - \psi_\ell(\tilde x_{\ell-1})|^2|\mathcal{F}_{t_{\ell-1}}) \le h_\ell L_3^2|x_{\ell-1} - \tilde x_{\ell-1}|^2$,
  (A4) *there exists* $I\!E|\psi_\ell(0)|^2 < \infty$.

Suppose that for all $\ell = 1, \ldots, N$ *the perturbations* d_ℓ^* , $\tilde d_\ell$ *are* $\mathcal{F}_{t_\ell}$ *-measurable and possess finite second moments* $I\!E|d_\ell^*|^2$, $I\!E|\tilde d_\ell|^2 < \infty$.
Then there exists a positive constant h^0, *and a grid-independent stability constant* S *such that the following holds true for all grids with maximal stepsize* $h := \max_{\ell=1,\ldots,N} h_\ell$ *with* $h \le h^0$ *and* $h \cdot N \le a \cdot (T - t_0)$ *with a uniform constant* $a \ge 1$:

- *the perturbed discrete system*

$$\tilde x_\ell = \tilde x_{\ell-1} + \varphi_\ell(\tilde x_{\ell-1}, \tilde x_\ell) + \psi_\ell(\tilde x_{\ell-1}) + \tilde d_\ell , \quad \ell = 1, \ldots, N , \quad \tilde x_0 \in I\!R^n \quad (3.3)$$

  possesses a unique solution $(\tilde x_\ell)_{\ell=1}^N$, *and*

- *the mean-square norm* $\varepsilon_\ell := \|x^*_\ell - \tilde{x}_\ell\|_{L_2}$ *of the differences of two solutions* $(x^*_\ell)_{\ell=1}^N$*, and* $(\tilde{x}_\ell)_{\ell=1}^N$ *of perturbed discrete systems related to perturbations* $(d^*_\ell)_{\ell=1}^N$*, and* $(\tilde{d}_\ell)_{\ell=1}^N$ *can be estimated by*

$$\max_{\ell=1,\dots,N} \varepsilon_\ell \leq S(\varepsilon_0 + \max_{\ell=1,\dots,N} \delta_\ell / h_\ell^{1/2} + \max_{\ell=1,\dots,N} \bar{\delta}_\ell / h_\ell) \;, \tag{3.4}$$

where

$$\delta_\ell := \|d^*_\ell - \tilde{d}_\ell\|_{L_2} \;, \quad \bar{\delta}_\ell := \|\mathbb{E}(d^*_\ell - \tilde{d}_\ell | \mathcal{F}_{t_{\ell-1}})\|_{L_2} \;.$$

Definition: If the scheme (3.2) for the SDE (3.1) fulfils the assertion of Theorem 3.1 we call it **numerically stable in the mean-square sense**.

Proof of Theorem 3.1: The proof is organized in three parts. First, we show the existence of unique solutions of the perturbed discrete systems. Second, we show that the second moments of these solutions exist. And third, we derive a stability inequality.
Throughout the second and third part we frequently use the following trivial implications of the Cauchy-Schwarz inequality:

$$\mathbb{E}(x^T y) \leq (\mathbb{E}|x|^2)^{1/2} \cdot (\mathbb{E}|y|^2)^{1/2} \leq \frac{1}{2}(\mathbb{E}|x|^2 + \mathbb{E}|y|^2) \quad , \tag{$*$}$$

$$\mathbb{E}|x+y|^2 = \mathbb{E}|x|^2 + 2\mathbb{E}(x^T y) + \mathbb{E}|y|^2 \leq 2\mathbb{E}|x|^2 + 2\mathbb{E}|y|^2 \quad . \tag{$**$}$$

Part 1 (existence of solutions $\tilde{x}_\ell$):
We consider the scheme (3.3). If the function φ_ℓ does not depend on the current solution value $\tilde{x}_\ell$, the scheme is explicit and the right-hand side of (3.3) gives the new iterate $\tilde{x}_\ell$ explicitly.
Otherwise, the scheme is implicit, and the new iterate $\tilde{x}_\ell$ is given by (3.3) only implicitly. Let us suppose that $\tilde{x}_{\ell-1}$ is a known $\mathcal{F}_{t_{\ell-1}}$ -measurable random variable, and let us express (3.3) as

$$\tilde{x}_\ell = \varphi_\ell(\tilde{x}_{\ell-1}, \tilde{x}_\ell) + \tilde{b}_\ell \;,$$

where, $\tilde{b}_\ell := \tilde{x}_{\ell-1} + \psi_\ell(\tilde{x}_{\ell-1}) + \tilde{d}_\ell$ is now a known $\mathcal{F}_{t_\ell}$ -measurable random variable. Considering the fixed point equation

$$x = \varphi_\ell(z, x) + b =: \eta_\ell(x; z, b) \tag{3.5}$$

we see that $\eta_\ell(x; z, b)$ is a globally contractive mapping with respect to x, since

$$|\eta_\ell(x; z, b) - \eta_\ell(\tilde{x}; z, b)| \leq h_\ell L_2 |x - \tilde{x}| \leq \frac{1}{2}|x - \tilde{x}| \quad \forall h_\ell \leq h \leq h^0 \leq \frac{1}{2L_2} \;.$$

Thus, $\eta_\ell(\cdot; z, b)$ has a globally unique fixed point $x = \xi_\ell(z, b)$. Inserting $(\tilde{x}_{\ell-1}, \tilde{b}_\ell)$ gives the unique solution $\tilde{x}_\ell$ of (3.3). Moreover, $\xi_\ell(z, b)$ depends Lipschitz-continuously on z

and b since

$$\begin{aligned}
|\xi_\ell(z,b)-\xi_\ell(\tilde z,\tilde b)| &= |\eta_\ell(\xi_\ell(z,b);z,b)-\eta_\ell(\xi_\ell(\tilde z,\tilde b);\tilde z,\tilde b)| \\
&= |\varphi_\ell(z,\xi_\ell(z,b))-\varphi_\ell(\tilde z,\xi_\ell(\tilde z,\tilde b))+b-\tilde b| \\
&\overset{A1}{\leq} h_\ell L_1|z-\tilde z|+h_\ell L_2|\xi_\ell(z,b)-\xi_\ell(\tilde z,\tilde b)|+|b-\tilde b| \\
&\leq hL_1|z-\tilde z|+\frac{1}{2}|\xi_\ell(z,b)-\xi_\ell(\tilde z,\tilde b)|+|b-\tilde b|\ ,
\end{aligned}$$

$$|\xi_\ell(z,b)-\xi_\ell(\tilde z,\tilde b)| \leq 2hL_1|z-\tilde z|+2|b-\tilde b|\ .$$

Part 2 (existence of finite second moments $\mathbb{E}|\tilde x_\ell|^2<\infty$):
Assume that $\mathbb{E}|\tilde x_{\ell-1}|^2<\infty$. To show the existence of $\mathbb{E}|\tilde x_\ell|^2<\infty$ we compare $\tilde x_\ell=\xi_\ell(\tilde x_{\ell-1},\tilde b_\ell)$ with $x_\ell^0:=\xi_\ell(0,0)$ using the Lipschitz continuity of the implicit function ξ_ℓ . It holds that

$$|\tilde x_\ell-x_\ell^0|=\xi_\ell(\tilde x_{\ell-1},\tilde b_\ell)-\xi_\ell(0,0)\leq 2hL_1|\tilde x_{\ell-1}|+2|\tilde b_\ell|$$

and, thus

$$\begin{aligned}
\mathbb{E}|\tilde x_\ell|^2 &= \mathbb{E}|(x_\ell-x_\ell^0)+x_\ell^0|^2 \overset{(**)}{\leq} 2\mathbb{E}|x_\ell-x_\ell^0|^2+2|x_\ell^0|^2 \\
&\leq 2\mathbb{E}(2hL_1|\tilde x_{\ell-1}|+2|\tilde b_\ell|)^2+2|x_\ell^0|^2 \\
&\overset{(**)}{\leq} 16(h^2L_1^2\mathbb{E}|\tilde x_{\ell-1}|^2+\mathbb{E}|\tilde b_\ell|^2)+2|x_\ell^0|^2\ .
\end{aligned}$$

It remains to show that $\mathbb{E}|\tilde b_\ell|^2<\infty$. It holds that

$$\begin{aligned}
\mathbb{E}|\tilde b_\ell|^2 &= \mathbb{E}|(\tilde x_{\ell-1}+\tilde d_\ell)+\psi_\ell(\tilde x_{\ell-1})|^2 \overset{(**)}{\leq} 2\mathbb{E}|\tilde x_{\ell-1}+\tilde d_\ell|^2+2\mathbb{E}|\psi_\ell(\tilde x_{\ell-1})|^2 \\
&= 2\mathbb{E}|\tilde x_{\ell-1}+\tilde d_\ell|^2+\mathbb{E}|(\psi_\ell(\tilde x_{\ell-1})-\psi_\ell(0))+\psi_\ell(0)|^2 \\
&\overset{(**)}{\leq} 4\mathbb{E}|\tilde x_{\ell-1}|^2+4\mathbb{E}|\tilde d_\ell|^2+2\mathbb{E}|\psi_\ell(\tilde x_{\ell-1})-\psi_\ell(0)|^2+2\mathbb{E}|\psi_\ell(0)|^2 \\
&\overset{A3}{\leq} 4\mathbb{E}|\tilde x_{\ell-1}|^2+4\mathbb{E}|\tilde d_\ell|^2+2hL_3^2\mathbb{E}|\tilde x_{\ell-1}|^2++2\mathbb{E}|\psi_\ell(0)|^2 \\
&\overset{A4}{<} \infty\ .
\end{aligned}$$

