On some questions related to the Krichever correspondence

Denis Osipov, Alexander Zheglov*

1 Introduction

The aim of this paper is to show several new properties and examples of the two-dimensional Krichever correspondence. These examples, from one hand side, make a general picture of the Krichever correspondence more clear, and from another hand side, they should show a possible way how to generalize the "one-dimensional" Krichever correspondence to the case of dimension 2.

Also we give some explicite examples for the one-dimensional Krichever correspondence, which explain some unclearness in the paper [3], and give some new KP-equations which appear as corollaries from generalized KP-hierarchy on two-dimensional local skew fields. These skew fields were first considered in [8] and then classified in [14].

First recall that the Krichever map for algebraic curves is the following. We take a data $(C, p, \mathcal{F}, e_p, t)$, where C is irredicible algebraic curve, p is a smooth k-point on C, t is a local formal parameter at p, $\mathcal{F}$ is a torsion free coherent sheaf of rank r on C, e_p is a local formal trivialization of the sheaf $\mathcal{F}$ at p. Consider the embedding from $H^0(C \setminus p, \mathcal{F})$ into $K_p \otimes \mathcal{F}$. This embedding gives us the map from our data to Fredholm subspaces of $k((t))^{\oplus r}$ by means of identifying $K_p \otimes \mathcal{F} \simeq k((t))^{\oplus r}$ after fixing e_p and t. This map is called the Krichever map, see [2], [4], [5].

Recall that the Euler characteristik of a Fredholm subspace $W \subset k((t))^{\oplus r}$ is equal to $\chi(W) = \dim_k W \cap \mathcal{O} - \dim_k k((t))^{\oplus r}/(W + \mathcal{O})$, where $\mathcal{O} = k[[t]]^{\oplus r}$.

Now recall the Krichever map for algebraic surfaces (see [9], [10]). We are starting from data $(X, C, p, \mathcal{F}, e_p, t, u)$, where X is a projective irreducible normal algebraic surface, C is an ample Cartier divisor on X, p is a smooth k-point on X and C, $\mathcal{F}$ is a rank r vector bundle on X, e_p is a local formal trivialization of $\mathcal{F}$ at p on X, u and t are local parameters at p such that $u = 0$ is a local equation of C on X near p. By such data is canonically constructed k-subspace W in two-dimensional local field $k((t))((u))^{\oplus r}$.

Shortly, this construction is the following. By the data $p \in C \subset X$ we can canonically build two-dimensional local field $K_{p,C}$, which is isomorphic to $k((t))((u))$ by fixing u

*Partially supported by the Graduiertenkolleg "Geometrie und Nichtlineare Analisys" of the DFG and by DFG-Schwerpunkt "Globale Methoden in der Komplexen Geometrie"

and t (see [7], [11], [1]). Consider the subfield $K_C \subset K_{p,C}$, which is the completion of the function field on X with respect to the valuation given by the curve C, and the subring $B_p = \lim\limits_{\longrightarrow n} \frac{\hat{\mathcal{O}}_p}{u^n}$, where $\hat{\mathcal{O}}_p$ is the complete local ring at p on X. (B_p does not depend on choosing of u). Now we have

$$W = \bigcap_{q \in C,\, q \neq p} (B_q \otimes \mathcal{F}) \cap (K_C \otimes \mathcal{F})$$

And from including of $K_C \otimes \mathcal{F}$ to $K_{p,C} \otimes \mathcal{F}$ we have an imbedding of W to $k((t))((u))^{\oplus r}$ by means of choosen e_p, t, and u.

Note also that the Krichever map was generalized to the algebraic varieties of higher dimensions in the work [6] by means of higher-dimensional local fields.

2 Some explicit examples to the two-dimensional Krichever correspondence

In this section we shall compute some examples of the two-dimensional Krichever correspondence on $\mathbf{P^2}$.

In [9] was given an example when $X = \mathbf{P^2}$ with homogeneous coordinates $(x : y : z)$, C is given by $z = 0$, p is $(1 : 0 : 0)$, $\mathcal{F} = \mathcal{O}_X$, $t = y/x$, $u = z/x$. Then $W = k[t^{-1}]((t^{-1}u))$.

For our examples we need the following lemma.

