Abelian approach to modular forms of neat 2-ball lattices: Dimension formulas

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Abstract

In previous papers [Ho86], [Ho00] we found neat Picard modular surfaces with abelian minimal model and, conversely, a divisor criterion on abelian surfaces A for such a situation. For the corresponding ball lattices Γ we prove dimension formulas for modular forms depending only on the intersection graph of the image on A of the compactification divisor of the Γ-quotient surface.

1 Introduction: Main results and motivations

We look for explicite structures of rings $R(\Gamma)$ of modular forms for Picard modular groups Γ , especially in cases when the corresponding Picard modular surfaces are well determined by explicitly known algebraic equations. The quotient surface $\Gamma\diagdown\mathbb{B}$, $\mathbb{B}$ the complex two- dimensional unit ball, can be compactified by means of finitely many cusp singularities to a (normal complex projective) algebraic surface $\widehat{\Gamma\diagdown\mathbb{B}}$, the Baily-Borel compactification. By Baily-Borel's theorem [B-B] one has

$$\widehat{\Gamma\diagdown\mathbb{B}} = Proj\, R(\Gamma).$$

Generators of $R(\Gamma)$ and relations between them define a projective model of $\widehat{\Gamma\diagdown\mathbb{B}}$. It is not a simple problem to discover the ring structure in connection with the algebraic equations assumed to be known.

If, moreover, Γ is a neat ball lattice, then we are in a comfortable situation. Namely, there is a natural ring isomorphism

$$R(\Gamma) \cong \bigoplus_{n=0}^{\infty} H^0(X'_\Gamma, \Omega^2(logT')) \tag{1}$$

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onto the ring of logarithmic pluricanonical forms of the (smooth compact) Picard modular surface X'_Γ with compactification divisor

$$T' = \sum_{j=1}^{h} T'_j, \tag{2}$$

which is a disjoint sum of elliptic curves. For a cofinite group extension Γ_1 of Γ, defined by an exact sequence of groups

$$1 \longrightarrow \Gamma \longrightarrow \Gamma_1 \longrightarrow G \longrightarrow 1 \tag{3}$$

with finite group G, we get isomorphisms

$$R(\Gamma_1) \cong R(\Gamma)^G \cong \bigoplus_{n=0}^{\infty} H^0(X'_\Gamma, \Omega^2(logT'))^G \tag{4}$$

Assume, we know the ring structure in the neat case and the representation of G on $R(\Gamma)$. Then it is "only" a matter of invariant theory for finite groups to get the structure of $R(\Gamma_1)$. For this latter step good software as "SINGULAR" or "GAP" should be used. For the first step, it is necessary to determine the dimensions

$$h^0(X'_\Gamma, \Omega^2(logT')) := \dim H^0(X'_\Gamma, \Omega^2(logT'))$$

Knowing some important cases (e.g. Picard modular surfaces of Gauss and Eisenstein numbers, see [Ho00]) we concentrate our attention in this paper to abelian ball quotient surface models X'_Γ with neat ball lattices Γ, which are not cocompact.

Definition 1.1 . *A ball lattice is called* coabelian *iff the corresponding compactified quotient surface is abelian up to birational equivalence.*

Remark 1.2 . *Neat coabelian ball lattices are not cocompact, because the quotient surfaces of neat cocompact lattices are known to be of general type.*

An abelian surface is the (unique) minimal model in its birational equivalence class (of smooth surfaces). Therefore, for any neat coabelian ball lattice Γ there exist birational morphisms

$$A \xleftarrow{\ \sigma\ } A' := X'_\Gamma \longrightarrow \hat{X}_\Gamma \xleftarrow{\ \iota\ } X_\Gamma := \Gamma\diagdown\mathbb{B} \tag{5}$$

where $\hat{X}_\Gamma$ is the (normal projective) Baily-Borel compactification of X_Γ with (minimal) singularity resolution X'_Γ, ι the natural embedding and A is an abelian surface. In [Ho00], Cor. 2.8, we proved the first part of

Proposition 1.3 . *The abelian surface A, which is a contracted ball quotient as described in (5), is isogeneous to $E \times E$ for a suitable elliptic curve E. If, moreover, E has complex multiplication, e.g. in the case of a Picard modular surface, then A is isomorphic to $E \times E$.*