Part 3 (stability inequality): With the notations

$$e_\ell:=x_\ell^*-\tilde x_\ell\quad,\quad d_\ell:=d_\ell^*-\tilde d_\ell$$

we have

$$e_\ell = e_{\ell-1}+\underbrace{\varphi_\ell(x_{\ell-1}^*,x_\ell^*)-\varphi_\ell(\tilde x_{\ell-1},\tilde x_\ell)}_{=:D\varphi_\ell}+\underbrace{\psi_\ell(x_{\ell-1}^*)-\psi_\ell(\tilde x_{\ell-1})}_{=:D\psi_\ell}+d_\ell$$

from the difference of the perturbed discrete systems, and

$$\begin{aligned}
\mathbb{E}|e_\ell|^2 = {} & \mathbb{E}|e_{\ell-1}|^2+\mathbb{E}|D\varphi_\ell|^2+\mathbb{E}|D\psi_\ell|^2+\mathbb{E}|d_\ell|^2+2\mathbb{E}(D\varphi_\ell)^TD\psi_\ell \\
&+2\mathbb{E}e_{\ell-1}^TD\varphi_\ell+2\mathbb{E}e_{\ell-1}^TD\psi_\ell+2\mathbb{E}e_{\ell-1}^Td_\ell+2\mathbb{E}(D\varphi_\ell)^Td_\ell+2\mathbb{E}(D\psi_\ell)^Td_\ell\ .
\end{aligned}$$

We now estimate the different terms in the equation above:

$$
\begin{aligned}
\mathbb{E}|D\varphi_\ell|^2 &\overset{A1}{\leq} h_\ell^2 \mathbb{E}(L_1|e_{\ell-1}| + L_2|e_\ell|)^2 \\
&= h_\ell^2(L_1^2\varepsilon_{\ell-1}^2 + 2L_1L_2\mathbb{E}|e_{\ell-1}||e_\ell| + L_2^2\varepsilon_\ell^2) \\
&\overset{C.S.}{\leq} h_\ell^2(L_1^2\varepsilon_{\ell-1}^2 + 2L_1L_2\varepsilon_{\ell-1}\varepsilon_\ell + L_2^2\varepsilon_\ell^2) = (h_\ell(L_1\varepsilon_{\ell_1} + L_2\varepsilon_\ell))^2 \\
&\leq h_\ell^2(L_1^2\varepsilon_{\ell-1}^2 + L_1L_2(\varepsilon_{\ell-1}^2 + \varepsilon_\ell^2) + L_2^2\varepsilon_\ell^2)
\end{aligned}
$$

$$
\mathbb{E}|D\psi_\ell|^2 = \mathbb{E}\mathbb{E}(|D\psi_\ell|^2|\mathcal{F}_{t_{\ell-1}}) \overset{A3}{\leq} h_\ell L_3^2 \mathbb{E}e_{\ell-1}^2 = h_\ell L_3^2 \varepsilon_{\ell-1}^2
$$

$$
\begin{aligned}
\mathbb{E}(D\varphi_\ell)^T D\psi_\ell &\overset{C.S.}{\leq} (\mathbb{E}|D\varphi_\ell|^2)^{1/2}(\mathbb{E}|D\psi_\ell|^2)^{1/2} \\
&\leq h_\ell(L_1\varepsilon_{\ell-1} + L_2\varepsilon_\ell)h_\ell^{1/2}L_3\varepsilon_{\ell-1} = h_\ell^{3/2}(L_1L_3\varepsilon_{\ell-1}^2 + L_2L_3\varepsilon_{\ell-1}\varepsilon_\ell) \\
&\leq h_\ell^{3/2}(L_1L_3\varepsilon_{\ell-1}^2 + L_2L_3\frac{1}{2}(\varepsilon_{\ell-1}^2 + \varepsilon_\ell^2))
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}e_{\ell-1}^T D\varphi_\ell &\leq \mathbb{E}|e_{\ell-1}||D\varphi_\ell| \overset{A1}{\leq} h_\ell \mathbb{E}|e_{\ell-1}|(L_1|e_{\ell-1}| + L_2|e_\ell|) \\
&= h_\ell L_1\varepsilon_{\ell-1}^2 + h_\ell L_2\mathbb{E}|e_{\ell-1}||e_\ell| \\
&\overset{C.S}{\leq} h_\ell L_1\varepsilon_{\ell-1}^2 + h_\ell L_2\varepsilon_{\ell-1}\varepsilon_\ell \leq h_\ell L_1\varepsilon_{\ell-1}^2 + h_\ell L_2\frac{1}{2}(\varepsilon_{\ell-1}^2 + \varepsilon_\ell^2)
\end{aligned}
$$

$$
\mathbb{E}e_{\ell-1}^T D\psi_\ell = \mathbb{E}\mathbb{E}(e_{\ell-1}^T D\psi_\ell|\mathcal{F}_{t_{\ell-1}}) = \mathbb{E}e_{\ell-1}^T\mathbb{E}(D\psi_\ell|\mathcal{F}_{t_{\ell-1}}) \overset{A2}{=} 0
$$

$$
\begin{aligned}
\mathbb{E}e_{\ell-1}^T d_\ell &= \mathbb{E}\mathbb{E}(e_{\ell-1}^T d_\ell|\mathcal{F}_{t_{\ell-1}}) = \mathbb{E}e_{\ell-1}^T\mathbb{E}(d_\ell|\mathcal{F}_{t_{\ell-1}}) \\
&\overset{C.S.}{\leq} (\mathbb{E}|e_{\ell-1}|^2)^{1/2}(\mathbb{E}|\mathbb{E}(d_\ell|\mathcal{F}_{t_{\ell-1}})|^2)^{1/2} \\
&= h_\ell^{1/2}\varepsilon_{\ell-1} \cdot \bar{\delta}_\ell h_\ell^{-1/2} \leq \frac{1}{2}(h_\ell\varepsilon_{\ell-1}^2 + \bar{\delta}_\ell^2/h_\ell)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(D\varphi_\ell)^T d_\ell &\overset{(*)}{\leq} \leq \frac{1}{2}(\mathbb{E}|D\varphi_\ell|^2 + \mathbb{E}|d_\ell|^2) \\
&\leq \frac{1}{2}(h_\ell^2(L_1^2\varepsilon_{\ell-1}^2 + L_1L_2(\varepsilon_{\ell-1}^2 + \varepsilon_\ell^2) + L_2^2\varepsilon_\ell^2) + \delta_\ell^2)
\end{aligned}
$$

$$
\mathbb{E}(D\psi_\ell)^T d_\ell \overset{(*)}{\leq} \frac{1}{2}(\mathbb{E}|D\psi_\ell|^2 + \mathbb{E}|d_\ell|^2) \;\leq\; \frac{1}{2}(h_\ell L_3^2\varepsilon_{\ell-1}^2 + \delta_\ell^2) \,.
$$

Summarizing we obtain the following inequality:

$$\begin{aligned}
\varepsilon_\ell^2 &\le \varepsilon_{\ell-1}^2\Big(1+h_\ell^2\frac{3}{2}(L_1^2+L_1L_2)+h_\ell^{3/2}(L_1L_3+\frac{1}{2}L_2L_3)+h_\ell(\frac{3}{2}L_3^2+L_1+\frac{1}{2}L_2+\frac{1}{2})\Big)\\
&\quad+\varepsilon_\ell^2\Big(h_\ell^2\frac{3}{2}(L_1L_2+L_2^2)+h_\ell^{3/2}L_2L_3\frac{1}{2}+h_\ell L_2\frac{1}{2}\Big) \;+\; 2\delta_\ell^2 \;+\; \frac{1}{2}\bar\delta_\ell^2/h_\ell\\
&\le \varepsilon_{\ell-1}^2(1+Lh_\ell)+\varepsilon_\ell^2L_2h_\ell \;+\; 2\delta_\ell^2 \;+\; \frac{1}{2}\bar\delta_\ell^2/h_\ell\\
&\le \varepsilon_{\ell-1}^2(1+Lh)+\varepsilon_\ell^2L_2h \;+h\max_{\ell=1,\ldots,N}(\;2\delta_\ell^2/h_\ell \;+\; \frac{1}{2}\bar\delta_\ell^2/h_\ell^2)
\end{aligned}$$

for all $h\in(0,h^0]$ with a sufficiently small h^0 and with a suitable constant L. Hence

$$\begin{aligned}
\varepsilon_\ell^2 &\le \frac{1+Lh}{1-L_2h}\varepsilon_{\ell-1}^2+\frac{h}{1-L_2h}\max_{\ell=1,\ldots,N}(2\delta_\ell^2/h_\ell+\frac{1}{2}\bar\delta_\ell^2/h_\ell^2)\\
&\le (\frac{1+Lh}{1-L_2h})^{\ell-1}\varepsilon_0^2+\frac{h}{1-L_2h}\sum_{i=0}^{\ell-1}(\frac{1+Lh}{1-L_2h})^i\max_{\ell=1,\ldots,N}(2\delta_\ell^2/h_\ell+\frac{1}{2}\bar\delta_\ell^2/h_\ell^2)\\
&\le \exp(\frac{L+L_2}{1-L_2h}\cdot hN)\big(\varepsilon_0^2+\frac{1}{L+L_2}\max_{\ell=1,\ldots,N}(2\delta_\ell^2/h_\ell+\frac{1}{2}\bar\delta_\ell^2/h_\ell^2)\big)\\
&\le S^2\big(\varepsilon_0^2+\max_{\ell=1,\ldots,N}(\delta_\ell^2/h_\ell+\bar\delta_\ell^2/h_\ell^2)\big)
\end{aligned}$$

for $L_2h^0\le 1/2$, where the stability constant is chosen such that

$$S^2 = 2\exp(2(L+L_2)a(T-t_0))\max(1,\frac{1}{L+L_2})\;.$$

As a trivial conclusion we obtain

$$\max_{\ell=1,\ldots,N}\varepsilon_\ell \le S\big(\varepsilon_0+\max_{\ell=1,\ldots,N}(\delta_\ell/h_\ell^{1/2}+\bar\delta_\ell/h_\ell)\big)\quad,$$

which completes the proof. #

Applying Theorem 3.1 to local discretization errors gives convergence results. First, we give the precise notions of strong (or mean-square) consistency and strong (or mean-square) convergence.