Lemma 1 *Suppose that there is an ample effective divisor D on X such that $D \cdot C = mp$ for some integer m, and $C \setminus p$ is given by the equation $f = 0$ on $X \setminus D$ for some $f \in k[X \setminus D]$. Choose any $y_1, \ldots, y_l \in k[X \setminus D]$ such that their images in $k[C \setminus p]$ generate this ring. (Such y_i always exist). Canonically $y_1, \ldots, y_l, f \in K_{p,C}$. Then we have*

$$W = k[y_1, \ldots, y_l]((f)) \subset K_{p,C}$$

.

Proof follows very easily from the definition of W and facts that $K_C = k(C)((f))$ and $B_q = \lim\limits_{\longrightarrow n} \frac{\hat{\mathcal{O}}_q}{f^n}$ for any point $q \in C$, $q \neq p$.

Now we shall aply this lemma to the case $X = \mathbf{P^2}$.

Consider the case of quadric curve.

Proposition 1 *Let $X = \mathbf{P^2}$ with the homogeneous coordinates $(x : y : z)$. Let the curve C be given by the equation $y^2 + z^2 - 2xz = 0$, and the point $p = (1 : 0 : 0)$. Let $t = y/x$, $u = (y/x)^2 + (z/x)^2 - 2z/x$. Then for the sheaf $\mathcal{F} = \mathcal{O}_X$ we have*

$$W = k[\alpha]((\frac{\alpha^2}{t^2} u)),$$

where $\alpha = \frac{t(1+(1+u-t^2)^{1/2})}{t^2-u}$.

Proof . For applying the lemma 1 consider the curve D which is the tangent line to C at the point p. So D is given by $z=0$, and $D \cdot C = 2p$. Now in the affine domain $z \neq 0$ we have equation $f = (y/z)^2 + 1 - 2x/z$. Note that $f = ux^2/z^2$, and $k[C \setminus p] = k[y/z]$. Denote $\alpha = y/z$. Then from lemma 1 we obtain that $W = k[\alpha]((u\alpha^2/t^2))$. Let us calculate α. From the expression for u we have that $u = t^2 + (t/\alpha)^2 - -2t/\alpha$. Hence $t/\alpha = 1 + (1+u-t^2)^{1/2}$ or $t/\alpha = 1 - (1+u-t^2)^{1/2}$. But $t/\alpha = z/x$ has zero of order 2 at the point p after the restriction to C. Therefore for t/α we have the second variant, and hence obtain the formula from the proposition. Proposition is proved.

Note that for any quadric on $\mathbf{P^2}$ we could take the tangent to the point and apply lemma 1.

Now consider the case of any curve C of degree m on $\mathbf{P^2}$. Suppose that there exists point $p \in C$ such that the tangent line D to the curve C at p has the property $D \cdot C = mp$.

For example, if C is a cubic curve then such point p always exists. (But for the case of a quartic curve this fact is not true anymore, for general quartic curve such point does not exist.)

Choose homogeneous coordinates $(x:y:z)$ on $\mathbf{P^2}$ such that D is given by $z=0$, and p is $(1:0:0)$. Then we have that $C \setminus p$ on $X \setminus D$ is given by the equation $f(x/z, y, z) = 0$. Therefore we can choose the local parameters $u = (z/x)^m f$ and $t = y/x$. Also the images of x/z and y/z generate the ring $k[C \setminus p]$. Hence by lemma 1 for the sheaf $\mathcal{F} = \mathcal{O}_X$ we obtain

$$W = k[\beta, \beta t]((\beta^m u)), \tag{1}$$

where $\beta = x/z$ is defined from the equation $f(\beta, \beta t) = 0$ and the condition that β after the restriction to C has a pole of oder m.

Define the support of a k-subspace W from $k((t))((u))$ as the closed k-subspace of $k((t))((u))$ generated by all monomials which are the lowest degrees of elements from W.

Note that the similar subspace plays important role for the case of Krichever correspondence for algebraic curves. For example, the "holes" between monomials of negative degrees are Weierstrass gaps, their quantity is the genus of the curve. Also in this case it is possible to reconstruct the cohomology groups of original sheaf on the curve from the support of a subspace arising from a geometrical data.

When we are interesting in k-subspaces from $k((t))((u))$ which are images of the Krichever correspondence for algebraic surfaces, it make sense to investigate the possible supports for such subspaces. But note at once that not all the properties will hold for the case of algebraic surfaces. So, from the support of the subspace we can not reconstruct the cohomology groupes of the original sheaf on a surface. (For example, we know not enough about H^1 from the support of the subspace.)

Now from explicit calculations above we have some examples of subspaces-supports which appear from the Krichever correspondence for algebraic surfaces.