$\square$

The second part follows from the first by a theorem of Shioda-Mitani [SM], see also [BL], X, Corollary (6.3). So the determination of the structure of the ring of Picard modular forms in the neat abelian case can be reduced to the theory of elliptic functions. Namely, looking back to (1) we get

$$R(\Gamma) \cong \bigoplus_{n=0}^{\infty} H^0(A', \Omega^2(logT')), \tag{6}$$

where $A' = X'_\Gamma$ is a blown up abelian surface $A \sim E \times E$, where "$\sim$" means isogeny. Using obvious notations (omitting $'$) the image divisor of the compactification divisor T'

$$T := \sigma(T') = \sum_{j=1}^{h} T_j \tag{7}$$

is an *elliptic divisor* on A. This means that T is a reduced divisor with elliptic curves as components. On the universal covering $\mathbb{C}^2$ of A they are lifted to affine complex lines. Therefore the components T_j intersect each other (at most) transversally. The set of all intersection points is the singular locus

$$S = S(T) := \bigcup_{j \neq k} S_j \cap S_k \tag{8}$$

of T. We consider also the subsets of S on the components

$$S_j = S_j(T) := S \cap T_j. \tag{9}$$

Remark 1.4 . *The morphism $\sigma =: \sigma_S$ in (5) is nothing else but the blowing up of all points of S.*

Surprisingly, abelian ball quotient surface models (A, T) can be recognized by an intersection property of the elliptic divisor T. Namely, we proved

Theorem 1.5 *([Ho00], Theorem 2.5). Let (A, T) be an abelian surface with an elliptic divisor T and $\sigma : A' \longrightarrow A$ the blowing up of A at the singular locus $S = S(T)$ of T with proper transform T' of T on A'. The following conditions are equivalent:*

(i) (A', T') *is a neat (coabelian) ball quotient surface with compactification divisor* T'*;*

(ii)

$$4s = \sum s_j$$

with cardinalities $s := \#S$, $s_j := \#S_j$ *and* S_j *defined in (9).*

$\square$

Let

$$L = L_1 + ... + L_s \tag{10}$$

be the exceptional divisor of $\sigma : A' \longrightarrow A$. It is a disjoint sum of s projective lines on A' with selfintersection index -1.

Theorem 1.6 . *Let* Γ *be a neat coabelian ball lattice with smoothly compactified quotient surface* $A' = X'_\Gamma \sim E \times E$, E *a suitable elliptic curve. With the notations around Theorem (1) the dimensions of spaces* $[\Gamma, n]$ *of* Γ*-automorphic forms of weight* n *are*

$$\dim[\Gamma, n] = \begin{cases} h^0(A', \mathcal{O}_{A'}(L + T')), & if \quad n = 1 \\ 3\binom{n}{2}s + h, & if \quad n > 1 \end{cases}$$

For the dimensions of spaces $[\Gamma, n]_0$ of Γ-cusp forms of weight n we get the following explicit formulas:

Proposition 1.7 . *In the situation of Theorem 1.6 it holds that*

$$\dim[\Gamma, n]_0 = 3\binom{n}{2}s + \delta_{n,1}, \ n \in \mathbb{N},$$

where $\delta_{n,1} \in \{0, 1\}$ *is the Kronecker symbol.*

Example 1 (Hirzebruch [Hi] and Holzapfel [Ho86], [Ho00]). Neat coabelian Picard modular group of Eisenstein numbers:
$A = E \times E$, E elliptic CM-curve with $K = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\omega)-$ multiplication, ω primitive 3-rd unit root, Γ commensurable with the full Picard modular groups $\mathbb{U}((2,1), \mathfrak{O}_K)$ of Eisenstein numbers

$$T' = T'_1 + ... + T'_4, \ h = 4, \qquad L = L_1, \ s = 1,$$

on $A' = (E \times E)'$;

$$\dim[\Gamma, 1] = h^0(\mathcal{O}_{(E \times E)'}(L_1 + T'_1 + ... + T'_4))$$
$$\dim[\Gamma, n] = \dim[\Gamma, n]_0 + 4 = 3\binom{n}{2} + 4, \quad n > 1.$$