We call the numerical scheme (3.2) for the SDE (3.1) **strongly (mean-square) consistent** with order $\gamma>0$ if the local error

$$l_\ell := x(t_\ell)-\check x_\ell\;,\ \text{where}\quad \check x_\ell := x(t_{\ell-1})+\varphi_\ell(x(t_{\ell-1}),x(t_\ell))+\psi_\ell(x(t_{\ell-1}))$$

satisfies

$$\|l_\ell\|_{L_2}\le c\cdot h_\ell^{\gamma+\frac{1}{2}}\;,\ \text{and}\quad \|E(l_\ell|\mathcal{F}_{t_{\ell-1}})\|_{L_2}\le\bar c\cdot h_\ell^{\gamma+1}\quad,\quad \ell=1,\ldots,N\;,$$

with constants $c\,,\,\bar c>0$ only depending on the SDE and its solution.

We call the numerical scheme (3.2) for the SDE (3.1) **strongly (mean-square) convergent** with order $\gamma > 0$ if the global error

$$e_\ell := x(t_\ell) - x_\ell$$

satisfies

$$\max_{\ell=1,\dots,N} \|e_\ell\|_{L_2} \leq C \cdot h^\gamma \ , \ \text{where} \quad h := \max_{\ell=1,\dots,N} h_\ell \ ,$$

with a grid-independent constant $C > 0$.

With these notions a scheme that is numerically stable in the mean-square sense and strongly (mean-square) consistent is strongly (mean-square) convergent. As a corollary from Theorem 3.1 it holds:

Theorem 3.2 *If the numerical scheme (3.2) for the SDE (3.1) is strongly (mean-square) consistent with order* $\gamma > 0$ *and the assumptions of Theorem 3.1 hold true, then (3.2) is strongly (mean-square) convergent with order* γ*. For the difference of the analytical solution* $x(t_\ell)$ *at the discrete time-points and the solution of the perturbed numerical scheme* $\tilde{x}_\ell$ *we have the estimate*

$$\max_{\ell=1,\dots,N} \|x(t_\ell) - \tilde{x}_\ell\|_{L_2} \leq S((c+\bar{c})h^\gamma + \max_{\ell=1,\dots,N} \tilde{\delta}_\ell / h_\ell^{1/2} + \max_{\ell=1,\dots,N} \bar{\tilde{\delta}}_\ell / h_\ell) \ , \tag{3.6}$$

$$\textit{where} \quad \tilde{\delta}_\ell := \|\tilde{d}_\ell\|_{L_2} \ , \quad \bar{\tilde{\delta}}_\ell := \|I\!\!E(\tilde{d}_\ell | \mathcal{F}_{t_{\ell-1}})\|_{L_2} \quad , \quad \textit{with} \quad \tilde{d}_\ell \quad \textit{from (3.3)} \ .$$

Proof: The assertion follows by applying the triangle inequality

$$\max_{\ell=1,\dots,N} \|x(t_\ell) - \tilde{x}_\ell\|_{L_2} \leq \max_{\ell=1,\dots,N} \|x(t_\ell) - x_\ell\|_{L_2} + \max_{\ell=1,\dots,N} \|x_\ell - \tilde{x}_\ell\|_{L_2}$$

and the stability estimate (3.4) once to $\{x(t_\ell)\, , \, x_\ell\}$ related to the perturbations $\{-l_\ell\, , \, 0\}$ and once again to $\{x_\ell\, , \, \tilde{x}_\ell\}$ related to the perturbations $\{0\, , \, \tilde{d}_\ell\}$.
The strong (mean-square) convergence follows as a special case of (3.6) for $\tilde{d}_\ell = 0$. #

These general results apply rather easily to well-known schemes for SDEs. We illustrate this for the family of drift-implicit Euler schemes

$$x_\ell = x_{\ell-1} + h_\ell\big(\alpha f(x_\ell, t_\ell) + (1-\alpha) f(x_{\ell-1}, t_{\ell-1})\big) + G(x_{\ell-1}, t_{\ell-1})\Delta w_\ell \ , \tag{3.7}$$

and for the family of drift-implicit Milstein schemes

$$\begin{aligned} x_\ell = x_{\ell-1} + h_\ell\big(\alpha f(x_\ell, t_\ell) + (1-\alpha) f(x_{\ell-1}, t_{\ell-1})\big) + G(x_{\ell-1}, t_{\ell-1})\Delta w_\ell \\ + \sum_{j=1}^{m} ((g_j)'_x \cdot G)(x_{\ell-1}, t_{\ell-1}) I_{(j),\ell} \ , \end{aligned} \tag{3.8}$$

where $\alpha \in [0,1]$ is a parameter, g_j denotes the j-th column of G, and
$I_{(j),\ell} = (I_{1,j;\ell}, \dots, I_{m,j;\ell})^T$ denotes the vector of double Itô-integrals .

Checking the suppositions of Theorem 3.1 we will see that these methods are numerically stable in the mean-square sense. It holds:

Lemma 3.3 *Let the functions f and G be Lipschitz-continuous with respect to x. Then the Euler schemes (3.7) are numerically stable in the mean-square sense.*
If, additionally, the partial derivatives $(g_j)'_x, j = 1, \dots, m$ exist and the functions $(g_j)'_x \cdot G$ are Lipschitz-continuous with respect to x, then the Milstein schemes (3.8) are numerically stable in the mean-square sense.

Proof: We will prove the numerical stability by applying Theorem 3.1. Let L_f, L_G denote the Lipschitz constants of f and G with respect to x. For the family of Euler schemes (3.7) we have

$$\varphi(z, x; t, h) := h\big(\alpha f(x, t+h) + (1-\alpha) f(z, t)\big) \quad , \quad \psi(z; t, h, \Delta w_\ell) = G(z, t)\Delta w_\ell \quad .$$

(A1) is fulfilled with $L_1 := (1-\alpha)l_f$, $L_2 := \alpha L_f$.
(A2) is fulfilled since the Wiener increments Δw_ℓ are independent of $\mathcal{F}_{t_{\ell-1}}$ and have zero expectation.
(A3) holds true since

$$\begin{aligned} I\!\!E(|\psi_\ell(x_{\ell-1}) - \psi_\ell(\tilde{x}_{\ell-1})|^2|\mathcal{F}_{t_{\ell-1}}) &= I\!\!E(|\big(G(x_{\ell-1}, t_{\ell-1}) - G(\tilde{x}_{\ell-1}, t_{\ell-1})\big)\Delta w_\ell|^2|\mathcal{F}_{t_{\ell-1}}) \\ &\le \|G(x_{\ell-1}, t_{\ell-1}) - G(\tilde{x}_{\ell-1}, t_{\ell-1})\|^2 \cdot I\!\!E(|\Delta w_\ell|^2|\mathcal{F}_{t_{\ell-1}}) \\ &\le h_\ell L_G^2 |x_{\ell-1} - \tilde{x}_{\ell-1}|^2 \ . \end{aligned}$$

Finally, (A4) holds true since

$$I\!\!E|\psi_\ell(0)|^2 = I\!\!E|G(0, t_{\ell-1})\Delta w_\ell|^2 \le \|G(0, t_{\ell-1})\|^2 I\!\!E|\Delta w_\ell|^2 = h_\ell \|G(0, t_{\ell-1})\|^2 < \infty \ .$$

For the family of Milstein schemes (3.8) we have the same function φ as for the Euler schemes and

$$\psi(z; t, h, \Delta w_\ell, I_{(1),\ell}, \dots, I_{(m),\ell}) = G(z, t)\Delta w_\ell + \sum_{j=1}^{m} ((g_j)'_x \cdot G)(z, t) I_{(j),\ell} \ .$$

(A1) is fulfilled as above. (A2) is fulfilled since Δw_ℓ and the double Itô-integrals $I_{(j),\ell}$ are independent of $\mathcal{F}_{t_{\ell-1}}$ and have zero expectation. (A3) holds true due to the assumed Lipschitz-continuity of the functions $(g_j)'_x G$. (A4) holds true analogously to the Euler schemes with additional terms involving $I\!\!E|I_{(j),\ell}|^2 = O(h_\ell^2)$. Hence, by Theorem 3.1, the Euler schemes (3.7) and the Milstein schemes (3.8) are numerically stable in the mean square sense. #

From the literature (see e.g. [24]) it is known that the Euler schemes (3.7) are strongly consistent with order 1/2 if, additionally, the coefficients are Hölder-continuous with exponent 1/2 with respect to t. The Milstein schemes are strongly consistent with order 1 if the functions f, G are sufficiently smooth. Applying Theorem 3.2 then gives the known strong (mean square) convergence of the Euler schemes with order 1/2 and the Milstein schemes with order 1.