Proposition 2 *Let the conditions be as in formula 1. Then the support of such W is the closed subspace generated by all monomials t^iu^j with $i \le -m^2j$ except the monomials $t^{-\alpha_1}u^{-m^2l}, \dots, t^{-\alpha_g}u^{-m^2l}$ where i, j and l are any integers, and natural $\alpha_1, \dots, \alpha_g$ are the Weierstrass gaps for the curve from our data.*

Proof follows by direct applying formula 1. And $\alpha_1, \dots, \alpha_g$ are numbers which can not be obtained by linear combinations of integers m and $m-1$ with positive coefficients (g is the genus of the curve).

3 Stabilizer ring of a k-subspace of a two-dimensional local field.

This section is one of the first steps to describe explicitly the images of the two-dimensional Krichever map, that is, to determine k-subspaces of $k((t))((u))$ which are in the image of the two-dimensional Krichever map.

Recall the case of the Krichever correspondence for the curves (see [2]). For any Fredholm k-subspace of $k((t))$, that is, for the k-subspace W such that $\dim W \cap \mathcal{O} < \infty$, and $\dim k((t))/(W + \mathcal{O}) < \infty$, where $\mathcal{O} = k[[t]]$ we consider the ring $A \stackrel{\mathrm{def}}{=} \{a \in k((t)) : a \cdot W \subset W\}$. Then A is finitely generated over k, and either $A = k$, or $trdeg A = 1$. In the last case W is in the image of the Krichever map, that is, the curve $C = \mathrm{Proj}(\bigoplus\limits_n A \cap t^{-n}\mathcal{O})$, the sheaf $\mathcal{F} = \mathrm{Proj}(\bigoplus\limits_n A \cap t^n\mathcal{O})$, the point p appears from the given filtration $A \cap t^n\mathcal{O}$ of the ring A, t is the formal local parameter at p, and $1 \in k((t))$ is the formal triviliazation of $\mathcal{F}$. Moreover, the original k-subspaces A and W appears as the images of the Krichever map of the construted data $(C, p, \mathcal{O}_C, 1, t)$ and $(C, p, \mathcal{F}, e_p, t)$ correspondingly. Note also that this description is connected with the description of commutative subrings in the ring of differential operators in one variable, that is "Schur pairs", see [2].

Consider now the case of a two-dimensional local field. Recall that a two-dimensional local field has, by definition, two valuations ν and $\bar{\nu}$, where ν is a discrete valuation of rank 1 on a two-dimensional local field and $\bar{\nu}$ is a discrete valuation of rank 1 on the residue field of a two-dimensional local field. Let $V = k((t))((u))$, and define $\mathcal{O}_1 = k((t))[[u]]$, $\mathcal{O}_2 = k[[t]]((u))$. For any integer n, for any k-subspace $W \subset V$ let $W(n) = (t^n\mathcal{O}_1 \cap W)/(t^{n+1}\mathcal{O}_1 \cap W)$.

Theorem 1 *Let W be a k-subspace of V such that for any integer n $W(n)$ is a Fredholm subspace in a one-dimensional local field, and $\chi(W(n)) = a + bn$, where $b < 0$. Let A be a k-subring of V such that $A \supset k$, $A \cdot W \subset W$. Then*

a) for any element $a \in A$ we have $\bar{\nu}(a) \le b\nu(a)$.

b) trdeg $(A \cap \mathcal{O}_2) \le 2$, and $A \cap \mathcal{O}_2$ is finitely generated over the ground field k.

Proof. a) Assume the converse. Then there exists an element $x \in A$ such that $\bar{\nu}(x) > b\nu(x)$. We have $x \cdot W \subset W$ and $x \cdot W(0) \subset W(m)$. It is easy to see that $\chi(x \cdot W(0)) = \chi(W(0)) + \bar{\nu}(x)$. Now we have

$$\chi(W(m)) = a + bm < a + \bar{\nu}(x) = \chi(W(0)) + \bar{\nu}(x) = \chi(x \cdot W(0)) \le \chi(W(m)),$$

a contradiction.

b) Instead of this case we will prove more general result:

Lemma 2 *Let B be a subring in a two dimensional local field $V = k((t))((u))$ such that $k \subset B$ and the following condition holds:*

for every $a \in B$ $0 \le \bar{\nu}(a) \le -\pi\nu(a)$, $\pi > 0$.

Then trdeg $B \le 2$ and B is finitely generated over the ground field k.