Example 2 ([Ho00]). Neat coabelian Picard modular group of Gauss numbers: $A = E \times E$, E elliptic CM-curve with $K = \mathbb{Q}(i)$- multiplication, Γ commensurable with the full Picard modular groups $\mathbb{U}((2,1), \mathfrak{O}_K)$ of Gauss numbers,

$$T' = T'_1 + ... + T'_8,\ h = 8, \qquad L = L_1 + ... + L_6,\ s = 6,$$

on $A' = (E \times E)'$;

$$\begin{aligned} \dim[\Gamma, 1] &= h^0(\mathcal{O}_{(E\times E)'}(L_1 + ... + L_6 + T'_1 + ... + T'_8)) \\ \dim[\Gamma, n] &= \dim[\Gamma, n]_0 + 8 = 9n^2 - 9n + 8, \quad n > 1. \end{aligned}$$

Example 3 (Vladov). Neat coabelian Picard modular group of Gauss numbers: $A = E \times E$, E elliptic CM-curve with $K = \mathbb{Q}(i)$- multiplication, Γ group extension of index 2 of the ball lattice in Example 2, hence also commensurable with the full Picard modular groups $\mathbb{U}((2,1), \mathfrak{O}_K)$ of Gauss numbers,

$$T' = T'_1 + ... + T'_6,\ h = 6, \qquad L = L_1 + ... + L_3,\ s = 3,$$

on $A' = (E \times E)'$;

$$\begin{aligned} \dim[\Gamma, 1] &= h^0(\mathcal{O}_{(E\times E)'}(L_1 + ... + L_3 + T'_1 + ... + T'_6)) \\ \dim[\Gamma, n] &= \dim[\Gamma, n]_0 + 6 = 9\binom{n}{2} + 6, \quad n > 1. \end{aligned}$$

In the forthcoming article [Ho] we compose lifted quotients of elliptic Jacobi theta functions to abelian functions on hyperbolic biproducts of elliptic curves. We are able to transform them to explicit Picard modular forms. Basic algebraic relations of basic forms come from different multiplicative decompositions of these abelian functions in simple ones of same lifted type. Especially, for Vladov's example we can show that the explicitly constructed basic modular forms yield a Baily-Borel embedding into $\mathbf{P}^{22}$ together with explicit relations (homogeneous equations) for the Picard modular image surface.

2 Proof of dimension formulas

For the sake of clearness we remember to precise definitions. By $\mathbb{U}((2,1), \mathbb{C})$ we denote the *unitary group* $\mathbb{U}(V)$ of a hermitian vector space $(V, <,>)$ with $\dim_{\mathbb{C}}(V) = 3$ and a hermitian form $<,>$ of signature $(2,1)$. The ball $\mathbb{B}$ appears as subspace

$$\mathbb{B} = \mathbb{P}V_- := \mathbb{P}\{v \in V; < v, v > < 0\} \subset \mathbb{P}V \cong \mathbb{P}^2(\mathbb{C})$$

of all complex lines in V generated by a "negative" vector v. The group $\mathbb{U}((2,1), \mathbb{C})$ acts on $\mathbb{B}$ via the natural composition

(11)
$$\mathbb{U}((2,1), \mathbb{C}) \subset \mathbb{G}l(V) \longrightarrow \mathbb{PG}l(V) = Aut_{hol}(\mathbb{P}V) \cong \mathbb{PG}l_3(\mathbb{C}) \cong Aut_{hol}\mathbb{P}^2(\mathbb{C}).$$