We conclude this chapter with applying the general results of the Theorems 3.1 and 3.2 to the family of split-step Euler schemes

$$x_\ell^* = x_{\ell-1} + h_\ell\big(\alpha f(x_\ell^*, t_\ell) + (1-\alpha) f(x_{\ell-1}, t_{\ell-1})\big) \ , \tag{3.9}$$
$$x_\ell = x_\ell^* + G(x_\ell^*, t_\ell)\Delta w_\ell \ , \tag{3.10}$$

where $\alpha \in [0,1]$ is a parameter. Unless $\alpha = 0$, (3.9) is an implicit deterministic equation in x_ℓ^*. For $\alpha = 1$ we obtain the split-step backward Euler scheme (SSBE), which is studied e.g. in [16] for autonomous SDEs. In [16] strong convergence of order 1/2 is proved under only one-sided Lipschitz conditions and a polynomial growth condition for the drift coefficient. The SSBE is also studied in [23], where it is shown to be effective for inheriting ergodicity in special applications.
From the numerical theory for ordinary differential equations (ODEs) [7, 15] it is known that the implicit equation

$$x - z - h\big(\alpha f(x, t+h) + (1-\alpha) f(z,t)\big) = 0 \tag{3.11}$$

possesses a unique solution $x = \chi_\alpha(z,t,h)$ for all $z \in I\!R^m, t \in \mathcal{J}, h$ with $h\alpha\mu \leq 1/2$ if f is continuous, has a continuous derivative with respect to x and fulfils the one-sided Lipschitz-condition

$$\langle f(x,t) - f(\tilde{x},t), x - \tilde{x}\rangle \leq \mu |x - \tilde{x}|^2 \quad \forall x, \tilde{x} \in I\!R^m, \, t \in \mathcal{J} \quad . \tag{3.12}$$

Hence, under the above conditions on f, the equation (3.9) possesses the unique solution $x_\ell^* = \chi_\alpha(x_{\ell-1}, t_{\ell-1}, h_\ell)$ and (3.9),(3.10) can be written as the formally explicit scheme

$$\begin{aligned} x_\ell = x_{\ell-1} \quad + \quad & h_\ell\big(\alpha f(\chi_\alpha(x_{\ell-1}, t_{\ell-1}, h_\ell), t_\ell) + (1-\alpha) f(x_{\ell-1}, t_{\ell-1})\big) \\ & + G(\chi_\alpha(x_{\ell-1}, t_{\ell-1}, h_\ell), t_\ell)\Delta w_\ell \,. \end{aligned} \tag{3.13}$$

With

$$\begin{aligned} \varphi(z,x;t,h) &:= h\big(\alpha f(\chi_\alpha(z,t,h), t+h) + (1-\alpha) f(z,t)\big) \,, \\ \psi(z;t,h,\Delta w) &:= G(\chi_\alpha(z,t,h), t+h)\Delta w \,, \end{aligned}$$

we see that the split-step Euler schemes are of the general form (3.2). Now, we are able to verify that (3.9),(3.10) is numerically stable in the mean square sense and strongly (mean square) consistent with order 1/2 if the coefficients are globally Lipschitz continuous with respect to x, and Hölder continuous with exponent 1/2 with respect to t. We formulate the results in the following lemmata:

Lemma 3.4 *Let f, G be continuous and Lipschitz continuous with respect to x. Then the split-step Euler methods are numerically stable in the mean square sense.*

Proof: It suffices to check the assumptions (A1)-(A4) of Theorem 3.1.
Let f, G be Lipschitz continuous with respect to x with constants L_f, L_G. Then f trivially fulfils a one-sided Lipschitz condition with $\mu = L_f$. For all $h \leq 1/(2\alpha L_f)$ the equation (3.11) has the unique solution $x = \chi_\alpha(z,t,h)$ and the implicit function χ_α is Lipschitz continuous with respect to z with a constant $L_\chi := 2(1 + h(1-\alpha)L_f)$. Now we see:
(A1) holds true with $L_1 := \alpha L_f L_\chi + (1-\alpha)L_f$ and $L_2 := 0$.
(A2) holds true since $x_\ell^* = \chi_\alpha(x_{\ell-1}, t_{\ell-1}, h_\ell)$ is $\mathcal{F}_{t_{\ell-1}}$-measurable.
(A3) holds true with $L_3 := L_G L_\chi$. Finally, (A4) holds true since

$$I\!E|\psi_\ell(0)|^2 = I\!E|G(\chi_\alpha(0, t_{\ell-1}, h_\ell), t_\ell)\Delta w_\ell|^2 \leq h_\ell \|G(\chi_\alpha(0, t_{\ell-1}, h_\ell), t_\ell)\| < \infty \,.$$

The assertion follows by Theorem 3.1. #

Lemma 3.5 *Let f, G be Lipschitz continuous with respect to x and Hölder continuous with exponent $1/2$ with respect to t. Then the split-step Euler methods are strongly (mean square) consistent with order $1/2$.*

Proof: Let us use the notations of Lemma 3.4 and, further, denote the Hölder constants of f, G with respect to t by H_f, H_G. f also fulfils a linear growth condition

$$|f(x,t)| \leq K_f + L_f|x| \quad \text{with} \quad K_f := \|f(0,\cdot)\|_\infty \,.$$

As in Lemma 3.4 the implicit function $\chi_\alpha(z,t,h)$ is Lipschitz continuous with respect to z for $h\alpha L_f < 1/2$ and fulfils a linear growth condition

$$|\chi_\alpha(z,h,t)| \leq K_\chi + L_\chi|x| \,, \quad K_\chi := 2hK_f \,.$$

Now we consider the local errors. For simplicity we will restrict ourselves to the SSBE scheme ($\alpha = 1$) and denote $\chi_\ell(z) := \chi_1(z, t_{\ell-1}, h_\ell)$. We have

$$\begin{aligned} l_\ell &:= x(t_\ell) - x(t_{\ell-1}) + h_\ell f(\chi_\ell(x(t_{\ell-1})), t_\ell) + G(\chi(x_\ell(t_{\ell-1})), t_\ell)\Delta w_\ell \\ &= \int_{t_{\ell-1}}^{t_\ell} f(x(s),s) - f(\chi_\ell(x(t_{\ell-1})), t_\ell)ds + \int_{t_{\ell-1}}^{t_\ell} G(x(s),s) - G(\chi_\ell(x(t_{\ell-1})), t_\ell)dw(s) \\ &=: \mathcal{I}_{\Delta f} \;\; + \;\; \mathcal{I}_{\Delta G} \,. \end{aligned}$$

For the difference $|x(s) - \chi_\ell(x(t_{\ell-1}))|$ we have

$$\begin{aligned} |x(s) - \chi_\ell(x(t_{\ell-1}))| &= |\chi_\ell(x(s)) - h_\ell f(\chi_\ell(x(s)), t_\ell) + -\chi_\ell(x(t_{\ell-1}))| \\ &\leq L_\chi|x(s) - x(t_{\ell-1})| + h_\ell(K_f + L_f|\chi(x(s))|) \\ &\leq L_\chi|x(s) - x(t_{\ell-1})| + h_\ell(K_f + L_f(2h_\ell K_f + L_\chi|x(s)|)) \,, \end{aligned}$$

and, hence,

$$\begin{aligned} |x(s) - \chi_\ell(x(t_{\ell-1}))|^2 &\leq c_1|x(s) - x(t_{\ell-1})|^2 + c_2h_\ell^2 + c_3h_\ell^2|x(s)|^2 \,, \\ \mathbb{E}|x(s) - \chi_\ell(x(t_{\ell-1}))|^2 &\leq c_4|s - t_{\ell-1}| + c_5h_\ell^2 \end{aligned}$$

with positive constants $c_1, \ldots, c_5$. It holds that

$$\begin{aligned} \delta_\ell^2 &:= \mathbb{E}|l_\ell|^2 = \mathbb{E}|\mathcal{I}_{\Delta f} + \mathcal{I}_{\Delta G}|^2 \leq 2\mathbb{E}|\mathcal{I}_{\Delta f}|^2 + 2\mathbb{E}|\mathcal{I}_{\Delta G}|^2 \,, \\ \bar{\delta}_\ell^{\,2} &:= \mathbb{E}|\mathbb{E}(l_\ell|\mathcal{F}_{t_{\ell-1}})|^2 = \mathbb{E}|\mathbb{E}(\mathcal{I}_{\Delta f}|\mathcal{F}_{t_{\ell-1}})|^2 \leq \mathbb{E}(\mathbb{E}(|\mathcal{I}_{\Delta f}|^2|\mathcal{F}_{t_{\ell-1}})) = \mathbb{E}|\mathcal{I}_{\Delta f}|^2 \,. \end{aligned}$$

As an example, we estimate the term $\mathbb{E}|\mathcal{I}_{\Delta G}|^2$ and obtain

$$\begin{aligned} \mathbb{E}|\mathcal{I}_{\Delta G}|^2 &= \int_{t_{\ell-1}}^{t_\ell} \mathbb{E}|G(x(s),s) - G(\chi_\ell(x(t_{\ell-1})), t_\ell)|^2 ds \\ &\leq 2\int_{t_{\ell-1}}^{t_\ell} (L_G^2\mathbb{E}|x(s) - \chi_\ell(x(t_{\ell-1}))|^2 + H_G^2|t_\ell - s|)ds \\ &= H_G^2h_\ell^2 + 2L_G^2\int_{t_{\ell-1}}^{t_\ell} \mathbb{E}|x(s) - \chi_\ell(x(t_{\ell-1}))|^2 ds \\ &\leq H_G^2h_\ell^2 + L_G^2(c_4h_\ell^2 + 2c_5h_\ell^3) = \mathcal{O}(h_\ell^2) \,. \end{aligned}$$

Similarly one obtains $\mathbb{E}|\mathcal{I}_{\Delta f}|^2 = \mathcal{O}(h_\ell^3)$ and, hence, $\delta_\ell = \mathcal{O}(h_\ell)$, $\bar{\delta}_\ell = \mathcal{O}(h_\ell^{3/2})$, which completes the proof. #

4 Discretization schemes for index 1 SDAEs

Starting with the Euler Maruyama scheme [20] a wide spectrum of numerical methods for SDEs has been developed. However, first deriving an inherent regular SDE and then applying numerical methods to this special SDE would be a very inefficient procedure for various reasons. In general, one would have to apply a numerical method to solve the constraints for the algebraic variables. First, it would be much more difficult to exploit special structures and sparseness of the given system. And second, implicit methods are necessary anyway if the underlying dynamics are stiff. Here we aim at numerical methods for SDAEs that should work directly on the given implicit structure, as in the case of deterministic DAEs.