Proof. Consider the subspace $\triangle(N) = \{a \in B | \nu(a) > -N\}$, where integer $N > 0$. Note that this subspace has a finite dimension over the field k and this dimension is not greater than πN^2.

Indeed, if this dimension is greater than πN^2, then B must contain an element x such that $\bar{\nu}(x) > -\pi\nu(x)$, a contradiction.

Now suppose trdeg $B > 2$. Take any three algebraically independent elements $a, b, c \in B$. Consider the subspace with a basis $a^{n_1}b^{n_2}c^{n_3}$, where $n_1, n_2, n_3 < n$. Then this subspace lies in the subspace $\triangle(N)$, where $N = -n(\nu(a)+\nu(b)+\nu(c))$. So, the dimension must be less than $\pi N^2 = \pi(\nu(a)+\nu(b)+\nu(c))^2 n^2$. But the dimension of this subspace is equal to n^3. So, we get a contradiction for n sufficiently large.

Now note that trdeg $B < 2$ iff $\bar{\nu}(a) = l\nu(a)$ for some constant l and all $a \in B$. Indeed, if there are two elements a, b such that the last condition does not hold, then they must be algebraically independent, because the first monomials of such elements are algebraically independent. Converse, if $\bar{\nu}(a) = l\nu(a)$ for all $a \in B$, then the supports (the monomials of lowest degrees) of all elements in B lie on a line, and one can apply arguments from, say, [2], Prop. 3.2. to obtain trdeg $B < 2$.

Suppose now trdeg $B = 2$. Take two elements $a, b \in B$ such that $\bar{\nu}(a)/\nu(a) \ne \bar{\nu}(b)/\nu(b)$. This means that their supports don't belong to a line which goes through zero point on a coordinate plane. These elements exist because trdeg $B = 2$ and because of the arguments in the previous paragraph. So, they are algebraically independent. We can evaluate the dimension of a subspace generated over k by powers of a, b which lie in the subspace $\triangle(N)$ for some N. It's clear that this dimension is greater than $eN^2 + lN + r$ for some real e, l, r and for every sufficiently large N.

Now, suppose B has infinite dimension over the ring $k[a, b]$. Take M linearly independent over $k[a, b]$ elements $a_1, \dots a_M$, where M satisfies the condition $Me > \pi$. Without loss of generality we can assume that $\nu(a_1) \le \nu(a_i)$ for all i. Since the dimension of a subspace generated over k by powers of a, b which lie in the subspace $\triangle(N + M\nu(a_1))$ is greater than $e(N + M\nu(a_1))^2 + l(N + M\nu(a_1)) + r$, we obtain that the dimension of a subspace generated over k by elements $a_1, \dots a_M$ multiplied to powers of a, b which lie in the subspace $\triangle(N + M\nu(a_1))$ is greater than $M(e(N + M\nu(a_1))^2 + l(N + M\nu(a_1)) + r)$. The last subspace is inside $\triangle(N)$. And since $Me > \pi$, we have for N sufficiently large $M(e(N + M\nu(a_1))^2 + l(N + M\nu(a_1)) + r) > \pi N^2 > \dim_k \triangle(N)$, a contradiction.

The lemma is proved.

The theorem is proved.

□

Remark 1. As we have seen in the proof of the lemma, $\mathrm{trdeg}\, B = 1$ iff the supports of all elements in B lie on a one line which goes through zero point on a coordinate plane. If $\mathrm{trdeg}\, B = 0$ then $B = k$.

Remark 2. The assertion b) in theorem 1 is not true for the ring A as the following example shows.

Consider the subspace $W = \{a \in V | \bar{\nu}(a) \le -\nu(a)\}$. One can check that W satisfies the conditions from theorem and that $A = W$. Since A contains the subfield $k((ut^{-1}))$, A has infinite transcendental degree over k.

Remark 3. We take the ring $A \cap \mathcal{O}_2$ in theorem 1, because this ring is the image of the affine ring $H^0(X \setminus C, \mathcal{O}_X)$ when the pair (A, W) is the image of data $(X, C, p, \mathcal{O}, 1, t, u)$, $(X, C, p, \mathcal{F}, e_p, t, u)$ under the Krichever map. And we need exactly this ring for the reconstruction theorem; namely, $X = Proj(\bigoplus_n (A \cap \mathcal{O}_2) \cap t^n \mathcal{O}_2)$, the curve C appears from the filtration induced by $t^n \mathcal{O}_2$ on $A \cap \mathcal{O}_2$, and the point p appears from the one-dimensional Krichever correspondence on C, which is given by $W(0)$ (see [9]).