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic number field, d a square-free positive integer, and $\mathfrak{O}_K$ the ring of integers in K. A *Picard modular group* (of the field K) is, by definition, commensurable with the the *full Picard modular group* $\mathbb{U}((2,1),\mathfrak{O}_K)$. All Picard modular groups are *ball lattices.* This means that they act proper discontineously on $\mathbb{B}$ and the volume of a Γ-fundamental domain with respect to the $\mathbb{G}(\mathbb{R})$-invariant hermitian (Bergmann) metric on $\mathbb{B}$ (uniqely determined up to a non-trivial constant factor) is finite. The quotient surface $\Gamma\diagdown\mathbb{B}$ can be compactified by means of finitely many cusp singularities to a (normal complex projective) algebraic surface $\widehat{\Gamma\diagdown\mathbb{B}}$, the *Baily-Borel compactification.* Now let $\Gamma \subset \mathbb{U}((2,1),\mathbb{C})$ be a ball lattice. It acts via $Aut_{hol}\mathbb{B} = \mathbb{PU}((2,1),\mathbb{C})$ on the $\mathbb{C}$-vector space $H^0(\mathbb{B},\mathcal{O}_\mathbb{B})$ of holomorphic functions on $\mathbb{B}$ corresponding to each $f(z_1,z_2)$ the function $\gamma^*(f)(z_1,z_2) = f(\gamma(z_1,z_2))$. For each n one gets a representation

$$(12) \qquad \rho_n : \Gamma \longrightarrow Aut\, H^0(\mathbb{B},\mathcal{O}_\mathbb{B}), \quad \Gamma \ni \gamma : f \mapsto j_\gamma^{-n} \cdot \gamma^*(f)$$

with the Jacobi determinants

$$j_\gamma(z_1,z_2) = det(\frac{\partial\gamma(z_1,z_2)}{\partial(z_1,z_2)})$$

Then $[\Gamma,n] \subset H^0(\mathbb{B},\mathcal{O}_\mathbb{B})$ is defined to be the eigensubspace of $\rho_n(\Gamma)$ of the eigenvalue 1, that means

$$(13) \qquad [\Gamma,n] = \{f \in H^0(\mathbb{B},\mathcal{O}_\mathbb{B});\ \gamma^*(f) = j_\gamma^n \cdot f\ for\ all\ \gamma \in \Gamma\}$$

Γ-*cusp forms* are Γ-automorphic forms which vanish at infinity, this means at the cusps. To be more precise, let us first interprete automorphic forms as holomorphic sections of sheaves of higher differential form bundles $\mathfrak{K} = \mathfrak{K}_\mathbb{B}^n := \mathfrak{K}_\mathbb{B}^{\otimes n}$ with the sheaf $\mathfrak{K}_\mathbb{B}$ of holomorphic differential forms on $\mathbb{B}$. The canonical action of Γ on $\mathbb{B}$ is defined by

$$\gamma :\ \omega = f dz_1 \wedge dz_2 \ \mapsto\ \gamma^*(\omega) = \gamma^*(f)\gamma^*(dz_1 \wedge dz_2) = \gamma^*(f) \cdot j_\gamma^{-n} \cdot dz_1 \wedge dz_2.$$

The embeddings

$$(14) \qquad H^0(\mathbb{B},\mathcal{O}_\mathbb{B}) \longrightarrow H^0(\mathbb{B},\mathfrak{K}^n), \quad f \mapsto f \cdot (dz_1 \wedge dz_2)^{\otimes n}$$

are compatible with the corresponding Γ-actions (ρ_n on the preimage space) and

$$(15) \qquad [\Gamma,n] \cong H^0(\mathbb{B},\mathfrak{K}^n)^\Gamma.$$

The latter space has the advantage to go down to the quotient space $\Gamma\diagdown\mathbb{B}$:

$$(16) \qquad H^0(\mathbb{B},\mathfrak{K}_\mathbb{B}^n)^\Gamma \subseteq H^0(\Gamma\diagdown\mathbb{B},\mathfrak{K}_{\Gamma\diagdown\mathbb{B}}^n),$$

if we assume that Γ acts freely on $\mathbb{B}$, that means $\mathbb{B} \longrightarrow \Gamma\diagdown\mathbb{B}$ is a universal covering. The *space of cusp forms* $[\Gamma,n]_0 \subseteq [\Gamma,n]$ is defined by corresponding to forms $\omega \in H^0(\Gamma\diagdown\mathbb{B}, \mathfrak{K}^n_{\Gamma\diagdown\mathbb{B}})$ which can be extended to zero at all boundary (cusp) points $P \in \widehat{\Gamma\diagdown\mathbb{B}} \setminus (\Gamma\diagdown\mathbb{B})$.