Not all of the discretization schemes for SDEs are suitable for SDAEs as well. Explicit numerical schemes are not suitable for the nonlinear equation solving task involved in SDAEs. The new iterates would not be uniquely determined by an explicit scheme. In this section we will consider the drift-implicit Euler scheme, the split-step backward Euler scheme, the trapezoidal rule, and the drift-implicit Milstein scheme. We will formulate them in such a way that the convergence properties of these methods for SDEs are preserved. Nevertheless, we will see that the iterates are influenced more critically by computational errors in the constraints.

4.1 The drift-implicit Euler scheme

We discretize

$$\begin{aligned} Ax'(t) + f(x(t),t) + G(x(t),t)\xi(t) &= 0 \quad , \quad t \in \mathcal{J} \; , && (4.1)\\ Ax(t_0) &= Ax^0 && (4.2) \end{aligned}$$

by the drift-implicit Euler-scheme

$$A\frac{x_\ell - x_{\ell-1}}{h_\ell} + f(x_\ell, t_\ell) + G(x_{\ell-1}, t_{\ell-1})\frac{\Delta w_\ell}{h_\ell} = 0 \; , \quad \ell = 1, \ldots, N \; , \qquad (4.3)$$

with a given consistent initial value x_0 , where $h_\ell = t_\ell - t_{\ell-1}$, $\Delta w_\ell = w(t_\ell) - w(t_{\ell-1})$ on the grid $0 = t_0 < t_1 < \ldots < t_N = T$.
The Jacobian of (4.3) with respect to the new iterate x_ℓ is (as in the deterministic case)

$$\frac{1}{h_\ell}A + f'_x(x_\ell, t_\ell) \; ,$$

which is non-singular for sufficiently small stepsizes h_ℓ. Its condition number behaves like $\mathcal{O}(\frac{1}{h_\ell})$ (see e.g.[12]). The crucial point for the good properties of the drift-implicit Euler-scheme for the SDAE (4.1) is that the iterates have to fulfil the constraints of the SDAE at the actual time-point

$$Rf(x_\ell, t_\ell) = 0 \; .$$

This allows to perform an analogous decoupling procedure as in Chapter 2 for the continuous problem. We will do this decoupling for the theoretical understanding now. Denote

$$u_\ell := Px_\ell \ , \ v_\ell := Qx_\ell \ , \ \ell = 0, \ldots, N \ .$$

Then the drift-implicit Euler scheme (4.3) with consistent initial value x_0 for the SDAE (4.1) is equivalent to the composition

$$x_\ell := u_\ell + \hat{v}(u_\ell, t_\ell) \ , \quad \ell = 0, \ldots, N \ ,$$

and the following scheme in the differential solution parts u_ℓ:

$$\begin{aligned} \frac{u_\ell - u_{\ell-1}}{h_\ell} \ &+ \ D(I-R)f(u_\ell + \hat{v}(u_\ell, t_\ell), t_\ell) \\ &+ \ DG(u_{\ell-1} + \hat{v}(u_{\ell-1}, t_{\ell-1}), t_{\ell-1}) \frac{\Delta w_\ell}{h_\ell} \ = \ 0 \end{aligned} \tag{4.4}$$

with initial value $u_0 := Px_0$. This is the drift-implicit Euler scheme applied to the inherent SDE (2.12). Thus, convergence results for (4.3) applied to the SDAE (4.1) can be deduced from convergence results of the drift-implicit Euler scheme applied to SDEs. We can trace back conditions on the coefficients of the inherent SDE to corresponding ones on the coefficients of the original SDAE. That way all known convergence results for the drift-implicit Euler scheme for SDEs apply to index 1 SDAEs, too.

For a more detailed look including also the numerical stability of the method we compare the exact solution $\{x_\ell\}$ of (4.3) with the solutions of a scheme perturbed by small errors $\{d_\ell\}$. Clearly, the measurement of errors in the discrete equations depends on the scaling of these equations. And, in contrast to ODEs or SDEs, there is no natural scaling of DAEs or SDAEs. Choosing errors in a setting corresponding to that of local truncation errors we consider the system

$$A(\tilde{x}_\ell - \tilde{x}_{\ell-1}) + h_\ell f(\tilde{x}_\ell, t_\ell) + G(\tilde{x}_{\ell-1}, t_{\ell-1})\Delta w_\ell = d_\ell \ , \quad \ell = 1, \ldots, N \ , \tag{4.5}$$

or

$$A\frac{\tilde{x}_\ell - \tilde{x}_{\ell-1}}{h_\ell} + f(\tilde{x}_\ell, t_\ell) + G(\tilde{x}_{\ell-1}, t_{\ell-1})\frac{\Delta w_\ell}{h_\ell} = \frac{d_\ell}{h_\ell} \ , \quad \ell = 1, \ldots, N \ ,$$

with initial value $\tilde{x}_0$. Let $\tilde{u}_\ell := P\tilde{x}_\ell$, $\tilde{v}_\ell := Q\tilde{x}_\ell$.
Multiplying by the projector R we obtain the perturbed constraints

$$Rf(\tilde{u}_\ell + \tilde{v}_\ell, t_\ell) = \frac{Rd_\ell}{h_\ell} \ , \tag{4.6}$$

which, together with the condition $P\tilde{v}_\ell = 0$, implicitly determine

$$\tilde{v}_\ell = \hat{v}(\tilde{u}_\ell, t_\ell, Rd_\ell / h_\ell) \ .$$

Now, for the differential parts $\{\tilde{u}_\ell\}$ we obtain the scheme

$$\begin{aligned} \tilde{u}_\ell - \tilde{u}_{\ell-1} \ &+ \ D(I-R)h_\ell f(\tilde{u}_\ell + \hat{v}(\tilde{u}_\ell, t_\ell, Rd_\ell/h_\ell), t_\ell) \\ &+ \ DG(\tilde{u}_{\ell-1} + \hat{v}(\tilde{u}_{\ell-1}, t_{\ell-1}, Rd_{\ell-1}/h_{\ell-1}), t_{\ell-1})\Delta w_\ell \ = \ D(I-R)d_\ell \ , \end{aligned} \tag{4.7}$$

which can be written in the form

$$\tilde u_\ell - \tilde u_{\ell-1} + h_\ell \hat f(\tilde u_\ell, t_\ell) + \hat G(\tilde u_{\ell-1}, t_{\ell-1})\Delta w_\ell \;=\; \hat d_\ell \;, \tag{4.8}$$

where

$$\begin{aligned}
\hat d_\ell &= D(I-R)\{d_\ell - h_\ell d_{f,\ell} - d_{G,\ell-1}\Delta w_\ell\}\;,\\
d_{f,\ell} &= f(\tilde u_\ell + \hat v(\tilde u_\ell, t_\ell, Rd_\ell/h_\ell), t_\ell) - f(\tilde u_\ell + \hat v(\tilde u_\ell, t_\ell, 0), t_\ell)\;,\\
d_{G,\ell-1} &= G(\tilde u_{\ell-1} + \hat v(\tilde u_{\ell-1}, t_{\ell-1}, Rd_{\ell-1}/h_{\ell-1}), t_{\ell-1}) - G(\tilde u_{\ell-1} + \hat v(\tilde u_{\ell-1}, t_{\ell-1}, 0), t_{\ell-1})\;.
\end{aligned}$$

Now, suppose that the perturbations d_ℓ are $\mathcal{F}_{t_\ell}$ - measurable with finite second moments and denote

$$\delta_\ell := \|d_\ell\|_{L_2}\;, \quad \bar\delta_\ell := \|\mathbb{E}(d_\ell|\mathcal{F}_{t_{\ell-1}})\|_{L_2}\;,$$

$$\hat\delta_\ell := \|\hat d_\ell\|_{L_2}\;, \quad \bar{\hat\delta}_\ell := \|\mathbb{E}(\hat d_\ell|\mathcal{F}_{t_{\ell-1}})\|_{L_2}\;, \quad \hat D := D(I-R)\;.$$