Also $W(n)$ is the image of the sheaf $\mathcal{F} \mid_C \otimes N_{C/X}^{\otimes -n}$ under the Krichever map on the curve C. And by the Riemann-Roch theorem we have the Euler characteristik $\chi(N_{C/X}^{\otimes -n}) = 1 - g(C) + deg(\mathcal{F}) - n \, deg N_{C/X}$. And $N_{C/X} = \mathcal{O}(C \cdot C)$. And from ampleness of C we have $deg N_{C/X} > 0$. This explains the condition $\chi(W(n)) = a + bn$, $b < 0$ from theorem 1.

Remark 4. If a k-subspace W is the image of data $(X, C, p, \mathcal{F}, e_p, t, u)$ under the Krichever map, then we have

$$W \cap \mathcal{O}_1 \cap \mathcal{O}_2 = H^0(X, \mathcal{F})$$

$$\frac{W \cap (\mathcal{O}_1 + \mathcal{O}_2)}{W \cap \mathcal{O}_1 + W \cap \mathcal{O}_2} = H^1(X, \mathcal{F})$$

$$\frac{V}{W + \mathcal{O}_1 + \mathcal{O}_2} = H^2(X, \mathcal{F})$$

And we know that the dimensions of these subspaces are finite as dimensions of cohomology groups. It would be interesting to add these finite-dimensionality conditions to conditions of theorem 1, and to obtain new corollaries for answering if a k-subspace $W \subset V$ is in the image of geometrical data under the Krichever map.

4 On an embedding of a projective line to an infinite-dimensional affine space

In [3], p.13 was posted that there exists an embedding of the Universal Sato Grassmanian into an infinite-dimensional affine space. In this section we clarify if this embedding is algebraic or not. Namely, we consider an explicite example where this embedding is applied to a projective line in the Universal Sato Grassmanian.

Recall some known facts about the Sato grassmanian and the map which plays a role of the embedding in [3].

Let $V = k((z))$ be a field of Laurent power series with filtration $V(n) = z^n k[[z]]$. Let $V_1 = V(0)$.

Denote by $Gr(V)$ the set of the subspaces W in V such that the complex

$$W \oplus V_1 \to V$$

is a Fredholm one. It is a (infinite-dimensional) projective variety with connected components marked by the Euler characteristik of the complex. We will work with "zero" component $Gr_0(V)$ of the grassmanian.

Let us introduce the skew field $P = k((x))((\partial^{-1}))$ of formal pseudo-differential operators with coeffitients from the field $k((x))$ as the left $k((x))$-module of all expressions $L = \sum_{i>\infty}^{n} a_i \partial^i$, $a_i \in k((x))$ with a multiplication defined according to the Leibnitz rule.

In the same way one can define the ring $E = k[[x]]((\partial^{-1}))$, $E \subset P$. It can be checked that P and E are associative rings (the details see in [8]).

There is a decomposition

$$E = E_+ + E_-,$$

where $E_- = \{L \in E : L = \sum_{n<0} a_n \partial^n\}$ and E_+ consists of the operators containing only ≥ 0 powers of ∂. The elements from $E_+ =: D$ are the differential operators and the elements from E_- are the Volterra operators.

The map $E \to E/Ex = V$ (we identify the image of ∂^{-1} with z) defines a linear action of the ring E on V and also on $Gr(V)$. The map $E \to V$ is called the Sato map.

Let's introduce the notion of a standart subspace:

Definition 1 *Consider a subset S in $\mathbb{Z}$, $S := \{\sigma(-1), \sigma(-2), \ldots\}$, where $\sigma(-i) \in \mathbb{Z}$, $\sigma(-i) = -i$ for $i \gg 0$.*

The subspaces $V^S := \bigoplus_{k\in\mathbb{N}} k \cdot z^{\sigma(-k)} \subset V$, $V^S \in Gr_0(V)$ are called the standart subspaces.

We will denote the subspace V^{S_0}, where $S_0 = \{-1, -2, \ldots\}$, by W_0.

In [13] the following lemma was proved.

Lemma 3 *There exists a unique operator R such that $RV^S = W_0$. Namely,*

$$R = \partial^{-m}(x\partial - (\sigma(-1)+1))(x\partial - (\sigma(-2)+2)) \ldots (x\partial - (\sigma(-m)+m))$$

where m is a maximal number such that $\sigma(-m) - m \neq 0$.