Now let Γ be a neat ball lattice and $X' = X'_\Gamma$ the corresponding minimal smoothly compactified ball quotient surface as in (5) (forgetting the arrow on the left-hand side there) with compactification divisor T', which has disjoint elliptic curve components. The link between sections of line bundles on X' with Γ-automorphic forms used in (1) is:

$$(17) \qquad [\Gamma, n] \cong H^0(X, (\mathfrak{K}_{X'} \otimes \mathfrak{T}')^n), \quad [\Gamma, n]_0 \cong H^0(X, \mathfrak{K}^n_{X'} \otimes \mathfrak{T}'^{n-1}),$$

where $\mathfrak{K}_{X'} = \mathcal{O}_{X'}(K)$ is the canonical bundle of X', K a canonical divisor, and $\mathfrak{T}' = \mathcal{O}_{X'}(T')$ the line bundle corresponding to T'. We refer to (33) of [Ho98] or, more originally to Hemperly [Hem]. The Riemann-Roch formula expresses the Euler characteristics for arbitrary line bundles $\mathfrak{V}$ on X' as

$$\chi(\mathfrak{V}) := \sum_j (-1)^j h^j(X', \mathfrak{V}) = \frac{1}{2}(\mathfrak{V} \cdot (\mathfrak{V} \otimes \mathfrak{K}^{-1}_{X'})) + \chi(X'),$$

where

$$\chi(X') = \sum_i (-1)^i h^i(\mathcal{O}_{X'}) = \sum_i (-1)^i \dim H^i(X', \mathcal{O}_{X'})$$

is the arithmetic genus of X'. Using intersections of divisors we want to calculate the Euler characteristics of

$$(18) \qquad \mathfrak{G}_n := (\mathfrak{K}_{X'} \otimes \mathfrak{T})^n \quad \textit{and} \quad \mathfrak{F}_n := \mathfrak{K}^n_{X'} \otimes \mathfrak{T}^{n-1}.$$

$$(19) \qquad \chi(\mathfrak{G}_n) = \chi(\mathfrak{F}_n) = \binom{n}{2}((K + T')^2) + \chi(X'),$$

Namely, by the above Riemann-Roch formula we have

$$\begin{aligned}
\chi(\mathfrak{G}_n) &= \frac{1}{2}(n(K + T') \cdot ((n-1)(K + T') + T')) \\
&= \frac{1}{2}n(n-1)(K + T')^2 + \frac{1}{2}n((K + T') \cdot T') \\
\chi(\mathfrak{F}_n) &= \frac{1}{2}((n(K + T') - T') \cdot (n-1)(K + T')) \\
&= \frac{1}{2}n(n-1)(K + T')^2 - \frac{1}{2}(n-1)(T' \cdot (K + T'))
\end{aligned}$$

For each neat ball quotient surface X' with (elliptic) compactification divisor T' it holds that $(T' \cdot (T' + K_{X'})) = 0$, see the proof of (iii) in the next propostion. So the second summands of both identities vanishes. This proves (19).

Now we concentrate our attention to neat coabelian ball lattices Γ and the corresponding quotient surfaces.

Proposition 2.1 . *Consider a neat coabelian ball quotient surface $(X'_\Gamma, T') = (A', T')$ with compactification divisor T' and exceptional divisor L of σ. With the notations around Theorem 1.5 it holds that*

(i) $K = K_{X'} = L$ *is a canonical divisor of* X';

(ii) $(K^2) = (L^2) = -s$;

(iii) $(T' \cdot (T' + K)) = (T' \cdot (T' + L)) = 0$;

(iv) $-(T^{'2}) = (L \cdot T^{''}) = (K \cdot T') = 4s$;

(v) $((K + T')^2) = ((L + T')^2) = (K^2) - (T^{'2}) = 3s$;

(vi) $((K + T') \cdot K) = ((L + T') \cdot K) = ((L + T') \cdot L) = 3s$.