Then $d_{f,\ell}$ is $\mathcal{F}_{t_\ell}$ - measurable, $d_{G,\ell-1}$ is $\mathcal{F}_{t_{\ell-1}}$ - measurable, and we can estimate

$$\begin{aligned}
\hat\delta_\ell &\le \|\hat D\|\big(\|d_\ell - h_\ell d_{f,\ell}\|_{L_2} + \|d_{G,\ell-1}\Delta w_\ell\|_{L_2}\big)\\
&\le \|\hat D\|\big((1+L_f L_{\hat v,d})\delta_\ell + \|d_{G,\ell-1}\|_{L_2}\cdot h_\ell^{1/2}\big)\\
&\le \|\hat D\|\big((1+L_f L_{\hat v,d})\delta_\ell + L_G L_{\hat v,d}\|Rd_{\ell-1}\|_{L_2}/h_\ell^{1/2}\big)\\
&= \mathcal{O}(\delta_\ell) + \mathcal{O}(\|Rd_{\ell-1}\|_{L_2}/h_\ell^{1/2})
\end{aligned}$$

and

$$\begin{aligned}
\bar{\hat\delta}_\ell &\le \|\hat D\mathbb{E}(d_\ell - h_\ell d_{f,\ell}|\mathcal{F}_{t_{\ell-1}})\|_{L_2} \le \|\hat D\|(\bar\delta_\ell + h_\ell\|\mathbb{E}(d_{f,\ell}|\mathcal{F}_{t_{\ell-1}})\|_{L_2})\\
&\le \|\hat D\|(\bar\delta_\ell + L_f L_{\hat v,d}\|Rd_\ell\|_{L_2}) \;=\; \mathcal{O}(\bar\delta_\ell) + \mathcal{O}(\|Rd_\ell\|_{L_2})\;.
\end{aligned}$$

We emphasize that both quantities $\hat\delta_\ell$, $\bar{\hat\delta}_\ell$ are effected more critically by perturbations of the constraints. Considering only local discretization errors these critical terms vanish since the values of the exact solution $x(t_\ell)$ fulfil the constraints exactly.
Theorem 3.2 for the drift-implicit Euler scheme applied to the inherent SDE gives the following estimate for the global errors of the differential components $Px(t_\ell) - P\tilde x_\ell$:

$$\begin{aligned}
\max_{\ell=1,\dots,N}\|Px(t_\ell) - P\tilde x_\ell\|_{L_2} &\le \hat S(\hat c\cdot h^{1/2} + \max_{\ell=1,\dots,N}\hat\delta_\ell/h_\ell^{1/2} + \max_{\ell=1,\dots,N}\bar{\hat\delta}_\ell/h_\ell)\;,\\
&\le \hat S(\hat c\cdot h^{1/2} + \max_{\ell=1,\dots,N}(c_1\delta_\ell/h_\ell^{1/2} + c_2\|Rd_{\ell-1}\|_{L_2}/h_\ell)\\
&\qquad + \max_{\ell=1,\dots,N}(c_3\bar\delta_\ell/h_\ell + c_4\|Rd_\ell\|_{L_2}/h_\ell))\\
&\le \hat S(\hat c\cdot h^{1/2} + c_1\max_{\ell=1,\dots,N}\delta_\ell/h_\ell^{1/2} + c_3\max_{\ell=1,\dots,N}\bar\delta_\ell/h_\ell\\
&\qquad + c_5\max_{\ell=0,\dots,N}\|Rd_\ell\|_{L_2}/h_\ell)\;.
\end{aligned} \tag{4.9}$$

By the composition of the solution (2.11) and due to the Lipschitz-continuity of the implicit function $\hat v$ we obtain:

Corollary 4.1 *Let the suppositions of Theorem 2.1 be fulfilled. Furthermore, let f, G be Hölder continuous with exponent $1/2$ with respect to t with a Hölder constant growing only linearly with x.*
Then the following estimate

$$\max_{\ell=1,\dots,N} \|x(t_\ell)-\tilde{x}_\ell)\|_{L_2} \le S(\hat{c}\cdot h^{1/2}+c_1 \max_{\ell=1,\dots,N} \delta_\ell/h_\ell^{1/2}+c_3 \max_{\ell=1,\dots,N} \bar{\delta}_\ell/h_\ell+c_6 \max_{\ell=0,\dots,N} \|Rd_\ell\|_{L_2}/h_\ell)$$

holds for the global errors $x(t_\ell) - \tilde{x}_\ell$ of the perturbed drift-implicit Euler scheme (4.5).

Proof: Similarly to the proof of Theorem 2.1 one can show that the smoothness assumptions on the coefficients f, G of the SDAE (4.1) carry over to corresponding smoothness properties of the coefficients $\hat{f}, \hat{G}$ of the inherent regular SDE. Then the drift-implicit Euler scheme applied to the inherent regular SDE is strongly (mean-square) consistent with order 1/2 (compare e.g. [24]) and the estimate (4.9) holds for the discretization of the inherent regular SDE. The solution of the SDAE (4.1) as well as the solution of the perturbed discrete system (4.5) are composed by

$$x(t) := Px(t) + \hat{v}(Px(t), t) , \quad \tilde{x}_\ell := P\tilde{x}_\ell + \hat{v}(P\tilde{x}_\ell, t_\ell, Rd_\ell/h_\ell) .$$

Due to the Lipschitz property of the implicit function $\hat{v}$ it follows that

$$|x(t_\ell) - \tilde{x}_\ell| \le (1 + L_{\hat{v}})|Px(t_\ell) - P\tilde{x}_\ell| + L_{\hat{v},d}|Rd_\ell|/h_\ell ,$$

and in combination with (4.9) we obtain the asserted estimate with $S := (1+L_{\hat{v}})\hat{S}$ and $c_6 := c_5 + L_{\hat{v},d}/S$. #

4.2 The split-step backward Euler scheme

At the end of Chapter 3 we considered the split-step backward Euler scheme (SSBE) for SDEs. Here, we intend to construct a scheme for the SDAE (4.1) that should realize the SSBE for the inherent regular SDE

$$u' + \hat{f}(u,t) + \hat{G}(u,t)\xi(t) := u' + D(I-R)f(u+\hat{v}(u,t),t) + DG(u+\hat{v}(u,t),t)\xi(t) = 0 .$$

The first step (3.9,$\alpha=1$) is realized straightforwardly by applying the backward Euler scheme to the deterministic part of the SDAE. But, realizing the second step (3.10) causes more effort than an assignment since we have to force the iterates to fulfil the constraints. Here we give a realization that explicitly uses a projector R along im A:

$$\begin{aligned} A(x_\ell^* - x_{\ell-1}) + h_\ell f(x_\ell^*, t_\ell) &= 0 , & (4.10)\\ A(x_\ell - x_\ell^*) + Rf(x_\ell, t_\ell) + G(x_\ell^*, t_\ell)\Delta w_\ell &= 0 , \quad \ell = 1, \dots, N . & (4.11) \end{aligned}$$

(4.10) as well as (4.11) are implicit equations in x_ℓ^* resp. x_ℓ. The Jacobian $A+h_\ell f'_x(x_\ell^*, t_\ell)$ of (4.10) has the same structure as for the drift-implicit Euler method. Its condition number behaves like $\mathcal{O}(1/h_\ell)$. The Jacobian $A+Rf'_x(x_\ell, t_\ell)$ of (4.11) is nonsingular with bounded condition number due to the index 1 condition.
Multiplying (4.10) as well as (4.11) by the projector R along im A we see that both, x_ℓ^* and x_ℓ are forced to fulfil the constraints. That is why (4.10), (4.11) realizes the SSBE

scheme for the inherent regular SDE. (4.10), (4.11) show the convergence properties stated in [16] if the inherent regular SDE meets the conditions stated there, where the one-sided Lipschitz condition for the drift-coefficient $\hat{f}$ should be considered on the invariant subspace im P only. In terms of the original SDAE that means

$$\langle Px - P\tilde{x}, D(I-R)(f(x,t) - f(\tilde{x},t))\rangle \leq \mu|Px - P\tilde{x}|^2 \quad \forall x, \tilde{x} \in \mathcal{M}(t) \quad ,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product and $\mathcal{M}(t) := \{z \in I\!R^m : Rf(z,t) = 0\}$ denotes the constraint manifold.
Alternatively, strong (mean square) convergence with order 1/2 is guaranteed if $\hat{f}, \hat{G}$ fulfil the suppositions of the Lemmata 3.4,3.5, that is, if f, G are Lipschitz continuous with respect to x and Hölder continuous with respect to t.

4.3 The trapezoidal rule

Due to its symmetric asymptotic stability behaviour the trapezoidal rule is widely used to integrate oscillatory solutions of ODEs. It is A-stable and of order 2 . It is also applied to index 1 DAEs of the form

$$Ax'(t) + f(x(t), t) = 0 \tag{4.12}$$

via the following reformulation: Formally transforming (4.12) to the augmented semi-explicit system

$$\begin{aligned} x'(t) - y(t) &= 0 \\ Ay(t) + f(x(t), t) &= 0 \quad , \end{aligned}$$

discretizing the differential equations by the trapezoidal rule

$$\begin{aligned} \frac{x_\ell - x_{\ell-1}}{h} - \frac{1}{2}\{y_\ell + y_{\ell-1}\} &= 0 \\ Ay_\ell + f(x_\ell, t_\ell) &= 0 \, , \end{aligned}$$

and reformulating this system to

$$y_\ell := -y_{\ell-1} + 2\frac{x_\ell - x_{\ell-1}}{h} \;, \quad A(-y_{\ell-1} + 2\frac{x_\ell - x_{\ell-1}}{h}) + f(x_\ell, t_\ell) = 0$$

implicitly realizes the trapezoidal rule for the inherent regular ODE. Formally the augmented system is no longer of index 1 since the constraints $Ay(t) + f(x(t), t) = 0$ are not solvable for the algebraic variables y of this augmented system. The components Qy are determined only after a hidden differentiation, but, this does not matter since these components do not enter the implicit formula in x_ℓ. They are of no interest here. Implementing requires only residuals.
As a stochastic counterpart of the trapezoidal rule for integrating SDEs (3.1) there is the special member of the family of Euler schemes (3.7) with parameter $\alpha = 1/2$:

$$\frac{x_\ell - x_{\ell-1}}{h_\ell} = \frac{1}{2}(f(x_\ell, t_\ell) + f(x_{\ell-1}, t_{\ell-1})) + G(x_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell}\Delta w_\ell \, . \tag{4.13}$$