In [12] a notion of quasiregular operator was introduced.

Definition 2 *Let $\mathcal{E}_K^{(0)}$, $K = k((x))$, be a group of monic operators of degree zero from the ring P.*

The operator $W \in \mathcal{E}_K^{(0)}$ is called quasiregular if there exists $m, n \in \mathbb{N}$ such that $x^m W$ and $W^{-1} x^n$ both belong to E.

Denote the set of all quasiregular operators by $\mathcal{R}$.

Now we can introduce the map from $\mathcal{R}$ to $Gr_0(V)$. Put

$$\gamma(W) = (W^{-1}x^n)W_0, \quad W \in \mathcal{R}$$

Since $x \cdot W_0 = W_0$, this definition is correct, i.e., it does not depend on n. It's clear that $\gamma(W) \in Gr_0(V)$.

In [13] the following theorem was proved.

Theorem 2 *The map $\gamma : \mathcal{R} \to Gr_0(V)$ is a bijection.*

In [3], p.13 was noted that $\mathcal{E}_K^{(0)}$ is an infinite dimensional affine space, so $\mathcal{R}$ is embedded into this space. It was noted that due to the theorem there is an embedding of the universal Sato grassmanian into this affine subspace. To check this assertion we give below an explicite computation of this embedding for a projective line which lies in the universal Sato grassmanian.

Consider the set of Fredholm subspaces

$$R = \{W(\alpha,\beta) \subset V, W(\alpha,\beta) = \bigoplus_{l=1}^{\infty} k \cdot z^{-l} \bigoplus k \cdot (\alpha + \beta z)\},$$

where $\alpha, \beta \in k$. It's clear that R is a projective line in $Gr_0(V)$ with coordinates $(\alpha : \beta)$.

Consider the operators $\tilde{S}(\alpha,\beta) = \alpha + \beta x + \beta\partial^{-1}$. So, $\tilde{S}(\alpha,\beta) \in E$. Note that $\tilde{S}(\alpha,\beta)W_0 = W(\alpha,\beta)$ for any pair (α,β). If $\alpha \neq 0$, define the operator $S(\alpha,\beta) = \alpha^{-1}\tilde{S}(\alpha,\beta)(1+\beta/\alpha x)^{-1}$. Since $(1+\beta/\alpha x)W_0 = W_0$, we have $S(\alpha,\beta)W_0 = W(\alpha,\beta)$. Note that $S \in \mathcal{E}_K^{(0)} \cap E$. If $\alpha = 0$, we can represent the operator $\beta^{-1}\tilde{S}(\alpha,\beta)$ in the form $H(\alpha,\beta)^{-1}x$, where $H(\alpha,\beta) \in \mathcal{E}_K^{(0)}$, $H(\alpha,\beta) = (1+\partial^{-1}x^{-1})^{-1} = 1 - x^{-1}\partial^{-1} + \ldots$. So, we get

$$\gamma^{-1}(W(\alpha,\beta)) = \begin{cases} S(\alpha,\beta)^{-1} = 1 - (\beta/\alpha - (\beta/\alpha)x + (\beta/\alpha)^2x^2 + \ldots)\partial^{-1} + \ldots & \text{if} \quad \alpha \neq 0 \\ H(\alpha,\beta) = 1 - x^{-1}\partial^{-1} + \ldots \quad \text{if} \quad \alpha = 0 \end{cases}$$

Now let's consider two coordinate functions. The affine space $\mathcal{E}_K^{(0)}$ has coordinates a_{ij}, where a_{ij} is a coefficient of a Laurent expansion $1 + \sum_{i,j} a_{ij}u^i\partial^{-j}$. For our projective line we have the following coordinates by the embedding:

$$a_{-1,1} = \begin{cases} 0 & \text{if} \quad \alpha \neq 0 \\ -1 & \text{if} \quad \alpha = 0 \end{cases}$$

$$a_{0,1} = \begin{cases} -\beta/\alpha & \text{if} \quad \alpha \neq 0 \\ 0 & \text{if} \quad \alpha = 0 \end{cases}$$

These functions are not continuous. This shows that the embedding of the universal Sato grassmanian into the affine space is only a set-theoretic map.