Proof. (i): The canonical divisor of the abelian surface A is trivial. The canonical divisor of a blown up surface is the sum of the exceptional divisor and the inverse image of the original surface, see [BPV], I, Theorem (9.1), (vii). This means in our situation: $K_{X'} = \sigma^*(O) + L = L$.
(ii) follows immediately from (i) and (10).
(iii) needs the adjunction formula (see e.g. [BPV], II.11, (16))

$$-(C \cdot (C + K_Y)) = e(C) \ (\textit{Euler number})$$

for smooth curves C on smooth compact surfaces Y. For the elliptic curves T'_j we get

$$0 = -e(T_j) = (T'_j \cdot (T'_j + K)),$$

hence

$$(T' \cdot (T' + K)) = \sum_{j=1}^{h} (T'_j \cdot (K + \sum_{m=1}^{h} T'_m) = \sum_{j=1}^{h} (T'_j \cdot (K + T'_j)) = 0.$$

(iv) The first two identities come from (iii) and (i). With the help of Theorem 1.5, (ii), we get

$$(T^{'2}) = \sum (T_j^{'2}) = -\sum S_j = -4s.$$

(v), (vi) follow immediately from the relations proved just before:

$$\begin{aligned} ((K + T')^2) &= (T' \cdot (T' + K)) + (K \cdot (T' + K)) \\ &= 0 + (K \cdot T') + (K^2) = 4s - s = 3s. \end{aligned}$$

□

The Hodge diamond $(h^{pq}) = (h^p(\Omega^q))$ of the abelian surface A is well-known to be

$$\begin{pmatrix} h^{00} & h^{01} & h^{02} \\ h^{10} & h^{11} & h^{12} \\ h^{20} & h^{21} & h^{22} \end{pmatrix}(A) = \begin{pmatrix} 1 & q & p \\ q & h^{11} & q \\ p & q & 1 \end{pmatrix}(A) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

Since the geometric genus p and the irregularity q are birational invariants and the Euler number $e = \sum(-1)^{p+q}h^{pq}$ increases by 1 after applying a σ-process at one point, we get the following Hodge diamond for $A' = X'$:

$$\begin{pmatrix} 1 & q & p \\ q & h^{11} & q \\ p & q & 1 \end{pmatrix}(X') = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4+s & 2 \\ 1 & 2 & 1 \end{pmatrix}. \tag{20}$$

Notice that

$$\begin{aligned} &\chi(X') = 1 - q + p = 0 \quad (\textit{arithmetic genus}) \\ &e(X') = 2 \cdot 1 + 2 \cdot p + h^{11} - 4 \cdot q = s \quad (\textit{Euler number}). \end{aligned} \tag{21}$$

Recall that

$$h^{p,0} = h^{0,p} = h^p(\Omega^0(Y)) = h^p(\mathcal{O}_Y)$$

for each compact complex algebraic manifold Y. By Serre duality our Hodge diamond contains also the following dimensions of cohomology groups:

$$\begin{aligned} h^2(\mathfrak{K}_{X'}) &= h^0(\mathcal{O}_{X'}) = h^{00} = 1, \\ h^1(\mathfrak{K}_{X'}) &= h^1(\mathcal{O}_{X'}) = h^{10} = q = 2, \\ h^0(\mathfrak{K}_{X'}) &= h^2(\mathcal{O}_{X'}) = h^{20} = p = 1. \end{aligned} \tag{22}$$

With (20) and (v) of Poposition 2.1 we make the relations of (20) more explicit:

Proposition 2.2 *For the line bundles $\mathfrak{F}_n$, $\mathfrak{G}_n$ defined in (18) on $X' = X'_\Gamma$, Γ a neat coabelian ball lattice, it holds that*

$$\chi(\mathfrak{G}_n) = \chi(\mathfrak{F}_n) = 3\binom{n}{2}s$$

for all $n \in \mathbb{N}_+$.

□

Proof of Proposition 1.7.