Like the other Euler methods the trapezoidal rule (4.13) is of strong order 1/2.
Trying to apply this method to index 1 SDAEs (4.1), a procedure analogous to that in the deterministic case would lead to an implicit discretization of the diffusion term.
A way out is to create explicit constraints. This can be done by a suitable scaling of (4.1): We scale the system by a suitable non-singular matrix $\tilde{D}$ such that

$$\tilde{D}A = \begin{pmatrix} \tilde{D}_1 A \\ \tilde{D}_2 A \end{pmatrix} = \begin{pmatrix} \tilde{A}_1 \\ 0 \end{pmatrix} \quad \text{or} \quad \tilde{D}R = \begin{pmatrix} 0 \\ \tilde{R}_2 \end{pmatrix} \quad , \ \mathrm{rank}\tilde{A}_1 = \mathrm{rank}A.$$

Then the trapezoidal rule is realized by the scheme

$$\begin{aligned} \tilde{A}_1 \tfrac{x_\ell - x_{\ell-1}}{h} + \tfrac{1}{2}\{\tilde{D}_1 f(x_\ell, t_\ell) + \tilde{D}_1 f(x_{\ell-1}, t_{\ell-1})\} + \tilde{D}_1 G(x_{\ell-1}, t_{\ell-1}) \tfrac{1}{h_\ell} \Delta w_\ell &= 0 \\ \tilde{D}_2 f(x_\ell, t_\ell) &= 0\,. \end{aligned} \tag{4.14}$$

The iterates fulfil the constraints at the current time-point. Hence, the trapezoidal rule for the inherent SDE is realized. Since the differential equations and the constraints are now decoupled, it is possible to use a different scaling for both parts, which leads to a better conditioned system:

$$\begin{aligned} \tilde{A}_1 (x_\ell - x_{\ell-1}) + \tfrac{h_\ell}{2}\{\tilde{D}_1 f(x_\ell, t_\ell) + \tilde{D}_1 f(x_{\ell-1}, t_{\ell-1})\} + \tilde{D}_1 G(x_{\ell-1}, t_{\ell-1}) \Delta w_\ell &= 0 \\ \tilde{D}_2 f(x_\ell, t_\ell) &= 0\,. \end{aligned} \tag{4.15}$$

The Jacobian of (4.15) with respect to the new iterate is (as in the deterministic case)

$$\begin{pmatrix} \tilde{A}_1 + h_\ell \tilde{D}_1 f'_x(x,t)/2 \\ \tilde{D}_2 f'_x(x,t) \end{pmatrix} ,$$

which is non-singular for sufficiently small stepsizes and whose condition number is bounded independently of the stepsizes. If the right-hand side of (4.15) is perturbed by a vector $\tilde{d} = \begin{pmatrix} \tilde{d}_1 \\ \tilde{d}_2 \end{pmatrix}$, we see that the factor $1/h_\ell$ in the perturbations of the constraints is avoided. Let $\tilde{x}_\ell$ denote the solution of this perturbed system and let the suppositions of Corollary 4.1 be fulfilled. Then an estimate

$$\max_{\ell=1,\dots,N} \|x(t_\ell) - \tilde{x}_\ell)\|_{L_2} \le S(\tilde{c}_0 \cdot h^{1/2} + \tilde{c}_1 \max_{\ell=1,\dots,N} \tilde{\delta}_\ell / h_\ell^{1/2} + \tilde{c}_3 \max_{\ell=1,\dots,N} \bar{\tilde{\delta}}_\ell / h_\ell) ,$$

holds, where

$$\tilde{\delta}_\ell := \|\tilde{d}_\ell\|_{L_2} , \quad \bar{\tilde{\delta}}_\ell := \|\mathbb{E}(\tilde{d}_\ell | \mathcal{F}_{t_{\ell-1}})\|_{L_2} .$$

4.4 The drift-implicit Milstein scheme

In Section 3 we considered also the family of Milstein schemes (3.8) and showed their numerical stability. Compared to the family of Euler schemes (3.7) including the trapezoidal rule they possess a higher order of strong (mean-square) convergence, namely order 1. This has to be paid for by the use of the double Itô-integrals as well as of the Jacobians

$(g_j)'_x$ in the scheme.

Our intention here is to construct a scheme for the SDAE (4.1) which should realize the drift-implicit Milstein scheme (3.8) with parameter $\alpha = 1$ applied to the inherent SDE

$$u' + \hat{f}(u,t) + \hat{G}(u,t)\xi(t) := u' + D(I-R)f(u+v(u,t),t) + DG(u+v(u,t),t)\xi(t) = 0 \ ,$$

i.e. ,

$$\frac{u_\ell - u_{\ell-1}}{h_\ell} + \hat{f}(u_\ell, t_\ell) + \hat{G}(u_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell}\Delta w_\ell - \sum_{j=1}^{m} (\hat{g}_j)'_x \hat{G}(u_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell} I_{(j),\ell} = 0 \ ,$$

where $\hat{G} = (\hat{g}_1, \ldots, \hat{g}_m)$, and $I_{(j),\ell} = (I^\ell_{j,i})_{i=1}^m$, $I^\ell_{j,i} = \int_{t_{\ell-1}}^{t_\ell} \int_{t_\ell}^{s} dw_i(\tau) dw_j(s)$.
This is realized by the scheme

$$A\frac{x_\ell - x_{\ell-1}}{h} + f(x_\ell, t_\ell) + G(x_{\ell-1}, t_{\ell-1})\frac{1}{h}\Delta w_\ell - \sum_{j=1}^{m} (g_j)'_x DG(x_{\ell-1}, t_{\ell-1})\frac{1}{h_\ell} I_{(j),\ell} = 0 \ , \quad (4.16)$$

since again the iterates fulfil the constraints at the current time-point. We point out the explicit use of a scaling D with $DA = P$ in the last term of (4.16). (4.16) is closely related to a scheme developed in [25] for the application in circuit simulation. There, an approximation to such a scaling is involved.
For problems with multiple Wiener processes ($m > 1$) $m(m-1)/2$ mixed double Itô-integrals have to be approximated in general and form a serious drawback. The terms involving these mixed double Itô-integrals disappear if the diffusion coefficients are commutative in the sense that

$$[Dg_i, Dg_j] = [\hat{g}_i, \hat{g}_j] := (\hat{g}_i)'_x \hat{g}_j - (\hat{g}_j)'_x \hat{g}_i = 0 \quad \forall i \neq j \ .$$

We remark that also this condition makes explicit use of the scaling D.
Due to the condition that the iterates fulfil the constraints at the current time-point we obtain an error estimation similarly to that in Corollary 4.1. If the coefficients f, G are smooth enough to guarantee strong (mean square) consistence with order 1 one obtains an estimate

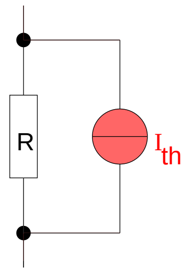
$$\max_{\ell=1,\ldots,N} \|x(t_\ell) - \tilde{x}_\ell)\|_{L_2} \leq S(c_0 h + c_1 \max_{\ell=1,\ldots,N} \delta_\ell / h_\ell^{1/2} + c_2 \max_{\ell=1,\ldots,N} \bar{\delta}_\ell / h_\ell + c_3 \max_{\ell=0,\ldots,N} \|Rd_\ell\|_{L_2} / h_\ell) \ ,$$

with positive constants c_0, c_1, c_2, c_3.

5 Application in circuit simulation

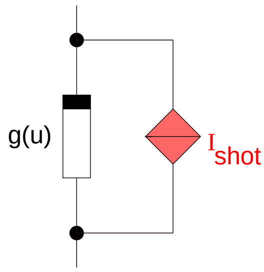
In industry, circuit analysis is a standard tool for the design of electric circuits. One of the most used techniques is the charge-oriented Modified Nodal Analysis (MNA). The equations are generated automatically by combining the network topology, Kirchhoff's Current Law, and the characteristic equations describing the physical behaviour of the network elements. This results in large systems of DAEs, whose special structure was analyzed in a number of papers, e.g. [10, 13, 29]. The increasing scale of integration of electric circuits, among other things, leads to decreasing signal to noise ratios. In special applications where linear noise analysis is no longer satisfactory transient noise analysis becomes necessary.

Here, we deal with models of thermal noise of resistors and shot noise of pn-junctions. Both are modelled as external Gaussian white noise sources in parallel to the original element (see Figure 1 and Figure 2).



thermal noise of a resistor

Figure 1



shot noise of a pn-junction

Figure 2

Nyquist's theorem (see e.g. [4, 8, 9, 31]) states that the current through an arbitrary linear resistor having a resistance R, maintained in thermal equilibrium at a temperature T, can be described as the sum of the deterministic current and a Gaussian white noise process with spectral density $S_{th} := \frac{2kT}{R}$, where k is Boltzmann's constant. Hence, the additional current is modelled as

$$I_{th} = \sigma_{th} \cdot \xi(t) = \sqrt{\tfrac{2kT}{R}} \cdot \xi(t) \ ,$$

where $\xi(t)$ is a standard Gaussian white noise process. In [30, 31] a thermo-dynamical foundation to apply this model to mildly nonlinear resistors and reciprocal networks is given.