5 New equations of KP-type on skew fields

In this section we give an answer on a question given in [8]. Namely, the classical KP-hierarchy is constructed by means of the ring of pseudo-differential operators $P = k((x))((\partial^{-1}))$. This ring is a skew field. The point is to consider other skew fields instead of this one. We will study if there exist some new non-trivial generalizations of the KP-hierarchy for a list of two-dimensional skew fields. In particular, we give a number of new partial differential equations of the KP-type.

Recall that in [8] A.N.Parshin pointed out a class of non-commutative local fields and showed that these skew fields possess many features of commutative fields. He defined a skew field of formal pseudo-differential operators in n variables and studied some of their properties. He raised a problem of classifying non-commutative local skew fields. In [14] this problem was solved for $n = 2$. The author got a list of skew fields, which contain also a classical ring of pseudodifferential operators P mentioned in the previous part of this paper. The following theorem was proved in [14]:

Theorem 3 *(I) Let K be a two-dimensional local skew field with a commutative residue skew field.*

It splits if the canonical automorphism α satisfy the condition $\alpha^n \neq Id$ for all n. If this condition does not hold, there are examples of non-spliting skew fields.

(II) Let K, K' be skew fields as in (I). Assume $\alpha^n \neq Id$, $\alpha'^n \neq Id$ for all n. Then
(a) K is isomorphic to a two-dimensional local skew field $\bar{K}((z))$ where $za = a^{\alpha}z$, $a \in \bar{K}$ and $\bar{K}$ is a one-dimensional local field with the residue field k.

(b) K and K' are isomorphic iff $k \cong k'$ and there exists an isomorphism $f : \bar{K} \mapsto \bar{K}'$ such that $\alpha = f^{-1}\alpha' f$.

(c) If $char K = char k$, $char K' = char k'$ and k, k' are algebraically closed fields of characteristik 0, then K is isomorphic to K' iff $k \cong k'$ and $(a_1, i_{\alpha}, y(\alpha)) = (a'_1, i_{\alpha'}, y(\alpha'))$.

(III) Let K, K' be two-dimensional splitting local skew fields of characteristik 0, $k \subset Z(K)$, $k' \subset Z(K')$, and $\alpha^n = Id$, $\alpha'^{n'} = Id$ for some natural $n, n' \geq 1$. Then (a) K is isomorphic to a two-dimensional local skew field $k((u))((z))$ where

$$zuz^{-1} = \xi u + u^{\delta'_{i_n}} z^{i_n} + u^{\delta'_{2i_n}} z^{2i_n},$$

where $\xi^n = 1$, $i_n = i_n(0, \ldots, 0)$,
$\delta'_{i_n}(u) = cu^{r_n}$, $c \in k^*/(k^*)^e$, $e = (r_n - 1, i)$,
$\delta'_{2i_n}(u) = (a_n(0, \ldots, 0) + r_n(i_n + 1)/2)u^{-1}(\delta'_{i_n}(u))^2$
(i_n, r_n, a_n were defined in [14]).
If $n = 1$, $i_n = \infty$, then K is commutative.

(b) K is isomorphic to K' iff $k \cong k'$ and the sets $(n, \xi, i_n, r_n, c, a_n)$, $(n', \xi', i'_n, r'_n, c', a'_n)$ coinside.

Corollary 1 *Every two-dimensional local skew field K with the ordered set*

$$(n,\xi,i_n,r_n,c,a_n)$$

is a finite-dimensional extension of a skew field with the ordered set $(1,1,1,0,1,a)$.

For every skew field from this list we can write down a decomposition $K=K_++K_-$, where $K_-=\{L\in K: ord(L)<0\}$ and K_+ consists of the operators containing only ≥ 0 powers of z, and a "KP-hierarchy" in the Lax form:

$$\frac{\partial L}{\partial t_n}=[(L^n)_+,L],$$

where $L\in z^{-1}+K_-\otimes k[[\dots,t_m,\dots]]$. Let $L=z^{-1}+u_1z+u_2z^2+\dots$, where $u_m=u_m(u,t_1,t_2,\dots)$. Further we will denote $\partial/\partial t_n$ as ∂_n.

One can check that if the canonical automorphism α in the classification theorem 3 is not trivial, then our "KP-hierarchy" becames trivial, i.e. it can be easily linearized and solvable. We omit calculations here. So, it can be assumed that $\alpha=id$. The same is true if $i>1$, because $[(L^n)_+,L]=-[(L^n)_-,L]=0 \mod \wp^i$ in this case, where $\wp$ is a maximal ideal of the first valuation in K. So, our "KP-hierarchy" again is linear and easily solvable in this case.