The case $n = 1$ is easy because $\mathfrak{F}_1$ is the canonical bundle $\mathfrak{K} = \mathfrak{K}_{X'}$. With (17) and (22) one gets

$$[\Gamma, 1]_0 = h^0(\mathfrak{F}_1) = h^0(\mathfrak{K}_{X'}) = 1,$$

in general. For $n > 1$ we need the following Kodaira vanishing result:

Proposition 2.3 *(see [Ho98], Prop. 3.6; [Hem], Thm. 9.1). For any neat ball quotient surface X'_Γ the invertible sheaves $\mathfrak{F}_n$, $n > 1$, are cohomologically trivial (acyclic) in the sense that the (higher) cohomology groups $H^j(X, \mathfrak{F}_n)$, $j > 0$, vanish.*

□

Together with (17) and Proposition 2.2 it follows that

$$\dim[\Gamma, n]_0 = h^0(\mathfrak{F}_n) = \chi(F_n) = 3\binom{n}{2}s$$

for all $n > 1$.

□

Proof of Theorem 1.6.

The second cohomology group of $\mathfrak{G}_n$ vanishes because of Serre duality:

$$H^2(X', \mathfrak{G}_n) \cong H^0(X', \mathfrak{K} \otimes \mathfrak{G}_n^{-1}) = H^0(X', \mathcal{O}(-nT' - (n-1)K) = 0, \ n > 0.$$

Namely, $-nT' - (n-1)K_{X'}$ is a negative divisor on X' because $T' > 0$ and also $K = L > 0$ by choice, see (i) of Proposition 2.1. Proposition 2.2, (17), the definitions of $\mathfrak{G}_n$ and Euler characteristics yield

$$\begin{aligned} \dim\,[\Gamma, n] &= h^0(X', \mathfrak{G}_n) = \chi(\mathfrak{G}_n) + h^1(\mathfrak{G}_n) \\ &= \chi(\mathfrak{F}_n) + h^1(\mathfrak{G}_n) = h^1(\mathfrak{G}_n) + 3\binom{n}{2}s. \end{aligned} \tag{23}$$

for all $n \in \mathbb{N}_+$.

We have to calculate the first cohomology group of $\mathfrak{G}_n$. Consider the exact residue sequence (see [BPV], II.1, (6)) of sheaves

$$0 \longrightarrow \mathfrak{K}_{X'} \longrightarrow \mathfrak{K}_{X'} \otimes \mathfrak{T}' \longrightarrow \omega_{T'} \longrightarrow 0$$

with canonical sheaf ω on a smooth curves (written as index). Since $T' = \sum \mathfrak{T}'_j$ is a disjoint sum of s elliptic curves can we identify

$$\omega_{\mathfrak{T}'} = \bigoplus \omega_{T'_j} = \bigoplus \mathcal{O}_{T'_j} = \mathcal{O}_{T'} \tag{24}$$

Tensor products with the sheaves

$$\mathfrak{G}_{n-1} = \mathfrak{K}_{X'}^{n-1} \otimes \mathfrak{T}'^{n-1} \cong \mathfrak{G}_1^{n-1}$$

yield the exact sequences

$$0 \longrightarrow \mathfrak{F}_n \longrightarrow \mathfrak{G}_n \longrightarrow \mathfrak{G}_{n-1} \otimes \mathcal{O}_{\mathfrak{T}'} \longrightarrow 0 \tag{25}$$

We deduce long exact sequences of cohomology groups:

(26)

$$\begin{array}{llllll} 0 & \longrightarrow & H^0(X',\mathfrak{F}_n) & \longrightarrow & H^0(X',\mathfrak{G}_n) & \longrightarrow & H^0(T',(\mathfrak{G}_1\otimes\mathcal{O}_{\mathfrak{T}'})^{n-1}) \\ & \longrightarrow & H^1(X',\mathfrak{F}_n) & \longrightarrow & H^1(X',\mathfrak{G}_n) & \longrightarrow & H^1(T',(\mathfrak{G}_1\otimes\mathcal{O}_{\mathfrak{T}'})^{n-1}) \\ & \longrightarrow & H^2(X',\mathfrak{F}_n) & \longrightarrow & H^2(X',\mathfrak{G}_n)=0. & & \end{array}$$