Shot noise of pn-junctions, caused by the discrete nature of current due to the elementary charge, is also modelled by a Gaussian white noise process, where the spectral density is proportional to the current I through the pn-junction: $S_{shot} := q|I|$, where q is the elementary charge. If the current through the pn-junction is described by a characteristic $I = g(u)$ in dependence on a voltage u, the additional current is modelled by

$$I_{shot} = \sigma_{shot}(u) \cdot \xi(t) = \sqrt{q|g(u)|} \cdot \xi(t) \ ,$$

where $\xi(t)$ is a standard Gaussian white noise process. For a discussion of the model assumptions we refer to [4, 8, 9, 30, 31].

Now, we consider an electrical network with n_C capacitances, n_R resistances, n_L inductances, $n_V + n_I$ possibly controlled voltage and current sources, and n_N additional noise sources. Each element corresponds to a branch connecting two nodes of the network. Let there be n_e nodes plus the datum node. The network model is determined by its topology, which is represented by means of the incidence matrix $(A_C, A_R, A_L, A_V, A_I, A_N)$, and the characteristic equations of its elements. Using vector-valued characteristics, this

extends to multi-ports, too. For a detailed description we refer to [10, 13].
Combining Kirchhoff's Current Law and the characteristic equations of the voltage-controlled elements, supplemented by the characteristic equations of inductances and voltage sources and the defining equations for the charges and fluxes, leads to the charge-oriented MNA system with the following structure (see [10, 13] for the deterministic case):

$$\begin{aligned} A_C q' + f_1(e, j_L, j_V, t) + A_N \otimes \sigma(A_N^T e, t)\xi(t) &= 0 & (5.1)\\ \phi' - A_L^T e &= 0 & (5.2)\\ A_V^T e - v_s(e, j_L, t) &= 0 & (5.3)\\ q - q_C(A_C^T e, t) &= 0 & (5.4)\\ \phi - \phi_L(j_L, t) &= 0\,, & (5.5)\end{aligned}$$

where $f_1(e, j_L, j_V, t) := A_R g(A_R^T e, t) + A_L j_L + A_V j_V + A_I i_s(e, j_L, j_I, t)$, and $q_C, g, \phi_L, v_s, i_s, \sigma$ are given functions. The vector of unknowns describing the system behaviour consists of all node potentials e, the branch currents of current-controlled elements (inductances and voltage sources) j_L, j_V, as well as the charges q of capacitances and the fluxes ϕ of inductances. ξ denotes an n_N-dimensional vector of independent standard Gaussian white noise processes, and $(A_N \otimes \sigma)_i = \sigma_i a_{N,i}$, $i = 1, \ldots, n_N$, where $a_{N,i}$ denotes the i-th column of A_N, and σ_i the i-th component of σ. In industry-relevant applications one has to deal with a large number of unknowns as well as of noise sources. The first block of equations (5.1) means a stochastic integral equation :

$$A_C q(s)\Big|_{t_0}^{t} + \int_{t_0}^{t} f_1(x(s), s)ds + \int_{t_0}^{t} G_1(x(s), s)dw(s) = 0\,,$$

where $G_1 := A_N \otimes \sigma$, and w denotes an n_N-dimensional Wiener process. With

$$A := \begin{pmatrix} A_C & & \\ & I_{n_L} & \\ & & 0 \end{pmatrix}, \quad G := \begin{pmatrix} A_N \otimes \sigma \\ 0 \\ 0 \end{pmatrix} \tag{5.6}$$

(5.1)-(5.5) forms a specially structured SDAE of the type (2.1) discussed in the previous chapters. We will start discussing the suppositions of Theorem 2.1 for the SDAE (5.1)-(5.5).

- The first condition we need in order to ensure that (5.1)-(5.5) is an index 1 SDAE is that the noise-sources do not appear in the constraints, i.e., $\operatorname{im} G(x,t) \subseteq \operatorname{im} A$: This is guaranteed if $\operatorname{im} A_N \subseteq \operatorname{im} A_C$ or, in terms of the network topology, if there is always a path of capacitances in parallel to a noise source.

- The second condition we need in order to ensure that (5.1)-(5.5) is an index 1 SDAE is that the constraints are globally uniquely solvable for the algebraic variables: This follows if the Jacobian $A + Rf'_x$ is globally bounded invertible, where R is a projector along $\operatorname{im} A$.
Let $Q_C \in I\!R^{n_e \times n_e}$ be a projector onto $\ker A_C^T$. Then $A_C^T Q_C = 0 \in I\!R^{n_C \times n_e}$ as well as $Q_C^T A_C = 0 \in I\!R^{n_e \times n_C}$ holds true, and Q_C^T is a projector along $\operatorname{im} A_C$. Hence, we can

choose

$$R := \begin{pmatrix} Q_C^T & & \\ & 0_{n_L} & \\ & & I \end{pmatrix} \tag{5.7}$$

as a projector along im A. Let us denote

$$C(u,t) := (q_C)'_u(u,t)\ , \quad G(u,t) := r'_u(u,t)\ , \quad L(j_L,t) := (\phi_L)'_{j_L}(j_L,t)\ .$$

If there are no controlled sources, i.e., $v_s(e,j_L,t) = v_s(t)$, $i_s(e,j_L,j_V,t) = i_s(t)$, we obtain a Jacobian with the following structure:

$$A + Rf'_x = \begin{pmatrix} A_C & & Q_C^T A_R G A_R^T & Q_C^T A_L & Q_C^T A_V \\ & I_{n_L} & & & \\ & & A_V^T & & \\ I_{n_C} & & -CA_C^T & & \\ & I_{n_L} & & -L & \end{pmatrix} .$$

Now, we suppose:

- The matrices $C(u,t), G(u,t), L(j_L,t)$ are symmetric and uniformly positive definite.
- The matrix $Q_C^T A_V$ has full column rank, which means that there are no loops of capacitances and voltage sources.
- The matrix (A_C, A_R, A_V) has full row rank, which means that there are no cut-sets of inductances or current sources.

Then the Jacobian $A + Rf'_x$ is non-singular and its inverse is uniformly bounded. Following the lines of [10] this result remains true if there are controlled sources as long as they fulfil certain conditions described there.

- f, G are globally Lipschitz continuous with respect to x and continuous with respect to t if this condition is true for the model functions $q_C, r, \phi_L, v_s, i_s, \sigma$.

Summarizing we have:

Corollary 5.1 *Suppose that the functions $q_C, r, \phi_L, v_s, i_s, \sigma$ are globally Lipschitz continuous with respect to the unknown variables and continuous with respect to t, and that the partial derivatives $C(u,t), G(u,t), L(j_L,t)$ are uniformly positive definite. Suppose that there is always a path of capacitances in parallel to a noise source, that there are no loops of capacitances and voltage sources, and no cut-sets of inductances or current sources, and that controlled voltage or current sources fulfil the conditions described in [10].*
Then the system (5.1)-(5.5) has a pathwise unique solution process with properties described in Theorem 2.1.

We complete this chapter discussing the discretization schemes from Chapter 4 for the charge-oriented MNA-system (5.1)-(5.5).
The drift-implicit Euler scheme can be implemented straightforwardly. Realizing the SSBE scheme (4.10),(4.11) requires a projector R along im A. Such a projector is given

in (5.7) based on a projector Q_C^T along im A_C. Using a scaling $\tilde{D}_C$ with $\tilde{D}_C A_C = \binom{\tilde{A}_C}{0}$ one obtains a scaling $\tilde{D}$ to implement the trapezoidal rule with splitted residuals as described in Chapter 4.3 . Furthermore, the scaling D with $DA = P$, which is needed in the Milstein scheme in Chapter 4.4, is determined by the matrix A from (5.6), basically by A_C already, and thus by the topology of the network.

Since the additional smoothness conditions on the coefficients f, resp. G, which are needed to guarantee the numerical stability and consistency of the considered discretization schemes, are transferred to corresponding smoothness conditions on the model functions q_C, r, ϕ_L, v_s, i_s, resp. σ, the following corollary holds true:

Corollary 5.2 *Let the assumptions of Corollary 5.1 hold true. Additionally, assume that the functions $q_C, r, \phi_L, v_s, i_s, \sigma$ are Hölder continuous with exponent $1/2$ with respect to t. Then the drift-implicit Euler scheme (4.3), the SSBE scheme (4.10), (4.11) and the trapezoidal rule (4.14) applied to the system (5.1)-(5.5) are numerically stable in the mean square sense and strongly (mean square) convergent with order $1/2$.*

Under further smoothness conditions an analogous result ensures that the drift-implicit Milstein scheme (4.16) applied to the system (5.1)-(5.5) is strongly (mean square) convergent with order 1.

6 Conclusions

In contrast to SDEs, in SDAEs the solution has to fulfil constraints and also the numerical approximations have to be forced to do so. In the present paper an approach to the numerical analysis of generally nonlinear DAEs driven by Gaussian white noise is developed. Hereby, the fact that the leading Jacobian A in front of the derivatives is constant is extensively used. It allows a decoupling of the SDAE as well as of its discretization underlying the theoretical results by multiplying by constant matrices, which is compatible to the Itô calculus. The drift-implicit methods discussed in Chapter 4 work directly on the given structure and are formulated in such a way that the convergence properties of these methods known for SDEs are preserved. The presented approach applies to weakly convergent schemes, too.

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