So, we assume $i=1$, hence, $r=0$ and $c=1$, and there is only only one non-trivial parameter a. If $a=0$, K is isomorphic to the ring P of pseudo-differential operators. Denote by $u',u'',\dots$ the subsequent derivatives by x.

First for $n=1$, we get

$$\partial_1u_1=u_1'$$

This means that we can take $t_1=x$ for u_1.

Now we write down the first two equations for $n=2$ and the first equation for $n=3$.

$$\partial_2u_1={u''}_1+2u_2' \tag{2}$$

$$\partial_2u_2=2u_3'+2u_1u_1'+{u''}_2+2ax^{-1}u_2' \tag{3}$$

$$\partial_3u_1={u'''}_1+3{u''}_2+3u_3'+6u_1u_1'+3a(x^{-1}{u''}_1-x^{-2}u_1') \tag{4}$$

Let us introduce the new notation: $u=u_1(x,y,t)$ with $y=t_2$, $t=t_3$. Also we use the standart notation $u_t,u_y,u_{yy},\dots$ for derivatives.

We can eliminate u_3' from equations 3 and 4 and then we get

$$3u_{2y}-2u_t=-6uu'-3{u''}_2-2u'''+6ax^{-1}u_2'-6ax^{-1}u''+6ax^{-2}u' \tag{5}$$

From 2 we find

$${u'''}_2=1/2({u''}_y-u'''),u_{2y}'=1/2(u_{yy}-{u''}_y)$$

Differentiating equation 5 by x and inserting these expressions we finally get new KP-equation

$$(4u_t-u'''-12uu')'=3u_{yy}+6a(5x^{-2}u''-x^{-2}u_y-3x^{-1}u'''+x^{-1}u_y'-4x^{-3}u')$$

One can see that if $a=0$, we get the usual KP-equation (see also explicite calculations in [10].

References

[1] Hüber A., *On the Parshin-Beilinson Adeles for Schemes*, Abh. Math. Sem. Univ. Hamburg, **61** (1991), 249-273.

[2] Mulase M., *Category of vector bundles on algebraic curves and infinite dimensional Grassmanians*, Int. J. Math., **1** (1990), 293-342.

[3] Mulase M., *Solvability of the super KP equation and a generalization of the Birkhoff decomposition*, Inv. Math., Vol. 92, Fasc. 1, 1988, 1-47

[4] Osipov D.V., *Stability of torsion free sheaves on curves and infinite-dimensional grassmanian manifold*, preprint IC/99/39 of the Abdus Salam International Centre for Theoretical Physics (ICTP), (April 1999, Miramare-Trieste, Italy), 13 p; e-print math.AG/9904152

[5] Osipov D.V., *The Krichever map and (semi)stable sheaves on curves*, Uspehi Matematicheskih Nauk, 54 (1999), n.3, p.177-178

[6] Osipov D. V., *Krichever correspondence for algebraic varieties*, Izvestija RAN: Ser. Mat., Izvestiya RAN: Ser. Mat., n.5, vol.65, 2001, p.91-128, e-print math.AG/0003188

[7] Parshin A. N., *On the arithmetic of two-dimensional schemes, I. Repartitions and residues*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 4, 736-773; English transl. in Math. USSR Izv. **10** (1976), no. 4, 695-729.

[8] Parshin A. N., *On a ring of formal pseudo-differential operators*, Proc. Steklov Math. Institute, **224** (1999), 266-280, e-print math.AG/9911098

[9] Parshin A. N., *Krichever correspondence for algebraic surfaces*, Funct. Anal., v.35 (2001), n.1, pp. 88-90, e-print alg-geom/9911098.

[10] Parshin A. N., *Integrable systems and local fields*, Communic. in Algebra, 2001 (to appear)

[11] Fimmel T., Parshin A.N., *An introduction to the higher adelic theory*, preprint

[12] Sato M., Sato Y., *Soliton Equations as Dynamical Systems on Infinite Dimensional Grassmann Manifold*, Lecture Notes in Num. Appl. Anal., **5** (1982), 259-271.

[13] Sato M. and Noumi M., *Soliton equations and universal Grassmann manifold* (in Japanese), Sophia Univ. Lec. Notes Ser. in Math. 18 (1984), pp. 74-75

[14] Zheglov A.B., *On the structure of two-dimensional local skew fields*, Izvestiya RAN: Ser. Mat., 1, 2001, 25-60