Especially, for n = 1 this sequence coincides with

$$\begin{array}{llllll} 0 & \longrightarrow & H^0(X',\mathfrak{K}) & \longrightarrow & H^0(X',\mathfrak{G}_1) & \longrightarrow & H^0(T',\mathcal{O}_{\mathfrak{T}'}) \\ & \longrightarrow & H^1(X',\mathfrak{K}) & \longrightarrow & H^1(X',\mathfrak{G}_1) & \longrightarrow & H^1(T',\mathcal{O}_{\mathfrak{T}'}) \\ & \longrightarrow & H^2(X',\mathfrak{K}) & \longrightarrow & H^2(X',\mathfrak{G}_1)=0 & & \end{array}$$

¿From (24) it is clear that

$$\begin{array}{lll} H^0(T',\mathcal{O}_{T'}) & \cong \bigoplus_{j=1}^h H^0(T_j,\mathcal{O}_{T'_j}) & \cong \mathbb{C}^h, \\ H^1(T',\mathcal{O}_{T'}) & \cong \bigoplus_{j=1}^h H^1(T_j,\mathcal{O}_{T'_j}) & \cong \bigoplus_{j=1}^h H^0(T_j,\omega_{T'_j}) \\ & \cong \bigoplus_{j=1}^h H^0(T_j,\mathcal{O}_{T'_j}) & \cong \mathbb{C}^h, \end{array} \tag{27}$$

Together with (22) our long exact sequence looks like

$$\begin{array}{l} 0 \longrightarrow \mathbb{C} \longrightarrow H^0(X',\mathfrak{G}_1) \longrightarrow \mathbb{C}^h \\ \longrightarrow \mathbb{C}^2 \longrightarrow H^1(X',\mathfrak{G}_1) \longrightarrow \mathbb{C}^h \\ \longrightarrow \mathbb{C} \longrightarrow H^2(X',\mathfrak{G}_1)=0. \end{array}$$

The alternating sum of dimensions of all vector spaces in an exact sequence vanishes. Therefore $h^1(\mathfrak{G}_1)=h^0(\mathfrak{G}_1)$ and finally

$$[\Gamma,1]=3\binom{1}{2}s+h^0(\mathfrak{G}_1)=h^0(X',\mathfrak{G}_1)$$

by (23), which proves together with (i) of Proposition 2.1 the case $n=1$ of Theorem 1.6.

For $n>1$ we remark that the canonical sheaf on T' is obtained by restriction

$$\omega_{T'}=\mathfrak{K}_{X'}\otimes\mathfrak{T}'\otimes\mathcal{O}_{T'}=\mathfrak{G}_1\otimes\mathcal{O}_{T'},$$

(adjunction formula, see [Ha], II.8, Proposition 8.20). This sheaf coincides with $\mathcal{O}_{T'}$ by (24). Taking tensor powers we get identifications

$$(\mathfrak{G}_1\otimes\mathcal{O}_{\mathfrak{T}'})^{n-1}=\mathcal{O}_{T'}^{n-1}=(\bigoplus_{j=1}^h\mathcal{O}_{T'_j})^{\otimes(n-1)}=\bigoplus_{j=1}^h\mathcal{O}_{T'_j}^{\otimes(n-1)}=\bigoplus_{j=1}^h\mathcal{O}_{T'_j}=\mathcal{O}_{T'}.$$

Taking also into account the vanishing of $H^p(X',\mathfrak{F}_n)$, $p=1,2$ (Proposition 2.3), the exact sequence (26) splits into two short exact sequences

$$\begin{array}{l} 0\longrightarrow H^0(X',\mathfrak{F}_n)\longrightarrow H^0(X',\mathfrak{G}_n)\longrightarrow H^0(T',\mathcal{O}_{\mathfrak{T}'})\cong\mathbb{C}^h\longrightarrow 0 \\ 0\longrightarrow H^1(X',\mathfrak{G}_n)\longrightarrow H^1(T',\mathcal{O}_{\mathfrak{T}'})\cong\mathbb{C}^h\longrightarrow 0 \end{array}$$

Now use the second row and (23) or the first row to get

$$[\Gamma, n] = h^0(X', \mathfrak{G}_n) = h^0(X', \mathfrak{F}_n) + h = 3\binom{n}{2} s + h.$$

which was to be proved.

$\square$

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